ANALYTICITY OF THE SRB MEASURE FOR HOLOMORPHIC FAMILIES OF QUADRATIC-LIKE COLLET-ECKMANN MAPS

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ABSTRACT. We show that if f_t is a holomorphic family of quadratic-like maps with all periodic orbits repelling so that for each real t the map f_t is a real Collet-Eckmann S-unimodal map then, writing μ_t for the unique absolutely continuous invariant probability measure of f_t, the map

$$t \mapsto \int \psi \, d\mu_t$$

is real analytic for any real analytic function ψ.

1. Introduction and statement of the theorem

If $t \mapsto f_t$ is a smooth one-parameter family of dynamics f_t so that f_0 admits a unique SRB measure μ_0, it is natural to ask whether the map $t \mapsto \mu_t$, where t ranges over a set Λ of parameters such that f_t has (at least) one SRB measure μ_t, is differentiable at 0. Differentiability should be understood in the sense of Whitney if Λ does not contain a neighbourhood of 0, as suggested by Ruelle [16]. Katok, Knieper, Pollicott, and Weiss [7] gave a positive answer to this differentiability question in the setting of C^3 families of transitive Anosov flows, showing that $t \mapsto \int \psi \, d\mu_t$ is differentiable, for all smooth ψ. If f_0 is a C^3 mixing Axiom A attractor and the family $t \mapsto f_t$ is C^3, Ruelle [15] not only proved that $t \mapsto \int \psi \, d\mu_t$ is differentiable, but also gave an explicit formula, the *linear response formula,* for the derivative. Of course, in the Anosov and Axiom A cases, Λ is a neighbourhood of 0.

Ruelle [16] suggested that this linear response formula, appropriately interpreted, should hold in much greater generality. Indeed, Dolgopyat [6] obtained the linear response formula for a class of partially hyperbolic diffeomorphisms. In a previous work [3, 4], we found that in the setting of piecewise expanding unimodal interval maps, the SRB measure is differentiable if and only if the path f_t is tangent to the topological class of f_0, that is, if and only if $\partial_t f_t|_{t=0}$ is horizontal. We emphasize that this setting is not structurally stable. When differentiability holds, Ruelle's candidate for the derivative, as interpreted in [2], gives the linear response formula. (We refer to [2, 3, 4], which also contain conjectures about smooth, not necessarily analytic, Collet–Eckmann maps, for more information and additional references.) Then, Ruelle [17] proved the linear response formula for a class of nonrecurrent

Date: May 28, 2008.

1991 *Mathematics Subject Classification.* 37C40 37C30 37D25 37E05.

V.B. is partially supported by ANR-05-JCJC-0107-01. D.S. is partially supported by CNPq 470957/2006-9 and 310964/2006-7, FAPESP 2003/03107-9. D. S. thanks the DMA of École Normale Supérieure for hospitality during a visit where a crucial part of this work was done. V.B. wrote part of this paper while visiting the Universidad Católica del Norte, Antofagasta, Chile, whose hospitality is gratefully acnowledged. We thank D. Sands for very helpful comments.

analytic unimodal interval maps f_t, assuming that all f_t stay in the topological class of f_0. (Recall that f_t is nonrecurrent if $\inf_k d(f_t^k(c), c) > 0$, where c denotes the critical point.)

In the present work, we consider holomorphic families f_t of quadratic-like holomorphic Collet–Eckmann maps. By holomorphic, we mean complex analytic. Our assumptions imply, using classical holomorphic motions, that all f_t lie in the same conjugacy class. Generalising one of the arguments in [7], we are able to show that $t \mapsto \int \psi \, d\mu_t$ is real analytic for any real analytic function ψ, our main result.

Let us now state our result more precisely. Let $I = [-1, 1]$. A C^3 map $f : I \to I$ is an *S-unimodal* map if it has $c = 0$ as unique critical point, and f has nonpositive Schwarzian derivative, that is $\frac{f'''}{f'} - \frac{3}{2}\big(\frac{f''}{f'}\big)^2 \le 0$ except at c. An S-unimodal map is called Collet-Eckmann if there exist $C > 0$ and $\lambda_c > 1$ so that $|(f^n)'(f(c))| \ge C\lambda_c^n$ for all $n \ge 1$. In this paper, we shall only consider S-unimodal maps with $f''(c) \ne 0$.

In Section 2 we shall define precisely the notion of a holomorphic (complex analytic) family of *quadratic-like* maps *in a neighbourhood of I* and what *all periodic orbits repelling* means for such maps, and prove the main result of this work:

Theorem 1.1. *Let $t \mapsto f_t$ be a holomorphic family of quadratic-like maps in a neighbourhood of I, with all periodic orbits repelling. Assume in addition that for each small real t the map f_t restricted to I is a (real) Collet-Eckmann S-unimodal map. Then there exists $\epsilon > 0$ so that for each real analytic $\psi : I \to \mathbb{C}$, the map*

$$t \mapsto \int \psi \rho_t \, dx ,$$

where ρ_t is the invariant density of f_t, is real analytic on $(-\epsilon, \epsilon)$.

The quadratic-like assumption implies that $f_t''(c) < 0$. The fact that periodic orbits are repelling implies that f_t is topologically conjugated with f_0: See our use of Mañé-Sad-Sullivan [10] in the beginning of the proof of the theorem in Section 2. Besides Mañé-Sad-Sullivan [10] the other main ingredient of our proof are the results and constructions of Keller and Nowicki [8] which allow us to exploit dynamical zeta functions, following the argument in the work of Katok–Knieper–Pollicott–Weiss [7, First proof of Theorem 1].

The extension from quadratic-like to polynomial-like is straightforward, and we stick to the nondegenerate case $f''(c) \ne 0$ for the sake of simplicity of exposition. As the proof uses only real-analyticity of the holomorphic motions $t \mapsto h_t$, it is conceivable that the conclusion of the theorem holds if f_t is a real analytic family of quadratic-like maps, using ideas of [1], but this generalisation appears to be nontrivial.

Lyubich's work [9] implies that there are *many* nontrivial families f_t satisfying the assumptions of our theorem. Constructing *examples* of such families f_t is in fact easier, and we sketch the procedure next: Start from two topologically conjugated Collet-Eckmann quadratic-like maps f and g which are not differentiably conjugated. By the result of Przytycki and Rohde [14], they are quasi-conformally conjugated. Hence, using a Beltrami path, one can construct a complex analytic family f_t of Collet-Eckmann maps containing both f and g, with $f_0 = f$, say. If f and g are real and conjugated in the real line, one can ensure that f_t is real for real parameters t. If f has negative Schwartzian derivative, then f_t has negative Schwartzian derivative for t close to 0. See also [5] for a specific example, and numerical experiments.

2. Proof of the Theorem

Before we prove the theorem, let us define precisely the objects we are studying:

Definition. We say that f_t is a *holomorphic family of quadratic-like maps in a neighbourhood of* I if there exists a complex neigbourhood U of I so that $t \mapsto f_t$ is a holomorphic map from a complex neighbourhood of zero to the Banach space $B(U)$ of holomorphic functions on U extending continously to $\overline{U}$, with the supremum norm, such that:

- For real t, the map f_t is real on $U \cap \mathbb{R}$, with $f_t(I) \subset I$ and $f_t(-1) = f_t(1) = -1$.
- There exist simply connected complex domains W and V, whose boundaries are analytic Jordan curves, with $I \subset V$, $\overline{V} \subset U$, $\overline{V} \subset W$, and so that $f_0 : V \mapsto W$ is a degree two ramified covering, with $c = 0$ as a unique critical point. That is, $f_0 : V \mapsto W$ is a quadratic-like restriction of f_0.

If f_t is a holomorphic family of quadratic-like maps in a neighbourhood of I then it is easy to see that for small complex t, denoting by V_t the connected component of $f_t^{-1}(W)$ containing 0, then $f_t : V_t \mapsto W$ is a quadratic-like restriction of f_t: Indeed, ∂W is an analytic Jordan curve, and f_0 has no critical point on ∂V. If $f_t \in B(U)$ is close to f_0, there is a simply connected domain V_t close to V such that $f_t(V_t) = W$, and the boundary of ∂V_t is a Jordan curve, by the implicit function theorem. Then $f_t : V_t \to W$ is a quadratic-like extension. We may then give another definition:

Definition. We say that f_t is a holomorphic family of quadratic-like maps in a neighbourhood of I *with all periodic orbits repelling,* if f_t is a holomorphic family of quadratic-like maps in a neighbourhood of I so that, for each small complex t, the map f_t only has repelling periodic orbits in V_t.

Proof. Since we assumed that all periodic points of f_t are repelling, [10, Theorem B] implies that there exists a holomorphic motion of the Julia set $K(f_0)$ of f_0, that is, a map $h : D \times K(f_0) \to C$ where $D = \{z \in \mathbb{C} \mid |z| < \epsilon_0\}$ for some $\epsilon_0 > 0$, such that for each $x \in K(f_0)$ the map $t \mapsto h_t(x)$ is holomorphic, and for every $t \in D$ the function $x \mapsto h_t(x)$ is continuous and injective on $K(f_0)$, with

$$h_t \circ f_0 = f_t \circ h_t \, .$$

In particular, h_t is a homeomorphism from $K(f_0)$ to $K(f_t)$. Note that [10, Theorem B] is quoted for polynomial maps, but the proof immediately extends to polynomial-like. Our assumptions imply that $[f_0^2(0), f_0(0)] = K(f_0) \cap \mathbb{R}$ and $h_t(K(f_0) \cap \mathbb{R}) = K(f_t) \cap \mathbb{R} = [f_t^2(0), f_t(0)]$. From now on, we only use real analyticity of $t \mapsto f_t(x)$ and $t \mapsto h_t(x)$ for $x \in [f^2(0), f(0)]$.

We next claim that our assumptions guarantee that each f_t satisfies the technical requirement needed by Keller and Nowicki [8, (1.2)]. Denoting by $\mathrm{var}_J \phi$ the total variation of a function ϕ on an interval J, and writing $f = f_t$, we need to check that there is that a constant $M > 0$ such that:

a. $M^{-1} < \sup_I \frac{|x-c|}{|f'(x)|} + \mathrm{var}_I \frac{|x-c|}{|f'(x)|} < M$,

b. $\mathrm{var}_{J_u} \frac{|f(x)-f(u)|}{|x-u||f'(x)|} < M$ where $J_u = [-1, u]$ if $u < c$ and $= [u, 1]$ if $u > c$.

Let $\delta_1 > 0$ be so that $|f''(y)| > |f''(c)|/2$ if $|y - c| < \delta_1$. It suffices to prove (a.) and (b.) for $|x - c| < \delta_1$ and $|u - c| < \delta_1$, and we restrict to such points. Noting that for every such $x \neq c$ there exist y_x, z_x, and $\tilde{z}_x$, between x and c, so that

$$\frac{|x-c|}{|f'(x)|} = -\frac{x-c}{f'(x)-f'(c)} = -\frac{1}{f''(y_x)}\,,$$

and, using $f''(x) = f''(c) + f^{(3)}(z_x)(x-c)$ and $f'(x) = f''(c)(x-c) + f^{(3)}(\tilde{z}_x)\frac{(x-c)^2}{2}$,

$$\partial_x \frac{|x-c|}{|f'(x)|} = \frac{-f'(x)+(x-c)f''(x)}{(f'(x))^2} = \frac{(x-c)^2}{(f'(x))^2}\big(f^{(3)}(z_x) - \frac{f^{(3)}(\tilde{z}_x)}{2}\big)\,,$$

the first two conditions hold because f is C^3. For the third condition, consider $x \geq u > c$ (the other case is symmetric). Since

$$\frac{f(x)-f(u)}{(x-u)f'(x)} = 1 + \frac{x-u}{f'(x)}\frac{f''(z_x)}{2} = 1 + \frac{x-u}{f'(x)}\frac{f''(z_x)}{2f''(y_x)}\,,$$

and $0 < -\frac{x-u}{f'(x)} < -\frac{x-c}{f'(x)}$, we get that $\big|\frac{f(x)-f(u)}{(x-u)f'(x)}\big|$ is bounded on $[u,1]$, uniformly in u. Finally, since

$$\partial_x \frac{x-u}{f'(x)} = \frac{f'(x)-(x-u)f''(x)}{(f'(x))^2}\,,$$

analyticity of f implies that $\partial_x \frac{x-u}{f'(x)}$ changes signs finitely many times, uniformly in u, proving (b.).

Also, the results of Nowicki–Sands [13] and Nowicki–Przytycki [12] ensure (see Appendix A) that there exist $\lambda_c > 1$, $\lambda_{per} > 1$, $\lambda_\eta > 1$, and $\epsilon_1 > 0$ so that, for each $|t| < \epsilon_1$, there is $C_t > 0$ with

$$|(f_t^n)'(f_t(0))| \geq C_t \lambda_c^n\,, \forall n \geq 1\,, \tag{1}$$

and so that for each $x \in I$ so that $f_t^p(x) = x$ for some $p \geq 1$, we have

$$|(f_t^p)'(x)| \geq C_t \lambda_{per}^p\,, \tag{2}$$

and, finally, setting

$$\lambda_\eta(t) := \liminf_{n\to\infty} \{|\eta|^{-1/n} \mid \eta \subset I \text{ is the biggest monotonicity interval of } f_t^n\}\,,$$

$$\inf_{|t|<\epsilon_1} \lambda_\eta(t) > \lambda_\eta\,. \tag{3}$$

In other words, the hyperbolicity constants are uniform in t, guaranteeing uniformity when applying the results of Keller and Nowicki [8]. (We choose $\epsilon_1 < \epsilon_0$.)

We now adapt the strategy used in the first proof of [7, Theorem 1]. Fix ψ and, for $x \in I$ so that $f_0^p(x) = x$ for $p \geq 1$, and for small real s and t, consider

$$g_{s,t}(x) = \frac{e^{s\psi(h_t(x))}}{|f_t'(h_t(x))|}\,. \tag{4}$$

Since ψ is real analytic, the analyticity of $t \mapsto h_t$ and of $t \mapsto f_t$ together with (2) imply that there is $\epsilon_2 > 0$ so that, for every periodic point $x \in I$ of period $p \geq 1$ for f, the function

$$(t,s) \mapsto g_{s,t}^{(p)}(x) := \frac{e^{s\sum_{k=0}^{p-1}\psi(h_t(f^k(x))}}{|(f_t^p)'(h_t(x))|}$$

is real analytic in $|s| < \epsilon_2$ and $|t| < \epsilon_2$, uniformly in x. We take $\epsilon_2 < \epsilon_1$.

Therefore, the dynamical zeta function defined by

$$\zeta(s,t,z) := \exp \sum_{p=1}^{\infty} \frac{z^p}{p} \sum_{x\in I:\, f_0^p(x)=x} g_{s,t}^{(p)}(x) \tag{5}$$

has the following property: There exists $\delta_2 > 0$ so that for each $|z| < \delta_2$ the function $\zeta(s,t,z)$ is real analytic in $|t| < \epsilon_2$, $|s| < \epsilon_2$, and so that for each (s,t) with $|t| < \epsilon_2$, $|s| < \epsilon_2$ the map $\zeta(s,t,z)$ is holomorphic and nonvanishing in $|z| < \delta_2$.

Now, $h_t \circ f_0 = f_t \circ h_t$ immediately implies

$$\zeta(s,t,z) = \exp \sum_{p=1}^{\infty} \frac{z^p}{p} \sum_{y\in I:\, f_t^p(y)=y} \frac{e^{s\sum_{k=0}^{p-1}\psi(f_t^k(y))}}{|(f_t^p)'(y)|} \, . \tag{6}$$

Before we proceed, we warn the reader that our parameter s is called t in [8], the parameter β in [8] is $\beta = 1$, and our parameter t corresponds to changing the dynamics.

Recall (1, 2, 3) and take $\Theta \in (0,1)$ with

$$\Theta^{-1} < \min\{\lambda_\eta, \sqrt{\min(\lambda_c, \lambda_{per})}\} \, .$$

Keller and Nowicki [8, Theorem 2.1] prove that, if $\epsilon_3 \in (0, \epsilon_2)$ is small enough, then for $|s| < \epsilon_3$ and $|t| < \epsilon_3$ the transfer operator

$$\mathcal{L}_{s,t}\varphi(x) = \sum_{\hat f_t(y)=x} \frac{\omega_t(y)}{\omega_t(x)} \frac{\exp(s\psi(y))}{|\hat f_t'(y)|} \varphi(y) \, ,$$

acting on functions of bounded variation on a suitable Hofbauer tower extension $\hat f_t : \hat I \to \hat I$ of f_t [8, Section 3], endowed with an appropriate [8, §6.2] cocycle ω_t, is a bounded operator. Note that the cocycle embodies the singularities along the postcritical orbit of f_t.

If $s = 0$ then the spectral radius $\lambda_{0,t}$ of $\mathcal{L}_{s,t}$ is equal to 1, it is a simple eigenvalue, whose eigenvector gives the invariant density ρ_t of f_t, and the rest of the spectrum is contained in a disc of strictly smaller radius. In addition, the essential spectral radius $\theta_{s,t}$ of $\mathcal{L}_{s,t}$ satisfies $\sup_{|t|<\epsilon_3, |s|<\epsilon_3} \theta_{s,t} < \Theta$, and for each $|t| < \epsilon_3$ the spectral radius $\lambda_{s,t} > \Theta$ of $\mathcal{L}_{s,t}$ is an analytic function [8, Prop. 4.2] of s.

Note that $\lambda_{s,t}$ is the exponential of the topological pressure of $s\psi - \log|f_t'|$ for f_t, and that $\rho_t\, dx$ is the equilibrium state for f_t and $-\log|f_t'|$. Now, perturbation theory gives (see [8, (5.2)])

$$\partial_s \log \lambda_{s,t}|_{s=0} = \int \psi \rho_t \, dx \, . \tag{7}$$

Keller and Nowicki also show [8, Theorem 2.2] that for $|t| < \epsilon_3$ and $|s| < \epsilon_3$ the power series $\zeta(s,t,z)$ defined by (6) extends meromorphically to the disc of radius Θ^{-1}, and its poles z_k in this disc are in bijection with the eigenvalues λ_k of $\mathcal{L}_{s,t}$, via $\lambda_k = z_k^{-1}$. In addition, the order of the pole coincides with the algebraic multiplicity of the eigenvalue. By [8, Prop. 4.3 and Lemma 4.5]) $\zeta(s,t,z)$ does not vanish in the disc of radius Θ^{-1}. It follows that $z \mapsto \zeta(s,t,z)^{-1}$ is holomorphic in the disc of radius Θ^{-1}. This disc contains $\lambda_{s,t}^{-1}$, which is a simple zero.

To end the proof, recalling (7), it suffices to see that $(s,t) \mapsto \lambda_{s,t}$ is real analytic, but this easily follows from Shiffman's [18] real analytic Hartogs' theorem (see Appendix B or [7, Thm p. 589]) applied to $d(s,t,z) = \zeta(s,t,z)^{-1}$, which implies that for each $(s,t) \in (-\epsilon_3, \epsilon_3) \times (-\epsilon_3, \epsilon_3)$ the map $z \mapsto d(s,t,z)$ is holomorphic in $|z| < \Theta^{-1}$. Indeed, by the implicit function theorem, the simple zeroes of $d(s,t,\cdot)$ depend real analytically on s and t.

We used the same ϵ_i discs for the s and t variables, but a more careful analysis shows that ϵ in the statement of the theorem may be selected independently of ψ. □

Appendix A. Uniformity of the hyperbolicity constants

Duncan Sands' explanations were instrumental towards writing this appendix, and we thank him for that.

We start with a preliminary observation: Let g be an S-unimodal Collet–Eckman map (with $g''(0) < 0$, say). Denote by $\lambda_c(g)$, $\lambda_{per}(g)$, and $\lambda_\eta(g)$ the constants defined by (1, 2, 3) (replacing f_t by g). Nowicki and Sands [13] proved that if g is an S-unimodal map and $\lambda_{per}(g) > 1$ then $\lambda_c(g) > 1$. A careful study of their proof shows that $\lambda_c(g) > \lambda_{per}(g)^\alpha$, where the exponent $\alpha > 0$ only depends on the maximum length $N(g)$ of "almost-parabolic funnels" of g (see [13, Lemma 6.6] for a definition of $N(g)$, which can be bounded by a function of $1/\log(\lambda_{per}(g))$ and $\sup |g'|$). Since $N(g)$ is in fact invariant under topological conjugacy and f_t is topologically conjugated to f_0, we conclude that $\lambda_c(f_t) > \lambda_{per}(f_t)^\alpha$, with $\alpha > 0$ uniform in small t.

Next, recall that Nowicki and Przytycki [12] proved that if g and $\tilde{g}$ are S-unimodal maps, with $g''(c) \neq 0$ and $\tilde{g}''(c) \neq 0$, say, conjugated by a homeomorphism of the interval and g is Collet–Eckmann, then $\tilde{g}$ is Collet–Eckmann. Take $g = f_0$ and $\tilde{g} = f_t$. In particular, f_t is C^2 close to f_0 and $t \mapsto h_t$ is smooth. Then it is not very difficult to see that the constants $M = M(f_t) > 0$, $P_4 = P_4(f_t) > 0$, and $\delta_4 = \delta_4(f_t) > 0$ from the topological characterisation ("finite criticality") of Collet–Eckmann in [12, (4) p. 35]) are uniform in small t.

Recall that our assumptions imply $f_t''(c) \neq 0$ for all small t, so that the constant denoted l_c in [12] is $l_c = 2$. Section 2 of [12], and in particular the use of the Koebe principle there, implies that there exists a (universal) function $q : \mathbb{R}_*^+ \times (0,1) \to (0,1)$ with $q(M, 1/4) < 1/2$ for any M (see [12, Lemma 2.2]), and so that $\lambda_{per}(f_t) > \big(1 - 2q(M(f_t), 1/4)\big)^{-1}$. Therefore, $\lambda_{per}(f_t) > 1$ is uniformly bounded away from 1 for small t. The preliminary observation then implies that $\lambda_c(f_t)$ is also uniformly bounded in t. By [11, Proposition 3.2] (see also [12, p. 35]), this implies a uniform lower bound for $\lambda_\eta(f_t)$. Indeed, in the notations of [11, §3], we have $\lambda_\eta = \lambda_5 = \lambda_4 \geq \lambda_3 = \lambda_1 \geq \sqrt{\lambda_c}$.

Appendix B. Shiffman's real analytic Hartogs' extension theorem

Theorem B.1. [18] *Let $\delta > 0$ and $0 < r < R$. Assume that*

$$d : (-\delta, \delta)^2 \times \{z \in \mathbb{C} \mid |z| < R\} \to \mathbb{C}$$

satisfies the following conditions:

- *For each $(s,t) \in (-\delta, \delta)^2$ the map $z \mapsto d(s,t,z)$ is holomorphic in $|z| < R$.*
- *For each $|z| < r$ the map $(s,t) \mapsto d(s,t,z)$ is real analytic in $(-\delta, \delta)^2$.*

Then $d(s,t,z)$ is real analytic on $(-\delta, \delta)^2 \times \{|z| < R\}$.

Note that the above theorem fails if real analyticity is replaced by C^k for $k \leq \infty$.

The theorem holds because $|z| < r$ is not pluripolar in $|z| < R$. Shiffman's result is based on deep work of Siciak [19]

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