Dynamical determinants and spectrum for hyperbolic diffeomorphisms

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Dedicated to Prof. Michael Brin on the occasion of his 60th birthday.

Abstract. For smooth hyperbolic dynamical systems and smooth weights, we relate Ruelle transfer operators with dynamical Fredholm determinants and dynamical zeta functions: First, we establish bounds for the essential spectral radii of the transfer operator on new spaces of anisotropic distributions, improving our previous results [**7**]. Then we give a new proof of Kitaev's [**17**] lower bound for the radius of convergence of the dynamical Fredholm determinant. In addition we show that the zeroes of the determinant in the corresponding disc are in bijection with the eigenvalues of the transfer operator on our spaces of anisotropic distributions, closing a question which remained open for a decade.

1. Introduction

1.1. Historical perspective. The spectral properties of transfer operators and their relations to analytic properties of dynamical Fredholm determinants and dynamical zeta functions are fascinating subjects in study of smooth dynamical systems. The basic idea about the relation is rather simple: The dynamical Fredholm determinant of a transfer operator $\mathcal{L}$ associated to a dynamical system T and a weight function g is formally defined by

$$d_{\mathcal{L}}(z) = \det(\mathrm{I}d - z\mathcal{L}) := \exp\left(- \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{T^m(x)=x} \frac{g^{(m)}(x)}{|\det(\mathrm{Id} - DT^m(x))|} \right).$$

Naturally, we would like that the inverse of each eigenvalue of the transfer operator $\mathcal{L}$ appears as a zero of the dynamical Fredholm determinant. To make mathematically rigorous statements, we first have to show that the transfer operator has nice spectral properties (similar to those of compact operators) on a suitable Banach space. Finding the right Banach space is thus one of the hurdles in this subject.

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Then, we have to give an interpretation of the sums over periodic points as (approximate) traces of transfer operators, recalling the formal relation

$$\det(\mathrm{Id} - zF) = \exp\left(-\sum_{m=0}^{\infty} \frac{z^m}{m} \mathrm{Tr} F^m\right) .$$

For analytic hyperbolic diffeomorphisms and weights, it has been known for a long time that $d_{\mathcal{L}}(z)$ is an entire function when the dynamical foliations are analytic: This is the content of the fundamental paper of Ruelle [**23**], who showed that the transfer operators are nuclear on a suitable space of functions using Grothendieck's theory of nuclear operators. More recently, Rugh [**25**] and Fried [**14**] studied $d_{\mathcal{L}}(z)$ in the hyperbolic analytic framework, but without any assumption on the foliations, giving a spectral interpretation of its zeroes (however, not quite as the eigenvalues of a natural transfer operator $\mathcal{L}$). In the case of finite differentiability r, the connection between transfer operators and dynamical determinants of *expanding* endomorphisms has been well understood by Ruelle (see [**24**]).

In a ground-breaking article [**17**] circulated as a preprint since 1995, Kitaev considered hyperbolic diffeomorphisms of finite differentiability C^r, and obtained a remarkable formula $\rho_r := \inf_{q<0<p, p-q<r-1} \rho^{p,q}(T,g)$ (see Section 1.2 for a definition of $\rho^{p,q}(T,g)$) as a lower bound for the radius of a disc in which $d_{\mathcal{L}}(z)$ admits a holomorphic extension. But Kitaev did not construct a Banach space and his approach does *not* give spectral results. Interpreting the zeroes of $d_{\mathcal{L}}(z)$ in the disc furnished by Kitaev as inverse eigenvalues of a transfer operator remained a challenging problem for over a decade.

The main contribution of the present paper is to close this problem (Theorems 1.1 and 1.5). Along the way, we give a new proof of Kitaev's result. In addition, we give a new variational-like interpretation of Kitaev's formula $\rho^{p,q}(T,g)$ as a kind of generalised topological pressure $Q^{p,q}(T,g)$ (Lemma 1.4).

Finding appropriate Banach spaces

The first reason why this problem remained open for so long is that there were until recently no good Banach spaces adapted to the transfer operators of hyperbolic dynamical systems in finite differentiability: For a long time, statistical properties of hyperbolic diffeomorphisms were investigated using symbolic dynamics via Markov partitions. Since the correspondence is not smoother than Hölder, the information thus obtained on the spectrum of transfer operator was severely limited, and this made it difficult to go beyond the results on dynamical zeta functions by Parry and Pollicott (see [**21**]). (See Section 2 for a discussion about dynamical zeta functions $\zeta_{T,g}(z)$.) Recently, in a pioneering work[**9**], Blank, Keller and Liverani introduced scales of Banach spaces of distributions on the manifold adapted to hyperbolic diffeomorphisms and proved that the transfer operators acting on those Banach spaces have a spectral gap. However, there were technical restrictions in the methods in [**9**], which did not allow them to go beyond Lipschitz smoothness. These restrictions were removed by Gouëzel and Liverani [**15**] and by the authors [**7**] independently, but using different kind of Banach spaces. The intuitive idea is the same for both kind of Banach spaces: They consist of distributions on the manifold, which are as smooth as C^p functions for some $p > 0$ in directions close to the unstable direction, and which are as rough as distributions of order $-q$ for some $q < 0$ in directions close to the stable direction. However the real construction of the Banach spaces in [**15**] and [**7**] are quite different. We refer to the original papers for details. (The reader-friendly survey [**8**] will be helpful to get ideas in the construction in [**7**].)

In our first main result (Theorem 1.1), we introduce yet another scale of Banach spaces, $\mathcal{C}^{p,q}$, which is a kind of hybrid of those in [**7**] and [**15**], and gives a better upper bound on the essential spectral radius. This upper bound coincides with Kitaev's formula $\rho^{p,q}(T,g)$ (Lemma 1.4). In view of the results [**16**] of Gundlach and Latushkin for expanding maps, we believe that our bound is optimal.

Introducing appropriate traces

The second difficulty to solve this problem in the case of hyperbolic C^r diffeomorphisms is to find an appropriate definition for the trace of transfer operators that are not even compact. Liverani [**19**] found a simple argument to relate eigenvalues of $\mathcal{L}$ with zeroes of the dynamical Fredholm determinant $d_{\mathcal{L}}(z)$, using the Banach spaces in [**15**]. More recently, Liverani and Tsujii [**20**] provided an abstract argument that is adaptable to both of the Banach spaces in [**15**] and [**7**] and that improves the result in [**19**]. Still, by technical reasons, the methods in [**19**] and [**20**] give the relation only on a strictly smaller disk (by the factor of one half, at least) than that given in Kitaev's [**17**] formula. Our second main result (Theorem 1.5) improves this point.

In this paper, we use the structure of our Banach spaces to define the trace. The basic idea in the construction of our Banach spaces is to view functions u on the manifold as superpositions of countably many parts u_γ, $\gamma \in \Gamma$, each of which is compactly supported in Fourier space (in charts). Accordingly we regard the transfer operator as a countable matrix of operators $\mathcal{L}_{\gamma\gamma'}$. Each operator $\mathcal{L}_{\gamma\gamma'}$ turns out to have a smooth kernel. Thus we may define the trace of $\mathcal{L}_{\gamma\gamma}$ as the integration of its kernel along the diagonal, and then the trace of the transfer operator $\mathcal{L}$ as the sum of the traces of the $\mathcal{L}_{\gamma\gamma}$. We found that hyperbolicity of the diffeomorphism ensures that this trace coincides with the expected sum over fixed points. Then, using the abstract notion of approximation numbers [**22**], we estimate the traces thus defined and get our second main result (Theorem 1.5). This implement the idea mentioned in the beginning for the case of C^r hyperbolic diffeomorphisms.

1.2. Main results. In the following, X denotes a d-dimensional C^∞ Riemann manifold and $T : X \to X$ is a diffeomorphism of class C^r for some $r > 1$. If r is not an integer, this means that the derivatives of T of order $[r]$ satisfy a Hölder condition of order $r - [r]$. Our standing assumption is that there exists a hyperbolic basic set $\Lambda \subset X$ for T, that is, a compact T-invariant subset that is hyperbolic, isolated and transitive. By definition there exist a compact isolating neighbourhood V such that $\Lambda = \cap_{m\in\mathbb{Z}} T^m(V)$ and an invariant decomposition $T_\Lambda X = E^u \oplus E^s$ of the tangent bundle over Λ, such that $\|DT^m|_{E^s}\| \le C\lambda^m$ and $\|DT^{-m}|_{E^u}\| \le C\lambda^m$, for all $m \ge 0$ and $x \in \Lambda$, with some constants $C > 0$ and $0 < \lambda < 1$. By transitivity, the dimensions of $E^u(x)$ and $E^s(x)$ are constant, which are denoted by d_u and d_s respectively. We will suppose that neither d_s nor d_u is zero.

For $s \ge 0$, let $C^s(V)$ be the set of complex-valued C^s functions on X with support contained in the interior of V. The Ruelle transfer operator with weight $g \in C^{r-1}(V)$ is defined by

$$\mathcal{L} = \mathcal{L}_{T,g} : C^{r-1}(V) \to C^{r-1}(V), \quad \mathcal{L}\varphi(x) = g(x) \cdot \varphi \circ T(x).$$

Our first theorem improves the results of [**7**] and [**15**] on the spectrum of $\mathcal{L}$. For a T-invariant Borel probability measure μ on Λ, we write $h_\mu(T)$ for the metric entropy of (μ, T), and $\chi_\mu(A) \in \mathbb{R} \cup \{-\infty\}$ for the largest Lyapunov exponent of a linear cocycle A over $T|_\Lambda$, with $(\log \|A\|)^+ \in L^1(d\mu)$. Let $\mathcal{M}(\Lambda, T)$ denote the set of T-invariant ergodic Borel probability measures on Λ. Then the theorem is stated as follows.

THEOREM 1.1. *For each real numbers $q < 0 < p$ so that $p - q < r - 1$, there exists a Banach space $\mathcal{C}^{p,q}(T, V)$ of distributions on V, containing $C^s(V)$ for any $s > p$, and contained in the dual space of $C^s(V)$ for any $s > |q|$, with the following property:*

For any $g \in C^{r-1}(V)$, the Ruelle operator $\mathcal{L}_{T,g}$ extends to a bounded operator on $\mathcal{C}^{p,q}(T, V)$ and the essential spectral radius of that extension is not larger than

$$Q^{p,q}(T, g) = \exp \sup_{\mu \in \mathcal{M}(\Lambda,T)} \Big\{ h_\mu(T) + \chi_\mu \left(\frac{g}{\det(DT|_{E^u})} \right) + \max\big\{ p\chi_\mu(DT|_{E^s}), |q|\chi_\mu(DT^{-1}|_{E^u}) \big\} \Big\} .$$

Note that, in the setting of C^r expanding endomorphisms, Gundlach and Latushkin [**11**, §8],[**16**] showed that the essential spectral radius of the transfer operator acting on $C^{r-1}(X)$ is given exactly by a variational expression analogous to $Q^{p,q}(T, g)$.

REMARK 1.2. By upper-semi-continuity of $\mu \mapsto h_\mu$, $\mu \mapsto \chi_\mu(A)$, the supremum in the expression for $Q^{p,q}(T, g)$ is a maximum. Also we have $\chi_\mu(g/\det(DT|_{E^u})) = \int \log|g|\, d\mu - \int \log|\det(DT|_{E^u})|\, d\mu$. This artificial expression as a Lyapunov exponent will make sense when we consider Ruelle operators on sections of vector bundles in the next section.

REMARK 1.3. Note that we have

$$Q^{p,q}(T, g) \le \lambda^{\min\{p,-q\}} \cdot Q^{0,0}(T, g) < Q^{0,0}(T, g) .$$

We shall see in Remark 1.6 that, if $g > 0$ on Λ, the spectral radius of $\mathcal{L}_{T,g}$ on $\mathcal{C}^{p,q}(T, V)$ coincides with $Q^{0,0}(T, g)$.

To compare the results in this paper with those in Kitaev's article [**17**], we next give an alternative expression for $Q^{p,q}(T, g)$. For $g \in C^0(V)$ and $m \ge 0$, we write

$$g^{(m)}(x) = \prod_{k=0}^{m-1} g(T^k(x)) .$$

We define local hyperbolicity exponents for $x \in \Lambda$ and $m \in \mathbb{Z}_+$ by[1]

$$\begin{aligned} \lambda_x(T^m) &= \sup\left\{ \frac{\|DT^m_x(v)\|}{\|v\|} \;\middle|\; DT^m_x(v) \in E^s(T^m(x)) \setminus \{0\} \right\} \le C\lambda^m, \\ \nu_x(T^m) &= \inf\left\{ \frac{\|DT^m_x(v)\|}{\|v\|} \;\middle|\; v \in E^u(x) \setminus \{0\} \right\} \ge C^{-1}\lambda^{-m}. \end{aligned} \tag{1.1}$$

For real numbers q and p, an integer $m \ge 1$ and $x \in \Lambda$, we set

$$\lambda^{(p,q,m)}(x) = \max\big\{ (\lambda_x(T^m))^p, (\nu_x(T^m))^q \big\} .$$

We may extend $E^s(x)$ and $E^u(x)$ to continuous bundles on V (which are not necessarily invariant), so that the inequalities in (1.1) hold for all $x \in \cap_{k=0}^{m-1} T^{-k}(V)$

[1]The definition of $\lambda_x(T^m)$ may look a bit strange. We need this formulation for the extension of $E^s(x)$ just below.

and for all $m \geq 0$, with some constant C. Taking such an extension[2], we extend the definition of $\lambda_x(T^m)$, $\nu_x(T^m)$, and $\lambda^{(p,q,m)}(x)$ to $\cap_{k=0}^{m-1} T^{-k}(V)$. Letting dx denote normalised Lebesgue measure on X, define for integers $m \geq 1$ and p, $q \in \mathbb{R}$

$$\rho^{p,q}(T,g,m) = \int_X |g^{(m)}(x)| \cdot \lambda^{(p,q,m)}(x)\, dx\,. \tag{1.2}$$

Kitaev[**17**] proved[3] that the limit

$$\rho^{p,q}(T,g) = \lim_{m\to\infty} (\rho^{p,q}(T,g,m))^{1/m}$$

exists for all $q \leq 0 \leq p$ in $\mathbb{R}$ and $g \in C^\delta(V)$ with $\delta > 0$. In Section 3, we show:

LEMMA 1.4. *For any $g \in C^\delta(V)$ with $\delta > 0$, we have $Q^{p,q}(T,g) = \rho^{p,q}(T,g)$ for all real numbers $q \leq 0 \leq p$.*

In [**7**] we proved a result similar to Theorem 1.1, with $\mathcal{C}^{p,q}(T,V)$ replaced by other spaces of anisotropic distributions $C_*^{p,q}(T,V)$, respectively $W_*^{p,q,t}(T,V)$ for $1 < t < \infty$, and with the bound $Q^{p,q}(T,g)$ replaced by $R^{p,q,\infty}(T,g)$, respectively $R^{p,q,t}(T,g)$, where

$$R^{p,q,t}(T,g) = \lim_{m\to\infty} \left(\sup_\Lambda |\det DT^m|^{-1/t}(x) \cdot |g^{(m)}(x)| \lambda^{(p,q,m)}(x) \right)^{1/m}\,.$$

Note that if $|\det DT| \leq 1$ then $\inf_{t\in[1,\infty]} R^{p,q,t}(T,g) = R^{p,q,\infty}(T,g)$. Since

$$\exp\left(\chi_\mu(g) + \max\{p\chi_\mu(DT|_{E^s}), |q|\chi_\mu(DT^{-1}|_{E^u})\} \right) \leq \lim_{m\to\infty} \left(\sup_\Lambda |g^{(m)}(x)| \lambda^{(p,q,m)}(x) \right)^{1/m}\,,$$

the variational principle tells that we have $Q^{p,q}(T,g) \leq R^{p,q,\infty}(T,g)$ in general and the equality holds only if the supremum in the definition of $Q^{p,q}(T,g)$ is attained by the SRB measure for T. Therefore Theorem 1.1 can be viewed as an improvement of our previous result [**7**]. In Appendix B we prove that, in general,

$$\rho^{p,q}(T,g) \leq \inf_{t\in[1,\infty]} R^{p,q,t}(T,g)\,, \tag{1.3}$$

where the inequality can be strict.

Another improvement on [**7**] is that we now have the same bounds for the essential spectral radii of the pull-back operator and the Perron-Frobenius operator, which are dual of each other: Take[4] $h \in C^\infty(V)$ so that $h \equiv 1$ on a neighbourhood of Λ, and consider the pull-back operator $\varphi \mapsto h \cdot \varphi \circ T$ on $B^{p,q}(T,V)$, and the Perron-Frobenius operator $\varphi \mapsto (h \cdot \varphi) \circ T^{-1} \cdot |\det(DT^{-1})|$ on $B^{-q,-p}(T^{-1},V)$. Exchanging the roles of E^s and E^u, the bounds in Theorem 1.1 for the essential spectral radii of these operators coincide:

$$Q^{p,q}(T,g) = Q^{-q,-p}(T^{-1}, g \cdot |\det(DT^{-1})|)\,.$$

[2] The choice of extensions is not essential.

[3] Kitaev used the notation $\rho^{p,-q}(\mathcal{L})$ for our $\rho^{p,q}(T,g)$.

[4] We need to multiply by h to localize functions to V. If T is Anosov, we may forget about h.

We next turn to dynamical Fredholm determinants. The dynamical Fredholm determinant $d_{\mathcal{L}}(z)$ corresponding to the Ruelle transfer operator $\mathcal{L} = \mathcal{L}_{T,g}$ is

$$d_{\mathcal{L}}(z) = \exp\left(- \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{T^m(x)=x} \frac{g^{(m)}(x)}{|\det(\mathrm{Id} - DT^m(x))|} \right) . \tag{1.4}$$

The power series in z which is exponentiated converges only if $|z|$ is sufficiently small. (See Remark 1.6.) Our main result is about the analytic continuation of $d_{\mathcal{L}}(z)$:

THEOREM 1.5. *Let $g \in C^{r-1}(V)$.*
(1) *The function $d_{\mathcal{L}}(z)$ extends holomorphically to the disc of radius $(Q_{r-1}(T,g))^{-1}$ with*

$$Q_{r-1}(T,g) = \inf_{q<0<p,\, p-q<r-1} Q^{p,q}(T,g) .$$

(2) *For any real numbers $q < 0 < p$ so that $p - q < r - 1$, and each z with $|z| < (Q^{p,q}(T,g))^{-1}$, we have $d_{\mathcal{L}}(z) = 0$ if and only if $1/z$ is an eigenvalue of $\mathcal{L}$ on $\mathcal{C}^{p,q}(T,V)$, and the order of the zero coincides with the algebraic multiplicity of the eigenvalue.*

REMARK 1.6. The sum over m in the right hand side of (1.4) converges when

$$|z| < \exp\Big(-P_{top}\big(T|_\Lambda, \log(|g|/|\det(DT|_{E^u})|)\big) \Big) = (Q^{0,0}(T,g))^{-1} ,$$

so that $d_{\mathcal{L}}(z)$ is a nowhere vanishing holomorphic function in this disc. To see this, note that there is $C \ge 1$ so that for all m and all $x \in \Lambda$ with $T^m(x) = x$

$$C^{-1} \le \frac{|\det(\mathrm{Id} - DT^m(x))|}{|\det(DT^m|_{E^u})(x)|} \le C ,$$

then use the Cauchy criterion for the convergence of a power series and the expression of topological pressure as an asymptotic weighted sum over periodic orbits (see, e.g., [**21**, Prop. 5.1]). If $g > 0$ on Λ, then it follows from Pringsheim's theorem on power series with positive coefficients [**18**, §17] that $d_{\mathcal{L}}(z)$ has a zero at $(Q^{0,0}(T,g))^{-1}$.

This paper is organized as follows. In Section 2, we discuss about transfer operators acting on sections of vector bundles, with applications to dynamical zeta functions. In Section 3, we present a key alternative expression for the bound $Q^{p,q}(T,g)$ (useful also to prove both main theorems), and we prove Lemma 1.4.

In Section 4, we consider the transfer operator L on $\mathbb{R}^d$ for a C^r diffeomorphism $\mathcal{T}$ and a C^{r-1} weight G. We first introduce the Banach space $\mathcal{C}^{\Theta,p,q}(K)$ of anisotropic distributions on a compact subset $K \subset \mathbb{R}^d$, slightly modifying the definitions in [**7**]: the L^∞ norm in the definition of anisotropic spaces in [**7**] is replaced by a mixed norm, which involves both the supremum norm and the L^1-norm along manifolds close to unstable manifolds. To study the action of the transfer operator L on this Banach space, we work with an auxiliary operator M, which is an infinite matrix of operators describing transitions induced by L between frequency bands in Fourier space. We observe that the operator M is naturally decomposed as $M_b + M_c$ with M_b having small spectral radius and M_c a compact operator. In Lemma 4.17, we give a simple estimate on the operator norm of M_b. In Subsection 4.3, we study the approximation numbers of M_c and show, in particular, that M_c is compact.

The use of approximation numbers to study dynamical transfer operators seems to be new.

In Section 5, we introduce the anisotropic Banach spaces $\mathcal{C}^{p,q}(T,V)$, and prove Theorem 1.1. Taking a system of local charts on V adapted to hyperbolic structure of T, we consider the system $\mathcal{K}$ of transfer operators that $\mathcal{L}$ induces on the local charts. Then we associate an auxiliary operator[5] $\mathcal{M}$ to $\mathcal{K}$, in the same manner as we associate M to L in Section 4. The spectral data of $\mathcal{K}$ and $\mathcal{M}$ turn out to be (almost) identical with that of $\mathcal{L}$. We will decompose $\mathcal{M}^m$ for $m \geq 1$ as $(\mathcal{M}^m)_b + (\mathcal{M}^m)_c$, where $(\mathcal{M}^m)_c$ is compact and $(\mathcal{M}^m)_b$ has norm smaller than $C(Q^{p,q}(T,g)+\epsilon)^m$, proving Theorem 1.1.

In Section 6, we introduce a formal trace $\mathrm{tr}^\flat(\mathcal{P})$, called the flat trace, and a formal determinant $\det^\flat(\mathrm{Id} - z\mathcal{P}) = \exp - \sum_{m\geq 1} \frac{z^m}{m} \mathrm{tr}^\flat(\mathcal{P}^m)$. The flat trace is a key tool inspired from [**5, 6**]. (The terminology was borrowed from Atiyah–Bott [**1**], but we do not relate our flat trace to theirs.) Our flat trace coincides, on the one hand, with the usual trace for finite rank operators and, on the other hand, with the dynamical trace for each $\mathcal{M}^m$:

$$\mathrm{tr}^\flat(\mathcal{M}^m) = \sum_{T^m x = x} \frac{g^{(m)}(x)}{|\det(\mathrm{Id} - DT^m(x))|}\,, \quad \text{so} \quad d_{\mathcal{L}}(z) = \det{}^\flat(\mathrm{Id} - z\mathcal{M})\,.$$

Also, the flat trace $\mathrm{tr}^\flat((\mathcal{M}^m)_b)$ vanishes for all large enough m.

In Section 7, we give the proof of Theorem 1.5. The basic idea of the proof is then to exploit the formal determinant identity[6]

$$\det{}^\flat(\mathrm{Id} - z\mathcal{M}) = \det{}^\flat(\mathrm{Id} - z\mathcal{M}_c(\mathrm{Id} - z\mathcal{M}_b)^{-1}) \cdot \det{}^\flat(\mathrm{Id} - z\mathcal{M}_b)\,. \tag{1.5}$$

If $r > d+1+p-q$ each operator $(\mathcal{M}^m)_c$ turns out to be an operator with summable approximation numbers, and our proof in this case is fairly simple, although we cannot apply (1.5) directly, since we only know that $\mathrm{tr}^\flat((\mathcal{M}^m)_b) = 0$ and that the spectral radius of $(\mathcal{M}^m)_b$ is smaller than $(Q^{p,q}(T,g)+2\epsilon)^m$ *for large* m. If $r \leq d+1+p-q$, we need more estimates since only some iterate of $(\mathcal{M}^m)_c$ has summable approximation numbers. Still the proof is straightforward. [7]

In Appendix A, we discuss about eigenvalues and eigenvectors of the transfer operator $\mathcal{L}$ on different Banach spaces.

2. Operators on vector bundles and dynamical zeta functions

We may generalize the statements and proofs of the main results to similar operators acting on spaces of sections of vector bundles. Since Ruelle zeta function is given as a product of the dynamical Fredholm determinants of such operators [**13, 23**], we can derive statements for Ruelle zeta functins from our main theorems. See also [**21**] for a presentation of classical results about dynamical zeta functions.

For $r > 1$, T, and V as in Section 1, let $\pi_B : B \to V$ be a finite dimensional complex vector bundle, and let $\mathbb{T} : B \to B$ be a C^{r-1} vector bundle endomorphism such that $\pi_B \circ \mathbb{T} = T^{-1} \circ \pi_B$. Denote the natural action of $\mathbb{T}$ on continuous sections of B by $\mathcal{L} = \mathcal{L}_{\mathbb{T}}$, that is, $\mathcal{L}u(x) = \mathbb{T}(u(T(x)))$. Then we can define $Q^{p,q}(T,\mathbb{T})$ in

[5]It is possible to work directly with $\mathcal{L}$, decomposing it into a compact term $\mathcal{L}_c$ and a bounded term $\mathcal{L}_b$, on $\mathcal{C}^{p,q}(T,V)$, in the spirit of [**8**]. Then the flat trace of $(\mathcal{L}^m)_b$ is not zero, but it decays exponentially, arbitrarily fast [**4**].

[6]The operator $\mathcal{D}(z) = z\mathcal{M}_c(\mathrm{Id} - z\mathcal{M}_b)^{-1}$ can be viewed as a kneading operator, [**6**], [**3**].

[7]See [**4**] for a “regularised determinant” alternative to the argument in Section 7.

parallel with the definition of $Q^{p,q}(T,g)$ in Section 1, replacing $\chi_\mu(g/\det(DT|E^u))$ by $\chi_\mu(\mathbb{T}/\det(DT|E^u))$. Putting, for $m \ge 1$,

$$|\mathbb{T}^{(m)}|(x) = \|\mathbb{T}^m_x : B_x \to B_{T^{-m}(x)}\| ,$$

we can define $\rho^{p,q}(T,\mathbb{T},m)$ by using the same formal expression as for $\rho^{p,q}(T,g,m)$.

The next statement is just a formal extension of Theorem 1.1 and Lemma 1.4:

Theorem 2.1. *Let $q < 0 < p$ be so that $p - q < r - 1$. There exists a Banach space $\mathcal{C}^{p,q}(T,B)$ of distributional sections of B, containing C^s sections for any $s > p$, so that the operator $\mathcal{L}_\mathbb{T}$ extends to a bounded operator on $\mathcal{C}^{p,q}(T,B)$, and its essential spectral radius on this space is not larger than $Q^{p,q}(T,\mathbb{T}) = \rho^{p,q}(T,\mathbb{T})$.*

Note that if B is the k-th exterior power of the cotangent bundle of X then $\mathcal{C}^{p,q}(T,B)$ is a space of currents on X.

The dynamical Fredholm determinant of $\mathcal{L} = \mathcal{L}_\mathbb{T}$ as above is defined by

$$d_\mathcal{L}(z) = \exp - \sum_{m=1}^\infty \frac{z^m}{m} \sum_{T^m(x)=x} \frac{\operatorname{tr} \mathbb{T}^m_x}{|\det(\mathrm{Id} - DT^m(x))|} .$$

A formal extension of Theorem 1.5 gives:

Theorem 2.2. *For any $q < 0 < p$ so that $p - q < r - 1$, the function $d_\mathcal{L}(z)$ extends holomorphically to the disc of radius $(Q^{p,q}(T,\mathbb{T}))^{-1}$, and its zeroes in this disc are exactly the inverses of the eigenvalues of $\mathcal{L}_\mathbb{T}$ on $\mathcal{C}^{p,q}(T,B)$, the order of the zero coinciding with the multiplicity of the eigenvalue.*

Let $\pi_L : L \to \Lambda$ be the orientation line bundle for the bundle $\pi_{E^u} : E^u \to \Lambda$, that is, the fiber of L at $x \in B$ is isomorphic to the real line whose unit vectors corresponding to an orientation on $E^u(x)$. By shrinking the isolating neighbourhood V, we may extend it to a continuous line bundle $\pi_L : L \to V$. Let $g \in C^{r-1}(V)$. For $k = 0, 1, \cdots, d$, let $\pi : B_k = (\wedge^k T^* X) \otimes L \to V$ and let $\mathbb{T}_k : B_k \to B_k$ be the vector bundle endomorphism defined by $\mathbb{T}_k(w) = (g \circ \pi) \cdot T^*(w)$. Let $\mathcal{L}_k$ be the natural action of $\mathbb{T}_k$ on the sections of B_k. Then the Ruelle zeta function

$$\zeta_{T,g}(z) = \exp \left(\sum_{m=1}^\infty \frac{z^m}{m} \sum_{T^m(x)=x} g^{(m)}(x) \right) . \tag{2.1}$$

can be written as

$$\zeta_{T,g}(z) = \prod_{k=0}^d d_{\mathcal{L}_k}(z)^{(-1)^{k+\dim E^u+1}} .$$

Thus we obtain as a corollary of Theorem 2.2:

Corollary 2.3. *The Ruelle zeta function $\zeta_{T,g}(z)$ extends as a meromorphic function to the disk of radius*

$$\min_{0\le k\le d} \sup \left\{ Q^{p,q}(T,\mathbb{T}_k)^{-1} \mid q < 0 < p,\ p - q < r - 1 \right\} .$$

3. Alternative expressions for the bound $Q^{p,q}(T,g)$

In this section, we introduce two more expressions, $Q^{p,q}_*(T,g)$ and $\rho^{p,q}_*(T,g)$, in addition to $Q^{p,q}(T,g)$ and $\rho^{p,q}(T,g)$, inspired by [**17**]. And we show that these four expressions are all equivalent, proving Lemma 1.4 especially. Along the way, we express $\log Q^{p,q}(T,g)$ as a double limit of topological pressures (Lemma 3.5). Note that the expression $Q_*^{p,q}(T,g)$ will play a central role in the proofs of Theorems 1.1 and 1.5 in the following sections.

In this section, $r>1$, T and $\Lambda \subset V$ are as in Section 1, but we only assume $g \in C^\delta(V)$ for some $\delta>0$ (and sometimes only that $g\in C^0(X)$), even if r is large.

Remark 3.1. Unlike the standard argument[**28**] on topological pressure, we consider the case where the function g may vanish at some points on Λ. If we assumed that g vanishes nowhere on Λ, the argument in this section should be simpler and partly follow form the standard argument.

3.1. The expression $Q_*^{p,q}(T,g)$ and topological pressure. Recall that, in Section 1, we extended the decomposition $T_xX = E^s(x)\oplus E^u(x)$ on Λ to V and defined $\lambda_x(T^m)$ and $\nu_x(T^m)$ for $x \in \cap_{k=0}^m T^{-k}(V)$. Using this extension, we also define

$$|\det(DT^m|_{E^u})|(x) \qquad \text{for } x\in \cap_{k=0}^m T^{-k}(V)\,, \tag{3.1}$$

as the expansion factor of the linear mapping $DT^m : E^u(x)\to DT^m(E^u(x))$, with respect to the volume induced by the Riemannian metric on each d_u-dimensional linear subspace. Note that, for each $g\in C^0(V)$, the sequences of functions $g^{(m)}$ and $|\det(DT^m|_{E^u})|$ are multiplicative, while $\lambda^{(p,q,m)}$ is submultiplicative in m for all real numbers $q\le 0\le p$. In particular, $|g^{(m)}|\cdot\lambda^{(p,q,m)}\cdot|\det(DT^m|_{E^u})|^{-1}$ is submultiplicative in m for such p and q.

We say that $\mathcal{W}$ is a cover of V if it is a finite cover $\mathcal{W}=\{W_i\}_{i\in\mathcal{I}}$ of V by open subsets of X and if, in addition, the union $\cup_{i\in\mathcal{I}}W_i$ is contained in a compact isolating neighbourhood V' of Λ. For such a cover $\mathcal{W}$ and integers $n<m$, put

$$\mathcal{W}_n^m = \{\cap_{k=n}^{m-1} T^{-k}(W_{i_k}) \mid (i_k)_{k=n}^{m-1}\in \mathcal{I}^{m-n}\}\,,$$

and set $\mathcal{W}^m=\mathcal{W}_0^m$ for $m\ge 1$. Then $\mathcal{W}^m$ is a cover of $V^m := \cap_{k=0}^{m-1}T^{-k}(V)$. We say that a cover $\mathcal{W}$ of V is generating if the diameter of $\mathcal{W}_{-m}^m$ tends to zero as $m\to\infty$. (Generating covers exist because $\cap_{k=-m}^m T^{-k}V$ is contained in a small neighbourhood of Λ for large m.) For real numbers p and q, an integer $m\ge 1$, a generating cover $\mathcal{W}$ of V, and $g\in C^0(X)$, we define

$$Q_*^{p,q}(T,g,\mathcal{W},m) = \min_{\mathcal{W}'}\left(\sum_{U\in\mathcal{W}'} \sup_U \frac{|g^{(m)}|\lambda^{(p,q,m)}}{|\det(DT^m|_{E^u})|}\right) \tag{3.2}$$

where the minimum $\min_{\mathcal{W}'}$ is taken over subcovers $\mathcal{W}'\subset\mathcal{W}^m$ of V^m. By submultiplicativity with respect to m, the following limits exist if $q\le 0\le p$:

$$Q_*^{p,q}(T,g,\mathcal{W}) = \lim_{m\to\infty}\left(Q_*^{p,q}(T,g,\mathcal{W},m)\right)^{1/m}\,.$$

The following lemma may not be new. But, since we did not find it in the literature, we provide a proof.

Lemma 3.2. *For any generating cover $\mathcal{W}$ of V and $g\in C^0(X)$ with $\inf_X|g|>0$, we have $\log Q_*^{0,0}(T,g,\mathcal{W}) = P_{top}\big(T|_\Lambda, \log(|g|/|\det(DT|_{E^u})|)\big)$.*

Proof. It is enough to show

$$\log Q_*^{0,0}(T,g,\mathcal{W}) \le P_{top}(T|_\Lambda, \log(|g|/|\det(DT|_{E^u})|))\,, \tag{3.3}$$

since the inequality in the opposite direction is clear. Let $\mathcal{W}=\{W_i\}_{i\in\mathcal{I}}$. Take another cover $\mathcal{U}=\{U_i\}_{i\in\mathcal{I}}$ of V, so that $U_i\Subset W_i$ for $i\in\mathcal{I}$. Consider small $\epsilon>0$ so that, for each $i\in\mathcal{I}$, the ϵ-neighbourhood of U_i is contained in W_i.

Let $W_{\vec{\imath}} := \bigcap_{k=0}^{m-1} T^{-k}(W_{i_k})$ and $U_{\vec{\imath}} := \bigcap_{k=0}^{m-1} T^{-k}(U_{i_k})$ for $\vec{\imath} = (i_k)_{k=0}^{m-1} \in \mathcal{I}^m$. For each $m \geq 1$, let $Q_\Lambda(T, g, \mathcal{U}, m)$ be the minimum of

$$\sum_{\vec{\imath}\in\mathcal{I}'} \sup_{U_{\vec{\imath}}\cap\Lambda} \frac{|g^{(m)}|}{|\det(DT^m|_{E^u})|}$$

over subsets $\mathcal{I}' \subset \mathcal{I}^m$ such that $\{U_{\vec{\imath}} \cap \Lambda \mid \vec{\imath} \in \mathcal{I}'\}$ is a cover of Λ. Let $\mathcal{I}' = \mathcal{I}'(m)$ be a subset of $\mathcal{I}^m$ that attains this minimum.

Since V is an isolating neighbourhood for the hyperbolic basic set Λ, we can take large N so that, if $T^k(x) \in V$ for $0 \leq k \leq n + 2N$, there exists a point $y \in \Lambda$ such that $d(T^{N+k}(x), T^k(y)) < \epsilon$ for all $0 \leq k < n$. This implies that

$$\{W_{\vec{\imath}} \mid \vec{\imath} = (i_k)_{k=0}^{m+2N-1} \in \mathcal{I}^{m+2N} \text{ and } (i_{k+N})_{k=0}^{m-1} \in \mathcal{I}'(m)\} \subset \mathcal{W}^{m+2N}$$

is a cover of V^{m+2N}. Therefore we have, for all $m \geq 0$,

$$\min_{\mathcal{W}'\subset\mathcal{W}^{m+2N}} \left(\sum_{U\in\mathcal{W}'} \inf_U \frac{|g^{(m+2N)}|}{|\det(DT^{m+2N}|_{E^u})|} \right) \leq C \cdot Q_\Lambda(T, g, \mathcal{U}, m)\,,$$

where the minimum is taken over subcovers $\mathcal{W}' \subset \mathcal{W}^{m+2N}$ of V^{m+2N}, and hence

$$\overline{\lim_{m\to\infty}} \frac{1}{m} \log \min_{\mathcal{W}'\subset\mathcal{W}^m} \left(\sum_{U\in\mathcal{W}'} \inf_U \frac{|g^{(m)}|}{|\det(DT^m|_{E^u})|} \right) \leq P_{top}\left(T|_\Lambda, \log \frac{|g|}{|\det(DT|_{E^u})|} \right).$$

Since g is continuous and positive and since $\mathcal{W}$ is a generating cover, the left hand side coincides with $\log Q_*^{0,0}(T, g, \mathcal{W})$. □

We next express $\log Q_*^{p,q}(T, g, \mathcal{W})$ as a limit of topological pressures under the condition $\inf_X |g| > 0$:

Lemma 3.3. *If $\mathcal{W}$ is a generating cover of V and if $q \leq 0 \leq p$, then for each $g \in C^0(X)$ such that $\inf_X |g| > 0$, we have*

$$\log Q_*^{p,q}(T, g, \mathcal{W}) = \lim_{m\to\infty} \frac{1}{m} P_{top}\left(T^m|_\Lambda, \log \frac{|g^{(m)}| \cdot \lambda^{(p,q,m)}}{|\det(DT^m|_{E^u})|} \right). \tag{3.4}$$

Proof. The topological pressures in the claim are well-defined because for each m the function $\log h_m$, with

$$h_m := |g^{(m)}| \cdot \lambda^{(p,q,m)} \cdot |\det(DT^m|_{E^u})|^{-1}\,,$$

is continuous on Λ. The limit in (3.4) exists by sub-multiplicativity of $m \mapsto h_m$.

For each $\epsilon > 0$, there exists $m \geq 1$ so that

$$(Q_*^{p,q}(T, g, \mathcal{W}) + \epsilon)^m \geq Q_*^{p,q}(T, g, \mathcal{W}, m) = Q_*^{0,0}(T^m, |g^{(m)}|\lambda^{(p,q,m)}, \mathcal{W}^m, 1)\,.$$

By Lemma 3.2, the right-hand side is not smaller than $\exp(P_{top}(T^m|_\Lambda, \log h_m))$. Hence

$$Q_*^{p,q}(T, g, \mathcal{W}) \geq \lim_{m\to\infty} \exp((1/m) P_{top}(T^m|_\Lambda, \log h_m)) - \epsilon\,.$$

Since $\epsilon > 0$ is arbitrary, this give the inequality in one direction.

We next show the inequality in the opposite direction. By sub-multiplicativity and Lemma 3.2, we have, for any integer $m > 0$, that

$$\begin{aligned}\log Q_*^{p,q}(T,g,\mathcal{W}) &= \lim_{k\to\infty}\frac{1}{mk}\log Q_*^{0,0}(T^{mk},|g^{(mk)}|\lambda^{(p,q,mk)},\mathcal{W}^{mk},1)\\ &\le \lim_{k\to\infty}\frac{1}{mk}\log Q_*^{0,0}(T^{m},|g^{(m)}|\lambda^{(p,q,m)},\mathcal{W}^{m},k)\\ &= \frac{1}{m}\log Q_*^{0,0}(T^{m},|g^{(m)}|\lambda^{(p,q,m)},\mathcal{W}^{m}) = \frac{1}{m}P_{top}(T^m|_\Lambda,\log h_m)\,.\end{aligned}$$

This gives the inequality in the opposite direction. □

To get rid of the assumption $\inf_X |g| > 0$, we shall use the following:

Lemma 3.4. *Let $\mathcal{W}$ be a generating cover of V, let $g \in C^0(X)$, and let $q \le 0 \le p$. If g_n is a sequence of functions in $C^0(X)$ so that $\inf_X g_n > 0$ with $g_n \ge g_{n+1} \ge |g|$ for all n, and $\lim_{n\to\infty}\|g_n - |g|\|_{L^\infty(V)} = 0$, then*

$$\lim_{n\to\infty} Q_*^{p,q}(T,g_n,\mathcal{W}) = Q_*^{p,q}(T,g,\mathcal{W})\,.$$

By Lemmas 3.3 and 3.4, the exponent $Q_*^{p,q}(T,g,\mathcal{W})$ for any $g \in C^0(X)$ does not depend on the generating cover $\mathcal{W}$. So it will be denoted by $Q_*^{p,q}(T,g)$.

Proof. We have only to show $\lim_{n\to\infty} Q_*^{p,q}(T,g_n,\mathcal{W}) \le Q_*^{p,q}(T,g,\mathcal{W})$. For any $\epsilon > 0$, we take large m such that $Q_*^{p,q}(T,g,\mathcal{W},m) \le (Q_*^{p,q}(T,g,\mathcal{W})+\epsilon)^m$. Then take n_0 such that $Q_*^{p,q}(T,g_n,\mathcal{W},m) \le (Q_*^{p,q}(T,g,\mathcal{W})+2\epsilon)^m$ for $n \ge n_0$. By sub-multiplicativity, we get $Q_*^{p,q}(T,g_n,\mathcal{W}) \le Q_*^{p,q}(T,g,\mathcal{W}) + 2\epsilon$ for $n \ge n_0$. □

3.2. A variational principle. Lemmas 3.3 and 3.4 allow us to prove:

Lemma 3.5. *$Q^{p,q}(T,g) = Q_*^{p,q}(T,g)$ for $q \le 0 \le p$ and $g \in C^0(X)$. In particular, for every sequence g_n as in Lemma 3.4, we have*

$$\log Q^{p,q}(T,g) = \lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{m}P_{top}\big(T^m|_\Lambda, \log(g_n^{(m)}\lambda^{(p,q,m)}|\det(DT^m|_{E^u})|^{-1})\big)\,.$$

Proof. We first show the claim when $\inf_X |g| > 0$. For simplicity, we put

$$\chi_p(\mu) = p\cdot\chi_\mu(DT|_{E^s}),\qquad \chi_q(\mu) = |q|\cdot\chi_\mu(DT^{-1}|_{E^u})$$

and

$$P(\mu) = h_\mu(T) + \int \frac{|g|}{\det(DT|_{E^u})}\,d\mu + \max\{\chi_p(\mu),\chi_q(\mu)\}\,,$$

so that $\log Q^{p,q}(T,g) = \sup_{\mu\in\mathcal{M}(\Lambda,T)} P(\mu)$. Next we put

$$P_m(\mu) = mh_\mu(T) + \int \log\Big(|g^{(m)}|\cdot\lambda^{(p,q,m)}\cdot|\det(DT^m|_{E^u})|^{-1}\Big)\,d\mu\,.$$

By the variational principle [**28**], Lemma 3.3 implies

$$\log Q_*^{p,q}(T,g) = \lim_{m\to\infty}\sup_{\mu\in\mathcal{M}(\Lambda,T)}\frac{1}{m}P_m(\mu)\,.$$

Note that, for any invariant probability measure μ, Oseledec's theorem[**28**] gives

$$\lim_{m\to\infty}\frac{1}{m}\log\lambda^{(p,q,m)}(x) = \max\{\chi_p(\mu),\chi_q(\mu)\}\qquad \text{for } \mu\text{-a.e. } x. \tag{3.5}$$

We first show $Q^{p,q}(T,g) \le Q^{p,q}_*(T,g)$. There exists a measure $\mu_0 \in \mathcal{M}(\Lambda, T)$ such that $P(\mu_0) = \log(Q^{p,q}(T,g))$. (See Remark 1.2.) By (3.5), we obtain

$$\log Q^{p,q}_*(T,g) \ge \lim_{m\to\infty} \frac{1}{m} P_m(\mu_0) = P(\mu_0) = \log Q^{p,q}(T,g)\,.$$

We next show $Q^{p,q}(T,g) \ge Q^{p,q}_*(T,g)$. For each m, we take $\mu_m \in \mathcal{M}(\Lambda,T)$ such that $P_m(\mu_m) = \sup_{\mu\in\mathcal{M}(\Lambda,T)} P_m(\mu)$. Then we take a subsequence $m(i) \to \infty$ such that $\mu_{m(i)}$ converges weakly to an invariant probability measure μ_∞ on Λ. By decomposing μ_∞ into ergodic components we see that $\log Q^{p,q}(T,g) \ge P(\mu_\infty)$. Thus, the claim $Q^{p,q}(T,g) \ge Q^{p,q}_*(T,g)$ follows if we show

$$P(\mu_\infty) \ge \lim_{i\to\infty} \frac{1}{m(i)} P_{m(i)}(\mu_{m(i)})\,. \tag{3.6}$$

By the upper semi-continuity of entropy, we have $h_{\mu_\infty}(T) \ge \lim_{i\to\infty} h_{\mu_{m(i)}}(T)$. By sub-multiplicativity of $\lambda^{(p,q,m)}$ and (3.5), we have

$$\limsup_{i\to\infty} \int \frac{\log \lambda^{(p,q,m(i))}}{m(i)} d\mu_{m(i)} \le \inf_{m\ge1} \int \frac{\log\lambda^{(p,q,m)}}{m} d\mu_\infty \le \max\{\chi_p(\mu_\infty), \chi_q(\mu_\infty)\}.$$

Therefore we get the inequality (3.6).

Finally we consider the case $\inf_X |g| = 0$. Take a sequence g_n as in Lemma 3.4. In view of Lemma 3.4 and the argument above, it remains to show $Q^{p,q}(T,g) = \lim_{n\to\infty} Q^{p,q}(T,g_n)$ for $q \le 0 \le p$. Note that the sequence $Q^{p,q}(T,g_n)$ is decreasing and we have $Q^{p,q}(T,g) \le \lim_{n\to\infty} Q^{p,q}(T,g_n)$ obviously. We show the inequality in the opposite direction. We write $P(g,\mu)$ for $P(\mu)$. For each n, take $\mu_n \in \mathcal{M}(\Lambda,T)$ such that $P(g_n,\mu_n) = Q^{p,q}(T,g_n)$ and then take a subsequence $n(i) \to \infty$ so that $\mu_{n(i)}$ converges weakly to some invariant probability measure μ_∞ on Λ. Then, by upper-semi-continuity of the entropy and of the largest Lyapunov exponent as a function of μ, we obtain

$$\lim_{n\to\infty} Q^{p,q}(T,g_n) \le \liminf_{n\to\infty} P(g_n,\mu_n) \le P(g,\mu_\infty) \le Q^{p,q}(T,g)\,.$$

□

We may now complete the first step towards the proof of Lemma 1.4:

LEMMA 3.6. *$\rho^{p,q}(T,g) \le Q^{p,q}_*(T,g)$ for any $q \le 0 \le p$ and $g \in C^0(V)$.*

PROOF. Take a generating cover $\mathcal{W} = \{W_i\}$ of V. Then, by a standard argument on hyperbolicity, we can show[8] that the Riemann volume of $U \in \mathcal{W}^m$ is bounded by $C/|\det(DT^m|_{E^u})(x)|$ for any $x \in U$, where C is a constant that does not depend on U, x or m. Then we have, for any subcover $\mathcal{W}' \subset \mathcal{W}^m$ of V^m,

$$\rho^{p,q}(T,g,m) \le \sum_{U\in\mathcal{W}'} \int_U |g^{(m)}(x)|\lambda^{(p,q,m)}(x)\,dx \le C\cdot \sum_{U\in\mathcal{W}'} \sup_{y\in U} \frac{|g^{(m)}(y)|\lambda^{(p,q,m)}(y)}{|\det(DT^m|_{E^u})(y)|}.$$

This implies $\rho^{p,q}(T,g,m) \le CQ^{p,q}_*(T,g,\mathcal{W},m)$ and hence the lemma. □

[8]To see this, we can use the "pinning coordinates" in [**17**, §3.3 and p. 163].

3.3. The expression $\rho_*^{p,q}(T,g)$. We next introduce an exponent $\rho_*^{p,q}(T,g)$ due to Kitaev [**17**], using partitions of unity. A finite family $\Phi=\{\phi_\omega\}_{\omega\in\Omega}$ of C^∞ functions on X is called a partition of unity for V if $0\le\phi_\omega(x)\le 1$ on X, and $\sum_{\omega\in\Omega}\phi_\omega(x)\equiv 1$ on V. The diameter of Φ is $\max_{\omega\in\Omega}\{\mathrm{diam}\big(\mathrm{supp}(\phi_\omega)\cap V\big)\}$. For a partition of unity Φ and an integer $m\ge 1$, set

$$\Phi^m=\left\{\prod_{k=0}^{m-1}\phi_{\omega_k}(T^k(x))\;\middle|\;(\omega_k)_{k=0}^{m-1}\in\Omega^m\right\},$$

which is a partition of unity for $\cap_{k=0}^{m-1}T^{-k}(V)$. For $g\in C^0(V)$, the sequence

$$\rho_*^{p,q}(T,g,\Phi,m)=\sum_{\phi\in\Phi^m}\|\phi\cdot g^{(m)}\cdot\lambda^{(p,q,m)}\cdot\det(DT^m|_{E^u})^{-1}\|_{L^\infty}\,,$$

is then submultiplicative with respect to m if $q\le 0\le p$, so that we may put

$$\rho_*^{p,q}(T,g,\Phi)=\lim_{m\to\infty}\left(\rho_*^{p,q}(T,g,\Phi,m)\right)^{1/m}\,.$$

An important estimate due to Kitaev is:

LEMMA 3.7 (Kitaev [**17**]). *Let $q\le 0\le p$. For every partition of unity Φ for V of sufficiently small diameter and each $g\in C^\delta(V)$ with $\delta>0$, we have*

$$\rho_*^{p,q}(T,g,\Phi)=\rho^{p,q}(T,g)\,. \tag{3.7}$$

This lemma implies that $\rho_*^{p,q}(T,g,\Phi)$ takes a constant value for any sufficiently fine partition of unity Φ. This value is denoted by $\rho_*^{p,q}(T,g)$.

REMARK 3.8. In [**17**, Lemma 2], the corresponding claim is actually stated for "regular mixed transfer operator (MTO)". To get Lemma 3.7, we apply that claim to the regular MTO induced by T and g, using local charts and partitions of unity. See [**17**] and Remark 5.1.

REMARK 3.9. In Lemma 3.7, we can prove $\rho_*^{p,q}(T,g,\Phi)\ge\rho^{p,q}(T,g)$ without much difficulty, using the argument as in the proof of Lemma 3.6. But the inequality in the opposite direction and exactness of the limit in the definition of $\rho^{p,q}(T,g)$ are not easy to prove. In general, the functions $\lambda^{(p,q,m)}(x)$ for large m depend on x irregularly, so that we may not use a simple argument.

We finally prove Lemma 1.4:

PROOF OF LEMMA 1.4. By Lemma 3.5 and 3.6, we have only to show that $\rho^{p,q}(T,g)\ge Q_*^{p,q}(T,g)$. We start by a preliminary observation: For any integer $k\ge 1$, we have

$$Q_*^{p,q}(T^k,g^{(k)})=(Q_*^{p,q}(T,g))^k\quad\text{and}\quad\rho^{p,q}(T^k,g^{(k)})=(\rho^{p,q}(T,g))^k\,. \tag{3.8}$$

The former follows from Lemma 3.5. The latter is a consequence of the definition.

We take a partition of unity $\Phi=\{\phi_\omega\}_{\omega\in\Omega}$ of small diameter so that the intersection multiplicity of the supports of ϕ_ω is less than some constant N_d that depends only on the dimension d of X. Then $\mathcal{W}=\{\phi_\omega^{-1}((N_d^{-1},1])\mid\omega\in\Omega\}$ is a cover of V. We may assume it to be generating. Hence

$$Q_*^{p,q}(T,g,\mathcal{W},m)\le N_d^m\cdot\rho_*^{p,q}(T,g,\Phi,m)\quad\text{for } m\ge 1$$

and, by Lemma 3.7,

$$Q_*^{p,q}(T,g)=Q_*^{p,q}(T,g,\mathcal{W})\le N_d\cdot\rho_*^{p,q}(T,g)=N_d\cdot\rho^{p,q}(T,g)\,.$$

We may apply this estimate to T^k and $g^{(k)}$ for $k \geq 1$. Finally, we use both claims of (3.8) for large k to obtain $Q_*^{p,q}(T,g) \leq \rho_*^{p,q}(T,g)$. $\square$

4. Spaces of anisotropic distributions and transfer operators on $\mathbb{R}^d$

In this section, we introduce Banach spaces of anisotropic distributions on $\mathbb{R}^d$ and then argue about the action of transfer operators on it. The argument in this section will be applied to iterates of our original diffeomorphism T and weight g, using suitable local charts and partition of unity.

For a subset $K \subset \mathbb{R}^d$ and $0 \leq s \leq \infty$, let $C^s(K)$ be the set of C^s functions $u : \mathbb{R}^d \to \mathbb{C}$ whose support is contained in K. Let $C_0^s(\mathbb{R}^d)$ be the set of $u \in C^s(\mathbb{R}^d)$ with compact support. Let $C_*^\infty(\mathbb{R}^d)$ be the set of functions in $C^\infty(\mathbb{R}^d)$ such that $\sup_{x\in\mathbb{R}^d} |\partial^\beta u(x)| < \infty$ for all $\beta \in (\mathbb{Z}_+)^d$. The Schwartz space $\mathcal{S}$ consists of all $u \in C^\infty(\mathbb{R}^d)$ that are *rapidly decaying*, that is, $\sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta u(x)| < \infty$ for all α, $\beta \in (\mathbb{Z}_+)^d$. So $C_0^\infty(\mathbb{R}^d) \subset \mathcal{S} \subset C_*^\infty(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$.

4.1. Definition of local spaces. The basic idea in the definition of our anisotropic spaces of distribution is to slightly modify the classical Littlewood-Paley dyadic decomposition of functions in Fourier space, by introducing some cones of directions, or "polarizations". This approach was introduced in our previous paper [**7**]. Below we modify the definitions in [**7**] slightly in order to get the improved bounds in Theorem 1.1. (See also [**12**] for a recent Fourier analysis approach in the analytic setting.)

For two cones $\mathbf{C}$ and $\mathbf{C}'$ in $\mathbb{R}^d$, we write $\mathbf{C} \Subset \mathbf{C}'$ if $\overline{\mathbf{C}} \subset$ interior $(\mathbf{C}') \cup \{0\}$. Let $\mathbf{C}_+$ and $\mathbf{C}_-$ be closed cones in $\mathbb{R}^d$ with nonempty interiors. Assume that $\mathbf{C}_+ \cap \mathbf{C}_- = \{0\}$ and that $\mathbf{C}_+$ and $\mathbf{C}_-$ contain some d_s- and d_u-dimensional subspaces, respectively. Let $\varphi_+, \varphi_- : \mathbf{S}^{d-1} \to [0,1]$ be C^∞ functions on the unit sphere $\mathbf{S}^{d-1}$ in $\mathbb{R}^d$ satisfying

$$\varphi_+(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \mathbf{C}_+, \\ 0, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \mathbf{C}_-, \end{cases} \qquad \varphi_-(\xi) = 1 - \varphi_+(\xi).$$

We shall work with combinations $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ as above, which are called *polarizations.*

To a polarization Θ as above, we associate the set $\mathcal{F} = \mathcal{F}(\Theta)$ of all C^1-submanifolds $F \subset \mathbb{R}^d$, of dimension d_u, so that the straight line connecting any two distinct points in F is normal to a d_s-dimensional subspace contained in $\mathbf{C}_+$.

Remark 4.1. Our assumption on $\mathcal{F}(\Theta)$ implies that, if we take a d_s-dimensional subspace E that is normal to a d_u-dimensional subspace $E^\perp \subset \mathbf{C}_-$, then the projection $\pi : \mathbb{R}^d \to \mathbb{R}^d/E$ is a C^1 diffeomorphism when restricted to $F \in \mathcal{F}(\Theta)$.

For $u \in C^\infty(\mathbb{R}^d)$ and $\mathcal{F} = \mathcal{F}(\Theta)$, we set

$$\|u\|_{L^1(\mathcal{F})} = \sup_{F\in\mathcal{F}} \|u\|_{L^1(\mu_F)} \in \mathbb{R} \cup \{\infty\}, \tag{4.1}$$

where μ_F is the Riemann volume on F induced by the standard metric on $\mathbb{R}^d$.

The following lemma will play the role that the usual Young inequality for convolution played in [**7**]:

Lemma 4.2. *Let $\mathcal{F} = \mathcal{F}(\Theta)$. Then we have*

$$\|A * u\|_{L^1(\mathcal{F})} \leq \|A\|_{L^1} \|u\|_{L^1(\mathcal{F})} \quad \textit{for } u \in C^\infty(\mathbb{R}^d) \textit{ and } A \in L^1(\mathbb{R}^d)$$

where $*$ *denotes the convolution,* $A * u(x) = \int_{\mathbb{R}^d} A(y)u(x-y)dy$.

PROOF. Take $F \in \mathcal{F}$ arbitrarily and let $F + x$ be the translation of F by $x \in \mathbb{R}^d$, which also belongs to $\mathcal{F}$. Then we have

$$\begin{aligned}\|(A * u)\|_{L^1(\mu_F)} &\le \int_F \left(\int_{\mathbb{R}^d} |A(y)| \cdot |u(x-y)|dy\right) d\mu_F(x) \\ &= \int_{\mathbb{R}^d} |A(y)| \left(\int_F |u(x-y)|d\mu_F(x)\right) dy \\ &\le \int |A(y)| \cdot \|u\|_{L^1(\mu_{F-y})}\, dy \le \|A\|_{L^1}\|u\|_{L^1(\mathcal{F})}\,,\end{aligned}$$

where we used that μ_{F-y} is a translation of μ_F. □

We next introduce some notation in view of performing a dyadic decomposition in the Fourier space. Let $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ be a polarization. Fix a C^∞ function $\chi : \mathbb{R} \to [0,1]$ with $\chi(s) = 1$ for $s \le 1$, and $\chi(s) = 0$ for $s \ge 2$. For $n \in \mathbb{Z}_+$, define $\chi_n : \mathbb{R}^d \to [0,1]$ by $\chi_n(\xi) = \chi(2^{-n}|\xi|)$, and put $\chi_{-1} \equiv 0$. Set $\psi_n : \mathbb{R}^d \to [0,1]$ to be $\psi_n(\xi) = \chi_n(\xi) - \chi_{n-1}(\xi)$, for $n \in \mathbb{Z}_+$. Let $\Gamma = \{(n,\sigma) \mid n \in \mathbb{Z}_+, \sigma \in \{+,-\}\}$. For $(n,\sigma) \in \Gamma$, we define

$$\psi_{\Theta,n,\sigma}(\xi) = \begin{cases}\psi_n(\xi)\varphi_\sigma(\xi/|\xi|), & \text{if } n \ge 1,\\ \psi_0(\xi)/2 = \chi_0(\xi)/2, & \text{if } n = 0.\end{cases}$$

Then the family of functions $\{\psi_{\Theta,n,\sigma}\}_{(n,\sigma)\in\Gamma}$ is a C^∞ partition of unity. Note that the inverse Fourier transform $\widehat{\psi}_{\Theta,n,\sigma}(x) = (2\pi)^{-d}\int e^{ix\xi}\psi_{\Theta,n,\sigma}(\xi)\, d\xi$ of each $\psi_{\Theta,n,\sigma}$ belongs to $\mathcal{S}$, and satisfies the following scaling law:

$$\widehat{\psi}_{\Theta,n,\sigma}(x) = 2^{d(n-1)}\widehat{\psi}_{\Theta,1,\sigma}(2^{n-1}x) \quad \text{if } n \ge 2. \tag{4.2}$$

In particular, we have

$$\sup_{(n,\sigma)\in\Gamma} \|\widehat{\psi}_{\Theta,n,\sigma}\|_{L^1(\mathbb{R}^d)} < \infty. \tag{4.3}$$

We may decompose $u \in C_0^\infty(\mathbb{R}^d)$ as $u = \sum_{(n,\sigma)\in\Gamma} u_{\Theta,n,\sigma}$, by setting

$$u_{\Theta,n,\sigma} := \psi_{\Theta,n,\sigma}(D)u = \widehat{\psi}_{\Theta,n,\sigma} * u \in \mathcal{S}\,.$$

REMARK 4.3. For $\psi \in \mathcal{S}$, we define the pseudodifferential operator $\psi(D) : \mathcal{S} \to \mathcal{S}$ by

$$\psi(D)u(x) := (2\pi)^{-d}\int e^{i\xi(x-y)}\psi(\xi)u(y)d\xi dy = \widehat{\psi} * u(x).$$

We may write this operation as $\psi(D) = \mathbb{F}^{-1} \circ M_\psi \circ \mathbb{F}$ using Fourier transform $\mathbb{F}$ and the multiplication operator M_ψ by ψ. From the expression as a convolution operator, we may extend it as an operator $\psi(D) : C_*^\infty(\mathbb{R}^d) \to C_*^\infty(\mathbb{R}^d)$. We will often use the fact that

$$\psi_1(D)\psi_2(D) = (\psi_1 \cdot \psi_2)(D), \quad \widehat{\psi_1} * \widehat{\psi_2} = \widehat{\psi_1 \cdot \psi_2} \quad \text{for } \psi_1, \psi_2 \in \mathcal{S}. \tag{4.4}$$

Here we quote the following lemma from [**7**], which tells roughly that the functions $\psi_{\Theta,n,\sigma}(D)u$ decay rapidly outside of the support of $u \in C_0^\infty(\mathbb{R}^d)$.

Lemma 4.4 ([**7**, Lemma 4.1]). *Let $K \subset \mathbb{R}^d$ be a compact subset. For any positive numbers b, c and ϵ, there exists a constant $C > 0$ such that*

$$|\psi_{\Theta,n,\sigma}(D)u(x)| \le C \cdot \frac{\sum_{(\ell,\tau)\in\Gamma} 2^{-c\max\{n,\ell\}} \|\psi_{\Theta,\ell,\tau}(D)u\|_{L^\infty}}{d(x, \mathrm{supp}(u))^b} \tag{4.5}$$

for any $(n,\sigma) \in \Gamma$, $u \in C^\infty(K)$ and $x \in \mathbb{R}^d$ satisfying $d(x, \mathrm{supp}(u)) > \epsilon$.

Remark 4.5. Since $\psi_{\Theta,\ell,\tau}(D)u = \chi_{\ell+1}(D)\psi_{\Theta,\ell,\tau}(D)u = \widehat{\chi}_{\ell+1} * \psi_{\Theta,\ell,\tau}(D)u$, we have

$$\|\psi_{\Theta,\ell,\tau}(D)u\|_{L^\infty} \le C2^{d\ell}\|\psi_{\Theta,\ell,\tau}(D)u\|_{L^1(\mathcal{F}(\Theta))} \quad \text{for any } (\ell,\tau) \in \Gamma \text{ and } u \in C_0^\infty(\mathbb{R}^d).$$

Therefore we may replace the L^∞ norm in (4.5) by the norm $\|\cdot\|_{L^1(\mathcal{F}(\Theta))}$.

For a polarization Θ, real numbers $q < 0 < p$ and $u \in C_0^\infty(\mathbb{R}^d)$, we define

$$\|u\|_{\mathcal{C}^{\Theta,p,q}} = \max\left\{ \sup_{n\ge 0} 2^{pn}\|u_{\Theta,n,+}\|_{L^1(\mathcal{F}(\Theta))}, \ \sup_{n\ge 0} 2^{qn}\|u_{\Theta,n,-}\|_{L^1(\mathcal{F}(\Theta))} \right\}. \tag{4.6}$$

Consider a non-empty compact subset $K \subset \mathbb{R}^d$. We first check that the definition above gives a norm on $C^\infty(K)$. Let $\|\cdot\|_{C^s}$ be the usual C^s norm on $C^s(K)$.

Lemma 4.6. *For any $s > p$, there exists a constant $C = C(s,K)$ such that $\|u\|_{\mathcal{C}^{\Theta,p,q}} \le C\|u\|_{C^s}$ for all $u \in C^\infty(K)$.*

Proof. We may assume that s is not an integer. Recall the following characterization of C^s norm in terms of Littlewood-Paley decomposition (see [**26**, Appendix A]): For non-integer $s > 0$, the C^s norm is equivalent to the norm defined by

$$\|u\|_{C^s_*} := \sup_{n\ge 0} \left(2^{sn}\|\psi_n(D)u\|_{L^\infty}\right).$$

Since $\psi_{\Theta,n,\sigma}(D)u = \sum_{m:|m-n|\le 1} \widehat{\psi}_{\Theta,m,\sigma} * (\psi_n(D)u)$ by (4.4), we have

$$\|\psi_{\Theta,n,\sigma}(D)u\|_{L^\infty} \le C\|\psi_n(D)u\|_{L^\infty} \quad \text{for any } (n,\sigma) \in \Gamma \tag{4.7}$$

by Young inequality and (4.3). Using Lemma 4.4 with (4.7), we estimate $\psi_{\Theta,n,\sigma}(D)u$ outside some neighborhood of K and obtain

$$\|\psi_{\Theta,n,\sigma}(D)u\|_{L^1(\mathcal{F})} \le C(s,K)\cdot 2^{-sn}\|u\|_{C^s_*} \quad \text{for any } (n,\sigma) \in \Gamma.$$

Clearly this implies the lemma. □

We may now give the definition of our anisotropic space of distributions.

Definition. For a polarization $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ and real numbers $q < 0 < p$, set $\mathcal{C}^{\Theta,p,q}(K)$ to be the completion of $C^\infty(K)$ (or, equivalently, that of $C^s(K)$ with $s > p$) with respect to $\|\cdot\|_{\mathcal{C}^{\Theta,p,q}}$.

Remark 4.7. The only difference between the space $C_*^{\Theta,p,q}(K)$ in our previous paper [**7**] and the space $\mathcal{C}^{\Theta,p,q}(K)$ in the present work is that, in [**7**], the norm $\|\cdot\|_{L^1(\mathcal{F})}$ in the definition above was the L^∞ norm.

Lemma 4.8. *For any $s > |q|$, the space $\mathcal{C}^{\Theta,p,q}(K)$ is contained in the space of distributions of order s supported on K.*

PROOF. We may assume that s is not an integer. Take any $u \in C^\infty(K)$ and $v \in C_0^\infty(\mathbb{R}^d)$ and decompose them as $u = \sum_{(n,\sigma)\in\Gamma} \psi_{\Theta,n,\sigma}(D)u$ and $v = \sum_{n\geq 0} \psi_n(D)v$ respectively. Since $\mathrm{supp}(\psi_{\Theta,n,\sigma}) \cap \mathrm{supp}(\psi_m) \neq \emptyset$ only if $|m-n| \leq 1$, we get

$$\int u \cdot v\, dx = \sum_{(n,\sigma)\in\Gamma} \sum_{m:|m-n|\leq 1} \int \psi_{\Theta,n,\sigma}(D)u(x) \cdot \psi_m(D)v(x)\, dx$$

by Parseval's identity. Using Lemma 4.4 with Remark 4.5, we estimate $\psi_{\Theta,n,\sigma}(D)u$ outside some neighborhood of K and obtain

$$\left| \int u \cdot v\, dx \right| \leq C\|u\|_{\mathcal{C}^{\Theta,p,q}} \|v\|_{C_*^s}.$$

This implies the claim of the lemma. □

The decomposition introduced above can be viewed as an operator

$$\mathcal{Q}_\Theta : C^\infty(K) \to \mathcal{S}^\Gamma, \quad u \mapsto \left(u_{\Theta,n,\sigma} := \psi_{\Theta,n,\sigma}(D)u\right)_{(n,\sigma)\in\Gamma}.$$

Below we set up some Banach spaces for the target of $\mathcal{Q}_\Theta$, in the place of $\mathcal{S}^\Gamma$ above. For an integer $n \geq 0$, we define

$$\mathbf{B}_n^\Theta = \{u \in C^\infty(\mathbb{R}^d) \mid \chi_n(D)u = u \text{ and } \|u\|_{L^1(\mathcal{F}(\Theta))} < \infty\} \subset C_*^\infty(\mathbb{R}^d).$$

For each $s \geq 0$ and $n \geq 0$, there exists a constant $C(s,n) > 0$ such that

$$\|u\|_{C^s} \leq C(s,n)\|u\|_{L^1(\mathcal{F}(\Theta))} \qquad \text{for any } u \in \mathbf{B}_n^\Theta, \tag{4.8}$$

because $\partial^\alpha u = (\partial^\alpha \widehat{\chi}_n) * u$. Hence $\mathbf{B}_n^\Theta$ is a Banach space with respect to the norm $\|\cdot\|_{L^1(\mathcal{F}(\Theta))}$. We have that $\mathbf{B}_{n-1}^\Theta \subset \mathbf{B}_n^\Theta$ by $\chi_n(D)\chi_{n-1}(D) = \chi_{n-1}(D)$.

DEFINITION. For a polarization Θ and real numbers $q < 0 < p$, we define

$$\mathbf{B}_\Gamma^{\Theta,p,q} = \left\{ (u_{n,\sigma})_{(n,\sigma)\in\Gamma} \;\middle|\; u_{n,\sigma} \in \mathbf{B}_{n+3}^\Theta,\ \lim_{n\to\infty} \max_{\sigma=+,-} 2^{c(\sigma)n}\|u_{n,\sigma}\|_{L^1(\mathcal{F}(\Theta))} = 0 \right\},$$

where $c(+) = p$ and $c(-) = q$. This is a Banach space with respect to the norm

$$\|(u_{n,\sigma})_{(n,\sigma)\in\Gamma}\|_{\mathbf{B}_\Gamma^{\Theta,p,q}} := \sup_{(n,\sigma)\in\Gamma} \left(2^{c(\sigma)n}\|u_{n,\sigma}\|_{L^1(\mathcal{F}(\Theta))}\right).$$

REMARK 4.9. The space $\mathbf{B}_\Gamma^{\Theta,p,q}$ above is a closed subspace of the Banach space

$$\widetilde{\mathbf{B}}_\Gamma^{\Theta,p,q} = \left\{ (u_{n,\sigma})_{(n,\sigma)\in\Gamma} \;\middle|\; u_{n,\sigma} \in \mathbf{B}_{n+3}^\Theta \text{ and } \|(u_{n,\sigma})_{(n,\sigma)\in\Gamma}\|_{\mathbf{B}_\Gamma^{\Theta,p,q}} < \infty \right\}$$

with the identical norm $\|\cdot\|_{\mathbf{B}_\Gamma^{\Theta,p,q}}$. The space $\mathbf{B}_\Gamma^{\Theta,p,q}$ is slightly more convenient than $\widetilde{\mathbf{B}}_\Gamma^{\Theta,p,q}$ for us. For instance the subset

$$\mathbf{B}_\Gamma^\Theta := \{(u_{n,\sigma})_{(n,\sigma)\in\Gamma} \mid u_{n,\sigma} \in \mathbf{B}_{n+3}^\Theta,\ \#\{(n,\sigma)\in\Gamma \mid u_{n,\sigma} \neq 0\} < \infty\} \tag{4.9}$$

is dense in $\mathbf{B}_\Gamma^{\Theta,p,q}$ though this is not true for $\widetilde{\mathbf{B}}_\Gamma^{\Theta,p,q}$. The difference will also make sense in Proposition 6.2 and its proof.

By (4.4), we have, for $k \geq 1$,

$$\chi_{n+k}(D)\psi_{\Theta,n,\sigma}(D) = \psi_{\Theta,n,\sigma}(D) \quad \text{on } C_*^\infty(\mathbb{R}^d). \tag{4.10}$$

This and Lemma 4.6 imply that $\mathcal{Q}_\Theta(C^\infty(K)) \subset \mathbf{B}_\Gamma^{\Theta,p,q}$. Thus, by the definitions of the norms, the operator $\mathcal{Q}_\Theta$ above extends to the isometric embedding

$$\mathcal{Q}_\Theta : \mathcal{C}^{\Theta,p,q}(K) \to \mathbf{B}_\Gamma^{\Theta,p,q}.$$

From Lemma 4.4, we can see that the image of the embedding $\mathcal{Q}_\Theta$ above is contained in much smaller subspaces than $\mathbf{B}_\Gamma^{\Theta,p,q}$. Indeed we can take a smaller Banch space $\widehat{\mathbf{B}}_\Gamma^{\Theta,p,q} \subset \mathbf{B}_\Gamma^{\Theta,p,q}$ that contains the image of $\mathcal{Q}_\Theta$ as follows. We set $\beta(x) = (1+|x|^2)^{(d+1)/2}$ and, for $n \geq 0$,

$$\widehat{\mathbf{B}}_n^\Theta = \{u \in C^\infty(\mathbb{R}^d) \mid \chi_n(D)u = u \text{ and } \|\beta \cdot u\|_{L^1(\mathcal{F}(\Theta))} < \infty\}.$$

In parallel to (4.8), there exists a constant $C(s,n) > 0$ for each $s \geq 0$ and $n \geq 0$ such that

$$\|\beta \cdot u\|_{C^s} \leq C(s,n)\|\beta \cdot u\|_{L^1(\mathcal{F}(\Theta))} \qquad \text{for any } u \in \mathbf{B}_n^\Theta. \tag{4.11}$$

In particular, $\widehat{\mathbf{B}}_n^\Theta$ is a Banach space with respect to the norm $u \mapsto \|\beta u\|_{L^1(\mathcal{F}(\Theta))}$.

Definition. For a polarization Θ and two real numbers $q < 0 < p$, we define

$$\widehat{\mathbf{B}}_\Gamma^{\Theta,p,q} = \left\{ (u_{n,\sigma})_{(n,\sigma)\in\Gamma} \;\middle|\; u_{n,\sigma} \in \mathbf{B}_{n+2}^\Theta,\ \lim_{n\to\infty} \max_{\sigma=+,-} 2^{c(\sigma)n}\|\beta \cdot u_{n,\sigma}\|_{L^1(\mathcal{F}(\Theta))} = 0 \right\}$$

where $c(+) = p$ and $c(-) = q$. This is a Banach space with respect to the norm

$$\|(u_{n,\sigma})_{(n,\sigma)\in\Gamma}\|_{\widehat{\mathbf{B}}_\Gamma^{\Theta,p,q}} := \sup_{(n,\sigma)\in\Gamma} \left(2^{c(\sigma)n}\|\beta \cdot u_{n,\sigma}\|_{L^1(\mathcal{F}(\Theta))} \right).$$

Obviously the inclusion $\iota : \widehat{\mathbf{B}}_\Gamma^{\Theta,p,q} \to \mathbf{B}_\Gamma^{\Theta,p,q}$ is non-expansive.

Lemma 4.10. *$\mathcal{Q}_\Theta(\mathcal{C}^{\Theta,p,q}(K)) \subset \widehat{\mathbf{B}}_\Gamma^{\Theta,p,q}$. $\mathcal{Q}_\Theta : \mathcal{C}^{\Theta,p,q}(K) \to \widehat{\mathbf{B}}_\Gamma^{\Theta,p,q}$ is bounded.*

Proof. By (4.10) for $k = 2$ and by Lemma 4.4 with Remark 4.5, we can see that $\psi_{\Theta,n,\sigma}(D)(C^\infty(K)) \subset \mathbf{B}_{n+2}^\Theta$ and that $\|Q_\Theta u\|_{\mathbf{B}_\Gamma^{\Theta,p,q}} \leq \|Q_\Theta u\|_{\widehat{\mathbf{B}}_\Gamma^{\Theta,p,q}} \leq C\|Q_\Theta u\|_{\mathbf{B}_\Gamma^{\Theta,p,q}}$ for all $u \in C^\infty(K)$, for some constant C. This implies the lemma. $\square$

4.2. Transfer operators associated to cone-hyperbolic maps. In this subsection, we define regular cone-hyperbolic maps on bounded open subsets of $\mathbb{R}^d$ and consider transfer operators associated to such maps $\mathcal{T}$ and C^{r-1} weights G.

Definition. Let U and U' be bounded open subsets in $\mathbb{R}^d$, and let $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ and $\Theta' = (\mathbf{C}'_+, \mathbf{C}'_-, \varphi'_+, \varphi'_-)$ be two polarizations[9]. A C^r diffeomorphism $\mathcal{T} : U' \to U$ is *regular cone-hyperbolic* with respect to polarizations Θ and Θ' if $\mathcal{T}$ extends to a bilipschitz C^1 diffeomorphism of $\mathbb{R}^d$ so that $D\mathcal{T}_x^{tr}(\mathbb{R}^d \setminus \mathbf{C}_+) \Subset \mathbf{C}'_-$ for each $x \in \mathbb{R}^d$ and, in addition, that there exists, for each $x, y \in \mathbb{R}^d$, a linear transformation $\mathbb{L}_{xy}$ satisfying $(\mathbb{L}_{xy})^{tr}(\mathbb{R}^d \setminus \mathbf{C}_+) \Subset \mathbf{C}'_-$ and $\mathbb{L}_{xy}(x-y) = \mathcal{T}(x) - \mathcal{T}(y)$. (We denote the transposed matrix of A by A^{tr}.)

If $\mathcal{T}$ is regular cone-hyperbolic, then the extension $\mathcal{T}$ to $\mathbb{R}^d$ maps each element of $\mathcal{F}(\Theta')$ to an element of $\mathcal{F}(\Theta)$, from both conditions in the definition.

Remark 4.11. The second condition on the extension of $\mathcal{T}$ in the definition above does not follow from the first condition. For example, consider a hyperbolic horseshoe map $\mathcal{T}$, and let U be a small neighbourhood of the entire invariant horseshoe.

[9] We view $\mathbf{C}_{\omega,\pm}$, $\mathbf{C}'_{\omega,\pm}$ as constant cone fields in the *cotangent* bundle $T^*\mathbb{R}^d$, so we apply the *transpose* of $D\mathcal{T}$ to the vectors in them.

In the rest of this section, we consider the transfer operator

$$L : C^{r-1}(U) \to C^{r-1}(U'), \quad Lu = G \cdot (u \circ \mathcal{T}) \tag{4.12}$$

associated to a regular cone-hyperbolic C^r diffeomorphism $\mathcal{T} : U' \to U$ with respect to polarizations Θ and Θ' as above and a C^{r-1} weight $G \in C^{r-1}(U')$.

We begin with a simple estimate on the operator norm of L with respect to the norms $\|\cdot\|_{L^1(\mathcal{F}(\Theta))}$ and $\|\cdot\|_{L^1(\mathcal{F}(\Theta'))}$. Define

$$|\det(D\mathcal{T}|_{\mathbf{C}'_+})|(x) := \inf_L |\det(D\mathcal{T}|_L)|(x) \quad \text{for } x \in U',$$

where $\inf_L$ denotes the infimum over all d_u-dimensional subspaces $L \subset \mathbb{R}^d$ with normal subspace contained in $\mathbf{C}'_+$, and $\det(D\mathcal{T}|_L)$ is defined as for (3.1). Then we have, for any $u \in C^{r-1}(\mathbb{R}^d)$,

$$\|Lu\|_{L^1(\mathcal{F}(\Theta'))} \le \|G\|_{L^\infty} \cdot \sup_{\mathrm{supp}(G)} \left(|\det D\mathcal{T}|_{\mathbf{C}'_+}|^{-1} \right) \cdot \|u\|_{L^1(\mathcal{F}(\Theta))} . \tag{4.13}$$

Fix real numbers $q < 0 < p$ satisfying $p - q < r - 1$ henceforth. Below we will introduce an auxiliary operator $M : \mathbf{B}^{\Theta,p,q}_\Gamma \to \mathbf{B}^{\Theta',p,q}_\Gamma$ and show that the following diagram of bounded operators commutes, with L an extension of (4.12):

$$\begin{array}{ccc} \mathbf{B}^{\Theta,p,q}_\Gamma & \xrightarrow{M} & \mathbf{B}^{\Theta',p,q}_\Gamma \\ {\scriptstyle \mathcal{Q}_\Theta}\Big\uparrow & & {\scriptstyle \mathcal{Q}_{\Theta'}}\Big\uparrow \\ \mathcal{C}^{\Theta,p,q}(\overline{U}) & \xrightarrow{L} & \mathcal{C}^{\Theta',p,q}(\overline{U'}) \end{array} \tag{4.14}$$

The operator M is an infinite matrix of operators, each of which describes the transition between "frequency bands" induced by L.

We recall some definitions from [**7**]. We associate, to $\mathcal{T}$ and G, two integers

$$h^+_{\max} = h^+_{\max}(\mathcal{T}, G) \quad \text{and} \quad h^-_{\min} = h^-_{\min}(\mathcal{T}, G)$$

by

$$h^+_{\max} = \left[\log_2 \left(\sup_{x \in \mathrm{supp}(G)} \sup_{\substack{\|\xi\|=1 \\ D\mathcal{T}^{tr}(\xi) \notin \mathbf{C}'_-}} \|D\mathcal{T}^{tr}_x(\xi)\| \right) \right] + 6$$

$$h^-_{\min} = \left[\log_2 \left(\inf_{x \in \mathrm{supp}(G)} \inf_{\substack{\|\xi\|=1 \\ \xi \notin \mathbf{C}_+}} \|D\mathcal{T}^{tr}_x(\xi)\| \right) \right] - 6 .$$

Remark 4.12. We will consider the situation $h^+_{\max} \ll 0 \ll h^-_{\min}$ in application.

This definition implies that, for $x \in \mathrm{supp}(G)$ and $\xi \in \mathbb{R}^d$,

$$\begin{aligned} &\|D\mathcal{T}^{tr}_x(\xi)\| < 2^{h^+_{\max}-5}\|\xi\| \qquad \text{if } D\mathcal{T}^{tr}_x(\xi) \notin \mathbf{C}'_-, \text{ and} \\ &2^{h^-_{\min}+5}\|\xi\| < \|D\mathcal{T}^{tr}_x(\xi)\| \qquad \text{if } \xi \notin \mathbf{C}_+. \end{aligned} \tag{4.15}$$

We next introduce the relation $\hookrightarrow = \hookrightarrow_{\mathcal{T},G}$ on Γ as follows: Write $(\ell,\tau) \hookrightarrow (n,\sigma)$, for $(\ell,\tau), (n,\sigma) \in \Gamma$, if either

- $(\tau,\sigma) = (+,+)$ and $n \le \ell + h^+_{\max}$, or
- $(\tau,\sigma) = (-,-)$ and $\ell + h^-_{\min} \le n$, or
- $(\tau,\sigma) = (+,-)$ and ($n \ge h^-_{\min}$ or $\ell \ge -h^+_{\max}$).

Otherwise we write $(\ell,\tau)\not\hookrightarrow(n,\sigma)$.

Take a closed cone $\widetilde{\mathbf{C}}_+ \Subset \mathbf{C}_+$ such that

$$DT_x^{tr}(\mathbb{R}^d\setminus\widetilde{\mathbf{C}}_+)\Subset\mathbf{C}'_- \quad \text{for } x\in\operatorname{supp}(G) \tag{4.16}$$

and another closed cone[10] $\widetilde{\mathbf{C}}_-\Subset\mathbf{C}_-$. Let $\tilde\varphi_+,\ \tilde\varphi_- : \mathbf{S}^{d-1}\to[0,1]$ be C^∞ functions satisfying

$$\tilde\varphi_+(\xi)=\begin{cases}1, & \text{if } \xi\notin\mathbf{S}^{d-1}\cap\mathbf{C}_-;\\ 0, & \text{if } \xi\in\mathbf{S}^{d-1}\cap\widetilde{\mathbf{C}}_-,\end{cases}\qquad \tilde\varphi_-(\xi)=\begin{cases}0, & \text{if } \xi\in\mathbf{S}^{d-1}\cap\widetilde{\mathbf{C}}_+;\\ 1, & \text{if } \xi\notin\mathbf{S}^{d-1}\cap\mathbf{C}_+.\end{cases}$$

Put $\tilde\psi_\ell(\xi)=\chi(2^{-\ell-1}\|\xi\|)-\chi(2^{-\ell+2}\|\xi\|)$ for $\ell\ge1$, and define, for $(\ell,\tau)\in\Gamma$,

$$\tilde\psi_{\Theta,\ell,\tau}(\xi)=\begin{cases}\tilde\psi_\ell(\xi)\tilde\varphi_\tau(\xi/\|\xi\|), & \text{if } \ell\ge1,\\ \chi(2^{-1}\|\xi\|), & \text{if } \ell=0.\end{cases}$$

Then $\tilde\psi_{\Theta,\ell,\tau}(\xi)=1$ for all $\xi\in\operatorname{supp}(\psi_{\Theta,\ell,\tau})$ and (4.3) holds with $\psi_{\Theta,n,\sigma}$ replaced by $\tilde\psi_{\Theta,n,\sigma}$. Further, by modifying the cone $\widetilde{\mathbf{C}}_+$ if necessary, we may assume that

$$2^{h^-_{\min}+4}\|\xi\|<\|DT_x^{tr}(\xi)\| \qquad \text{for any } x\in\operatorname{supp}(G) \text{ and any } \xi\notin\widetilde{\mathbf{C}}_+. \tag{4.17}$$

From (4.15–4.17), there exists a constant $C(T,G)>0$ such that, if $(\ell,\tau)\not\hookrightarrow(n,\sigma)$ and $\max\{n,\ell\}\ge C(T,G)$ for $(\ell,\tau),(n,\sigma)\in\Gamma$, then we have, for all $x\in\operatorname{supp}(G)$,

$$d(\operatorname{supp}(\psi_{\Theta',n,\sigma}),DT_x^{tr}(\operatorname{supp}(\tilde\psi_{\Theta,\ell,\tau})))\ge 2^{\max\{n,\ell\}-C(T,G)}\,. \tag{4.18}$$

For each $(\ell,\tau),(n,\sigma)\in\Gamma$, we define the operator $S^{\ell,\tau}_{n,\sigma}:\mathbf{B}^{\Theta}_{\ell+3}\to\mathbf{B}^{\Theta'}_{n+1}$ by

$$S^{\ell,\tau}_{n,\sigma}u:=\psi_{\Theta',n,\sigma}(D)\circ L\circ\tilde\psi_{\Theta,\ell,\tau}(D)u\,.$$

We begin with defining the operator M formally by

$$M((u_{n,\sigma})_{(n,\sigma)\in\Gamma})=\left(\sum_{(\ell,\tau)\in\Gamma}S^{\ell,\tau}_{n,\sigma}u_{\ell,\tau}\right)_{(n,\sigma)\in\Gamma}\,. \tag{4.19}$$

To check that this formal definition gives a bounded operator $M:\mathbf{B}^{\Theta,p,q}_\Gamma\to\mathbf{B}^{\Theta',p,q}_\Gamma$, we recall from [**7**] a few estimates on the operators $S^{\ell,\tau}_{n,\sigma}$. Define the positive-valued integrable function $b:\mathbb{R}^d\to\mathbb{R}$ by

$$b(x)=1 \quad \text{if } \|x\|\le1, \qquad b(x)=\|x\|^{-d-1} \quad \text{if } \|x\|>1. \tag{4.20}$$

For $m>0$, we set

$$b_m:\mathbb{R}^d\to\mathbb{R}, \qquad b_m(x)=2^{dm}\cdot b(2^m x)\,, \tag{4.21}$$

so that $\|b_m\|_{L^1}=\|b\|_{L^1}$. There exists a constant $C>0$ such that

$$b_n*b_m(x)\le C\cdot b_{\min\{n,m\}}(x) \quad \text{for any } x\in\mathbb{R}^d \text{ and any } n,m\ge0. \tag{4.22}$$

By (4.2), there exists a constant $C>0$ such that, for any $x\in\mathbb{R}^d$ and $(n,\sigma)\in\Gamma$,

$$|\widehat{\psi}_{\Theta,n,\sigma}(x)|<C\cdot b_n(x) \quad \text{and} \quad |\widehat{\tilde\psi}_{\Theta,n,\sigma}(x)|<C\cdot b_n(x)\,. \tag{4.23}$$

[10] Actually $\widetilde{\mathbf{C}}_-$ will not play any roll in the following. One may set $\widetilde{\mathbf{C}}_-=\emptyset$.

LEMMA 4.13 ([**7**, (27)]). *There exists a constant $C(\mathcal{T}, G) \geq 1$, which may depend on $\mathcal{T}$ and G, so that, if $(\ell, \tau) \not\hookrightarrow (n, \sigma)$ for $(\ell, \tau), (n, \sigma) \in \Gamma$, then we have*

$$|S^{\ell,\tau}_{n,\sigma} u(x)| \leq C(\mathcal{T}, G) \cdot 2^{-(r-1)\max\{n,\ell\}} \int_{\mathbb{R}^d} b_{\min\{n,\ell\}}(x-y) \cdot |u(\mathcal{T}(y))| \, dy \,, \tag{4.24}$$

for any $u \in C^\infty_*(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

The proof is just a few applications of integration by parts using the estimate (4.18). For convenience of the reader, we give the proof in the case when r is an integer in Appendix C.

LEMMA 4.14. *There is a constant $C > 1$, which does not depend on $\mathcal{T}$ nor G, so that, for any $(\ell, \tau), (n, \sigma) \in \Gamma$ and any $u \in \mathbf{B}^\Theta_{\ell+3}$, it holds*

$$\|\beta \cdot S^{\ell,\tau}_{n,\sigma} u\|_{L^1(\mathcal{F}(\Theta'))} \leq C \cdot \|G\|_{L^\infty} \cdot \sup_{\mathrm{supp}(G)} \left(|\det D\mathcal{T}|_{\mathbf{C}'_+}|^{-1}\right) \cdot \|u\|_{L^1(\mathcal{F}(\Theta))} \,. \tag{4.25}$$

Further there is a constant $C(\mathcal{T}, G) > 1$ so that, if $(\ell, \tau) \not\hookrightarrow (n, \sigma)$ in addition, then

$$\|\beta \cdot S^{\ell,\tau}_{n,\sigma} u\|_{L^1(\mathcal{F}(\Theta'))} \leq C(\mathcal{T}, G) \cdot 2^{-(r-1)\max\{n,\ell\}} \|u\|_{L^1(\mathcal{F}(\Theta))} \,. \tag{4.26}$$

PROOF. Lemma 4.2 and (4.3) give the estimate

$$\|\psi_{\Theta',\ell,\tau}(D)u\|_{L^1(\mathcal{F}(\Theta'))} = \|\widehat{\psi}_{\Theta',\ell,\tau} * u\|_{L^1(\mathcal{F}(\Theta'))} \leq C\|u\|_{L^1(\mathcal{F}(\Theta'))}$$

and the parallel estimate for $\tilde{\psi}_{\Theta,\ell,\tau}(D)$. The claim (4.25) with β replaced by 1 follows from these estimates and (4.13). The claim (4.26) with β replaced by 1 follows from Lemma 4.2, 4.13 and (4.13). To put back the factor β, use[11] the following consequence of Lemma 4.4 and Remark 4.5: For any $\epsilon, c, b > 0$, there exists a constant $C > 0$ such that

$$|S^{\ell,\tau}_{n,\sigma} u(x)| \leq C \cdot d(x, \overline{U'})^{-b} \cdot \sum_{(n',\sigma') \in \Gamma} 2^{-c\max\{n,n'\}} \|S^{\ell,\tau}_{n',\sigma'} u\|_{L^1(\mathcal{F}(\Theta'))}$$

for all $x \in \mathbb{R}^d$ with $d(x, \overline{U'}) > \epsilon$. □

REMARK 4.15. The constant C in (4.25) depends only on the polarization Θ' and the family of functions $\{\widetilde{\psi}_{\Theta,\ell,\tau}\}_{(\ell,\tau)\in\Gamma}$. On the contrary, the constant $C(\mathcal{T}, G)$ in (4.26) and (4.24) depends heavily on $\mathcal{T}$ and G.

COROLLARY 4.16. *The formal definition (4.19) gives a bounded linear operator $M : \mathbf{B}^{\Theta,p,q}_\Gamma \to \widehat{\mathbf{B}}^{\Theta',p,q}_\Gamma$. The transfer operator L extends boundedly to $L : \mathcal{C}^{\Theta,p,q}(\overline{U}) \to \mathcal{C}^{\Theta',p,q}(\overline{U'})$. The diagram (4.14) commutes and $M(\mathbf{B}^{p,q}_\Gamma) \subset \mathcal{Q}_{\Theta'}(\mathcal{C}^{\Theta',p,q}(\overline{U'}))$.*

PROOF. The first claim is an immediate consequence of Lemma 4.14 and the definition of the relation $\hookrightarrow$. It is then easy to check that $M \circ \mathcal{Q}_\Theta = \mathcal{Q}_{\Theta'} \circ L$ on $C^\infty(K)$. Recalling that $\mathcal{Q}_{\Theta'} : \mathcal{C}^{\Theta',p,q}(\overline{U'}) \to \mathbf{B}^{\Theta',p,q}_\Gamma$ is an isometric embedding, we get the second claim and the commutative diagram (4.14). Since $M(\mathbf{B}^\Theta_\Gamma) \subset \mathcal{Q}_{\Theta'}(C^\infty(\overline{U'}))$ for $\mathbf{B}^\Theta_\Gamma$ defined in (4.9), we get the last claim by density. □

[11] For (4.26), use also the fact that there exists constants $C(\mathcal{T}, G) < C'(\mathcal{T}, G)$ such that the relation $(\ell, \tau) \hookrightarrow (n, \sigma)$ holds if $\sigma n - \tau\ell < C(\mathcal{T}, G)$ and only if $\sigma n - \tau\ell < C'(\mathcal{T}, G)$.

In view of the argument above, it is natural to decompose the operator M into

$$M_b((u_{n,\sigma})_{(n,\sigma)\in\Gamma}) = \left(\sum_{(\ell,\tau):(\ell,\tau)\hookrightarrow(n,\sigma)} S^{\ell,\tau}_{n,\sigma} u_{\ell,\tau} \right)_{(n,\sigma)\in\Gamma}$$

and

$$M_c((u_{n,\sigma})_{(n,\sigma)\in\Gamma}) = \left(\sum_{(\ell,\tau):(\ell,\tau)\not\hookrightarrow(n,\sigma)} S^{\ell,\tau}_{n,\sigma} u_{\ell,\tau} \right)_{(n,\sigma)\in\Gamma} .$$

By the same argument as in the case of M, we can check that the above definitions in fact gives bounded operators $M_b, M_c : \mathbf{B}^{\Theta,p,q}_{\Gamma} \to \widehat{\mathbf{B}}^{\Theta',p,q}_{\Gamma}$ and $M = M_b + M_c$. Using Lemma 4.14 and the definition of the relation $\hookrightarrow$ more carefully, we get[12]

Lemma 4.17. *There exists a constant $C > 0$, which does not depend on $\mathcal{T}$ nor G such that the operator norm of $M_b : \mathbf{B}^{\Theta,p,q}_{\Gamma} \to \widehat{\mathbf{B}}^{\Theta',p,q}_{\Gamma}$ is bounded by*

$$C \cdot \|G\|_{L^\infty} \cdot \left(\sup_{\mathrm{supp}(G)} |\det D\mathcal{T}|_{\mathbf{C}'_+}|^{-1} \right) \cdot 2^{\max\{p\cdot h^+_{\max}, q\cdot h^-_{\min}\}}.$$

4.3. Approximation numbers. We shall study approximation numbers of the operator M_c and show, in particular, that M_c is compact. First we recall some basic definitions and facts about the approximation number from [**22**]. Suppose that $\mathcal{B}$ and $\hat{\mathcal{B}}$ are Banach spaces. For $k \in \mathbb{Z}_+$, we define the k-th approximation number of a bounded linear operator $\mathcal{P} : \mathcal{B} \to \hat{\mathcal{B}}$ by

$$a_k(\mathcal{P}) = \inf\{\|\mathcal{P} - F\|_{\mathcal{B}} : F : \mathcal{B} \to \hat{\mathcal{B}},\ \mathrm{rank}(F) < k\} .$$

For $1 \le t \le \infty$, let $\mathcal{L}^{(a)}_t(\mathcal{B}, \hat{\mathcal{B}})$ be the set of bounded linear operators $\mathcal{P} : \mathcal{B} \to \hat{\mathcal{B}}$ such that $(a_k(\mathcal{P}))_{k\in\mathbb{Z}_+} \in \ell^t(\mathbb{Z}_+)$. For each $\mathcal{P} \in \mathcal{L}^{(a)}_t(\mathcal{B}, \hat{\mathcal{B}})$, we set $\|\mathcal{P}\|^{(a)}_t := \|(a_k(\mathcal{P}))\|_{\ell^t}$.

Suppose that $\mathcal{P}_1 : \mathcal{B}_1 \to \mathcal{B}_2$ and $\mathcal{P}_2 : \mathcal{B}_2 \to \mathcal{B}_3$ are bounded linear operators on Banach spaces. If $\mathcal{P}_1 \in \mathcal{L}^{(a)}_t(\mathcal{B}_1, \mathcal{B}_2)$ (resp. $\mathcal{P}_2 \in \mathcal{L}^{(a)}_t(\mathcal{B}_2, \mathcal{B}_3)$), then $\mathcal{P}_2\mathcal{P}_1 \in \mathcal{L}^{(a)}_t(\mathcal{B}_1, \mathcal{B}_3)$ and we have

(4.27)

$$\|\mathcal{P}_2\mathcal{P}_1\|^{(a)}_t \le \|\mathcal{P}_2\|_{L(\mathcal{B}_2,\mathcal{B}_3)} \|\mathcal{P}_1\|^{(a)}_t \quad (\text{resp. } \|\mathcal{P}_2\mathcal{P}_1\|^{(a)}_t \le \|\mathcal{P}_2\|^{(a)}_t \|\mathcal{P}_1\|_{L(\mathcal{B}_1,\mathcal{B}_2)})$$

where $\|\mathcal{P}\|_{L(\mathcal{B},\hat{\mathcal{B}})}$ denotes the operator norm of a linear operator $\mathcal{P} : \mathcal{B} \to \hat{\mathcal{B}}$. For $t, t', s \in [1, \infty]$ such that $1/t + 1/t' = 1/s$, there is a constant $C > 0$ so that

(4.28)

$$\|\mathcal{P}_2\mathcal{P}_1\|^{(a)}_s \le C\|\mathcal{P}_2\|^{(a)}_{t'} \cdot \|\mathcal{P}_1\|^{(a)}_t \qquad \text{for } \mathcal{P}_1 \in \mathcal{L}^{(a)}_t(\mathcal{B}_1, \mathcal{B}_2),\ \mathcal{P}_2 \in \mathcal{L}^{(a)}_{t'}(\mathcal{B}_2, \mathcal{B}_3) .$$

The next lemma tells that the operators in $\mathcal{L}^{(a)}_1(\mathcal{B}, \hat{\mathcal{B}})$ have nuclear representations:

Lemma 4.18 ([**22**, Proposition 2.3.11]). *There is a constant $C > 0$ such that, if $\mathcal{P} \in \mathcal{L}^{(a)}_1(\mathcal{B}, \hat{\mathcal{B}})$, there exist sequences $v_i \in \hat{\mathcal{B}}$ and $v^*_i \in \mathcal{B}^*$, $i \in \mathbb{Z}_+$, such that $\mathcal{P} = \sum_i v_i \otimes v^*_i$ and $\sum_i \|v_i\|_{\hat{\mathcal{B}}} \|v^*_i\|_{\mathcal{B}} \le C\|\mathcal{P}\|^{(a)}_1$.*

Remark 4.19. We refer [**22**, 2.2–2.3] for more explanation about approximation number. In particular, we refer 2.3.3 and 2.2.9 of [**22**] for (4.27) and (4.28) respectively.

We now return to the operator M_c.

[12] We refer the proof of [**7**, Theorem 6.1] for details, though it will not be necessary.

PROPOSITION 4.20. *The operator* $M_c : \mathbf{B}_\Gamma^{\Theta,p,q} \to \widehat{\mathbf{B}}_\Gamma^{\Theta',p,q}$ *in the last subsection belongs to* $\mathcal{L}_t^{(a)}(\mathbf{B}_\Gamma^{\Theta,p,q}, \widehat{\mathbf{B}}_\Gamma^{\Theta',p,q})$ *for any* $t > d/(r-p+q-1)$*, and is hence compact.*

For the proof, we prepare the following approximation lemma:

LEMMA 4.21. *Let* $K \subset \mathbb{R}^d$ *be a compact subset and let* Θ *be a polarization. Let* $s > 0$ *and* $\epsilon > 0$ *be positive real numbers. Then there exists a constant* $C > 0$ *such that, for each* $N > 0$ *and* $(n,\sigma) \in \Gamma$ *with* $n < N$*, there exists an operator* $F_{n,\sigma,N} : C^{r-1}(K) \to \widehat{\mathbf{B}}_{n+2}^\Theta$ *of rank at most* $2^{(1+\epsilon)dN}$*, so that, for any* $u \in C^{r-1}(K)$*,*

$$\|\beta \cdot (\psi_{\Theta,n,\sigma}(D)u - F_{n,\sigma,N}\, u)\|_{L^1(\mathcal{F}(\Theta))} \le C 2^{-sN} \|u\|_{L^1(\mathcal{F}(\Theta))}\,. \tag{4.29}$$

PROOF. We may assume that $K \subset (-1,1)^d$. Let $\phi : \mathbb{R}^d \to [0,1]$ be a C^∞ function so that $\phi \equiv 1$ on $[-1,1]^d$ and $\phi \equiv 0$ on outside of $[-2,2]^d$, and put $\phi_a(\xi) = \phi(2^{-a}\xi)$ for $a > 0$. Take arbitrary $\epsilon > 0$ and consider $N > 0$ and $(n,\sigma) \in \Gamma$ with $n < N$.

For $u \in C^{r-1}(K)$, put $H(u) = \phi_{\epsilon N} \cdot \psi_{\Theta,n,\sigma}(D)u = \phi_{\epsilon N} \cdot \widehat{\psi}_{\Theta,n,\sigma} * u$. Since the distance between K and $\operatorname{supp}(1-\phi_{\epsilon N})$ is greater than $2^{\epsilon N} - 1$, there exists a constant $C_s > 0$ for any $s > 0$ so that

$$\|\beta \cdot (H(u) - \psi_{\Theta,n,\sigma}(D)u)\|_{L^1(\mathcal{F}(\Theta))} \le C_s \cdot 2^{-sN} \|u\|_{L^1(\mathcal{F}(\Theta))} \quad \text{for } u \in C^{r-1}(K). \tag{4.30}$$

Since $H(u)$ for $u \in C^{r-1}(K)$ is supported on $(-2^{\epsilon N+1}, 2^{\epsilon N+1})^d$, we may regard it as a function on $\mathbb{R}^d/(2^{\epsilon N+2}\mathbb{Z})^d$ and consider the discrete Fourier coefficients

$$c_\alpha(u) = (2^{\epsilon N+2})^{-d/2} \int_{\mathbb{R}^d} e^{-i\alpha x} H(u)(x)dx \qquad \text{for } \alpha \in (2^{-\epsilon N-1}\pi) \cdot \mathbb{Z}^d\ .$$

Set

$$F(u)(x) = \phi_{\epsilon N+1}(x) \cdot \sum_{|\alpha| \le 2^{N+5}} c_\alpha(u) \cdot e^{i\alpha x} \qquad \text{for } u \in C^{r-1}(K).$$

Then the difference $H(u) - F(u)$ is supported on $(-2^{\epsilon N+2}, 2^{\epsilon N+2})^d$ and satisfies

$$\|\beta \cdot (H(u) - F(u))\|_{L^\infty} \le \left(\sup_{(-2^{\epsilon N+2}, 2^{\epsilon N+2})^d} \beta \right) \cdot \sum_{|\alpha| > 2^{N+5}} |c_\alpha(u)|\,. \tag{4.31}$$

We may write the coefficient $c_\alpha(u)$ for $\alpha \in (2^{-\epsilon N-1}\pi) \cdot \mathbb{Z}^d$ as

$$(2^{\epsilon N+2})^{-d/2} \cdot \mathbb{F}(H(u))(\alpha)(2^{\epsilon N+2})^{-d/2} \cdot (\mathbb{F}(\phi_{\epsilon N}) * (\psi_{\Theta,n,\sigma} \cdot \mathbb{F}(u)))(\alpha)$$

where $\mathbb{F}$ denotes Fourier transfrom. We have that $\mathbb{F}(\phi_{\epsilon N})(\xi) = 2^{\epsilon N d} \cdot \mathbb{F}\phi(2^{\epsilon N}\xi)$ with $\mathbb{F}\phi \in \mathcal{S}$ and that $\|\mathbb{F}(u)\|_{L^\infty} \le \|u\|_{L^1} \le C(K)\|u\|_{L^1(\mathcal{F}(\Theta))}$. Also we have $|\alpha - \xi| > 2^{-1}|\alpha| > 2^{N+4}$ if $\xi \in \operatorname{supp}(\psi_{\Theta,n,\sigma})$ and $|\alpha| > 2^{N+5}$. Therefore, for any $s > 0$, there exists a constant $C_s > 0$ such that

$$|c_\alpha| \le C_s \cdot |\alpha|^{-s} \|u\|_{L^1(\mathcal{F}(\Theta))} \quad \text{if } |\alpha| > 2^{N+5}.$$

Using this estimate in (4.31) and recalling (4.30), we find a constant $C_s > 0$ for each $s > 0$ so that

$$\|\beta \cdot (\psi_{\Theta,n,\sigma}(D)u - F(u))\|_{L^1(\mathcal{F}(\Theta))} \le C_s \cdot 2^{-sN} \|u\|_{L^1(\mathcal{F}(\Theta))} \quad \text{for } u \in C^{r-1}(K). \tag{4.32}$$

Finally put $F_{n,\sigma,N}(u) := \chi_{n+1}(D)(F(u))$ for $u \in C^{r-1}(K)$. The rank of the operator $F_{n,\sigma,N}$ or that of F is bounded by

$$\#\{\alpha \in 2^{-\epsilon N-1}\pi \cdot \mathbb{Z}^d \mid |\alpha| \le 2^{N+5}\} < C2^{(1+\epsilon)dN}.$$

It is not difficult to see that there exists a constant $C > 0$ such that

$$\|\beta \cdot \chi_{n+1}(D)v\|_{L^1(\mathcal{F}(\Theta))} \le C\|\beta \cdot v\|_{L^1(\mathcal{F}(\Theta))} \quad \text{for any } n \ge 0 \text{ and } v \in C_0^\infty(\mathbb{R}^d).$$

Thus the claim (4.29) follows from (4.10) and (4.32):

$$\begin{aligned}\|\beta \cdot (\psi_{\Theta,n,\sigma}(D)u - F_{n,\sigma,N}(u))\|_{L^1(\mathcal{F}(\Theta))} &\le C \cdot \|\beta \cdot (\psi_{\Theta,n,\sigma}(D)u - F(u))\|_{L^1(\mathcal{F}(\Theta))} \\ &\le C_s \cdot 2^{-sN}\|u\|_{L^1(\mathcal{F}(\Theta))}\,.\end{aligned}$$

From (4.29) and the relation $\chi_{n+2}(D)\chi_{n+1}(D) = \chi_{n+1}(D)$, the image of $F_{n,\sigma,N}$ is contained in $\widehat{\mathbf{B}}^{\Theta}_{n+2}$. □

Proof of Proposition 4.20. We first approximate the operators $S^{\ell,\tau}_{n,\sigma}$ defined in the last subsection by finite rank operators. By Lemma 4.2 and (4.13), we have

$$\|L\circ\tilde{\psi}_{\Theta,\ell,\tau}(D)u\|_{L^1(\mathcal{F}(\Theta'))} \le C(\mathcal{T}, G)\cdot\|u\|_{L^1(\mathcal{F}(\Theta))} \quad \text{for any } (\ell,\tau) \in \Gamma \text{ and } u \in \mathbf{B}^{\Theta}_{\ell+3}.$$

Take arbitrary $\epsilon > 0$ and let $N > 0$. Applying Lemma 4.21 to approximate the post-composition of $\psi_{\Theta',n,\sigma}(D)$, we find an operator $F^{\ell,\tau}_{n,\sigma} : \mathbf{B}^{\Theta}_{\ell+3} \to \widehat{\mathbf{B}}^{\Theta'}_{n+2}$ of rank at most $2^{(1+\epsilon)dN}$ for each $(n,\sigma), (\ell,\tau) \in \Gamma$ with $n < N$, such that

$$\|\beta \cdot (S^{\ell,\tau}_{n,\sigma}(u) - F^{\ell,\tau}_{n,\sigma}(u))\|_{L^1(\mathcal{F}(\Theta'))} \le C(\mathcal{T}, G) \cdot 2^{-(r-1)N}\|u\|_{L^1(\mathcal{F}(\Theta))}. \tag{4.33}$$

Define $\mathbb{P}_N : \mathbf{B}^{p,q}_Z \to \widehat{\mathbf{B}}^{p,q}_Z$ by

$$\mathbb{P}_N((u_{\ell,\tau})_{(\ell,\tau)\in\Gamma}) = \left(\sum_{(\ell,\tau)\in\Gamma} P^{\ell,\tau}_{n,\sigma}(u_{\ell,\tau})\right)_{(n,\sigma)\in\Gamma}$$

where $P^{\ell,\tau}_{n,\sigma} = F^{\ell,\tau}_{n,\sigma}$ if $\max\{n,\ell\} < N$ and $(\ell,\tau) \not\hookrightarrow (n,\sigma)$, and $P^{\ell,\tau}_{n,\sigma} = 0$ otherwise. The rank of $\mathbb{P}_N$ is bounded by $C \cdot N^2 \cdot 2^{(1+\epsilon)dN}$. By (4.33) and the claim (4.26) of Lemma 4.14, we obtain

$$\|M_c - \mathbb{P}_N\|_{L(\mathbf{B}^{\Theta,p,q}_{\Gamma}, \widehat{\mathbf{B}}^{\Theta',p,q}_{\Gamma})} \le C(\mathcal{T}, G, \epsilon) \cdot 2^{-(r-p+q-1-\epsilon)N}.$$

This implies that $a_k(M_c) < C2^{-(r-p+q-1-\epsilon)N}$ for $k = [C \cdot N^2 \cdot 2^{(1+\epsilon)dN}] + 1$, so that $\|(a_k(M_c))\|_{\ell^t}$ is bounded for any $t > d(1+\epsilon)/(r-p+q-1-\epsilon)$. Since $\epsilon > 0$ is arbitrary, we get the proposition. □

5. The transfer operator $\mathcal{L}$ and its extensions

In this section, we study the transfer operator $\mathcal{L} = \mathcal{L}_{T,g}$ for a C^r diffeomorphism $T : X \to X$ and a weight $g \in C^{r-1}(V)$, within the setting in Section 1. Using local charts and a partition of unity, we associate to $\mathcal{L}$ a system $\mathcal{K}$ of transfer operators on local charts and then introduce a key auxiliary operator $\mathcal{M}$. Once we define the operators $\mathcal{K}$ and $\mathcal{M}$ and check their relations to $\mathcal{L}$, the proof of Theorem 1.1 is an immediate consequence of the argument in the last section.

5.1. Local charts adapted to the hyperbolic structure. We first set up a finite system of C^∞ local charts on V, and of polarizations on each of the local charts, so that they are adapted to the hyperbolic structure of the dynamical system T. Consider C^∞ local charts $\{(V_\omega, \kappa_\omega)\}_{\omega\in\Omega}$, with open subsets $V_\omega \subset X$ and maps $\kappa_\omega : V_\omega \to \mathbb{R}^d$ such that $V \subset \cup_\omega V_\omega$, and consider also a system of polarizations on those local charts $\{\Theta_\omega = (\mathbf{C}_{\omega,+}, \mathbf{C}_{\omega,-}, \varphi_{\omega,+}, \varphi_{\omega,-})\}_{\omega\in\Omega}$. Since T is hyperbolic on Λ, we may assume that the following conditions hold:

(a) $\mathcal{V} = \{V_\omega\}_{\omega\in\Omega}$ is a generating cover of V and there is no strict subcover.
(b) $U_\omega = \kappa_\omega(V_\omega)$ is a bounded open subset of $\mathbb{R}^d$ for each $\omega \in \Omega$.
(c) If $x \in V_\omega \cap \Lambda$, the cone $(D\kappa_\omega)^*_x(\mathbf{C}_{\omega,+})$ contains the normal subspace of $E^u(x)$, and the cone $(D\kappa_\omega)^*_x(\mathbf{C}_{\omega,-})$ contains the normal subspace of $E^s(x)$.
(d) If $V_{\omega'\omega} = T^{-1}(V_\omega) \cap V_{\omega'} \neq \emptyset$, the map in charts

$$T_{\omega'\omega} = \kappa_\omega \circ T \circ \kappa_{\omega'}^{-1} : \kappa_{\omega'}(V_{\omega'\omega}) \to U_\omega$$

is a C^r regular cone-hyperbolic diffeomorphism with respect to the polarizations Θ_ω and $\Theta_{\omega'}$.

Let $\Phi = \{\phi_\omega\}$ be a C^∞ partition of unity for V subordinate to the cover $\{V_\omega\}_{\omega\in\Omega}$, that is, the support of each $\phi_\omega : X \to [0,1]$ is contained in V_ω, and we have $\sum_{\omega\in\Omega} \phi_\omega(x) = 1$ for all $x \in V$. We will henceforth fix the local charts, the system of polarizations and the partition of unity as above. We may now define the space $\mathcal{C}^{p,q}(T,V)$ of distributions:

Definition. The Banach space $\mathcal{C}^{p,q}(T,V)$ is the completion of $C^\infty(V)$ for the norm

$$\|\varphi\|_{\mathcal{C}^{p,q}(T,V)} = \max_{\omega\in\Omega} \|(\phi_\omega \cdot \varphi) \circ \kappa_\omega^{-1}\|_{\mathcal{C}^{\Theta_\omega,p,q}}$$

where the norms $\|\cdot\|_{\mathcal{C}^{\Theta_\omega,p,q}}$ are those defined by (4.6).

By Lemma 4.6 and 4.8, the space $\mathcal{C}^{p,q}(T,V)$ contains $C^s(V)$ for each $s > p$ and contained in the dual of $C^s(X)$ for each $s > |q|$.

We decompose the iterates $\mathcal{L}^m$ of $\mathcal{L}$ as follows. Take [13] a positive-valued C^{r-1} function $\tilde{g} : X \to \mathbb{R}$ such that $\tilde{g}(x) > |g(x)|$ for $x \in X$. For each $m \geq 1$, choose a subset $\Omega_m \subset \Omega^m$ so that

$$\mathcal{V}_m := \{V_{\vec{\omega}} := \cap_{i=0}^{m-1} T^{-i}(V_{\omega_i}) \mid \vec{\omega} = (\omega_0, \omega_1, \ldots, \omega_{m-1}) \in \Omega_m\}$$

is a cover of $\bigcap_{k=0}^{m-1} T^{-k}(V)$ by non-empty open sets and that (recall (3.2))

$$Q_*^{p,q}(T, \tilde{g}, \mathcal{V}, m) = \sum_{\vec{\omega}\in\Omega_m} \sup_{V_{\vec{\omega}}} \left(\frac{|\tilde{g}^{(m)}| \cdot \lambda^{(p,q,m)}}{|\det(DT^m|_{E^u})|} \right).$$

Take a C^∞ partition of unity $\Phi_m = \{\phi_{\vec{\omega}} \in C^\infty(V_{\vec{\omega}}) \mid \vec{\omega} \in \Omega_m\}$ for $\bigcap_{k=0}^{m-1} T^{-k}(V)$ subordinate to $\mathcal{V}_m$. Then we have $\mathcal{L}^m = \sum_{\vec{\omega}\in\Omega_m} \mathcal{L}^m_{\vec{\omega}}$ for the operators

$$\mathcal{L}^m_{\vec{\omega}} : C^{r-1}(V) \to C^{r-1}(V), \qquad \mathcal{L}^m_{\vec{\omega}}\varphi = \phi_{\vec{\omega}} \cdot g^{(m)} \cdot \varphi \circ T^m .$$

5.2. The system of transfer operators on local charts. We introduce the operator $\mathcal{K}$ as follows. For each $\omega \in \Omega$, take a C^∞ function $h_\omega \in C^\infty(U_\omega)$ so that $0 \leq h_\omega \leq 1$ on $\mathbb{R}^d$ and that $h_\omega \equiv 1$ on $\kappa_\omega(\mathrm{supp}(\phi_\omega))$. Set $C^{r-1}_\Omega = \bigoplus_{\omega\in\Omega} C^{r-1}(\overline{U_\omega})$. We define the operators $\Phi_* : C^{r-1}(V) \to C^{r-1}_\Omega$ and $H : C^{r-1}_\Omega \to C^{r-1}(V)$ by

$$\Phi_*(u) = ((\phi_\omega \cdot u) \circ \kappa_\omega^{-1})_{\omega\in\Omega} \quad \text{and} \qquad H((u_\omega)_{\omega\in\Omega}) = \sum_{\omega\in\Omega} (h_\omega \cdot u_\omega) \circ \kappa_\omega .$$

Obviously we have $H \circ \Phi_* = \mathrm{Id}$. For each $m \geq 1$, we define

$$\mathcal{K}^m = \Phi_* \circ \mathcal{L}^m \circ H : C^{r-1}_\Omega \to C^{r-1}_\Omega .$$

Remark 5.1. The operator $\mathcal{K}^m$ can be regarded as a regular MTO in the sense of Kitaev [**17**].

[13] We need to take the function $\tilde{g}$ in treating the case $\inf |g| = 0$. Otherwise we may set $\tilde{g} = g$.

Then $\mathcal{K}^m$ is the m-th iterate of $\mathcal{K} := \mathcal{K}^1$ and the following diagram commutes:

$$\begin{array}{ccc} C^{r-1}(V) & \xrightarrow{\Phi_*} & C^{r-1}_\Omega \\ {\scriptstyle \mathcal{L}^m}\downarrow & & {\scriptstyle \mathcal{K}^m}\downarrow \\ C^{r-1}(V) & \xrightarrow{\Phi_*} & C^{r-1}_\Omega . \end{array} \tag{5.1}$$

Likewise, for $m \geq 1$ and $\vec{\omega} \in \Omega_m$, we define the operator $\mathcal{K}^m_{\vec{\omega}}$ by replacing $\mathcal{L}$ by $\mathcal{L}^m_{\vec{\omega}}$ in the definition of $\mathcal{K}^m$. Then we have $\mathcal{K}^m = \sum_{\vec{\omega}\in\Omega_m} \mathcal{K}^m_{\vec{\omega}}$ and the commutative diagram above holds with $\mathcal{L}^m$ and $\mathcal{K}^m$ replaced by $\mathcal{L}^m_{\vec{\omega}}$ and $\mathcal{K}^m_{\vec{\omega}}$ respectively.

We can describe the operator $\mathcal{K}^m_{\vec{\omega}}$ as follows. Set

$$U_{\vec{\omega},\omega'\omega} := \kappa_{\omega'}(V_{\omega'} \cap V_{\vec{\omega}} \cap T^{-m}(V_\omega))$$

and define $T^m_{\omega'\omega} : U_{\vec{\omega},\omega'\omega} \to U_\omega$ and $G_{\vec{\omega},\omega'\omega} \in C^{r-1}(U_{\vec{\omega},\omega'\omega})$ by

$$T^m_{\vec{\omega},\omega'\omega} = \kappa_\omega \circ T^m \circ \kappa_{\omega'}^{-1}, \quad G_{\vec{\omega},\omega'\omega} = ((\phi_{\omega'} \cdot \phi_{\vec{\omega}} \cdot g^{(m)}) \circ \kappa_{\omega'}^{-1}) \cdot (h_\omega \circ T^m_{\vec{\omega},\omega'\omega}).$$

For $\omega, \omega' \in \Omega$, we define

$$(\mathcal{K}^m_{\vec{\omega}})_{\omega\omega'} : C^{r-1}(\mathbb{R}^d) \to C^{r-1}(U_{\vec{\omega},\omega'\omega}), \qquad (\mathcal{K}^m_{\vec{\omega}})_{\omega\omega'} u = G_{\vec{\omega},\omega'\omega} \cdot (u \circ T^m_{\vec{\omega},\omega'\omega}).$$

Then these operators are $\omega\omega'$-components of $\mathcal{K}^m_{\vec{\omega}}$:

$$\mathcal{K}^m_{\vec{\omega}}((u_\omega)_{\omega\in\Omega}) = \left(\sum_{\omega\in\Omega} (\mathcal{K}^m_{\vec{\omega}})_{\omega\omega'} u_\omega \right)_{\omega'\in\Omega} .$$

We will apply the argument in the last section to $L = (\mathcal{K}^m_{\vec{\omega}})_{\omega\omega'}$, setting

$$\begin{gathered} \mathcal{T} = T^m_{\vec{\omega},\omega'\omega},\ G = G_{\vec{\omega},\omega'\omega},\ \Theta = \Theta_\omega,\ \Theta' = \Theta_{\omega'}, \\ U' = U_{\vec{\omega},\omega'\omega},\ U = T^m_{\vec{\omega},\omega'\omega}(U_{\vec{\omega},\omega'\omega}). \end{gathered} \tag{5.2}$$

For this purpose, we have to choose cones $\widetilde{\mathbf{C}}_{\omega,+} \Subset \mathbf{C}_{\omega,+}$, $\widetilde{\mathbf{C}}_{\omega,-} \Subset \mathbf{C}_{\omega,-}$ for each $\omega \in \Omega$, so that, for any $m \geq 1$ and $\vec{\omega} \in \Omega_m$, if we set

$$\widetilde{\mathbf{C}}_+ = \widetilde{\mathbf{C}}_{\omega,+} \quad \text{and} \quad \widetilde{\mathbf{C}}_- = \widetilde{\mathbf{C}}_{\omega,-} \tag{5.3}$$

in addition to (5.2), the conditions (4.16) and (4.17) hold. Clearly this is possible if we take $\widetilde{\mathbf{C}}_{\omega,+}$ sufficiently close to $\mathbf{C}_{\omega,+}$. We then choose C^∞ functions $\tilde{\varphi}_{\omega,+}, \tilde{\varphi}_{\omega,-} : \mathbf{S}^{d-1} \to [0,1]$ and define $\tilde{\psi}_{\Theta_\omega,n,\sigma} \in C^\infty_0(\mathbb{R}^d)$ in the way parallel to that in the definitions of $\tilde{\varphi}_+$, $\tilde{\varphi}_-$ and $\tilde{\psi}_{\Theta,n,\sigma}$ in Subsection 4.2. When we refer the setting (5.2) in the following, we understand that it includes the additional setting (5.3) and

$$\tilde{\varphi}_+ = \tilde{\varphi}_{\omega,+}, \quad \tilde{\varphi}_- = \tilde{\varphi}_{\omega,-} \quad \text{and} \quad \tilde{\psi}_{\Theta,n,\sigma} = \tilde{\psi}_{\Theta_\omega,n,\sigma} \quad \text{for } (n,\sigma) \in \Gamma. \tag{5.4}$$

Consider the Banach space

$$\mathcal{C}^{p,q}_\Omega = \bigoplus_{\omega\in\Omega} \mathcal{C}^{\Theta_\omega,p,q}(\overline{U_\omega})$$

with the norm $\|(u_\omega)_{\omega\in\Omega}\|_{\mathcal{C}^{p,q}_\Omega} = \max_{\omega\in\Omega} \|u_\omega\|_{\mathcal{C}^{\Theta_\omega,p,q}}$. By the definitions of the norms, the operator Φ_* extends to an isometric embedding $\Phi_* : \mathcal{C}^{p,q}(V,T) \to \mathcal{C}^{p,q}_\Omega$.

Corollary 4.16 applied to the setting (5.2) tells that the diagram (5.1) extends to the following commutative diagram of bounded operators:

$$\begin{array}{ccc} \mathcal{C}^{p,q}(V,T) & \xrightarrow{\Phi_*} & \mathcal{C}^{p,q}_{\Omega} \\ {\scriptstyle \mathcal{L}^m_{\vec\omega}}\big\downarrow & & {\scriptstyle \mathcal{K}^m_{\vec\omega}}\big\downarrow \\ \mathcal{C}^{p,q}(V,T) & \xrightarrow{\Phi_*} & \mathcal{C}^{p,q}_{\Omega} \end{array}$$

Taking the sum with respect to $\vec\omega$, we get the same commutative diagram with $\mathcal{L}^m_{\vec\omega}$ and $\mathcal{K}^m_{\vec\omega}$ replaced by $\mathcal{L}^m$ and $\mathcal{K}^m$.

5.3. The auxiliary operator $\mathcal{M}$. We next introduce the auxiliary operator $\mathcal{M}$ as follows. Recall the Banach spaces $\mathbf{B}^{\Theta,p,q}_{\Gamma}$ in the last section and consider the Banach spaces

$$\mathbf{B}^{p,q}_Z := \bigoplus_{\omega\in\Omega} \mathbf{B}^{\Theta_\omega,p,q}_{\Gamma}, \qquad \widehat{\mathbf{B}}^{p,q}_Z := \bigoplus_{\omega\in\Omega} \widehat{\mathbf{B}}^{\Theta_\omega,p,q}_{\Gamma}.$$

with the norms

$$\|(\mathbf{u}_\omega)_{\omega\in\Omega}\|_{\mathbf{B}^{p,q}_Z} := \max_{\omega\in\Omega} \|\mathbf{u}_\omega\|_{\mathbf{B}^{\Theta_\omega,p,q}_{\Gamma}}, \quad \|(\mathbf{u}_\omega)_{\omega\in\Omega}\|_{\widehat{\mathbf{B}}^{p,q}_Z} := \max_{\omega\in\Omega} \|\mathbf{u}_\omega\|_{\widehat{\mathbf{B}}^{\Theta_\omega,p,q}_{\Gamma}} .$$

Let $\mathcal{Q} : \mathcal{C}^{p,q}_{\Omega} \to \mathbf{B}^{p,q}_Z$ be the isometric embedding defined by

$$\mathcal{Q}((u_\omega)_{\omega\in\Omega}) = (\mathcal{Q}_{\Theta_\omega}(u_\omega))_{\omega\in\Omega} .$$

Applying the construction in Subsection 4.2 to $L = (\mathcal{K}^m_{\vec\omega})_{\omega\omega'}$ in the setting (5.2), we define the operator

$$M = (\mathcal{M}^m_{\vec\omega})_{\omega\omega'} : \mathbf{B}^{\Theta_\omega,p,q}_{\Gamma} \to \widehat{\mathbf{B}}^{\Theta_{\omega'},p,q}_{\Gamma}$$

for $\vec\omega \in \Omega_m$ and $\omega, \omega' \in \Omega$, so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}^{\Theta_\omega,p,q}(\overline{U_\omega}) & \xrightarrow{\mathcal{Q}_{\Theta_\omega}} & \mathbf{B}^{\Theta_\omega,p,q}_{\Gamma} \\ {\scriptstyle (\mathcal{K}^m_{\vec\omega})_{\omega\omega'}}\big\downarrow & & {\scriptstyle (\mathcal{M}^m_{\vec\omega})_{\omega\omega'}}\big\downarrow \\ \mathcal{C}^{\Theta_{\omega'},p,q}(\overline{U_{\omega'}}) & \xrightarrow{\mathcal{Q}_{\Theta_{\omega'}}} & \mathbf{B}^{\Theta_{\omega'},p,q}_{\Gamma} \end{array} \tag{5.5}$$

We define the bounded operator $\mathcal{M}^m_{\vec\omega} : \mathbf{B}^{p,q}_Z \to \mathbf{B}^{p,q}_Z$ by

$$\mathcal{M}^m_{\vec\omega}((\mathbf{u}_\omega)_{\omega\in\Omega}) = \left(\sum_{\omega\in\Omega} (\mathcal{M}^m_{\vec\omega})_{\omega\omega'} \mathbf{u}_\omega \right)_{\omega'\in\Omega}$$

and put $\mathcal{M}^m = \sum_{\vec\omega\in\Omega_m} \mathcal{M}^m_{\vec\omega}$. Then we obtain the following commutative diagram of bounded operators:

$$\begin{array}{ccccc} \mathcal{C}^{p,q}(T,V) & \xrightarrow{\Phi_*} & \mathcal{C}^{p,q}_{\Omega} & \xrightarrow{\mathcal{Q}} & \mathbf{B}^{p,q}_Z \\ {\scriptstyle \mathcal{L}^m_{\vec\omega}}\big\downarrow & & {\scriptstyle \mathcal{K}^m_{\vec\omega}}\big\downarrow & & {\scriptstyle \mathcal{M}^m_{\vec\omega}}\big\downarrow \\ \mathcal{C}^{p,q}(T,V) & \xrightarrow{\Phi_*} & \mathcal{C}^{p,q}_{\Omega} & \xrightarrow{\mathcal{Q}} & \mathbf{B}^{p,q}_Z \end{array} \tag{5.6}$$

and the same diagram with $\mathcal{L}^m_{\vec\omega}$, $\mathcal{K}^m_{\vec\omega}$ and $\mathcal{M}^m_{\vec\omega}$ replaced by $\mathcal{L}^m$, $\mathcal{K}^m$ and $\mathcal{M}^m$.

By using continuity of $\mathcal{M}^m$, we can check that $\mathcal{M}^m$ is the m-th iteration of $\mathcal{M} := \mathcal{M}^1$ and that[14]

$$\mathcal{K}^m(\mathcal{C}^{p,q}_\Omega) \subset \Phi_*(\mathcal{C}^{p,q}(T,V)) \quad \text{and} \quad \mathcal{M}^m(\mathbf{B}^{p,q}_Z) \subset \mathcal{Q}(\mathcal{C}^{p,q}_\Omega).$$

This and (5.6) imply that the spectral properties of $\mathcal{L}$ on $\mathcal{C}^{p,q}(T,V)$, $\mathcal{K}$ on $\mathcal{C}^{p,q}_\Omega$ and $\mathcal{M}$ on $\mathbf{B}^{p,q}_Z$ are (almost) identical. More precisely, the essential spectral radii and the eigenvalues of modulus larger than the essential spectral radius coincide, including multiplicity, with an isometric bijection between the generalised eigenspaces.

Recalling Subsection 4.2, we decompose the operator $M = (\mathcal{M}^m_{\vec\omega})_{\omega\omega'}$ into

$$M_b = ((\mathcal{M}^m_{\vec\omega})_{\omega\omega'})_b \quad \text{and} \quad M_c = ((\mathcal{M}^m_{\vec\omega})_{\omega\omega'})_c : \mathbf{B}^{\Theta_\omega,p,q}_\Gamma \to \widehat{\mathbf{B}}^{\Theta_{\omega'},p,q}_\Gamma.$$

From Proposition 4.20, the operator $((\mathcal{M}^m_{\vec\omega})_{\omega\omega'})_c$ is compact. From Lemma 4.17, it follows

$$\|((\mathcal{M}^m_{\vec\omega})_{\omega\omega'})_b\|_{L(\mathbf{B}^{\Theta_\omega,p,q}_\Gamma,\widehat{\mathbf{B}}^{\Theta_{\omega'},p,q}_\Gamma)} \le C \cdot \frac{\sup_{V_{\vec\omega}} |g^{(m)}|}{\inf_{V_{\vec\omega}} \det(DT^m|_{E^u})} \cdot \sup_{V_{\vec\omega}} \lambda^{(p,q,m)} .$$

We decompose $\mathcal{M}^m_{\vec\omega} : \mathbf{B}^{p,q}_Z \to \widehat{\mathbf{B}}^{p,q}_Z$ into $(\mathcal{M}^m_{\vec\omega})_b, (\mathcal{M}^m_{\vec\omega})_c : \mathbf{B}^{p,q}_Z \to \widehat{\mathbf{B}}^{p,q}_Z$, by setting

$$(\mathcal{M}^m_{\vec\omega})_b((\mathbf{u}_\omega)_{\omega\in\Omega}) = \left(\sum_{\omega\in\Omega} ((\mathcal{M}^m_{\vec\omega})_{\omega\omega'})_b \mathbf{u}_\omega\right)_{\omega'\in\Omega}$$

and similarly for $(\mathcal{M}^m_{\vec\omega})_c$. Finally we put

$$(\mathcal{M}^m)_b = \sum_{\vec\omega} (\mathcal{M}^m_{\vec\omega})_b \quad \text{and} \quad (\mathcal{M}^m)_c = \sum_{\vec\omega} (\mathcal{M}^m_{\vec\omega})_c \ : \mathbf{B}^{p,q}_Z \to \widehat{\mathbf{B}}^{p,q}_Z$$

so that $\mathcal{M}^m = (\mathcal{M}^m)_b + (\mathcal{M}^m)_c$. Then the operator $(\mathcal{M}^m)_c$ is compact and

$$\|(\mathcal{M}^m)_b\|_{L(\mathbf{B}^{p,q}_Z,\widehat{\mathbf{B}}^{p,q}_Z)} \le C \cdot \sum_{\vec\omega\in\Omega} \frac{\sup_{V_{\vec\omega}} |g^{(m)}|}{\inf_{V_{\vec\omega}} \det(DT^m|_{E^u})} \cdot \sup_{V_{\vec\omega}} \lambda^{(p,q,m)} . \tag{5.7}$$

5.4. The end of the proof of Theorem 1.1. Since the spectral properties of $\mathcal{L}$ on $\mathcal{C}^{p,q}(T,V)$ and $\mathcal{M}$ on $\mathbf{B}^{p,q}_Z$ are (almost) identical as we noted, it is enough for the proof of Theorem 1.1 to show that the essential spectral radius of $\mathcal{M}$ on $\mathbf{B}^{p,q}_Z$ is bounded by $Q^{p,q}(T,g) = Q^{p,q}_*(T,g)$. Recall the positive-valued C^{r-1} function $\tilde g$ taken just before the definition of the subsets Ω_m. From standard argument in hyperbolic dynamical systems, there exists a constant $C(T,\tilde g) > 0$ such that

$$\frac{\sup_{V_{\vec\omega}} \tilde g^{(m)}}{\inf_{V_{\vec\omega}} |\det(DT^m|_{E^u})|} \le C(T,\tilde g) \cdot \inf_{V_{\vec\omega}} \left(\frac{\tilde g^{(m)}}{|\det(DT^m|_{E^u})|}\right) \tag{5.8}$$

for all $\vec\omega \in \Omega_m$ and $m \ge 1$. It follows

$$\frac{\sup_{V_{\vec\omega}} g^{(m)}}{\inf_{V_{\vec\omega}} \det(DT^m|_{E^u})} \cdot \sup_{V_{\vec\omega}} \lambda^{(p,q,m)} \le C(T,\tilde g) \cdot \sup_{V_{\vec\omega}} \left(\frac{\tilde g^{(m)} \cdot \lambda^{(p,q,m)}}{|\det(DT^m|_{E^u})|}\right).$$

Therefore we have, from (5.7),

$$\|(\mathcal{M}^m)_b\|_{L(\mathbf{B}^{p,q}_Z,\widehat{\mathbf{B}}^{p,q}_Z)} \le C(T,\tilde g) \cdot Q^{p,q}_*(T,\tilde g,\mathcal{V},m) . \tag{5.9}$$

Since $(\mathcal{M}^m)_c$ is compact, the essential spectral radius of $\mathcal{M} : \mathbf{B}^{p,q}_Z \to \mathbf{B}^{p,q}_Z$ is bounded by $(C(T,\tilde g)\cdot Q^{p,q}_*(T,\tilde g,\mathcal{V},m))^{1/m}$ and hence by $Q^{p,q}_*(T,\tilde g)$, letting $m \to \infty$.

[14] See the proof of Corollary 4.16 for the second inclusion.

This holds for any C^{r-1} function $\tilde{g}$ such that $\tilde{g}(x) > |g(x)|$ on X. Therefore, by Lemma 3.4, the essential spectral radius of $\mathcal{M}$ is bounded by $Q_*^{p,q}(T, g)$.

REMARK 5.2. We took a positive-valued function $\tilde{g}$ (instead of $|g|$) so that (5.8) holds. See Remark 3.9 also.

6. The flat trace

In this section, we discuss about a flat trace for operators $\mathcal{P} : \mathbf{B}_Z^{p,q} \to \widehat{\mathbf{B}}_Z^{p,q}$ and give some related results.

6.1. Definition of the flat trace. Set $Z = \Omega \times \Gamma$. For $\zeta = (\omega, n, \sigma) \in Z$, write

$$n(\zeta) = n\,, \quad \sigma(\zeta) = \sigma\,, \quad \omega(\zeta) = \omega \quad \text{and} \quad \mathcal{F}(\zeta) = \mathcal{F}(\Theta_{\omega(\zeta)}), \quad \Theta(\zeta) = \Theta_{\omega(\zeta)}.$$

Then the Banach space $\mathbf{B}_Z^{p,q}$ introduced in the last section is written as

$$\mathbf{B}_Z^{p,q} := \left\{ (u_\zeta)_{\zeta \in Z} \;\middle|\; u_\zeta \in \mathbf{B}_{n(\zeta)+3}^{\Theta(\zeta)} \text{ and } \lim_{n(\zeta)\to\infty} \left(2^{c(\sigma(\zeta))n(\zeta)} \|u_\zeta\|_{L^1(\mathcal{F}(\zeta))} \right) = 0 \right\}$$

where $c(+) = p$ and $c(-) = q$. We will regard each element $\mathbf{u}$ of $\mathbf{B}_Z^{p,q}$ as a family $(u_\zeta)_{\zeta \in Z}$ of functions with index set Z, and each u_ζ will be called the ζ-component of $\mathbf{u}$. For $\zeta \in Z$, let $\mathbf{B}_\zeta$ (resp.$\widehat{\mathbf{B}}_\zeta$) be the closed subspace of $\mathbf{B}_Z^{p,q}$ (resp.$\widehat{\mathbf{B}}_Z$) that consists of elements $(u_{\zeta'})_{\zeta' \in Z}$ such that $u_{\zeta'} = 0$ if $\zeta' \neq \zeta$, equipped with the restriction of the norm $\|\cdot\|_{\mathbf{B}_Z^{p,q}}$ (resp. $\|\cdot\|_{\widehat{\mathbf{B}}_Z^{p,q}}$).

Consider a bounded operator $\mathcal{P} : \mathbf{B}_Z^{p,q} \to \widehat{\mathbf{B}}_Z^{p,q}$. For $\zeta, \zeta' \in Z$, let $\mathcal{P}_{\zeta\zeta'} : \mathbf{B}_\zeta \to \widehat{\mathbf{B}}_{\zeta'}$ be the bounded operator that send $u \in \mathbf{B}_\zeta$ to the ζ'-component of $\mathcal{P}(u)$. Observe that the restriction of $\mathcal{P}_{\zeta\zeta'}$ to $\widehat{\mathbf{B}}_\zeta$ is written as an integral operator with kernel

$$K_{\zeta\zeta'}(x, y) = \mathcal{P}_{\zeta\zeta'}(\widehat{\chi}_{n(\zeta)+2}(y - \cdot))(x)\,.$$

Indeed, for $u \in \widehat{\mathbf{B}}_\zeta$, we have

$$\mathcal{P}_{\zeta\zeta'} u(x) = \mathcal{P}_{\zeta\zeta'}(\widehat{\chi}_{n(\zeta)+2} * u)(x) = \int \mathcal{P}_{\zeta\zeta'}(\widehat{\chi}_{n(\zeta)+2}(\cdot - y))(x) \cdot u(y) dy.$$

Since $\widehat{\chi}_{n(\zeta)+2}(\cdot - y)$ belongs to $\mathbf{B}_\zeta$ and depends on $y \in \mathbb{R}^d$ continuously, the kernel $K_{\zeta\zeta'}(x, y)$ is continous with respect to x and y. If $K_{\zeta\zeta}(x, x)$ is integrable with respect to x, we say that $\mathcal{P}_{\zeta\zeta}$ admits a flat trace and put

$$\mathrm{tr}^\flat(\mathcal{P}_{\zeta\zeta}) = \int_{\mathbb{R}^d} K_{\zeta\zeta}(x, x) dx.$$

REMARK 6.1. The operator $\mathcal{P}_{\zeta\zeta}$ may be expressed as integral operators with different kernels. And the different choice of kernels may give different traces for $\mathcal{P}_{\zeta\zeta}$.

DEFINITION. We say that a bounded operator $\mathcal{P} : \mathbf{B}_Z^{p,q} \to \widehat{\mathbf{B}}_Z^{p,q}$ admits a flat trace if $\mathcal{P}_{\zeta\zeta}$ for each $\zeta \in Z$ admits a flat trace and if the following limit exists:

$$\mathrm{tr}^\flat \mathcal{P} := \lim_{n\to\infty} \sum_{\zeta : n(\zeta) \le n} \mathrm{tr}^\flat(\mathcal{P}_{\zeta\zeta}).$$

If $\mathcal{P}^m$ admits a flat trace for all $m \ge 1$, we define the flat determinant of $\mathcal{P}$ to be the formal power series

$$\det{}^\flat(\mathrm{Id} - z\mathcal{P}) = \exp - \sum_{m\ge 1} \frac{z^m}{m} \mathrm{tr}^\flat(\mathcal{P}^m)\,. \tag{6.1}$$

Clearly, if $\mathrm{tr}^\flat(\mathcal{P}_1)$ and $\mathrm{tr}^\flat(\mathcal{P}_2)$ are well-defined, then so is $\mathrm{tr}^\flat(\mathcal{P}_1 + \mathcal{P}_2)$, and $\mathrm{tr}^\flat(\mathcal{P}_1) + \mathrm{tr}^\flat(\mathcal{P}_2) = \mathrm{tr}^\flat(\mathcal{P}_1 + \mathcal{P}_2)$.

PROPOSITION 6.2. *Suppose that* $\mathcal{P} : \mathbf{B}^{p,q}_Z \to \widehat{\mathbf{B}}^{p,q}_Z$ *is a bounded operator and has a nuclear representation* $\mathcal{P} = \sum_{i=1}^\infty v_i \otimes v_i^*$ *where* $v_i \in \widehat{\mathbf{B}}^{p,q}_Z$ *and* $v_i^* \in (\mathbf{B}^{p,q}_Z)^*$ *satisfy* $\sum_{i=1}^\infty \|v_i\|_{\widehat{\mathbf{B}}^{p,q}_Z} \cdot \|v_i^*\|_{\mathbf{B}^{p,q}_Z} < \infty$. *Then* $\mathcal{P}$ *admits a flat trace. Further it holds*

$$\mathrm{tr}^\flat \mathcal{P} = \sum_{i=1}^\infty v_i^*(v_i) \quad \textit{and} \quad |\mathrm{tr}^\flat \mathcal{P}| \le \sum_{i=1}^\infty \|v_i\|_{\widehat{\mathbf{B}}^{p,q}_Z} \cdot \|v_i^*\|_{\mathbf{B}^{p,q}_Z} < \infty\,.$$

PROOF. Put $\mathcal{P}^{(i)} = v_i \otimes v_i^*$. Let $v_{i,\zeta}$ for $\zeta \in Z$ be the ζ-component of v_i. Also let $v_{i,\zeta}^*$ be the functional on $\mathbf{B}_\zeta$ that v_i^* induces. Then we have $\mathcal{P}^{(i)}_{\zeta\zeta} = v_{i,\zeta} \otimes v_{i,\zeta}^*$ and also [15]

$$\sup_{\zeta\in Z} \|v_{i,\zeta}\|_{\mathbf{B}^{p,q}_Z} = \|v_i\|_{\mathbf{B}^{p,q}_Z} \quad \text{and} \quad \sum_{\zeta \in Z} \|v_{i,\zeta}^*\|_{\mathbf{B}^{p,q}_Z} = \|v_i^*\|_{\mathbf{B}^{p,q}_Z}. \tag{6.2}$$

By definition we have

$$\mathrm{tr}^\flat \mathcal{P}^{(i)}_{\zeta\zeta} = \int K^{(i)}_{\zeta\zeta}(x,x)\,dy \quad \text{where} \quad K^{(i)}_{\zeta\zeta}(x,y) = v_{i,\zeta}(x) v_{i,\zeta}^*(\widehat{\chi}_{n(\zeta)+2}(\cdot - y)).$$

Since $\widehat{\chi}_{n(\zeta)+2}(\cdot - y)$ for $y \in \mathbb{R}^d$ is uniformly bounded in $\mathbf{B}_\zeta$, we have, by (4.11), that

$$\int |K^{(i)}_{\zeta\zeta}(x,x)| dx \le C(\zeta) \cdot \|v_i\|_{\widehat{\mathbf{B}}^{p,q}_Z} \cdot \|v_i^*\|_{\mathbf{B}^{p,q}_Z} \quad \text{for all } 1 \le i < \infty.$$

This implies that $\sum_i K^{(i)}_{\zeta\zeta}(x,x)$ is integrable with respect to x, that is, $\mathcal{P}_{\zeta\zeta}$ admits a flat trace. Since $\widehat{\chi}_{n(\zeta)+2} * v_{i,\zeta} = v_{i,\zeta}$ for $v_{i,\zeta} \in \mathbf{B}^{\Theta(\zeta)}_{n(\zeta)+2}$, it holds

$$\mathrm{tr}^\flat \mathcal{P}^{(i)}_{\zeta\zeta} = v_{i,\zeta}^* \left(\int v_{i,\zeta}(x) \widehat{\chi}_{n(\zeta)+2}(\cdot - x)\,dx \right) = v_{i,\zeta}^*(v_{i,\zeta})$$

for each $\zeta \in Z$ and $i \ge 1$. It follows from (6.2) that

$$\sum_i \sum_{\zeta\in Z} |\mathrm{tr}^\flat \mathcal{P}^{(i)}_{\zeta\zeta}| \le \sum_i \sum_{\zeta \in Z} \|v_{i,\zeta}\|_{\widehat{\mathbf{B}}^{p,q}_Z} \|v_{i,\zeta}^*\|_{\mathbf{B}^{p,q}_Z} \le \sum_i \|v_i^*\|_{\widehat{\mathbf{B}}^{p,q}_Z} \|v_i\|_{\mathbf{B}^{p,q}_Z} < \infty\,.$$

Therefore we conclude that $\mathrm{tr}^\flat \mathcal{P}$ exists and

$$\mathrm{tr}^\flat \mathcal{P} = \lim_{n\to\infty} \sum_{n(\zeta)\le n} \sum_i \mathrm{tr}^\flat \mathcal{P}^{(i)}_{\zeta\zeta} = \sum_i \sum_{\zeta\in Z} v_{i,\zeta}^*(v_{i,\zeta}) = \sum_i v_i^*(v_i)\,.$$

The inequality for $|\mathrm{tr}^\flat \mathcal{P}|$ is then obvious. □

[15] For the second equality, recall Remark 4.9. This equality would not hold if we used the Banach space $\tilde{\mathbf{B}}^{\Theta,p,q}_\Gamma$ in the place of $\mathbf{B}^{\Theta,p,q}_\Gamma$

6.2. The flat trace of the operators $\mathcal{M}^m$. We next consider the flat traces of the operators $\mathcal{M}^m$, $(\mathcal{M}^m)_b$ and $(\mathcal{M}^m)_c$ introduced in the last section. The flat trace of $\mathcal{M}^m$ coincides with the dynamical trace:

PROPOSITION 6.3. *The operator $\mathcal{M}^m : \mathbf{B}_Z^{p,q} \to \widehat{\mathbf{B}}_Z^{p,q}$ for $m \geq 1$ admits a flat trace and holds*

$$\operatorname{tr}^\flat(\mathcal{M}^m) = \sum_{T^m(x)=x} \frac{g^{(m)}(x)}{|\det(\mathrm{Id} - DT^m(x))|}.$$

PROOF. Consider $\mathcal{M}^m$ for $m \geq 1$. Take $\omega \in \Omega$ and $\vec{\omega} \in \Omega_m$ and recall the definition of the operator $(\mathcal{M}^m_{\vec{\omega}})_{\omega\omega}$. Then we see that, for each $\zeta \in Z$ with $\omega(\zeta) = \omega$, the flat trace $\operatorname{tr}^\flat(\mathcal{M}^m_{\zeta\zeta})$ is defined as the integral

$$\int \widehat{\psi}_{\Theta(\zeta),n(\zeta),\sigma(\zeta)}(x-y) \cdot G(y) \cdot \widehat{\tilde{\psi}}_{\Theta(\zeta),n(\zeta),\sigma(\zeta)}(\mathcal{T}(y) - z) \cdot \widehat{\chi}_{n(\zeta)+2}(z-x) dx dy dz$$

where $\mathcal{T}$ and G are those in the setting (5.2) with $\omega' = \omega$. Since

$$\widehat{\tilde{\psi}}_{\Theta(\zeta),n(\zeta),\sigma(\zeta)} * \widehat{\chi}_{n(\zeta)+2} * \widehat{\psi}_{\Theta(\zeta),n(\zeta),\sigma(\zeta)} = \widehat{\psi}_{\Theta(\zeta),n(\zeta),\sigma(\zeta)}$$

by (4.4), we see that $\mathcal{M}^m_{\zeta\zeta}$ admits a flat trace and

$$\operatorname{tr}^\flat(\mathcal{M}^m_{\zeta\zeta}) = \int \widehat{\psi}_{\Theta(\zeta),n(\zeta),\sigma(\zeta)}(\mathcal{T}(x) - x) \cdot G(x)\, dx.$$

Thus, for each integer n_0, we have

$$\sum_{\zeta : n(\zeta) \leq n_0; \omega(\zeta) = \omega} \operatorname{tr}^\flat(\mathcal{M}^m_{\zeta\zeta}) = \int \widehat{\chi}_{n_0}(\mathcal{T}(x) - x) \cdot G(x)\, dx. \tag{6.3}$$

The function $\widehat{\chi}_{n_0}$, regarded as a distribution, converges to the Dirac measure at 0 as $n_0 \to \infty$. Note that there is at most one fixed point of $\mathcal{T}$ in $\operatorname{supp}(G)$ because the covering $\mathcal{V}$ is assumed to be generating. If there is no fixed point in $\operatorname{supp}(G)$, the sum (6.3) converges to zero as $n_0 \to \infty$. If there is one fixed point x_0 in $\operatorname{supp}(G)$, that fixed point should be hyperbolic by hyperbolicity of T and hence we may perform a local change of variable $z = \mathcal{T}(x) - x$ in its small neighborhood to obtain

$$\lim_{n_0 \to \infty} \int \widehat{\chi}_{n_0}(\mathcal{T}(x) - x) \cdot G(x) dx = \frac{G(x_0)}{|\det(\mathrm{Id} - D\mathcal{T}(x_0))|}.$$

Recalling the definition of $\mathcal{T}$, G and h_ω, we see that the operator $\mathcal{M}^m_{\vec{\omega}}$ admits a flat trace and

$$\operatorname{tr}^\flat(\mathcal{M}^m_{\vec{\omega}}) = \sum_{\omega \in \Omega} \sum_{T^m(x)=x} \frac{\phi_\omega(x) \cdot \phi_{\vec{\omega}}(x) \cdot g^{(m)}(x)}{|\det(\mathrm{Id} - DT^m(x))|} = \sum_{T^m(x)=x} \frac{\phi_{\vec{\omega}}(x) \cdot g^{(m)}(x)}{|\det(\mathrm{Id} - DT^m(x))|}.$$

Taking the sum with respect to $\vec{\omega} \in \Omega_m$, we obtain the proposition. □

The following property of $(\mathcal{M}^m)_b$ is important in the proof of Theorem 1.5.

PROPOSITION 6.4. *There is $L = L(T, g) \geq 1$ so that, if $m_j \geq L$ for $1 \leq j \leq J$, then*

$$\left(\prod_{j=1}^{J} (\mathcal{M}^{m_j})_b \right)_{\zeta\zeta} = 0 \quad \textit{for all } \zeta \in Z,$$

and in particular

$$\mathrm{tr}^\flat\left(\prod_{j=1}^{J}(\mathcal{M}^{m_j})_b\right) = 0 \ .$$

REMARK 6.5. We read the expression $\Pi_{i=1}^{k-1}\mathcal{P}_i$ as the product $\mathcal{P}_{k-1}\mathcal{P}_{k-2}\cdots\mathcal{P}_1$, not as $\mathcal{P}_1\mathcal{P}_2\cdots\mathcal{P}_{k-1}$.

PROOF. By hyperbolicity of T, there exists $L \geq 1$, such that, for all $\omega, \omega' \in \Omega$ and $\vec{\omega} \in \Omega_m$ with $m \geq L$, we have $h^+_{\max}(\mathcal{T}, G) < 0$ and $h^-_{\min}(\mathcal{T}, G) > 0$ in the setting (5.2). This and the definition of the relation $\hookrightarrow$ imply that $(\mathcal{M}^m)_b$ for $m \geq L$ is "strictly lower triangular" as a matrix of operators in the sense that $((\mathcal{M}^m)_b)_{\zeta\zeta'} \neq 0$ only if $\sigma(\zeta')n(\zeta') < \sigma(\zeta)n(\zeta)$. Clearly this gives the claim of the lemma. □

For the operator $(\mathcal{M}^m)_c$, we have the following: Recall Subsection 4.3 and put

$$k_* = k_*(d, r, p, q) := \left[1 + \frac{d}{r - p + q - 1}\right] \ . \tag{6.4}$$

PROPOSITION 6.6. *The operator $(\mathcal{M}^m)_c$ belongs to the class $\mathcal{L}^{(a)}_t(\mathbf{B}^{p,q}_Z, \widehat{\mathbf{B}}^{p,q}_Z)$ for any $t > d/(r-p+q-1)$. Further, given any bounded operators $\mathcal{P}_j : \mathbf{B}^{p,q}_Z \to \widehat{\mathbf{B}}^{p,q}_Z$ for $0 \leq j \leq k_*$ and any integers $m_j \geq 1$ for $1 \leq j \leq k_*$, we have*

$$\left\|\mathcal{P}_0\prod_{j=1}^{k_*}((\mathcal{M}^{m_j})_c\mathcal{P}_j)\right\|_1^{(a)} \leq C\cdot\|\mathcal{P}_0\|_{L(\mathbf{B}^{p,q}_Z, \widehat{\mathbf{B}}^{p,q}_Z)}\prod_{j=1}^{k_*}\left(\|(\mathcal{M}^{m_j})_c\|^{(a)}_{k_*}\cdot\|\mathcal{P}_j\|_{L(\mathbf{B}^{p,q}_Z, \widehat{\mathbf{B}}^{p,q}_Z)}\right)$$

PROOF. The first claim is a consequence of Proposition 4.20 and the definition of the operator $(\mathcal{M}^m)_c$. The second then follows from (4.27) and (4.28). □

7. Dynamical determinants: Proof of Theorem 1.5

In this section, we prove Theorem 1.5. Let $q < 0 < p$ be so that $p - q < r - 1$. As we noted in Subsection 5.3, the operators $\mathcal{M}$ on $\mathbf{B}^{p,q}_Z$ and $\mathcal{L}$ on $\mathcal{C}^{p,q}(T, V)$ share almost same spectral data. And we have proved in Subsection 5.4 that the essential spectral radius of $\mathcal{M}$ is not larger than $Q^{p,q}_*(T, g) = Q^{p,q}(T, g)$. Fix $\epsilon > 0$ arbitrarily and set $\rho := Q^{p,q}(T, g) + \epsilon$. By Proposition 6.3 we have $d_{\mathcal{L}}(z) = \det^\flat(\mathrm{Id} - z\mathcal{M})$ as formal power series. Therefore, in order to prove Theorem 1.5, it suffices to show that $d_{\mathcal{L}}(z) = \det^\flat(\mathrm{Id} - z\mathcal{M})$ extends holomorphically to the disc of radius ρ^{-1}, and that $d_{\mathcal{L}}(z)$ vanishes at order n_z in this disc if and only if $1/z$ is an eigenvalue of algebraic multiplicity n_z for $\mathcal{M}$ on $\mathbf{B}^{p,q}_Z$. This is the content of the present section.

Consider the spectral projector $\mathcal{P}_0$ for $\mathcal{M} : \mathbf{B}^{p,q}_Z \to \mathbf{B}^{p,q}_Z$ associated to eigenvalues of modulus larger than or equal to ρ. We have[16] $\mathcal{M}^m = \mathcal{M}^m_0 + \mathcal{M}^m_1$, with $\mathcal{M}_0 = \mathcal{M}\mathcal{P}_0$ and $\mathcal{M}_1 = \mathcal{M}(\mathrm{Id} - \mathcal{P}_0)$. For each $m \geq 1$, the operator $\mathcal{M}^m_0$ is of finite rank and its image is contained in $\widehat{\mathbf{B}}^{p,q}_Z$. By Proposition 6.2, $\mathcal{M}^m_0$ admits a flat trace and its flat trace coincides with the usual trace defined for finite rank operators. By Proposition 6.3, we also see that $\mathcal{M}^m_1 = \mathcal{M}^m - \mathcal{M}^m_0$ admits a flat trace. Hence we may decompose

$$\det{}^\flat(\mathrm{Id} - z\mathcal{M}) = \det{}^\flat(\mathrm{Id} - z\mathcal{M}_0)\det{}^\flat(\mathrm{Id} - z\mathcal{M}_1)$$

[16]Since $(\mathcal{M}_0)^m = \mathcal{M}^m\mathcal{P}_0 = (\mathcal{M}^m)_0$, we write $\mathcal{M}^m_0$ without risk of confusion, similarly for $\mathcal{M}_1$.

and the factor $\det^\flat(\mathrm{Id} - z\mathcal{M}_0)$ is a polynomial which vanishes exactly at the inverse eigenvalues of $\mathcal{M}\mathcal{P}_0$, with order equal to the multiplicity of the eigenvalue. To prove Theorem 1.5, it thus suffices to show that $\det^\flat(\mathrm{Id} - z\mathcal{M}_1)$ is holomorphic and nowhere zero in the disc of radius ρ^{-1}, i.e., for any $\epsilon' > 0$, there exists $C > 0$ such that

$$|\mathrm{tr}^\flat(\mathcal{M}_1^m)| < C(\rho + \epsilon')^m \quad \text{for all } m \geq 1. \tag{7.1}$$

Since the proof is much simpler in the case $r > d+p-q+1$, we will discuss about such case first in Subsections 7.1 and consider the other case later in Subsection 7.2. From (5.9), we may take an integer $m_0 \geq 1$ so that

(7.2)
$$\|(\mathcal{M}^m)_b\|_{L(\mathbf{B}_Z^{p,q}, \widehat{\mathbf{B}}_Z^{p,q})} \leq \rho^m \quad \text{and} \quad \|\mathcal{M}_1^m\|_{L(\mathbf{B}_Z^{p,q}, \widehat{\mathbf{B}}_Z^{p,q})} \leq \rho^m \quad \text{for } m \geq m_0.$$

For $m \geq 1$, we put $(\mathcal{M}^m)_a := \mathcal{M}_1^m - (\mathcal{M}^m)_b = (\mathcal{M}^m)_c - \mathcal{M}_0^m$ so that

$$\mathcal{M}_1^m = (\mathcal{M}^m)_a + (\mathcal{M}^m)_b. \tag{7.3}$$

7.1. The case $r > p - q + d + 1$. By Proposition 6.6, the operators $(\mathcal{M}^m)_c$ and $(\mathcal{M}^m)_a$ both belong to $\mathcal{L}_1^{(a)}(\mathbf{B}_Z^{p,q}, \widehat{\mathbf{B}}_Z^{p,q})$ in this case. Write a large integer n as a sum $n = m(1) + m(2) + \cdots + m(k)$ with $m_0 \leq m(i) \leq 2m_0$. Then, using (7.3), we may write $\mathcal{M}_1^n = \prod_{i=1}^k \mathcal{M}_1^{m(i)}$ as

$$\mathcal{M}_1^n = \prod_{i=1}^k (\mathcal{M}^{m(i)})_b + \sum_{j=1}^k \left(\prod_{i=j+1}^k \mathcal{M}_1^{m(i)} \cdot (\mathcal{M}^{m(j)})_a \cdot \prod_{i=1}^{j-1} (\mathcal{M}^{m(i)})_b \right).$$

By Proposition 6.4, we have $\mathrm{tr}^\flat\left(\prod_{i=1}^k (\mathcal{M}^{m(i)})_b\right) = 0$ for the first term. For the other terms, we have, by Proposition 6.6 with $k_* = 1$ and (7.2), that

$$\begin{aligned}
&\left| \mathrm{tr}^\flat \left(\prod_{i=j+1}^k \mathcal{M}_1^{m(i)} \cdot (\mathcal{M}^{m(j)})_a \cdot \prod_{i=1}^{j-1} (\mathcal{M}^{m(i)})_b \right) \right| \\
&\qquad \leq \prod_{i=j+1}^k \|\mathcal{M}_1^{m(i)}\|_{\mathbf{B}_Z^{p,q}} \cdot \|(\mathcal{M}^{m(j)})_a\|_1^{(a)} \cdot \prod_{i=1}^{j-1} \|(\mathcal{M}^{m(i)})_b\|_{\mathbf{B}_Z^{p,q}} \leq C(m_0)\rho^n.
\end{aligned}$$

Therefore we obtain the claim (7.1).

7.2. The case $r \leq d+p-q+1$. In this case, the operators $(\mathcal{M}^m)_c$ and $(\mathcal{M}^m)_a$ may not belong to $\mathcal{L}_1^{(a)}(\mathbf{B}_Z^{p,q}, \widehat{\mathbf{B}}_Z^{p,q})$, so we have to modify the simple proof in the last subsection. Recall the integer $k_* = k_*(d, r, p, q) \geq 2$ from (6.4).

Consider a large integer n and write it as a sum $n = m(1) + m(2) + \cdots + m(k)$ with $m_0 \leq m(i) \leq 2m_0$. Using (7.3), we write the product $\mathcal{M}_1^n = \prod_{i=1}^k \mathcal{M}_1^{m(i)}$ as

$$\begin{aligned}
\mathcal{M}_1^n = \prod_{i=1}^k (\mathcal{M}^{m(i)})_b \; &+ \sum_{\nu < k_*} \sum_{1 \leq j(1) < j(2) < \cdots < j(\nu) \leq k} M(\{j(\ell)\}_{\ell=1}^{\nu}) \\
&+ \sum_{1 \leq j(1) < j(2) < \cdots < j(k_*) \leq k} M'(\{j(\ell)\}_{\ell=1}^{k_*}),
\end{aligned} \tag{7.4}$$

where, setting $j(0)=0$,

$$M(\{j(\ell)\}_{\ell=1}^{\nu}) = \prod_{i=j(\nu)+1}^{k} (\mathcal{M}^{m(i)})_b \cdot \prod_{\ell=1}^{\nu}\Bigg((\mathcal{M}^{m(j(\ell))})_a \cdot \Bigg(\prod_{i=j(\ell-1)+1}^{j(\ell)-1} (\mathcal{M}^{m(i)})_b \Bigg)\Bigg),$$

and

$$M'(\{j(\ell)\}_{\ell=1}^{k_*}) = \prod_{i=j(k_*)+1}^{k} \mathcal{M}_1^{m(i)} \cdot \prod_{\ell=1}^{k_*}\Bigg((\mathcal{M}^{m(j(\ell))})_a \cdot \Bigg(\prod_{i=j(\ell-1)+1}^{j(\ell)-1} (\mathcal{M}^{m(i)})_b \Bigg)\Bigg).$$

Remark 7.1. The decomposition above is obtained as follows. Consider the process to expand $\mathcal{M}_1^n = \prod_{i=1}^k \mathcal{M}_1^{m(i)}$ using (7.3) for $m=m(i)$ in the turn $i = 1, 2, , \dots, k$. For instance, we have

$$\mathcal{M}_1^n = \left(\prod_{i=2}^{k} \mathcal{M}_1^{m(i)} \right) \cdot (\mathcal{M}^{m(1)})_a + \left(\prod_{i=2}^{k} \mathcal{M}_1^{m(i)} \right) \cdot (\mathcal{M}^{m(1)})_b$$

for the first step. When we find a term that contains $(\mathcal{M}_1^{m(i)})_a$ for k_* times, proceeding in this way, we stop expanding that term, obtaining the terms in the second sum in (7.4). The other resulting terms are collected in the first sum.

From Proposition 6.4, the flat trace of the first term on the right hand side of (7.4) is zero. Therefore, to prove (7.1), it suffices to show the following estimates for the other terms:

$$|\mathrm{tr}^\flat M'(\{j(\ell)\}_{\ell=1}^{k_*})| \le C\rho^n \tag{7.5}$$

and

$$|\mathrm{tr}^\flat M(\{j(\ell)\}_{\ell=1}^{\nu})| \le C\rho^n\,. \tag{7.6}$$

By Proposition 6.6 and (7.2), we can see that $M'(\{j(\ell)\}_{\ell=1}^{k_*}) \in \mathcal{L}_1^{(a)}(\mathbf{B}_Z^{p,q}, \widehat{\mathbf{B}}_Z^{p,q})$ and that the estimate (7.5) holds. Since $(\mathcal{M}^{m(i)})_c - (\mathcal{M}^{m(i)})_a = \mathcal{M}_0^{m(i)}$ is of finite rank, the estimate (7.6) follows if we show

$$\left|\mathrm{tr}^\flat\Bigg(\prod_{i=j(\nu)+1}^{k} (\mathcal{M}^{m(i)})_b \cdot \prod_{\ell=1}^{\nu}\Bigg((\mathcal{M}^{m(j(\ell))})_c \cdot \prod_{i=j(\ell-1)+1}^{j(\ell)-1} (\mathcal{M}^{m(i)})_b \Bigg)\Bigg)\right| \le C\rho^n\,. \tag{7.7}$$

In the following, we will work directly with kernels of operators to prove (7.7). Although the notation become a little complex, the argument is straightforward. Let $\mathcal{Y}$ be the set of sequences $\{\omega(i), \vec{\omega}(i)\}_{i=1}^k$ with $\omega(i) \in \Omega$ and $\vec{\omega}(i) \in \Omega_{m(i)}$. For $Y = \{\omega(i), \vec{\omega}(i)\}_{i=1}^k \in \mathcal{Y}$ and $1 \le i \le k$, we define the relation $\hookrightarrow_{Y,i}$ on Γ as the relation $\hookrightarrow_{\mathcal{T},G}$ defined in the setting (5.2) with $\omega = \omega(i)$, $\omega' = \omega(i+1)$ and $\vec{\omega} = \vec{\omega}(i)$. (Put $\omega(k+1) = \omega(1)$ in case $i=k$.) For $Y = \{\omega(i), \vec{\omega}(i)\}_{i=1}^k \in \mathcal{Y}$, let $\mathcal{Z}(Y)$ be the set of sequences $\{\zeta(i)\}_{i=1}^{k+1}$ in Z^{k+1} such that $\zeta(k+1) = \zeta(1)$, that $\omega(\zeta(i)) = \omega(i)$ for all $1 \le i \le k$ and that

$$(n(\zeta(i)), \sigma(\zeta(i))) \hookrightarrow_{Y,i} (n(\zeta(i+1)), \sigma(\zeta(i+1))) \quad \text{iff} \quad i \notin J := \{j(\ell)\}_{\ell=1}^{\nu}\,.$$

Then, to prove (7.7), it is enough to show

$$\sum_{Y=\{\omega(i),\vec{\omega}(i)\}_{i=1}^k \in \mathcal{Y}} \; \sum_{\{\zeta(i)\}_{i=1}^{k+1} \in \mathcal{Z}(Y)} \left| \mathrm{tr}^\flat \left(\prod_{i=1}^{k} (\mathcal{M}_{\vec{\omega}(i)}^{m(i)})_{\zeta(i)\zeta(i+1)} \right) \right| \le C\rho^n\,. \tag{7.8}$$

Let $Y = \{\omega(i), \vec{\omega}(i)\}_{i=1}^k \in \mathcal{Y}$ and $\{\zeta(i) = (\omega(i), n(i), \sigma(i))\}_{i=1}^{k+1} \in \mathcal{Z}(Y)$. By the definition, we have

$$\operatorname{tr}^\flat\left(\prod_{i=1}^k (\mathcal{M}_{\vec{\omega}(i)}^{m(i)})_{\zeta(i)\zeta(i+1)}\right) = \int \left(\prod_{i=1}^k K_i(x_i, x_{i+1})\right) dx_1 dx_2 \dots dx_k$$

where we read $x_{k+1} = x_1$ and put

$$K_i(x,y) = \int \widehat{\psi}_{\Theta_{\omega(i+1)}, n(i+1), \sigma(i+1)}(y-w) \cdot G_i(w) \cdot \widehat{\widehat{\psi}}_{\Theta_{\omega(i)}, n(i), \sigma(i)}(\mathcal{T}_i(w) - x) dw$$

with $\mathcal{T}_i := \mathcal{T}^{m(i)}_{\vec{\omega}(i), \omega(i+1)\omega(i)}$ and $G_i := G_{\vec{\omega}(i), \omega(i+1)\omega(i)}$. (Recall Subsection 5.2.) Here we canceled the term $\widehat{\chi}_{n(1)+2}$ by using $\widehat{\widehat{\psi}}_{\Theta_{\omega(1)}, n(1), \sigma(1)} * \widehat{\chi}_{n(1)+2} = \widehat{\widehat{\psi}}_{\Theta_{\omega(1)}, n(1), \sigma(1)}$.

For all $1 \le i \le k$, we have, from (4.23),

$$|K_i(x,y)| \le C \cdot \|G_i\|_{L^\infty} \cdot \int b_{n(i+1)}(y-w) \cdot b_{n(i)}(\mathcal{T}_i(w) - x)\, dw\,.$$

If $i \in J$, we have, from (4.24),

$$|K_i(x,y)| \le C(\mathcal{T}_i, G_i) \cdot 2^{-(r-1)\max\{n(\zeta(i)), n(\zeta(i+1))\}} b_{\min\{n(\zeta(i)), n(\zeta(i+1))\}}(y-x).$$

Therefore, using (4.22), we see that $|\operatorname{tr}^\flat(\prod_{i=1}^k (\mathcal{M}_{\vec{\omega}(i)}^{m(i)})_{\zeta(i)\zeta(i+1)})|$ is bounded by

$$C^k \cdot C(m_0)^{k_*} \cdot \prod_{i \in J} 2^{-(r-1)\max\{n(i+1), n(i)\}} \prod_{i \notin J} \sup_{V_{\vec{\omega}(i)}} |g^{(m(i))}|$$
$$\cdot \int b_{\nu(k)}(\mathcal{T}_k(x_0) - x_k) \cdots b_{\nu(2)}(\mathcal{T}_2(x_3) - x_2) \cdot b_{\nu(1)}(\mathcal{T}_1(x_2) - x_1)\, dx_1 \cdots dx_k\,,$$

where the constant C does not depend on the choice of m_0 while $C(m_0)$ may, and

$$\nu(i) = \begin{cases} \min\{n(i), n(i+1)\}, & \text{if } i \in J\ ; \\ n(i), & \text{if } i \notin J\ . \end{cases}$$

From hyperbolicity of T and the assumption that the covering $\mathcal{V}$ is generating, we may choose the extensions of the diffeomorphisms $\mathcal{T}_i$ so that the mapping

$$S : (\mathbb{R}^d)^k \to (\mathbb{R}^d)^k, \quad (x_i)_{i=1}^k \mapsto (\mathcal{T}_1(x_2) - x_1, \mathcal{T}_2(x_3) - x_2, \dots, \mathcal{T}_k(x_0) - x_k)\,,$$

is a diffeomorphism and satisfy

$$\inf_{\mathbb{R}^d} |\det DS| \ge C^{-k} \cdot \prod_{i=1}^k \left(\sup_{V_{\vec{\omega}(i)}} |\det DT^{m(i)}|_{E^u}|\right)$$

for a constant $C > 0$ that does not depend on the choice of m_0. Hence we get the following estimate for each term in (7.8):

$$\left|\operatorname{tr}^\flat\left(\prod_{i=1}^k (\mathcal{M}_{\vec{\omega}(i)}^{m(i)})_{\zeta(i)\zeta(i+1)}\right)\right| \le C^k \cdot C(m_0)^{k_*} \prod_{i \in J} 2^{-(r-1)\max\{n(i+1), n(i)\}} \prod_{i \notin J} \frac{\sup_{V_{\vec{\omega}(i)}} |g^{(m(i))}|}{\inf_{V_{\vec{\omega}(i)}} |\det DT^{m(i)}|_{E^u}|}\,. \tag{7.9}$$

Putting $c(-) = q$ and $c(+) = p$, we have, for $i \notin J$,

$$c(\sigma(i+1))n(i+1) - c(\sigma(i))n(i) \le \min\{p \cdot h^+_{\max}(\mathcal{T}_i, G_i), q \cdot h^-_{\min}(\mathcal{T}_i, G_i)\}\,.$$

We may assume that the right hand side is negative, taking larger m_0 if necessary. Since $\zeta(k+1)=\zeta(1)$, we have

(7.10)

$$\begin{aligned}
&-\sum_{i\in J}(r-1)\max\{n(i+1),n(i)\}\\
&\quad\le -\left(\sum_{i\in J}(c(\sigma(i+1))n(i+1)-c(\sigma(i))n(i))\right)+(r-p+q-1)\max_{1\le i\le k} n(i)\\
&\quad\le \left(\sum_{i\notin J}(c(\sigma(i+1))n(i+1)-c(\sigma(i))n(i))\right)+(r-p+q-1)\max_{1\le i\le k} n(i).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\Bigg|\sum_{\{\zeta(i)\}_{i=1}^{k+1}\in\mathcal{Z}(Y)} \operatorname{tr}^\flat\left(\prod_{i=1}^{k}(\mathcal{M}^{m(i)}_{\vec\omega(i)})_{\zeta(i)\zeta(i+1)}\right)\Bigg|\\
&\le C^k\cdot C(m_0)^{k_*}\cdot\prod_{i\notin J}\left(\frac{\sup_{V_{\vec\omega(i)}}|g^{(m(i))}|}{\inf_{V_{\vec\omega(i)}}|\det DT^{m(i)}|_{E^u}|}\cdot 2^{\min\{ph^+_{\max}(\mathcal{T}_i,G_i),qh^-_{\min}(\mathcal{T}_i,G_i)\}}\right)\\
&\le C^k\cdot C(m_0)^{k_*}\cdot\prod_{i\notin J}\left(\frac{\sup_{V_{\vec\omega(i)}}|g^{(m(i))}|}{\inf_{V_{\vec\omega(i)}}|\det DT^{m(i)}|_{E^u}|}\cdot\sup_{V_{\vec\omega(i)}}\lambda^{(p,q,m)}\right).
\end{aligned}$$

Recalling the function $\tilde g$ and the definition of Ω_m in Subsection 5.1 and using (5.8), we conclude that (7.8) is bounded by

$$C(\tilde g)^k\cdot C(m_0)^{k_*}\cdot\prod_{i\notin J}(Q^{p,q}_*(T,\tilde g,\mathcal{W},m(i)))\,.$$

By Lemma 3.3, we may take $\tilde g$ so that $Q^{p,q}_*(T,\tilde g)<\rho$. Since the constant $C(\tilde g)$ above does not depend on the choice of m_0, we obtain (7.7) by taking large m_0. We finished the proof of Theorem 1.5.

Appendix A. Eigenvalues and eigenvectors for different Banach spaces

In Theorem 1.1, we may choose a variety of p and q. Besides, as we will see in the proof, the space $\mathcal{C}^{p,q}(T,V)$ depends on many objects, such as the system of local charts. Moreover, in [**7**] and [**15**], other Banach spaces of distribution were introduced, for which the analogue of Theorem 1.1 holds with different bounds on the essential spectral radius. So one may ask to what extent the eigenvalues of the Ruelle transfer operator on different Banach spaces coincide. Theorem 1.5 gives one answer to this question because the dynamical Fredholm determinant does not depend on the choice of Banach spaces. The following simple abstract lemma, which may not be new, gives a more direct answer.

LEMMA A.1. *Let B be a separated topological linear space and let $(B_1,\|\cdot\|_1)$ and $(B_2,\|\cdot\|_2)$ be Banach spaces that are continuously embedded in B. Suppose further that there is a subspace $B_0\subset B_1\cap B_2$ that is dense both in the Banach spaces $(B_1,\|\cdot\|_1)$ and $(B_2,\|\cdot\|_2)$. Let $L:B\to B$ be a continuous linear map, which preserves the subspaces B_0, B_1 and B_2. Suppose that the restriction of L to B_1 and B_2 are bounded operators whose essential spectral radii are both strictly smaller* *than some number $\rho > 0$. Then the eigenvalues of $L : B_1 \to B_1$ and $L : B_2 \to B_2$ in $\{z \in \mathbb{C} \mid |z| > \rho\}$ coincide. Further the corresponding generalized eigenspaces coincide and are contained in the intersection $B_1 \cap B_2$.*

PROOF. First, we show that the essential spectral radius $r_{ess}(\mathcal{L})$ of an operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ on a Banach space $\mathcal{B}$ can be expressed as

$$\inf\{\, r(\mathcal{L}|_W) \mid W \subset \mathcal{B} \text{ is a closed } \mathcal{L}\text{-invariant subspace of finite codimension.}\},$$

where $r(\mathcal{L}|_W)$ is the spectral radius of the restriction of $\mathcal{L}$ to W. Indeed, take any $\tilde{\rho} > r_{ess}(\mathcal{L})$, and let W be the image of the spectral projector corresponding to the part of spectrum in the disk $\{|z| < \tilde{\rho}\}$, then we see that the infimum above is not greater than $\tilde{\rho}$ and hence not greater than $r_{ess}(\mathcal{L})$. Next let W be an arbitrary closed $\mathcal{L}$-invariant subspace of finite codimension, and let W' be a complementary subspace of W in $\mathcal{B}$ of finite dimension. Let $\pi : \mathcal{B} \to W$ and $\pi' : \mathcal{B} \to W'$ be the projections corresponding to the decomposition $\mathcal{B} = W \oplus W'$. Then we can decompose $\mathcal{L}$ as $\mathcal{L} = \mathcal{L} \circ \pi + \mathcal{L} \circ \pi'$, where $\mathcal{L} \circ \pi'$ is of finite rank. This implies that the essential spectral radius of $\mathcal{L}$ is bounded by $r(\mathcal{L} \circ \pi) = r(\mathcal{L}|_W)$ and hence by the infimum above.

The intersection $B_1 \cap B_2$ is a Banach space with respect to the norm $\|\cdot\|_1 + \|\cdot\|_2$. From the definition above, we can see that the essential spectral radius of the restriction $L : B_1 \cap B_2 \to B_1 \cap B_2$ is bounded by the maximum of those of $L : B_1 \to B_1$ and $L : B_2 \to B_2$. Thus, to prove the lemma, we may and do assume $B_1 \subset B_2$ and $\|\cdot\|_2 \le \|\cdot\|_1$.

Consider $\rho > 0$ as in the statement of the lemma. Let $E \subset B_1$ be the finite dimensional subspace that is the sum of generalized eigenspaces of $L : B_1 \to B_1$ for eigenvalues in $\{z \in \mathbb{C} \mid |z| \ge \rho\}$. Replacing B_1 and B_2 by their factor space by E respectively, we may and do assume that $E = \{0\}$ or that the spectral radius of $L : B_1 \to B_1$ is strictly smaller than ρ.

We can now complete the proof by showing that $L : B_2 \to B_2$ has no eigenvalues greater than or equal to ρ in absolute value. Suppose that it were not true. Then we could take an eigenvector for $L : B_2 \to B_2$ corresponding to an eigenvalue λ so that $|\lambda|$ is equal to the spectral radius of $L : B_2 \to B_2$ and is not less than ρ. Since B_0 is dense in B_2, this would imply that there exists a vector $v \in B_0 \subset B_1$ such that $\|L^n v\|_1 \ge \|L^n v\|_2 \ge |\lambda|^n$ for all $n \ge 0$. This contradicts the fact that the spectral radius of $L : B_1 \to B_1$ is strictly smaller than ρ. □

Since the spaces of functions in this paper and [**7**], as well as those in [**15**], are completions of the space of C^{r-1} functions and embedded in the space of distributions, the lemma above tells that the part of spectrum of $\mathcal{L}$ outside of the essential spectral radius does not depend on those choices of Banach spaces.

In view of Lemma A.1, it is natural to ask whether there exists a Banach space containing $C^{r-1}(V)$ on which $\mathcal{L}$ is bounded and has essential spectral radius strictly smaller than $Q_{r-1}(T, g)$. We expect that there may be such Banach spaces if $d \ge 3$ but not if $d = 2$. (For hyperbolic endomorphisms, we can find examples of such Banach spaces in [**2**, Thereom 3].)

Appendix B. Proof of the inequality (1.3)

We show here the claim in Remark 1.3 that $\rho^{p,q}(T,g) \le \inf_{t\in[1,\infty]} R^{p,q,t}(T,g)$. The proof will imply that the inequality can be strict. Put

$$R^{p,q,t}(T,g,m) = \sup_X |\det DT^m|^{-1/t}(x)|g^{(m)}(x)|\lambda^{(p,q,m)}(x)$$

for $m \ge 1$ and $t \in [1,\infty]$. Since $\mathrm{supp}(g)$ is contained in V, we have

$$R^{p,q,t}(T,g) = \lim_{m\to\infty} \left(R^{p,q,t}(T,g,m)\right)^{1/m} . \tag{B.1}$$

For each $m \ge 0$, we put $V^m = \cap_{k=0}^{m-1} T^{-k}(V)$, $V_+^m = \{x \in V^m; |\det DT^m(x)| \ge 1\}$ and $V_-^m = V^m - V_+^m$. Then we have

$$\begin{aligned}\int_X (|g^{(m)}| \cdot \lambda^{(p,q,m)})(x)dx \le \int_{V_-^m} &(|g^{(m)}| \cdot \lambda^{(p,q,m)})(x)dx \\ &+ \int_{T^m(V_+^m)} (|\det DT^m|^{-1} \cdot |g^{(m)}|\lambda^{(p,q,m)})(T^{-m}(y))dy .\end{aligned}$$

Since $|\det DT^m|^{-1/t} \ge 1$ on V_-^m and $|\det DT^m|^{-1/t} \ge |\det DT^m|^{-1}$ on V_+^m, and since $V_-^m \subset V$ and $T^m(V_+^m) \subset T(V)$, we obtain

$$\int_X (|g^{(m)}| \cdot \lambda^{(p,q,m)})(x)dx \le C \cdot R^{p,q,t}(T,g,m)$$

for any $m \ge 1$ and $t \in [1,\infty]$. Now the claim follows from (B.1) and the definition of $\rho^{p,q}(T,g)$.

Appendix C. The key estimate (4.24) for integer r

For the convenience of the reader, we reproduce here the proof of (4.24) from [**7**], when $r \ge 2$ is an integer.

By "integration by parts on w", we will mean application, for $f \in C^2(\mathbb{R}^d)$ and $g \in C_0^1(\mathbb{R}^d)$, with $\sum_{j=1}^d (\partial_j f)^2 \ne 0$ on $\mathrm{supp}(g)$, of the formula[17]

$$\int_{\mathbb{R}^d} e^{if(w)} g(w)\, dw = i \cdot \int_{\mathbb{R}^d} e^{if(w)} \cdot \sum_{k=1}^d \partial_k \left(\frac{\partial_k f(w) \cdot g(w)}{\sum_{j=1}^d (\partial_j f(w))^2} \right) dw .$$

Rewrite the operator $S_{n,\sigma}^{\ell,\tau}$ as $S_{n,\sigma}^{\ell,\tau}(u)(x) = \int_{\mathbb{R}^d} V_{n,\sigma}^{\ell,\tau}(x,y)|\det D\mathcal{T}|u(\mathcal{T}(y))\, dy$, where

$$V_{n,\sigma}^{\ell,\tau}(x,y) = (2\pi)^{-2d} \int e^{i(x-w)\xi + i(\mathcal{T}(w)-\mathcal{T}(y))\eta} G(w)\psi_{\Theta',n,\sigma}(\xi)\tilde\psi_{\Theta,\ell,\tau}(\eta)\, dw\, d\xi d\eta . \tag{C.1}$$

The required estimate thus follows if we show, for some $C(\mathcal{T},G) > 0$, that

$$|V_{n,\sigma}^{\ell,\tau}(x,y)| \le C(\mathcal{T},G) 2^{-(r-1)\max\{n,\ell\}} \cdot 2^{d\min\{n,\ell\}} b(2^{\min\{n,\ell\}}(x-y)) \tag{C.2}$$

for all $(\ell,\tau),(n,\sigma) \in \Gamma$ with $(\ell,\tau) \not\hookrightarrow (n,\sigma)$. Recall the constant $C(\mathcal{T})$ in (4.18). W may and do assume $\max\{n,\ell\} > C(\mathcal{T})$ in proving (C.2). Integrating (C.1) by parts $(r-1)$ times on w, we obtain

$$V_{n,\sigma}^{\ell,\tau}(x,y) = (2\pi)^{-2d} \int e^{i(x-w)\xi + i(\mathcal{T}(w)-\mathcal{T}(y))\eta} F(\xi,\eta,w)\psi_{\Theta',n,\sigma}(\xi)\tilde\psi_{\Theta,\ell,\tau}(\eta)\, dw\, d\xi\, d\eta$$

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[17] To handle noninteger $r > 1$, we may use "regularised integration by parts" instead. See [**7**].

where $F(\xi,\eta,w)$ is C^∞ in the variables ξ and η, continuous in w and supported on $\mathbb{R}^d\times\mathbb{R}^d\times\mathrm{supp}(G)$. Using (4.18), we can see that, if $\psi_{\Theta',n,\sigma}(\xi)\tilde\psi_{\Theta,\ell,\tau}(\eta)\neq 0$, then, for each multi-indices α, β and for $\xi,\eta\in\mathbb{R}^d$,

$$\|\partial_\xi^\alpha\partial_\eta^\beta F(\xi,\eta,\cdot)\|_{C^0}\le C_{\alpha,\beta}(\mathcal{T},G)2^{-n|\alpha|-\ell|\beta|-(r-1)\max\{n,\ell\}}\,. \tag{C.3}$$

Put $H_{n,\ell}(\xi,\eta,w)=F(\xi,\eta,w)\psi_{\Theta',n,\sigma}(\xi)\tilde\psi_{\Theta,\ell,\tau}(\eta)$, and consider the scaling

$$\widetilde H_{n,\ell}(\xi,\eta,w)=H_{n,\ell}(2^n\xi,2^\ell\eta,w)\,.$$

The estimate (C.3) implies that for all α and β

$$\|\partial_\xi^\alpha\partial_\eta^\beta\widetilde H_{n,\ell}(\xi,\eta,\cdot)\|_{C^0}\le C_{\alpha,\beta}(\mathcal{T},G)2^{-(r-1)\max\{n,\ell\}}\,.$$

Denote by $\mathbb{F}^{-1}_{\xi\eta}$ the inverse Fourier transform with respect to the variables ξ and η. Then the estimate above on $\widetilde H_{n,\ell}$ implies

$$\begin{aligned}|(\mathbb{F}^{-1}_{\xi\eta}H_{n,\ell})(x,y,w)|&=|(\mathbb{F}^{-1}_{\xi\eta}\widetilde H_{n,\ell})(2^nx,2^\ell y,w)|\\&\le C_{\alpha,\beta}(\mathcal{T},G)\cdot 2^{-(r-1)\max\{n,\ell\}}\cdot b_n(x)\cdot b_\ell(y)\end{aligned}$$

where b_m is the function defined in (4.21). Therefore we obtain

$$\begin{aligned}|V^{n,\sigma}_{\ell,\tau}(x,y)|&=\left|\int_{\mathrm{supp}(G)}(\mathbb{F}^{-1}_{\xi\eta}H_{n,\ell})(x-w,\mathcal{T}(w)-\mathcal{T}(y),w)dw\right|\\&\le C(\mathcal{T},G)\cdot 2^{-(r-1)\max\{n,\ell\}}\cdot\int b_n(x-w)\cdot b_\ell(w-y)dw\,,\end{aligned}$$

where we used the fact that $\mathcal{T}$ is bilipschitz to replace $\mathcal{T}(w)-\mathcal{T}(y)$ by $w-y$. Now (C.2) follows from (4.22).

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