Low temperature scaling of linear response coefficients in strongly disordered systems

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Motivations

- Low temperature electron transport in strongly disordered media
- Strong disorder: presence of many (infinite) relevant energetic scales

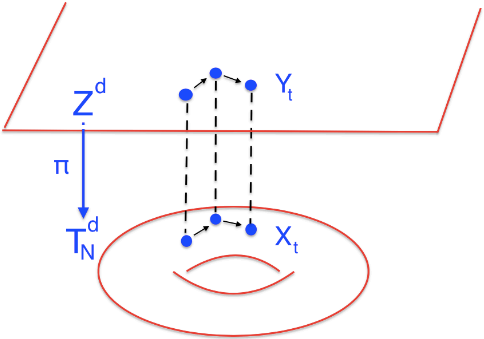
Message: *Linear response coefficients are not explicit and are given as solutions of variational problems on infinite dimensional spaces. Tuning a parameter (e.g. temperature going to zero) can amplify the separation of the energetic scales and a proper universal scaling behavior can emerge*

Random walk X_t on the discrete torus

- $\mathbb{T}_N^d = \mathbb{Z}^d / N\mathbb{Z}^d$
- $(X_t)_{t\geq 0}$: continuous time random walk on $\mathbb{T}_N^d$
- $c(x,y)$: probability rate for a jump $x \curvearrowright y$
- $c(x,y) = 0$ if $|x-y| \neq 1$
- $\mathbb{P}\big(X_{t+dt} = y | X_t = x\big) = c(x,y)dt + o(dt)$

Random walk Y_t on $\mathbb{Z}^d$

- $\pi : \mathbb{Z}^d \to \mathbb{T}_N^d$ canonical projection
- $(Y_t)_{t\geq 0}$ random walk on $\mathbb{Z}^d$ with N–periodic jump probability rates $c(x,y)$
- $\pi(Y_t) = X_t$



Assumption

Assumption

$(X_t)_{t\geq 0}$ is **irreducible** *and its unique invariant distribution μ is* **reversible***, i.e. μ satisfies the detailed balance condition*

$$\mu(x)c(x,y) = \mu(y)c(y,x) \qquad \forall x, y \in \mathbb{T}_N^d\,.$$

Reversibility: if X_0 has law μ, then $\forall T > 0$ the random trajectories

$$\big(X_t\big)_{0\leq t\leq T} \quad \text{and} \quad \big(X_{T-t}\big)_{0\leq t\leq T}$$

have the same law.

External field

- **External field**$=\lambda w$, $\lambda > 0$ tuning parameter, $w \in \mathbb{R}^d$
- $(X_t^\lambda)_{t\geq 0}$: perturbed random walk on $\mathbb{T}_N^d$
- $(Y_t^\lambda)_{t\geq 0}$: perturbed random walk on $\mathbb{Z}_N^d$
- $c_\lambda(x,y)$: perturbed jump probability rate

Local detailed balance

- Local detailed balance:

$$\frac{\mathbf{c}_\lambda(\mathbf{x},\mathbf{y})}{\mathbf{c}_\lambda(\mathbf{y},\mathbf{x})} = \frac{\mathbf{c}(\mathbf{x},\mathbf{y})}{\mathbf{c}(\mathbf{y},\mathbf{x})}\mathbf{e}^{\beta\boldsymbol{\Delta}(\mathbf{x},\mathbf{y})},$$

- $\beta = 1/kT$: inverse temperature
- $\Delta(x,y)$: work done by the field on the particle jumping $x \curvearrowright y$
- $\Delta(x,y) = \lambda w \cdot (y-x)$
- Example: $c_\lambda(x,y) = e^{\lambda\frac{\beta}{2}w\cdot(y-x)}c(x,y)$

Asymptotic velocity

- External field=λw, $\lambda > 0$, $w \in \mathbb{R}^d$
- μ_λ: unique invariant distribution of $(X_t^\lambda)_{t\geq 0}$
- $\psi_\lambda(x) := \sum_{|e|=1} c_\lambda(x, x+e)e$ instantaneous local drift
- $v_\lambda(w) := \mu_\lambda(\psi_\lambda)$

Fact

$v_\lambda(w)$ is the asymptotic velocity:

$$v_\lambda(w) = \lim_{t\to\infty} \frac{d}{dt}\mathbb{E}[X_t^\lambda] = \lim_{t\to\infty} \frac{d}{dt}\mathbb{E}[Y_t^\lambda],$$

$$v_\lambda(w) = \lim_{t\to\infty} \frac{Y_t^\lambda}{t} \qquad a.s.$$

Mobility matrix

- $\psi(x) := \sum_{|e|=1} c(x, x+e)e$ instantaneous local drift at equilibrium
- $\psi(x) = (\psi_1(x), \dots, \psi_d(x))$
- $\mathcal{L}f(x) = \sum_{|e|=1} c_{x,x+e} \left(f(x+e) - f(x)\right)$ Markov generator
- $\mathcal{L}$: symmetric operator in $L^2(\mu)$ (positive spectral gap)

Fact

It holds

$$\partial_{\lambda=0} v_\lambda(w) = \sigma w \qquad \forall w \in \mathbb{R}^d ,$$

where σ *is the* $d \times d$ *symmetric matrix*

$$\sigma_{ij} := \beta\mu\big(c(\cdot, \cdot + e_i)\big)\delta_{i,j} - \beta \int_0^\infty \langle \psi_i, e^{s\mathcal{L}}\psi_j \rangle_{L^2(\mu)} ds$$

σ: mobility matrix

Diffusion matrix and Einstein's relation

Fact

As $\varepsilon \downarrow 0$ *the diffusively rescaled random walk* $(\varepsilon Y_{\varepsilon^{-2}t})_{t \geq 0}$ *converges to a Brownian motion with diffusion matrix* D*, satisfying the Einstein relation:*

$$\sigma = \frac{\beta}{2} D$$

$D_{\text{phys}} = D/2$, hence $\sigma = \beta D_{\text{phys}}$

In what follows, we add the subindex N: σ_N, D_N

Disordered systems: random conductance model

- $\mathbb{E}^d$ = {unoriented edges of $\mathbb{Z}^d$}
- $\omega = \big(\omega(x,y) : \{x,y\} \in \mathbb{E}^d\big)$ stationary ergodic random field
- $\omega(x,y) > 0$ and $\mathbb{E}[\omega(x,y)] < +\infty$
- Fixed ω and N, $\omega^{(N)}$ is the N-periodization of ω

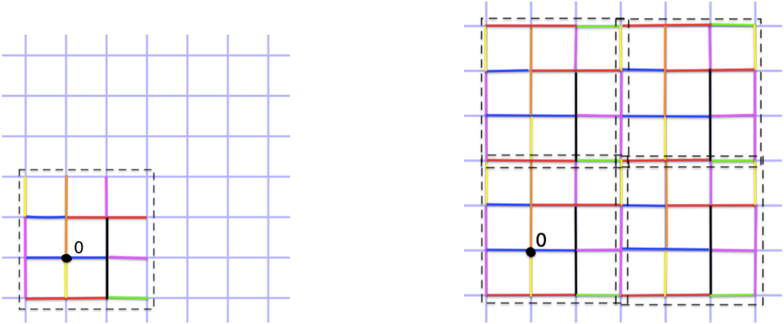


Figure: Left: ω. Right: $\omega^{(N)}$. $N = 3$.

N–mobility and N–diffusion matrix

- Given ω, N, let $X_t, X_t^\lambda, Y_t, Y_t^\lambda$ as before, where

$$c(x,y) := \omega^{(N)}(x,y)\,.$$

- $\sigma_N(\omega)$: mobility matrix
- $D_N(\omega)$: diffusion matrix
- Recall: $\sigma_N(\omega) = \frac{\beta}{2} D_N(\omega)$
- $\mathbb{P}$: law of the random field ω

Infinite volume limit

Theorem (Homogenization)

For $\mathbb{P}$–*a.a.* ω,

$$\begin{cases} \lim_{N\to\infty} \sigma_N(\omega) = \sigma\,, \\ \lim_{N\to\infty} D(\omega) = D\,, \end{cases}$$

where $\sigma = \frac{\beta}{2} D$ *and* D *is the* $d \times d$ *symmetric matrix*

$$a \cdot Da = \inf_{f \in L^\infty(\mathbb{P})} \frac{1}{2} \sum_{e:|e|=1} \int d\mathbb{P}(\omega)\omega(0,e)\left(a \cdot e - \nabla_e f(\omega)\right)^2$$

$\forall a \in \mathbb{R}^d$, *where* $\nabla_e f(\omega) := f(\tau_e \omega) - f(\omega)$.

σ, D enter into play in other problems

- Random walk on $\mathbb{Z}^d$
- $\omega(x,y)$=probability rate for a jump $x \curvearrowright y$
- $\omega(x,y)$ bounded from above.

Fact (P. Mathieu, JSP 2008)

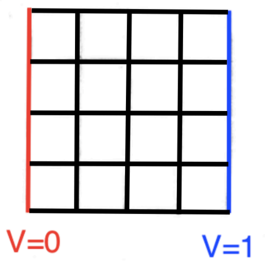
Diffusion behavior: *For $\mathbb{P}$–a.a. ω the diffusively rescaled random walk converges to a Brownian motion with diffusion matrix D*

Fact ($d=1$ H.C. Lam, J Depauw, SPA 2016; $d \geq 2$ N.Gantert, X.Guo, J.Nagel, AP 2017)

Linear response: *Under the field λw, for $\mathbb{P}$–a.a. ω the asymptotic velocity $v_\lambda(w)$ of the perturbed random walk is ω-independent and satisfies $\partial_{\lambda=0} v_\lambda(w) = \sigma w$.*

Random resistor network

- Fix ω
- node set: $\Lambda_L := [-L, L]^d \cap \mathbb{Z}^d$
- filaments: each edge $\{x, y\}$, with $x, y \in \Lambda_L$, has conductivity $\omega(x, y)$
- $C_L(\omega)$: effective conductivity along $1st$ direction (V=electrical potential)



Limit behavior of $C_L(\omega)$

For simplicity let D be diagonal

Theorem (A.F. 2019+)

Let $\mathbb{E}[\omega(0, e_i)^2] < +\infty$ *for all* $i = 1, 2, \dots, d$. *Then,* $\mathbb{P}$*–a.a.* ω*,*

$$\lim_{L \to \infty} L^{2-d} C_L(\omega) = D_{1,1} \,.$$

Strongly disordered systems

- $\omega(x,y) = e^{-\beta A(x,y)}$
- $(A(x,y) : \{x,y\} \in \mathbb{E}^d)$: β–independent i.i.d. random variables
- Energetic barriers
- Interesting regime: $T \ll 1$, i.e. $\beta \gg 1$
- $D = D(\beta)$

Low temperature scaling limit of $D(\beta)$

- Take isotropic medium. Hence $D(\beta) = D_{1,1}(\beta)\mathbb{I}$
- $\omega(x,y) = e^{-\beta A(x,y)}$
- $G(\theta)$: graph with edges $\{x,y\}$ such that $A(x,y) \leq \theta$, i.e. $\omega(x,y) \geq e^{-\beta\theta}$

Theorem (A.F. 2019+)

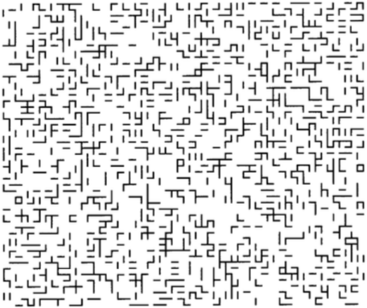
Let $\theta_c := \inf\{\theta \, : G(\theta) \text{ percolates}\}$. *Then*

$$\lim_{\beta\to\infty} -\frac{1}{\beta} \ln D_{1,1}(\beta) = \theta_c \,.$$

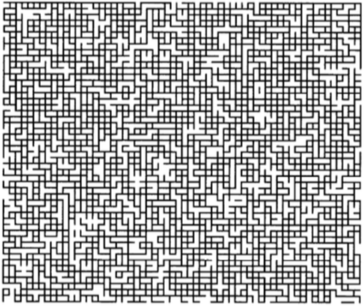
$D(\beta) \approx e^{-\theta_c \beta}\mathbb{I}$

Bond percolation on $\mathbb{Z}^d$ with parameter p

- We build a random graph by keeping each edge of $\mathbb{Z}^d$ with probability p (otherwise we erase it), independently for each edge
- $p_c(d) \in (0,1)$: critical parameter p for the existence of infinite connected component



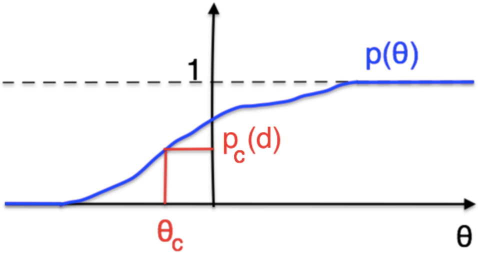
(a) $p = 0.25$



(d) $p = 0.75$

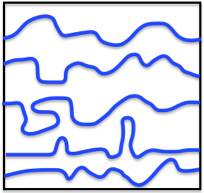
θ_c and bond percolation

- $G(\theta)$: graph with edges $\{x, y\}$ such that $A(x, y) \leq \theta$
- $G(\theta)$: bond percolation on $\mathbb{Z}^d$ with parameter $p(\theta) := \mathbb{P}\left(A(x, y) \leq \theta\right)$
- $\theta_c := \inf\{\theta \ : G(\theta) \text{ percolates}\}$



$D_{1,1}(\beta) \geq e^{-\theta_c \beta(1+o(1))}$

1. $D_{1,1}(\beta) = \lim_{L\to\infty} L^{2-d} C_L(\omega)$
2. $C_L(\omega) \geq$ effective conductivity of $G(\theta_c + \varepsilon)$ on Λ_L
3. Edges in $G(\theta_c + \varepsilon)$ have conductivity $\geq e^{-\beta(\theta_c+\varepsilon)}$
4. $G(\theta_c + \varepsilon)$ percolates
   $\Longrightarrow$ on Λ_L it has $O(L^{d-1})$ edge-disjoint left right crossings



$\Longrightarrow$ effective conductivity of $G(\theta_c + \varepsilon) \geq O(L^{d-2}) e^{-\beta(\theta_c+\varepsilon)}$

$D_{1,1}(\beta) \leq e^{-\theta_c \beta (1+o(1))}$

Variational characterization:

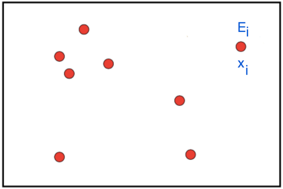
$$a \cdot Da = \inf_{f \in L^\infty(\mathbb{P})} \frac{1}{2} \sum_{e:|e|=1} \int d\mathbb{P}(\omega)\omega(0,e)\left(a \cdot e - \nabla_e f(\omega)\right)^2$$

Strategy: Take a clever test function f

Mott variable range hopping

- **Doped semiconductors**: crystalline solids with inserted atoms of a different type, called **impurities**
- Electron wavefunctions are **localized** around impurities and can **hop** by quantum tunneling
- In the regime of **low impurity density**, one considers independent random walks and encodes the electron interactions into the jump rates.
- Final object: **random walk on a marked simple point process**

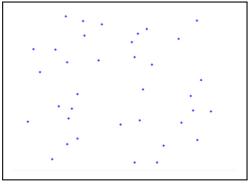
Marked point process



- $\{\bullet\} = \{x_i\}$: simple point process,
  random locally finite subset of $\mathbb{R}^d$
- E_i: mark of x_i, real random variable
- $\omega = \{(x_i, E_i)\}$ marked simple point process

Marked Poisson point process

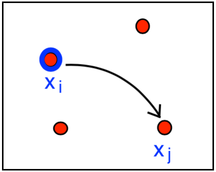
- $\{x_i\}$ is Poisson point process on $\mathbb{R}^d$ of intensity λ
  1. $\lambda = \mathbb{E}\left[\left|\{x_i\} \cap [0,1]^d\right|\right]$
  2. $A, B \subset \mathbb{R}^d$ and $A \cap B = \emptyset \Longrightarrow$
     $\left|\{x_i\} \cap A\right|$ and $\left|\{x_i\} \cap B\right|$ are independent random variables



- Points x_i are marked by i.i.d. random variables E_i

Random walk $(X_t^\omega)_{t\geq 0}$

- $\omega = \{(x_i, E_i)\}$ marked point process
- $(X_t^\omega)_{t\geq 0}$ continuous time random walk
- Markov chain such that
  - state space $= \{x_i\}$
  - $P(X_{t+dt}^\omega = x_j \,|\, X_t^\omega = x_i) = c_{x_i,x_j}(\omega)dt + o(dt), \;\; i \neq j$



Jump probability rates

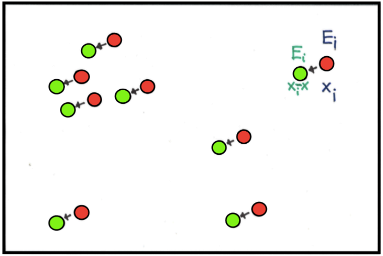
$$c_{x_i,x_j}(\omega) = \exp\Big\{ -\frac{2}{\gamma}|x_i - x_j| - \frac{\beta}{2}(|E_i| + |E_j| + |E_i - E_j|)\Big\}$$

- γ =localization length
- $\beta = \frac{1}{kT}$
- k =Boltzmann constant
- T =absolute temperature

Group of translations

Given $x \in \mathbb{R}^d$ and $\omega = \{(x_i, E_i)\}$,
define the x–translated configuration as

$$\tau_x \omega := \{(x_i - x, E_i)\}$$



$\mathbb{P}$: law of $\omega = \{(x_i, E_i)\}$. $\mathbb{P}$ stationary and ergodic.

Palm distribution

$\mathbb{P}_0$: Palm distribution associated to $\mathbb{P}$

- Probability measure with support $\{\omega : 0 \in \{x_i\}\}$,
- Roughly, $\mathbb{P}_0 = \mathbb{P}\big(\cdot\,|0 \in \{x_i\}\big)$
- By ergodicity,

$$\mathbb{P}_0 = \lim_{k\to\infty} \mathrm{Av}_{\substack{x:|x|\leq k\\ x\in\{x_i\}}} \delta_{\tau_x \omega}\,.$$

Effective diffusion matrix D

- Given $x \in \mathbb{R}^d$, $\nabla_x f(\omega) := f(\tau_x \omega) - f(\omega)$
- D: $d \times d$ symmetric matrix such that, $\forall a \in \mathbb{R}^d$,

$$a \cdot Da = \inf_{f \in L^\infty(\mathbb{P}_0)} \frac{1}{2} \int d\mathbb{P}_0(\omega) \sum_i c_{0,x_i}(\omega) \left(a \cdot x_i + \nabla_{x_i} f\right)^2$$

- $\mathbb{P}_0$: Palm distribution

Under very general assumptions:

Theorem (A.F., P.Caputo, T. Prescotti, AIHP 2013)

For $\mathbb{P}$–a.a. ω, as $\varepsilon \downarrow 0$ the diffusively rescaled random walk $\left(\varepsilon X^\omega_{\varepsilon^{-1/2}t}\right)_{t \geq 0}$ converges to a Brownian motion with diffusion matrix D.

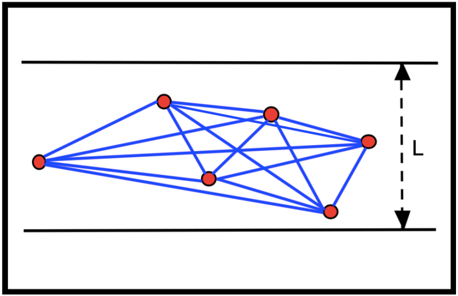
Linear response

One expects that, when applying an external field λw, the perturbed random walk $X_t^{\omega,\lambda}$ has asymptotic velocity $v_\lambda(w)$ satisfying

$$\begin{cases} \partial_{\lambda=0} v_\lambda(w) = \sigma w\,, \\ \sigma = \frac{\beta}{2} D\,. \end{cases}$$

For $d = 1$ it is proved (A.F., N. Gantert, M. Salvi AIHP, 2019)

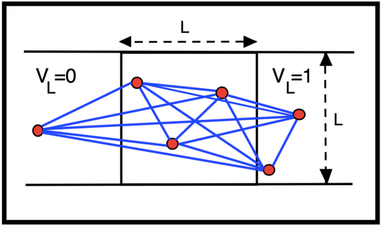
Miller-Abrahams random resistor network



Electrical filament with conductivity $c_{x_i,x_j}(\omega)$

Infinite volume effective conductivity

For simplicity: take D diagonal



$C_L(\omega)$: conductivity along the first coordinate
(electrical current in the box, along first direction)

Theorem (A.F. 2019+)

For $\mathbb{P}$–a.a. ω we have

$$\lim_{L\to\infty} L^{2-d} C_L(\omega) = D_{1,1}\,.$$

Mott law

Mott variable range hopping:

$$c_{x_i,x_j}(\omega) = \exp\Big\{ -\frac{2}{\gamma}|x_i - x_j| - \frac{\beta}{2}(|E_i| + |E_j| + |E_i - E_j|)\Big\}$$

$D_{1,1} = D_{1,1}(\beta)$

Physics:

1. $\{E_i\}$ i.i.d. random variables, law $c|E|^\alpha dE$ around zero, $\alpha \geq 0$
2. Mott law: $D_{1,1}(\beta) \approx e^{-C\,\beta^{\frac{\alpha+1}{\alpha+1+d}}}, \;\beta \gg 1$
3. C=?

Mott law

Homogenization + Percolation results (with A.H.Mimun):

Theorem (A.F. 2019+)

Let ω *be a marked Poisson point process of intensity* λ *and let* $E_i \geq 0$, *then*

$$\lim_{\beta \to +\infty} -\beta^{-\frac{\alpha+1}{\alpha+1+d}} \ln D_{1,1}(\beta) = (c_0/\lambda)^{\frac{1}{\alpha+1+d}}$$

where c_0 *admits a percolative characterization*

- $E_i \geq 0 \Longrightarrow$ FKG inequality
- **Universality**:
  Stationary and ergodic simple point processes are in the domain of attraction of PPP by thinning+rescaling

- $E_i \geq 0 \implies$ FKG inequality
- **Universality**:
  - Let ξ be a stationary ergodic simple point process of intensity $\lambda = \mathbb{E}\big[|\xi \cap [0,1]^d|\big]$
  - Given $p > 0$, let

$$[\xi]_p := \text{ } p\text{–thinning of } \xi$$

    i.e. $[\xi]_p$= site percolation on ξ with parameter p
  - Then

$$\lim_{p \downarrow 0} p^{-1/d}[\xi]_p = \text{ PPP with intensity } \lambda$$