Linear response for stochastic dynamics

P. Mathieu

Université d'Aix-Marseille

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Introduction

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Consider a stochastic dynamics $(X(t)\,;\,t \geq 0)$ "at equilibrium", with law $\mathbb{P}$.

Introduce a family of "perturbed dynamics" $(X^\lambda(t)\,;\,t \geq 0)$, with law $\mathbb{P}^\lambda$. ($\lambda \in [0,1]$ is the "strength" of the perturbation.
If $\lambda = 0$, then $X^\lambda = X$. We care about small λ's.)

Let $(A(t)\,;\,t \geq 0))$ be an additive functional. (Observable.)
Note $\mathbb{E}[A(t)] = t\mathbb{E}[A(1)]$.

AIM: Compare $\frac{1}{T}\mathbb{E}^\lambda[A(T)]$ and $\mathbb{E}[A(1)]$.
In particular: express the linear response $\partial_{\lambda=0}\frac{1}{T}\mathbb{E}^\lambda[A(T)]$ as a correlation or variance or covariance.

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We assume that: X is reversible.

Model to start with: diffusions on a torus.

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional torus.
Let $(X_x(t)\,;\, t \geq 0, x \in \mathbb{T}^d)$ be the solution of the sde:

$$dX_x(t) = b(X_x(t))dt + \sigma(X_x(t))dW_t\,;\, X_x(0) = x\,. \tag{1}$$

Here $(W_t\,;\, t \geq 0)$ is a $\mathbb{T}^d$-valued Brownian motion defined on some probability space $(\mathcal{W}, \mathcal{A}, P)$.
$\sigma = (\sigma(x)\,;\, x \in \mathbb{T}^d)$ is a smooth field of symmetric non-negative matrices over $\mathbb{T}^d$; $b = (b(x)\,;\, x \in \mathbb{T}^d)$ is a smooth vector field.

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The increment $X_x(t + \Delta t) - X_x(t)$ follows a Gaussian law

$$\mathcal{N}\big(b(X_x(t))\Delta t; a(X_x(t))\Delta t\big)$$

independent of the past before time t. ($a = \sigma^2$.)
The stochastic process $(X_x(t)\,;\, t \geq 0, x \in \mathbb{T}^d)$ is Markov with generator

$$\mathcal{L}f(x) = \frac{1}{2}\sum_{j,k} a_{j,k}(x)\nabla_j f(x)\nabla_k f(x) + \sum_j b_j(x)\nabla_j f(x)\,.$$

Model to start with: diffusions on a torus.

Recall

$$dX_x(t) = b(X_x(t))dt + \sigma(X_x(t))dW_t \, ; \; X_x(0) = x \, .$$

We assume that

$$a(x) = \sigma(x)^2 \, ; \; b(x) = \frac{1}{2} div(a(x)) \, ; \; \mathcal{L} = \frac{1}{2} div(a\nabla) \, .$$

The normalized Lebesgue measure on $\mathbb{T}^d$, say π, is a reversible and invariant probability measure.

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Reversible means that, for all T, if $x = X_x(0)$ has distribution π, then the forward evolution $(X_x(t)\,;\, 0 \le t \le T)$ and backward evolution $(X_x(T - t)\,;\, 0 \le t \le T)$ have the same distribution on path space. In particular $X_x(0)$ and $X_x(T)$ have the same law.

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The normalized Lebesgue measure on $\mathbb{T}^d$, say π, is a reversible and invariant probability measure.

Notation: $\mathcal{C}(\mathbb{R}_+;\mathbb{T}^d)$ be the set of continuous functions from $\mathbb{R}_+$ to $\mathbb{T}^d$. We use the notation $\mathbb{P}_x$ to denote the law of $(X_x(t)\,;\; t \geq 0)$. Similarly for $\mathbb{P}_\pi$ (if x has distribution π). So $\mathbb{P}_\pi$ is a measure on path space. We use the notation $(X(t)\,;\; t \geq 0)$ for a continuous path in $\mathcal{C}(\mathbb{R}_+;\mathbb{T}^d)$.

Perturbations

To define the perturbed process, let us consider a smooth vector field $\mathcal{V}$ defined on $\mathbb{T}^d$ and a real parameter $\lambda > 0$ and let $(X_x^\lambda(t)\,;\; t \geq 0, x \in \mathbb{T}^d)$ be the solution of the stochastic differential equation:

$$X_x^\lambda(0) = x\,,$$

$$dX_x^\lambda(t) = b(X_x^\lambda(t))dt + \lambda\sigma(X_x^\lambda(t))\mathcal{V}(X_x^\lambda(t))dt + \sigma(X_x^\lambda(t))dW_t\,. \tag{2}$$

Then $(X_x^\lambda(t)\,;\; t \geq 0, x \in \mathbb{T}^d)$ is Markov with generator

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$$\mathcal{L}^\lambda = \frac{1}{2}div(a\nabla) + \lambda\sigma\mathcal{V}\cdot\nabla\,.$$

Notation: as before $\mathbb{P}_x^\lambda$ is the law of $(X_x^\lambda(t)\,;\, t \geq 0)$ on path space. Also $\mathbb{P}_\pi^\lambda$ when we are at equilibrium.

Additive functionals (observables)

We may consider

symmetric additive functionals e.g. $A(t) = \int_0^t f(X(s))\, ds$,
$f : \mathbb{T}^d \to \mathbb{R}$,
or
anti-symmetric additive functionals e.g. let $(Z(t)\,;\, t \geq 0)$ be the lift of $(X(t)\,;\, t \geq 0)$ to $\mathbb{R}^d$. More generally

$$A(t) = \int_0^t g(X(s)) \circ dX(s)\,,$$

$g : \mathbb{T}^d \to \mathbb{R}^d$ a vector field.

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The words symmetric vs anti-symmetric refer to symmetry vs anti-symmetry with respect to time reversal.
Anti-symmetric additive functionals correspond to currents in physics.

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Observe that all anti-symmetric additive functionals have zero mean: for all t, $\mathbb{E}_\pi[\int_0^t g(X(s)) \circ dX(s)] = 0\,.$

Additive functionals

For diffusion processes (as in Equation (1)), symmetric and anti-symmetric additive functionals are related by

$$\int_0^t g(X_x(s)) \circ dX_x(s) = m_t + \frac{1}{2} \int_0^t div(ag)(X_x(s))\, ds\,,$$

where m is the martingale

$$m_t = \int_0^t (\sigma g)(X_x(s)) \cdot dW_s$$

under P.

Martingales are always easy to deal with. Results about anti-symmetric additive functionals transfer to symmetric additive functionals easily with the formula above. But ...

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observe we only get symmetric additive functionals where f is of the form

$$f = div(ag)\,.$$

Such f's form the H_{-1} space.

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In the rest of the talk we focus on
anti-symmetric additive functionals.

Linear response with a fixed time horizon

Fix a time horizon T.
On the space of trajectories $\mathcal{C}([0,T];\mathbb{T}^d)$, the two measures $\mathbb{P}_\pi$ and $\mathbb{P}_\pi^\lambda$ are equivalent and the Radon-Nikodym derivative is given by the Girsanov weight:

$$E\big[F(X_x^\lambda([0,T]))\big] = E\big[F(X_x([0,T]))e^{\lambda B(T)-\frac{\lambda^2}{2}\langle B\rangle(T)}\big]\,,$$

where

$$B(t) = \int_0^t \mathcal{V}(X_x(s))\cdot dW_s\,;\ \langle B\rangle(T) = \int_0^T |\mathcal{V}(X_x(s))|^2\,ds\,.$$

So

$$\partial_{\lambda=0}\frac{1}{T}\mathbb{E}_\pi^\lambda[A(T)] = \frac{1}{T}\mathbb{E}_\pi\big[A(T)B(T)\big]\,.$$

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Indeed a covariance.

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$$\partial_{\lambda=0}\frac{1}{T}\mathbb{E}_\pi^\lambda[A(T)] = \frac{1}{T}\mathbb{E}_\pi\left[A(T)B(T)\right].$$

Note at this point, we do not need the anti-symmetry of A. Not even the fact that we are at equilibrium.

We first need to understand the $T \to +\infty$ limit in the formula:

$$\partial_{\lambda=0} \frac{1}{T} \mathbb{E}_\pi^\lambda[A(T)] = \frac{1}{T} \mathbb{E}_\pi \left[A(T) B(T) \right].$$

Central limit Theorem:

Assume $A(t) = \int_0^t g(X(s)) \circ dX(s)$ with $\int g^2 \, d\pi < \infty$ i.e. $\mathbb{E}_\pi[A(1)^2] < \infty$. When T tends to $+\infty$, then the law of the vector $\frac{1}{\sqrt{T}}(A(T), B(T))$ under $\mathbb{P}_\pi$ converges to a Gaussian law with a certain covariance $\mathcal{S}$ and

$$\frac{1}{T} \mathbb{E}_\pi \left[A(T) B(T) \right] \to \mathcal{S}.$$

The CLT only holds in this form if π is ergodic. Otherwise we get a mixture of Gaussian laws but the existence of $\mathcal{S}$ still holds.

When $T \to +\infty$

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$$\frac{1}{T} \mathbb{E}_\pi \left[A(T) B(T) \right] \to \mathcal{S} .$$

Can we exchange the two limits?

$$\limsup_{T \to +\infty} \left| \frac{1}{\lambda T} \mathbb{E}_\pi^\lambda [A(T)] - \mathcal{S} \right| \to_{\lambda \to 0} 0 \, ?$$

LR scaling

J. Lebowitz, H. Rost '94.

The limit in the scaling $\lambda^2 T = 1$ still exists:

$$\frac{1}{\lambda T}\mathbb{E}^{\lambda}_{\pi}[A(T)] = \frac{1}{\lambda T}\mathbb{E}_{\pi}\left[A(T)e^{\lambda B(T) - \frac{\lambda^2}{2}\langle B\rangle(T)}\right]$$

$$= \mathbb{E}_{\pi}\left[\frac{1}{\sqrt{T}}A(T)e^{\frac{1}{\sqrt{T}}B(T) - \frac{1}{2T}\langle B\rangle(T)}\right]$$

$$\to E[\bar{A}e^{\bar{B} - \frac{1}{2}E[\bar{B}^2]}] = E[\bar{A}\bar{B}] = \mathcal{S}\,,$$

where $(\bar{A}, \bar{B})$ is the Gaussian vector given by the C.L.T. with covariance $E[\bar{A}\bar{B}] = \mathcal{S}$.

Once again the CLT only holds when π is ergodic. Otherwise we have a mixture of Gaussian laws and the conclusion still holds.

$T = +\infty$?

Conclusions so far:

Linear response holds for fixed times.

Linear response holds for larger times in the Lebowitz-Rost scaling for observables that satisfy the CLT.

What about infinite times?

On $\mathbb{R}^d$.

$$dZ_x(t) = b^\omega(Z_x(t))dt + \sigma^\omega(Z_x(t))dW_t\,;\; Z_x(0) = x\,. \qquad (3)$$

$$a^\omega(x) = \sigma^\omega(x)^2\,;\; b^\omega(x) = \frac{1}{2}div(a^\omega(x))\,;\; \mathcal{L}^\omega = \frac{1}{2}div(a^\omega\nabla)\,.$$

Now the coefficients $(\sigma^\omega(x)\,;\, x \in \mathbb{R}^d)$ are random i.e. depend on some $\omega \in \Omega$ with law $\mathbb{Q}$ on Ω.
We assume: $\mathbb{Q}$ is stationary and ergodic (w.r.t. translations in $\mathbb{R}^d$).

Reversible diffusions in a random environment

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Perturbation

$$dZ_x^\lambda(t) = b^\omega(Z_x^\lambda(t))dt + \lambda\sigma^\omega(Z_x^\lambda(t))\mathcal{V}^\omega(Z_x^\lambda(t))dt + \sigma^\omega(Z_x^\lambda(t))dW_t\,. \qquad (4)$$

$$\mathcal{V}^\omega(x) = \sigma^\omega(x)\cdot e_1\,.$$

(Constant force in direction e_1.)

The C.L.T. holds for the process $(Z_x(t)\,;\,t \geq 0)$:

$$u\cdot\Sigma v = \lim_{T\to+\infty} \frac{1}{T}\big(E[u\cdot Z_x(T) v\cdot Z_x(T)] - E[u\cdot Z_x(T)]\,E[v\cdot Z_x(T)]\big)\,.$$

Σ is the asymptotic covariance matrix of Z.

$$\ell(\lambda) = \lim_{T\to+\infty} \frac{1}{T} Z_x^{\lambda}(T)$$

(Does the limit exist?)

$$\partial_{\lambda=0}\ell(\lambda) = \Sigma e_1\,.$$

Einstein relation.

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C.L.T. was proved in the 80's (Kipnis-Varadhan, DeMasi-Ferrari-Goldstein-Wick ...). Since $\mathbb{Q}$ is ergodic, Σ is deterministic.

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The existence of the speed $\ell(\lambda)$ is non trivial (Off-equilibrium problem.) and not known in general.

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Einstein relation.

This model is infinite dimensional. Very non-hyperbolic.

Einstein relation: results

We establish the Einstein relation

$$\partial_{\lambda=0}\ell(\lambda) = \Sigma e_1 \,.$$

under two extra assumptions:
- σ^ω is uniformly elliptic (bounded from below and above),
- σ^ω has finite range of correlation (values of $\sigma^\omega(x)$ and $\sigma^\omega(y)$ are independent when $d(x, y) \geq R$ for some fixed R.).
(Ref Gantert-Mathieu-Piatnitski 2012.)

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The existence of the speed $\ell(\lambda)$ was obtained by Shen in 2003.

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Key step of the proof is: show the perturbed diffusion reaches equilibrium by time $T = 1/\lambda^2$; then compare with LR scaling.

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Key step of the proof is: show the perturbed diffusion reaches equilibrium by time $T = 1/\lambda^2$; then compare with LR scaling.

This is an example where, the smallest the perturbation, the largest the equilibrium time. See later for a more elementary finite dimensional example.

Continuity of the variance

Let

$$u\cdot\Sigma(\lambda)v = \lim_{T\to+\infty}\frac{1}{T}\left(E[u\cdot Z_x^\lambda(T)v\cdot Z_x^\lambda(T)]-E[u\cdot Z_x^\lambda(T)]\,E[v\cdot Z_x^\lambda(T)]\right),$$

be the asymptotic covariance of the perturbed diffusion Z^λ.

We also establish the Continuity of variance:

$$\lim_{\lambda\to 0}\Sigma(\lambda)=\Sigma\,.$$

(Ref: Mathieu-Piatnitski 2018.)

The continuity of variance is stronger than linear response.

Continuity of the variance

Let

$$u \cdot \Sigma(\lambda) v = \lim_{T \to +\infty} \frac{1}{T} \left(E[u \cdot Z_x^\lambda(T) v \cdot Z_x^\lambda(T)] - E[u \cdot Z_x^\lambda(T)] \, E[v \cdot Z_x^\lambda(T)] \right),$$

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$$\lim_{\lambda \to 0} \Sigma(\lambda) = \Sigma \, .$$

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The continuity of variance is stronger than linear response.

Many questions (and few results) to extend to time-dependent perturbations.

Diffusions on a torus

Back to diffusions on a torus:

$$dX_x(t) = b(X_x(t))dt + \sigma(X_x(t))dW_t\,;\; X_x(0) = x\,.$$

$$dX_x^\lambda(t) = b(X_x^\lambda(t))dt + \lambda\sigma(X_x^\lambda(t))\mathcal{V}(X_x^\lambda(t))dt + \sigma(X_x^\lambda(t))dW_t\,.$$

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Questions (in increasing order of difficulty)

1. Continuity of 'steady state': is $\frac{1}{T}\mathbb{E}_\pi^\lambda[A(T)]$ uniformly close to $\frac{1}{T}\mathbb{E}_\pi[A(T)] = \mathbb{E}_\pi[A(1)]$?

2. Linear response: compute $\partial_{\lambda=0}\frac{1}{T}\mathbb{E}_\pi^\lambda[A(T)]$ for large T?

3. Continuity of variance: what can we say about the variance of $A(T)$ under $\mathbb{P}_\pi^\lambda$ as $T \to +\infty$ and for small λ?

Diffusions on a torus

Back to diffusions on a torus:

$$dX_x(t) = b(X_x(t))dt + \sigma(X_x(t))dW_t\,;\; X_x(0) = x\,.$$

$$dX_x^\lambda(t) = b(X_x^\lambda(t))dt + \lambda\sigma(X_x^\lambda(t))\mathcal{V}(X_x^\lambda(t))dt + \sigma(X_x^\lambda(t))dW_t\,.$$

Questions (in increasing order of difficulty)

1. Continuity of 'steady state': is $\frac{1}{T}\mathbb{E}_\pi^\lambda[A(T)]$ uniformly close to $\frac{1}{T}\mathbb{E}_\pi[A(T)] = \mathbb{E}_\pi[A(1)]$?

2. Linear response: compute $\partial_{\lambda=0}\frac{1}{T}\mathbb{E}_\pi^\lambda[A(T)]$ for large T?

3. Continuity of variance: what can we say about the variance of $A(T)$ under $\mathbb{P}_\pi^\lambda$ as $T \to +\infty$ and for small λ?

Below partial answers to 1.

A degenerate example

On the interval $[0, 1]$.

$$dX_x(t) = b(X_x(t))dt + \sigma(X_x(t))dW_t \, ; \; X_x(0) = x \, .$$

$$a = \sigma^2 \, ; \; b = \frac{1}{2} diva \, .$$

Choose $a(x) = x^\alpha$ for small positive x and $\alpha > 2$ and a bounded from below elsewhere.
Choose: $\mathcal{V}(x) = -1$ for small positive x.

$$dX_x^\lambda(t) = \frac{\alpha}{2}(X_x^\lambda(t))^{\alpha-1}dt - \lambda(X_x^\lambda(t))^{\frac{\alpha}{2}}dt + (X_x^\lambda(t))^{\frac{\alpha}{2}}dW_t \, .$$

Note $x^{\frac{\alpha}{2}}$ is larger than $x^{\alpha-1}$ for small x.

A degenerate example

On the interval $[0, 1]$.

$$dX_x(t) = b(X_x(t))dt + \sigma(X_x(t))dW_t\,;\; X_x(0) = x\,.$$

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Then, for all positive λ, for almost all trajectories,

$$X_x^\lambda(T) \to 0\,.$$

For any $\lambda > 0$, the steady state is δ_0!

Lipschitz bounds

Recall $A(t) = \int_0^t g(X(s)) \circ dX(s)$.

THEOREM Choose g continuous.

$$\mathbb{E}_\pi^\lambda[A(T)] \le 4\lambda T\left(\|\mathcal{V}\|_\infty + \frac{2\gamma_2}{\lambda\sqrt{T}}\right)\|\sigma\cdot g\|_\infty \,.$$

In particular

$$\limsup_{\lambda\to 0}\frac{1}{\lambda}\limsup_{T\to+\infty}\frac{1}{T}\left|\mathbb{E}_\pi^\lambda[A(T)]\right| \le 4\|\mathcal{V}\|_\infty\|\sigma\cdot g\|_\infty \,.$$

The result says the steady state is always Lipschitz continuous: not as a measure (See example before.) but as a distribution on anti-symmetric additive functionals or, equivalently, on H_{-1}.

Lipschitz bounds

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THEOREM Choose g continuous.

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In particular

$$\limsup_{\lambda\to 0} \frac{1}{\lambda} \limsup_{T\to+\infty} \frac{1}{T} \big| \mathbb{E}_\pi^\lambda[A(T)] \big| \leq 4 \|\mathcal{V}\|_\infty \|\sigma \cdot g\|_\infty \, .$$

Proof is based on forward-backward martingale decomposition for A and concentration inequalities (Large deviations).

Application to diffusions in a random environment

Recall

$$dZ_x^\lambda(t) = b^\omega(Z_x^\lambda(t))dt + \lambda\sigma^\omega(Z_x^\lambda(t))\mathcal{V}^\omega(Z_x^\lambda(t))dt + \sigma^\omega(Z_x^\lambda(t))dW_t \,.$$

$$\mathcal{V}^\omega(x) = \sigma^\omega(x) \cdot e_1 \,.$$

Coefficients $(\sigma^\omega(x)\,;\, x \in \mathbb{R}^d)$ are random stationary.
Let

$$\ell(\lambda) = \lim_{T \to +\infty} \frac{1}{T} Z_x^\lambda(T)$$

(if it exists.)
Then

$$\big|\ell(\lambda)\big| \leq 4\lambda \|\sigma \cdot e_1\|_\infty^2 \,.$$

Optimistic Lipschitz bounds

Recall $A(t) = \int_0^t g(X(s)) \circ dX(s)$.

Choose g continuous.

$$\limsup_{\lambda \to 0} \frac{1}{\lambda} \limsup_{T \to +\infty} \frac{1}{T} \left| \mathbb{E}_\pi^\lambda [A(T)] \right| \leq 4 \|\mathcal{V}\|_\infty \|\sigma \cdot g\|_\infty .$$

Optimistic Lipschitz bounds

Recall $A(t) = \int_0^t g(X(s)) \circ dX(s)$.

Choose g continuous.

$$\limsup_{\lambda \to 0} \frac{1}{\lambda} \limsup_{T \to +\infty} \frac{1}{T} \left| \mathbb{E}_\pi^\lambda [A(T)] \right| \leq 4 \|\mathcal{V}\|_\infty \|\sigma \cdot g\|_\infty .$$

The bound also holds for time-dependent parameters σ or perturbation $\mathcal{V}$.

THEOREM Assume $\mathcal{V}$ depends on time and that

$$\lim_{T \to +\infty} \sup_x \|\mathcal{V}_T(x)\|_\infty = 0 .$$

Then

$$\lim_{T \to +\infty} \frac{1}{T} \mathbb{E}_\pi^\lambda [A(T)] = 0 .$$

End

Concluding remark:

Martingale techniques always useful for stochastic dynamics: Girsanov transforms, CLT's, deviation inequalities ... and flexible enough to adapt to time-dependent models.

End of the talk.