Periodic points and linear response - Souvenir programme

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1 Pseudo-Introduction

Given complicated real world problems one can try to reduce them to (often intractable) mathematical models. However, if one further reduces these to toy ergodic theory questions it may improve the chances of proving something, albeit at the risk of making the results less relevent.

A convenient principle (after Gallavotti). Let us assume that chaotic systems are uniformly hyperbolic (e.g., expanding maps, transitive Anosov diffeomorphisms and flows, etc.)

Assume for notational convenience that $T : X \to X$ is a transitive C^∞ Markov piecewise expanding map of the unit interval X, with the hope that the natural generalizations hold for Anosov systems.

2 A little dynamics

2.1 Orbits for the transformation T

Points $x \in X$ in the phase space "evolve in time" as described by the orbit $x, T(x), T^2(x), \cdots$. Ergodic theory provides a way to understand this long term behaviour as $n \to +\infty$. If we are (un)lucky the orbit may be *periodic*, i.e., there exists $n \geq 1$ such that $T^n(x) = x$ and the orbit consists of finitely many points. This happens rarely for hyperbolic systems, but there are always a countable infinity of such points.

2.2 Interesting quantitative values

There is a natural invariant measure μ_T called the Sinai-Ruelle-Bowen or SRB measure. [1] Associated to μ we have the following quantities.

1. The *Lyapunov exponent* (for expanding maps $\int \log |T'(x)| d\mu_T(x)$) which measures the "sensitivity on initial conditions" of the orbit of a typical point x.

2. The *linear response* describes how the measure μ_{T_λ} changes for a family T_λ, with $\lambda \in (-\epsilon, \epsilon)$, say, perhaps measured in terms of the integral $\int g d\mu_{T_\lambda}$ with respect to a suitable reference function $g : X \to \mathbb{R}$

3. The *rate of mixing*, measured through the spectrum of the transfer operator (e.g., eigenvalues close to unit circle might suggest being close to a tipping point)

Aim. To undertand these quantities in terms of the periodic orbits.

[1] For an expanding map this is (merely) the unique absolutely continuous invariant probability measure. But we could always construct μ from the periodic points by defining for $n \geq 1$ a probability measure $\mu_n = \frac{1}{\mathrm{Card}\{x \,:\, T^n x = x\}} \sum_{T^n x = x} 1/|(T^n)'(x)| \delta_x$ living on periodic orbits and then $\lim_{n \to +\infty} \mu_n = \mu$.

3 A little complex analysis: Determinants

It is convenient to package up the information from individual the periodic points into a single (generating) complex function. Given a smooth function $G : X \to \mathbb{R}$ we can write

$$D(z) = D_{G,T}(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{\exp\left(\sum_{i=0}^{n-1} G(T^i x)\right)}{1 - 1/(T^n)'(x)}\right), \quad z \in \mathbb{C},$$

where the middle summation is over the fixed points for T^n.

3.1 Radius of onvergence of $D(z)$

We can see that this converges to an analytic function provided the series converges, i.e., $|z|e^{P(G)} < 1$ where

$$e^{P(G)} = \lim_{n\to+\infty} \left| \sum_{T^n x = x} \frac{\exp\left(\sum_{i=0}^{n-1} G(f^i x)\right)}{1 - 1/(T^n)'(x)} \right|^{1/n} \left(= \lim_{n\to+\infty} \left| \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} G(f^i x)\right) \right|^{1/n} \right)$$

(where $|1/(T^n)'(x)| < 1$). In particular, writing $D(z)$ as a power series

$$D(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \tag{1}$$

with coefficients a_n depending on T and G, we see it has radius of convergence $e^{-P(G)}$.

Note I. Since T is piecewise C^∞ then $D(z)$ is analytic in all of $\mathbb{C}$ and thus $\lim_{n\to+\infty} |a_n|^{1/n} = 0$, i.e., for any $0 < \theta < 1$ there exists $C > 0$ such that $|a_n| \leq C\theta^n$.

Note II. The value $z = e^{-P(G)}$ is a zero for $D(z)$ in this extension.

3.2 $D(z)$ and linear response (following P.-Vytnova)

First a very basic question.

Question. What does the function $D(z)$ do for us?

An immediate answer is that its zero $e^{-P(G)}$ also gives us a connection to linear response following an exercise of Ruelle[2]:

(a) For $G_t(x) = -\log|T'(x)| + tg(x)$ we have that $\frac{de^{P(G_t)}}{dt}|_{t=0} = \int g d\mu_T$; and so

(b) For $G_{t,\lambda}(x) = -\log|T'_\lambda(x)| + tg(x)$ then $\frac{\partial^2 e^{P(G_t)}}{\partial t \partial \lambda}|_{t=0} = \frac{d}{d\lambda} \int g d\mu_{T\lambda}$

But by Note I we have that $D_{G_t T}(e^{P(G_t)}) = 0$ and so by (a) and the implicit function theorem:

$$\int g d\mu_T = \frac{\partial D_{G_t,T}}{\partial t} \Big/ \frac{\partial D_{G_t,T}}{\partial z}|_{z=1,t=0}.$$

Using the power series expansion (1) of $D(z)$ we can formally write

$$\int g d\mu_T = -\frac{\sum_{n=1}^{\infty} \frac{\partial a_n}{\partial t}|_{t=0}}{\sum_{n=1}^{\infty} n a_n}$$

[2]Exercise 5a) on page 99 of the 1978 edition of *Thermodynamic Formalism*

Finally, by replacing T by T_λ, differentiating both sides in λ and then using (b) we get an expression for the linear response in terms of the (derivatives of the) coefficients a_n.

Question. Can we get better (and more expicit) esimates on a_n?

We can if we assume that T is piecewise C^ω (i.e., real analytic) by employing a little operator theory.

4 A little operator theory (after Jenkinson-P.)

If T is piecewise C^ω then we can assume that we can choose a neighbourhood $X \subset U \subset \mathbb{C}$ so that T extends analytically to U. Let $\mathcal{H}$ be a Hilbert space of analytic function on U

Example (Hardy spaces). Let $U = \{z \in \mathbb{C} : |z - \frac{1}{2}| < 1\}$ be the unit disk then $f(z) = \sum_{n=0}^{\infty} b_n (z - \frac{1}{2})^n$ has norm $\|f\|^2 = \sum_{n=0}^{\infty} |b_n|^2$.

Let $\mathcal{L} = \mathcal{L}_{G,T} : \mathcal{H} \to \mathcal{H}$ be a transfer operator. [3] We can then assume that $\mathcal{L}$ is a trace class operator and a simple Lidskii-type identity relates periodic orbits to the spectrum

$$\mathrm{trace}(\mathcal{L}^n) = \sum_{T^n x = x} \frac{\exp\left(\sum_{i=0}^{n-1} G(T^i x)\right)}{1 - 1/(T^n)'(x)}$$

and then $D(z) = \det(I - z\mathcal{L}) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{trace}(\mathcal{L}^n)\right)$, where both sides depend on G and T.

Strategy. We want to use the operator description of $D(z)$ to get estimates on the a_n.

4.1 Approximation numbers

The *approximation numbers* for the operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ are a sequence of real numbers

$$A_l = \inf \{\|\mathcal{L} - K\| : K \text{ bounded linear operator of finite rank } l\}, \quad l \geq 1,$$

which measures how the operator norm can be reduced by a finite rank operator. [4]

- Let $(e_n)_{n=0}^{\infty}$ be a complete orthonormal family for $\mathcal{H}$.
- For $f = \sum_{n=0}^{\infty} c_n e_n$ with $\|f\| = \sqrt{\sum_{n=0}^{\infty} |c_n|^2} = 1$ then for each $l \geq 1$ let $K_l f = \sum_{n=0}^{l-1} c_n \mathcal{L}(e_n)$

and thus

$$\begin{aligned} A_l \leq & \|(\mathcal{L} - K_l) f\| \leq \sum_{n=l}^{\infty} |c_n| \|\mathcal{L}(e_n)\| \\ & \leq \sqrt{\sum_{l=n}^{\infty} |c_n|^2} \sqrt{\sum_{n=l}^{\infty} \|\mathcal{L}(e_n)\|^2} \leq \sqrt{\sum_{n=l}^{\infty} \|\mathcal{L}(e_n)\|^2} \end{aligned} \tag{2}$$

using the Cauchy-Schwarz inequality.

Example revisited. If $D = \{z \in \mathbb{C} : |z - \frac{1}{2}| < 1\}$ then $e_n(z) = \frac{(z - \frac{1}{2})^n}{\sqrt{n}}$ $(n \geq 0)$

[3] For example, let $T : X \to X$ be defined by $T(x) = 2x + \lambda \sin(2\pi x)$ (mod 1) then for $|\lambda|$ sufficiently small the inverse branches $T_0, T_1 : X \to X$ satisfy $T_0(U), T_1(U) \subset U$ and

$$\mathcal{L} w(z) = e^{G(T_0 z)} w(T_0 z) + e^{G(T_1 z)} w(T_1 z)$$

gives a well defined transfer operator.

[4] In the example, if $T_0(U), T_1(U) \subset \{z \in \mathbb{C} : |z| < r\}$ for some $0 < r < 1$ then it is easy to show there exists $C > 0$ with $A_l \leq C r^l$, for $l \geq 1$. But the bound in the text is better .

4.2 Weyl-Allakhverdiev inequality

The connection between the coefficients a_n and the approximation numbers comes from the bound

$$|a_n| \leq \sum_{l_1 < l_2 < \cdots < l_n} A_{l_1} A_{l_2} \cdots A_{l_n}. \tag{3}$$

5 Summary

We can write the linear response (or lyapunov exponents, variance, etc.) in terms of explicit absolutely convergent series defined in terms of the periodic points.

5.1 Basic convergence

The basic convergence (which follows from work of Ruelle, after Grothendieck) only needs:

Exercise (Euler Inequality). If $A_l \leq Cr^l$ then $|a_n| \leq \frac{C^n r^{n(n+1)/2}}{(1-r)(1-r^2)\cdots(1-r^n)} = O(r^{n(n+1)/2})$.

In particular, the coefficients a_n tend to zero fast enough to make the series for the expressions for quantitative to converge.

5.2 Better estimates

We can also use this as a method for approximating the numerical value in examples (although this wasn't the original purpose of the approach nor is it necessarily a good approach).

1. Fix $N > 0$ and compute $a_1, a_2, \cdots, a_N$ using the periodic points of period at most N (where N is chosen depending on the limitations of our computer).
2. Fix $M > N$ and then we can bound $|a_n|$ $(n \geq N)$ using (3) and the bounds:
   (a) for $N < l < M$ bound A_l using (2) (where M is chosen depending on the limitations of our computer).
   (b) For $M < l$ we can bound A_l using the Euler inequality.

5.3 Generalizations

If we have a C^ω Anosov diffeomorphism T then it is more appropriate to write

$$D(z) = D_{G,T}(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{\exp\left(\sum_{i=0}^{n-1} G(T^i x)\right)}{\det(I - (DT)^{-1}(x))} \right), \quad z \in \mathbb{C},$$

An extra feature here is that we only need to take $G_t(x) = tg(x)$ because of the contributions from $\det(I - (DT)^{-1}(x))$.

One can adapt this approach using either the less fashionable device of Markov partitions and the approach of Rugh, or using Anisotropic spaces of functions when applicable (for example on the torus in the context of Faure-Roy). They key point is simply to find a setting to which the functional analysis applies.

For C^ω Anosov flows it is perhaps simpler to use Markov sections to accommodate the operator theory being used.