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Kneading Determinants and Transfer Operators in Higher Dimensions

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Abstract. This note is based on a talk given at the First Joint Mathematical International Meeting AMS-SMF held in Lyon in July 2001. It aims to serve both as a very short introduction to the kneading approach to dynamical zeta functions and as a research announcement for [**BB**]: Ruelle-transfer operators $\mathcal{M}_k$ acting on k-forms in $\mathbb{R}^n$ are associated to C^r transversal local diffeomorphisms of $\mathbb{R}^n$ (the dynamics), and C^r compactly supported functions (the weights). A formal trace constructed with the fixed points of the diffeomorphisms, together with the weight, and the Lefschetz signs at these points, yields a formal Ruelle-Lefschetz determinant $\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M})$. The Milnor-Thurston-Kitaev equality recently proved by Baillif makes it possible to relate zeroes and poles of the Ruelle-Lefschetz determinant with spectra of the transfer operators $\mathcal{M}_k$ on suitable Banach spaces, under additional assumptions. In particular, this gives a new proof of a result of Ruelle on smooth expanding dynamics.

1. Introduction

Dynamical zeta functions.

Let f be a transformation (on a compact space M, say) and let g be a complex-valued "weight" function. Assume for now that each set $\mathrm{Fix}\, f^m = \{x \in M \mid f^m(x) = x\}$ is finite. A weighted (Ruelle) dynamical zeta function is associated to f and g as follows:

$$\zeta_{f,g}(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}\,(f^m)} \prod_{j=0}^{m-1} g(f^j(x)) \,, \tag{1.1}$$

If $f : \Sigma^+ \to \Sigma^+$ is a one-sided subshift of finite type associated to a (say, aperiodic and irreducible) Markov transfer matrix A, and g is constant, it is an easy algebraic

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exercise, using the formula (for an arbitrary finite matrix A)

$$\mathrm{Det}\,(\mathrm{Id}\, - zA) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{tr}\, A^m\,, \tag{1.2}$$

to see that $\zeta_{f,g}(z) = 1/\det(\mathrm{Id} - zA)$. In particular, the a priori formal power series $\zeta_{f,g}(z)$ is in fact a rational function, more precisely it is the inverse of the polynomial whose roots are exactly the inverses of the non-zero eigenvalues of the transfer matrix A. The largest eigenvalue of this matrix is the exponential of the topological entropy, while the other eigenvalues are related to the correlation functions of the measure of maximal entropy. In order to study other interesting invariant measures especially in a dynamical setting which includes some smoothness, it is natural to consider nonconstant weights g. In the case of the subshift of finite type, let us assume that g is not constant but that it is Lipschitz for a metric $d_\theta(x,y) = \sum_{j=0}^{\infty} \theta^j (1 - \delta_K(x_j, y_j))$ with $\theta < 1$, and δ_K the Kronecker delta (in other words, the variations on small cylinders decay exponentially). The Markov transfer matrix is then replaced by a transfer operator

$$\mathcal{L}\varphi(x) = \sum_{f(y)=x} g(y)\varphi(y)\,. \tag{1.3}$$

This operator is not compact when acting on the Banach space

$$\mathrm{Lip} = \{\varphi : \Sigma^+ \to \mathbb{C} \mid \exists K_\varphi\,,\, |\varphi(x) - \varphi(y)| \le K_\varphi d_\theta(x,y)\}\,, \|\varphi\|_{\mathrm{Lip}} = K_\varphi + \sup|\varphi|\,,$$

of Lipschitz functions on the one-sided shift space, but the noncompactness can be quantified: Pollicott [**Po**] proved that the essential spectral radius $\rho_{ess}(\mathcal{L})$ of $\mathcal{L}$ is bounded above by $\theta \exp(P(\log(|g|)))$ where P is the topological pressure, i.e. a weighted topological entropy. Since $\exp(P(\log(|g|)))$ coincides with the spectral radius $\rho(\mathcal{L})$ of $\mathcal{L}$ on Lip if g is positive, we get a sharp inequality between the spectral radius and the essential spectral radius, the so-called quasicompactness property. Haydn [**Ha**] then proved (using crucial steps by Ruelle and Pollicott) that $\zeta_{f,g}(z)$ is meromorphic in the disc of radius $\theta^{-1}\exp(-P\log(|g|))$, where its poles are inverse eigenvalues of $\mathcal{L}$ on Lip. (Throughout this introduction, we refer to the survey [**Ba1**] and the book [**Ba3**] for more information and references.)

Using Markov methods (symbolic dynamics via Markov partitions, covers, tower-like extensions, etc.), similar results were obtained for weighted dynamical zeta functions associated to smooth data on manifolds: Axiom A diffeomorphisms (Ruelle [**Ru2**], Pollicott [**Po**], Haydn [**Ha**]), one-dimensional piecewise monotone maps (Hofbauer-Keller, Baladi-Keller [**BaKe**], Keller-Nowicki [**KN**]), and higher-dimensional piecewise smooth and injective maps (Buzzi-Keller [**BuKe**]). There, one assumes not only (piecewise) smoothness of the dynamics f and the weight g but also some "hyperbolicity" (or expansion) properties in the dynamics.

For smooth expanding maps on manifolds, Ruelle [**Ru1, Ru3**] and Fried [**Fr1, Fr2, Fr3**] studied dynamical Fredholm-type determinants, the definitions of which are slightly different than that of $1/\zeta_{f,g}(z)$:

$$\mathrm{Det}^{\flat}(\mathrm{Id}\, - z\mathcal{L}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}\, f^m} \prod_{\ell=0}^{m-1} \frac{g(f^\ell(x))}{\mathrm{Det}\,(\mathrm{Id}\, - D_x f^{-m})}\,. \tag{1.4}$$

The above determinant only plays a formal part in the setting of [**BB**]. Let us just mention here for completeness that if the data is analytic, then the transfer operator $\mathcal{L}$ is nuclear on a suitable space of bounded holomorphic functions. In this case ([**Ru1**], [**Fr1, Fr2**]) the dynamical determinant $\mathrm{Det}^\flat(\mathrm{Id} - z\mathcal{L})$ not only deserves to be called a flat determinant in the sense of Atiyah and Bott [**AB**], but can further be interpreted as an honest Grothendieck-Fredholm determinant. (See also Mayer [**Ma**], Keller [**Ke**], Rugh [**Rug**].)

Relating kneading operators, dynamical zeta functions, and spectrum in dimension one.

In the past ten years, a new, so-called kneading, approach was launched (see [**BaRu**] and references therein), initially for piecewise monotone interval maps. The original motivation was to understand the links between, on the one hand the Milnor-Thurston [**MT**] identity relating the kneading matrix and the (unweighted) dynamical zeta function, and on the other hand the spectral interpretation of the zeroes and poles of (weighted) dynamical zeta functions. This kneading approach is now quite well understood in one real dimension. There, instead of a single piecewise monotone interval map f, the dynamical data consists in a (say finite) set of local homeomorphisms $\psi_\omega : U_\omega \to \psi_\omega(U_\omega)$ (U_ω an open interval of $\mathbb{R}$). The ψ_ω could be for example the inverse branches of a single piecewise monotone f, but they could also be totally unrelated. Note also that no contraction assumption whatsoever is required of the ψ_ω. Recall that the variation norm $\mathrm{var}\,(\varphi)$ of a function $\varphi : \mathbb{R} \to \mathbb{C}$ is defined by

$$\mathrm{var}\,\varphi = \sup\{\sum_{i=1}^{N} |\varphi(t_i - \varphi(t_{i-1})| \mid N \geq 1\,,\, t_0 < t_2 < \ldots < t_N\}\,.$$

The weight functions g_ω are assumed to be continuous, of bounded variation, and to have support inside U_ω (see Gouëzel [**Go**] for a weakening of the continuity assumption). For example, one can take $g_\omega = g \circ \psi_\omega$. In this notation, the transfer operator is expressed as

$$\mathcal{M}\varphi = \sum_\omega g_\omega \cdot (\varphi \circ \psi_\omega)\,. \tag{1.5}$$

Ruelle [**Ru4**] obtained an estimate $\widehat{R}$ for the essential spectral radius of $\mathcal{M}$ acting on the Banach space BV of functions of bounded variation, generalising results of Keller. In [**BaRu**], we linked the eigenvalues of $\mathcal{M}$ (outside of the disc of radius $\widehat{R}$), acting on BV, with the zeroes of the "sharp determinant"

$$\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{tr}^{\#}\mathcal{M}^m\,, \tag{1.6}$$

where (with the understanding that $y/|y| = 0$ if $y = 0$)

$$\mathrm{tr}^{\#}\mathcal{M} = \sum_\omega \int \frac{1}{2} \frac{\psi_\omega(x) - x}{|\psi_\omega(x) - x|}\, dg_\omega(x)\,. \tag{1.7}$$

We emphasize that $\mathrm{tr}^{\#}\mathcal{M}$ is a formal trace, and that $\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M})$ is a formal determinant. The trace and determinant notations are justified, however, by the results in [**BaRu**]. If the ψ_ω are strict contractions which form the set of inverse branches of a piecewise monotone interval map f, and if $g_\omega = g \circ \psi_\omega$, then integration by parts together with the key property that

$$d\frac{x}{2|x|} = \delta\,, \text{ the Dirac delta at the origin of } \mathbb{R}\,, \tag{1.8}$$

show that $\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = 1/\zeta_{f,g}(z)$. If one only assumes that the graph of each admissible composition $\psi^m_{\vec\omega}$ of m successive ψ_ωs (with $m \ge 1$) intersects the diagonal transversally, then
(1.9)

$$\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{\text{admissible } \psi^m_{\vec\omega}} \sum_{x \in \mathrm{Fix}\, \psi^m_{\vec\omega}} L(x, \psi^m_{\vec\omega}) \prod_{j=0}^{m-1} g_{\omega_j}(\psi^j_{\vec\omega}(x))\,,$$

where $L(x,\psi) \in \{-1, 1\}$ is the Lefschetz number of a transversal fixed point $x = \psi(x)$. Thus, it makes sense to call $\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M})$ a Ruelle-Lefschetz (dynamical) zeta function.

The crucial step in the proof of the spectral interpretation of the zeroes of this Ruelle-Lefschetz zeta function consists in establishing the following continuous version of the discrete Milnor-Thurston identity [**MT**]:

$$\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = \mathrm{Det}^{*}(\mathrm{Id} + \widehat{\mathcal{D}}(z))\,. \tag{1.10}$$

In (1.10) the "kneading operator" $\widehat{\mathcal{D}}(z)$ replaces the finite kneading matrix of Milnor and Thurston. If $|z| < 1/R$ then $\widehat{\mathcal{D}}(z)$ is a Hilbert-Schmidt operator on a suitable L^2 space (its kernel is in fact bounded and compactly supported), thus allowing the use of regularised determinants of order two. In fact, $\mathrm{Det}^{*}(\mathrm{Id} + \widehat{\mathcal{D}}(z))$ is the product of this regularised determinant with the exponential of the average of the kernel of $\widehat{\mathcal{D}}(z)$ along the diagonal, which is well-defined. Another kneading operator, $\mathcal{D}(z)$, is essential. If $1/z$ is not in the spectrum of $\mathcal{M}$ (on BV) then $\mathcal{D}(z)$ is also Hilbert-Schmidt, and we have $\mathrm{Det}^{*}(\mathrm{Id} + \widehat{\mathcal{D}}(z)) = \mathrm{Det}^{*}(\mathrm{Id} + \mathcal{D}(z))^{-1}$. We refer to [**BaRu**] for the original definition of $\mathcal{D}(z)$, $\widehat{\mathcal{D}}(z)$, but will sketch next a more recent definition.

The kneading operator approach to study Ruelle-Lefschetz dynamical zeta functions in one complex dimension is implemented in [**BKRS**]. There, a more conceptual definition of $\mathcal{D}(z)$ was suggested, which was later implemented in one real dimension [**Ba2**]:

$$\mathcal{D}(z) = \mathcal{N}(\mathrm{Id} - z\mathcal{M})^{-1}\mathcal{S}\,, \tag{1.11}$$

where $\mathcal{N}$ is an auxiliary transfer operator and $\mathcal{S}$ is the convolution (μ is an auxiliary nonnegative finite measure)

$$\mathcal{S}\varphi(x) = \int \frac{1}{2}\frac{x-y}{|x-y|}\varphi(y)d\mu\,. \tag{1.12}$$

The kneading approach we just described not only extends previous results on piecewise monotone interval maps, but, through its conceptual simplicity, it sheds a new light on weighted dynamical zeta functions associated to smooth maps: The mechanism relating the zeroes of the zeta function with eigenvalues is literally visible in the definition (1.11) which embodies the fact that the kneading operator is a regularised (through the convolution $\mathcal{S}$) object which describes (through the resolvent there and by the identity (1.10)) part of the inverse spectrum of the transfer operator. In cases where the essential spectral radius has been estimated on a space compatible with the kernel of $\mathcal{D}(z)$ (see "Choosing a Banach space" in Section 2 for more on this point), this mechanism will apply automatically. As a side-effect, the sometimes cumbersome Markov partition tool is bypassed.

Relating kneading operators and dynamical zeta functions in higher dimensions.

Implementing this strategy in higher dimensions is a natural goal: Assume that the U_ω are (finitely many, say) open subsets of $\mathbb{R}^n$ and that the $\psi_\omega : U_\omega \to \psi_\omega(U_\omega)$ are local C^r homeomorphisms or diffeomorphisms, while the g_ω are C^r functions, for $r \geq 1$. In 1995, A. Kitaev (one of the authors of [**BKRS**]) wrote a two-page sketch of the (statement and) proof of a higher-dimensional Milnor-Thurston formula. He made an additional transversality assumption: For each $x \in \mathbb{R}^n$ and $m \geq 1$, $\omega_1, \ldots, \omega_m$ for which the composition $\psi^m_{\vec\omega} = \psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1}$ is well defined and fixes $\psi^m_{\vec\omega}(x) = x$, the derivative $D_x(\psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1})$ does not have 1 as an eigenvalue. Transversality guarantees that the set of fixed points of each fixed period m is finite, allowing us to define a Ruelle-Lefschetz zeta function

(1.13)

$$\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{\text{admissible } \psi^m_{\vec\omega}} \sum_{x \in \mathrm{Fix}\, \psi^m_{\vec\omega}} L(x, \psi^m_{\vec\omega}) \prod_{j=0}^{m-1} g_{\omega_j}(\psi^j_{\vec\omega}(x)) \, .$$

Turning Kitaev's unpublished note into a mathematical proof was the subject of Baillif's Ph.D. thesis [**Bai**], see also [**Ba2**], where the following "Milnor-Thurston-Kitaev" formula is proved:

$$\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = \prod_{k=0}^{n-1} \mathrm{Det}^{\flat}(\mathrm{Id} + \mathcal{D}_k(z))^{(-1)^{k+1}} \, . \tag{1.14}$$

We refer the reader to Section 2 for the definition of the kneading operators $\mathcal{D}_k(z)$ and a brief explanation of the meaning of the "flat determinant" notation in the right-hand-side of (1.14). We just mention here that the $\mathcal{D}_k(z)$ are kernel operators acting on $k+1$ forms, constructed with the resolvent $(\mathrm{Id} - z\mathcal{M}_k)^{-1}$ (where the transfer operator $\mathcal{M}_k$ acts on k forms via the pull-back ψ^*_ω), together with a convolution operator $\mathcal{S}_k$, mapping $k+1$-forms to k-forms and which satisfies the homotopy equation $d\mathcal{S} + \mathcal{S}d = 1$. The kernel $\sigma_k(x, y)$ of $\mathcal{S}_k$ has singularities of the type $(x-y)/\|x-y\|^n$ (recall that $n \geq 2$ is the dimension). Ruelle noted that the homotopy operator $\mathcal{S}_k$ proposed by Kitaev and used by Baillif can be expressed as $d^*_{k+1}\Delta^{-1}_{k+1}$ where d^* is the dual to the exterior derivative mapping $k+1$ forms to k forms, and Δ^{-1}_{k+1} is the inverse of the Laplacian acting on compactly supported $k+1$-forms.

Although Baillif did not give a spectral interpretation of zeroes or poles of the sharp determinant (1.13), he did notice that, for $|z|$ very small, suitably high iterates of the $\mathcal{D}_k(z)$ are trace-class on $L^2(\mathbb{R}^n)$ (the underlying measure is Lebesgue measure from now on), showing that the corresponding regularised determinant of $\mathcal{D}_k(z)$ has a nonzero radius of convergence [**Bai**].

In [**BB**] we carry out the spectral interpretation of the zeroes and poles of the sharp determinant (1.13) in arbitrary dimension, under additional assumptions. In the second part of this note, we give precise statements of the main result and its corollary there and discuss very briefly the methods of proof.

2. Statement of our results

The kneading operators.

We write $\mathcal{A}_{k,C^m}$, $\mathcal{A}^C_{k,C^m}$ for the vector spaces of k-forms on $\mathbb{R}^n$ with C^m coefficients ($m \geq 0$ integer), respectively compactly supported C^m coefficients. If $\mathcal{B}$ is a Banach space of functions or distributions on $\mathbb{R}^n$, then $\mathcal{A}_{k,\mathcal{B}}$ denotes the Banach space of k-forms with coefficients in $\mathcal{B}$.

For $k = 0, \dots, n$ the transfer operators may be written

$$\mathcal{M}_k : \mathcal{A}_{k,C^{r-1}} \to \mathcal{A}^C_{k,C^{r-1}}\,, \qquad \mathcal{M}_k\phi(x) = \sum_\omega g_\omega(x)\,(\psi^*_\omega\phi)(x)\,. \tag{2.1}$$

Each kneading operator $\mathcal{D}_k(z)$ ($k = 0, \dots, n-1$) will be defined as a power series in z with coefficients operators $\mathcal{A}^C_{k+1,C^{r-1}} \to \mathcal{A}^C_{k+1,C^{r-1}}$. For some values of z, these operators will have a kernel and, for a suitable iterate $\mathcal{D}^{m(n)}_k(z)$, this kernel will be in $L^2(\mathbb{R}^n)$, so that Hilbert-Schmidt theory will furnish a regularised determinant of order two. The operators $\mathcal{D}_k(z)$ were first defined in [**Bai**], following an idea of Kitaev. They involve the auxiliary transfer operators $\mathcal{N}_k : \mathcal{A}_{k,C^{r-1}} \to \mathcal{A}^C_{k+1,C^{r-1}}$:

$$\mathcal{N}_k = d_k\mathcal{M}_k - \mathcal{M}_{k+1}d_k\,. \tag{2.2}$$

The fact that $\mathcal{N}_k$ does not lower the regularity comes, e.g., from the equivalent expression

$$\mathcal{N}_k\phi(x) = \sum_\omega dg_\omega(x) \wedge \psi^*_\omega\phi(x)\,. \tag{2.3}$$

The "good" properties of the kernels of the kneading operators $\mathcal{D}_k(z)$ are due to the presence in their definition of the already mentioned convolution operators $\mathcal{S}_k : \mathcal{A}^C_{k+1,C^{r-1}} \to \mathcal{A}_{k,C^{r-1}}$. Indeed, the kneading operators are defined (in the sense of formal power series with coefficients linear operators) as

$$\mathcal{D}_k(z) = z\mathcal{N}_k(\mathrm{Id} - z\mathcal{M}_k)^{-1}\mathcal{S}_k = \sum_{\ell=0}^{\infty} z^{\ell+1}\mathcal{N}_k\mathcal{M}^\ell_k\mathcal{S}_k\,, \quad k = 0, \dots, n-1\,. \tag{2.4}$$

The transversality assumption and elementary properties of the $\sigma_k(x, y)$ allow Baillif ([**Bai**] see also § 3 in [**Ba2**]) to define a formal trace of $\mathcal{D}_k(z)$ as a power series with coefficients obtained by integrating kernels along the diagonal (with a sign depending on k and n), and then to interpret this trace as a flat trace in the sense of Atiyah and Bott [**AB1, AB2**], whence the notation $\mathrm{Det}^\flat$ for the corresponding (flat) determinant in the right-hand-side of (1.14).

The main statement.

The main result in [**BB**] says that *if* (Axiom 1 below) there are suitable Banach spaces $\mathcal{B}_k$ of coefficients for k-forms on which (Axiom 2 below) the essential spectral radii of all transfer operators $\mathcal{M}_k$ are smaller than some finite number R, and (Axiom 3) the action of their resolvents on kernels from the convolution operators $\mathcal{S}_k$ gives rise to a meromorphic trace, *then*:

Theorem 1 (Main Theorem). *Under Axioms 1 and 2–3 (see below), the (formal) sharp determinant $\mathrm{Det}^\#(Id - z\mathcal{M})$ is meromorphic in the disc $\{|z| < 1/R\}$. The order of z as a zero/pole of $\mathrm{Det}^\#(Id - z\mathcal{M})$ coincides with*

• *the sum over* $0 \le 2k \le n$ *of the algebraic multiplicities of* $1/z$ *as an eigenvalue of* $\mathcal{M}_{2k}$ *acting on* $2k$ *forms with coefficients in* $\mathcal{B}_{2k}$

minus

• *the sum over* $1 \le 2k+1 \le n$ *of the algebraic multiplicities of* $1/z$ *as an eigenvalue of* $\mathcal{M}_{2k+1}$ *acting on* $2k+1$*-forms with coefficients in* $\mathcal{B}_{2k+1}$.

Let us next state the axioms more formally. Set

$$K = \{x \in \mathbb{R}^n \mid |x| < T\}\,,\ K' = \{x \in \mathbb{R}^n \mid |x| < 2T\}\,,$$

with T such that $\cup_\omega U_\omega \subset K$, and $\cup_\omega \psi_\omega(U_\omega) \subset K$, while $\chi_{K,K'}$ is a C^∞ function supported in K' and identically equal to 1 on K. For $k = 0, \ldots, n$ and $1 < t < \infty$:

Axiom 1: There are Banach spaces $\mathcal{B}_{k,t}$, containing C^∞ functions, with $\mathcal{B}_{k,t} \subset \mathcal{B}_{k,t'}$ boundedly if $t > t'$, and such that, if $k \le n-1$:

1. The operator $\chi_{K,K'}\mathcal{S}_k : \mathcal{A}_{k+1,\mathcal{B}_{k,t}} \to \mathcal{A}_{k,\mathcal{B}_{k,tn/(n-t)}}$ is bounded if $t < n$.
2. The two operators $\chi_{K,K'}\partial_j\mathcal{S}_k : \mathcal{A}_{k+1,\mathcal{B}_{k+1,t}} \to \mathcal{A}_{k,\mathcal{B}_{k+1,t}}$ and $\chi_{K,K'}\mathcal{S}_k : \mathcal{A}_{k+1,\mathcal{B}_{k+1,t}} \to \mathcal{A}_{k,\mathcal{B}_{k,t}}$ are bounded for each $1 \le j \le n$.
3. There is an integer $0 \le s \le \min(r,n)$ so that the operators $d_k : \mathcal{A}_{k,\mathcal{B}_{k,t}} \to \mathcal{A}_{k+1,\mathcal{B}_{k+1,t}}$ are bounded for all $k \le s$.
4. For all $k \le s$ we have the bounded inclusions

$$(\mathrm{Id} - \Delta)^{(r-k)/2}\mathcal{B}_{k,t} \subset L^t(K')\,, \qquad \chi_{K,K'}\mathcal{J}_{r-k}(L^t(K')) \subset \mathcal{B}_{k,t}\,, \tag{2.5}$$

where $\mathcal{J}_\ell$ is the ℓth iterate of the Bessel potential $\mathcal{J}_1$.

Axiom 2: There is $0 \le R < \infty$ so that

1. $\mathcal{M}_k$ is bounded on $\mathcal{A}_{k,\mathcal{B}_{k,t}}$ with

$$\begin{cases} \rho_{ess}(\mathcal{M}_k|_{\mathcal{B}_{k,t}}) < R \\ \quad \text{and there are no eigenvalues of modulus } R\,, & \forall k = 0,\ldots,s\,, \forall t\,, \\ \rho(\mathcal{M}_k|_{\mathcal{B}_{k,t}}) < R\,, & \forall k = s+1,\ldots,n\,,\forall t\,. \end{cases} \tag{2.6}$$

2. Let $\Pi_{k,t}$ be the spectral projector on $\mathcal{A}_{k,\mathcal{B}_{k,t}}$ associated to the spectrum of $\mathcal{M}_k$ inside the disc of radius R. The operator $\mathcal{N}_k^{(j)} = d\mathcal{M}_k^j - \mathcal{M}_{k+1}^j d$ is bounded from $\mathcal{A}_{k,\mathcal{B}_{k,t}}$ to $\mathcal{A}_{k+1,\mathcal{B}_{k+1,t}}$ for all $j \ge 0$, with

$$\sup_{t,k} \lim\sup_j \|\mathcal{N}_k^{(j)} \circ (\mathrm{Id} - \Pi_{k,t})\|^{1/j} < R\,. \tag{2.7}$$

Axiom 3: For each $0 \le k \le n$ there are a and C so that for each $j \ge 1$ we may decompose $(\mathrm{Id} - \Delta)^a\mathcal{M}_k^j\mathcal{J}_a(\sigma_k)(x,y) = \mathcal{H}_j(x,y) + \Sigma_j(x,y)$, where $z \mapsto \sum_j z^j\chi_K(y)\mathcal{H}_j(x,y)$ extends meromorphically to $|z| < 1/R$ with values in compactly supported n-forms with continuous coefficients; while $\Sigma_j(\cdot,y)$ is such that, for every $\mathcal{Q} = \sum_{\ell\ge1} z^\ell\mathcal{Q}_\ell : \mathcal{A}_k \to \mathcal{A}_{k+1}$ with $\mathcal{Q}_\ell\varphi = \sum_{\vec\omega\in\Omega^\ell} h^{(\ell)}_{\ell,\vec\omega} \wedge (\psi^\ell_{\vec\omega})^*\varphi$, where the $h_{\ell,\omega}$ are 1-forms with continuous coefficients supported in the domains of the ψ_ω, we have ($\mathcal{T}$ denotes the restriction to the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$)

$$|\int_x \mathcal{T}\mathcal{Q}_x\Sigma_j(x,y)| \le CR^j \cdot \sum_\ell |z|^\ell \sum_{\vec\omega\in\Omega^\ell} \sup|h^{(\ell)}_{\ell,\vec\omega}|\,.$$

If $s = n$, Axiom 1 essentially requires the $\mathcal{B}_{k,t}$ to be the Sobolev spaces

$$W^{r-k,t}(K') = \{\varphi \in L^t(K') \,|\, D^\alpha \varphi = \partial_{\alpha_1} \cdots \partial_{\alpha_n} \varphi \in L^t(K')\,, \\ \forall \alpha \text{ with } |\alpha| = \sum \alpha_i \le r-k\}\,.$$

When the ψ_ω are the inverse branches of a differentiable, (locally) uniformly expanding map on a compact manifold and the g_ω are differentiable, Axioms 1 and 2 are indeed satisfied by taking the $\mathcal{B}_{k,t}$ to be the Sobolev spaces $W^{r-k,t}(K')$. We then recover a result of Ruelle [**Ru3**]. After having stated this result precisely, we shall end this note by attempting to give a more intrinsic motivation for Axiom 1 and mentioning some features of our proof.

A new proof of a result of Ruelle.

Let M be a C^r ($r \ge 1$) compact manifold of dimension $n \ge 2$, and let $f : M \to M$ be C^r and (locally) uniformly expanding, that is, there is $\theta < 1$ such that $\|D_x f(v)\| \ge \theta^{-1}\|v\|$ for all v in the tangent space $T_x M$. Let $g : M \to \mathbb{C}$ be C^r. The transfer operator $\mathcal{L}_0 = \mathcal{L}_{f,g}$, acting e.g. on the Banach space of C^r functions $M \to \mathbb{C}$, is

$$\mathcal{L}_0 \phi(x) = \sum_{y: f(y)=x} g(y)\phi(y) = \sum_j g_j(x)\phi(\psi_j(x))\,, \tag{2.8}$$

where the ψ_j are the finitely many local inverse branches of f, and $g_j = g \circ \psi_j$. We also have operators acting on Banach spaces of k-forms with C^{r-1} coefficients:

$$\mathcal{L}_k \phi = \sum_j g_j \cdot (\psi_j^*)\phi\,. \tag{2.9}$$

An expanding map satisfies our strong quantified transversality requirement, and the Lefschetz number of the inverse map at each periodic orbit is positive, so that the Ruelle zeta function associated to f and g

$$\zeta_{f,g}(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}\, f^m} \prod_{\ell=0}^{m-1} g(f^\ell(x))\,, \tag{2.10}$$

can be viewed as a Lefschetz-Ruelle zeta function, in fact an inverse sharp determinant $(\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}))^{-1}$. Theorem 1.3 and Corollaries 1.5–1.6 in [**Ru3**] give:

THEOREM 2 (Ruelle [**Ru3**]). *For $r \ge 1$, let M be a C^r compact manifold of dimension $n \ge 2$, let $g : M \to \mathbb{C}$ be C^r, and let $f : M \to M$ be a C^r and (locally) uniformly expanding transformation. Write $P \in \mathbb{R}$ for the topological pressure of $(\log |g|, f)$.*

1. *The spectral radius of $\mathcal{L}_0$ acting on C^m is at most e^P while its essential spectral radius is at most $\theta^m e^P$ for $0 \le m \le r$. The spectral radius of $\mathcal{L}_k$ acting on forms with C^m coefficients is at most $\theta^k e^P$ while its essential spectral radius is at most $\theta^{m+k} e^P$, for $1 \le k \le n$ and $0 \le m \le r-1$.*
2. *The power series $\zeta_{f,g}(z)$ defines a meromorphic function in the disc of radius $\theta^{-r} e^{-P}$. In this disc, the order of z as a zero/pole of $\zeta_{f,g}(z)$ coincides with the sum of the algebraic multiplicities of $1/z$ as an eigenvalue of the $\mathcal{L}_{2k+1}$ acting on C^m for any $r-(2k+1) \le m \le r-1$ and $1 \le 2k+1 \le n$, minus the sum of the algebraic multiplicities of $1/z$ as an eigenvalue of the $\mathcal{L}_{2k}$ acting on C^m for $r-2k \le m \le r-1$ and $0 \le 2k \le n$.*

(See [**GuLa**] for a recent sharpening of the first statement.)

We recover Ruelle's result if the system is in fact $2\max(r,n)+r$ *times differentiable, and for* $g(y)=1/|\det Df(y)|$ *(i.e., the weight giving rise to the absolutely continuous invariant measure), and a weaker result, replacing* e^P *by* $\theta^{-n+1}e^P$*, in the case of an arbitrary smooth weight* g. We refer to [**BB**] for details, in particular regarding the technical step needed to go from Sobolev spaces to C^m spaces.

Choosing Banach spaces.

In dimension one, the kernel of the convolution operator $\mathcal{S}$ is a function of bounded variation. This is very handy, because the essential spectral radius of the transfer operator on BV can be estimated, and in many cases shown to be strictly smaller than the spectral radius ("quasicompactness"). To adapt the strategy of [**BaRu**] to our higher-dimensional setting, it is desirable to find Banach spaces which, on the one hand, contain the kernels $\sigma_k(\cdot,y)$ (or some "primitive" of the kernels, see below and Axiom 1) and, on the other hand, allow us to obtain quasicompactness of the transfer operators.

Higher-dimensional analogues of the space of functions of bounded variation are the Sobolev space $W^{1,1}(\mathbb{R}^n)$ and the Banach space

$$BV(\mathbb{R}^n)=\{\varphi\in L^1(\mathbb{R}^n)\mid \forall|\alpha|=1\,,D^\alpha\varphi \text{ is a measure with finite total variation}\}\,.$$

Since the spaces $W^{1,1}(\mathbb{R}^n)\subset BV(\mathbb{R}^n)$ are embedded in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ [**Ad, Gi**] while $\sigma_k(\cdot,y)$ is in $L^t_{loc}(\mathbb{R}^n)$ for all $1\le t<n/(n-1)$ but *not* in $L^{\frac{n}{n-1}}_{loc}(\mathbb{R}^n)$ (except for $n=1$), the kernels $\sigma_k(\cdot,y)$ cannot belong to $W^{1,1}(\mathbb{R}^n)_{loc}$. It turns out that these convolution kernels belong to a fractional Sobolev space $W^{\alpha,p}$ for $0<\alpha<1$ and small enough $p>1$ (see [**Ad**] for a definition). Instead of working with the limited "Hölder" type fractional regularity of the kernels themselves, our strategy is to improve their smoothness. We were inspired by a technique used by Ruelle [**Ru4**] in dimension one, involving modified kneading operators $\mathcal{D}^{(r)}(z)$ (with smoother kernels obtained by taking a "primitive" of the σ_k) which have the same traces as the original operators. This step allows us to let the transfer operators act on forms whose coefficients are (locally supported and) in a Sobolev space $L^t_m(\mathbb{R}^n)=W^{m,t}(\mathbb{R}^n)$, i.e., have their derivatives in $L^t(\mathbb{R}^n)$, up to some finite order $m\ge 0$. To achieve this goal, we need an analogue of the one-dimensional concept of "primitive." An obvious tool [**St**] is the inverse square root of the Laplacian (Riesz potential), or its better behaved version at infinity, the Bessel potential $\mathcal{J}_m$. Although all our objects are locally supported we use the Bessel potential for convenience.

This exploitation of the Bessel potential is emblematic of the most important new feature of this higher-dimensional version of [**BaRu**]: the *intervention of harmonic analysis.* Indeed, the elementary properties of convolution used in [**Bai**] are not sufficient to study the $d_k\mathcal{S}_k$ and classical Calderón-Zygmund results are behind the fact that the operators $d_k\mathcal{S}_k$ are bounded on $L^p(\mathbb{R}^n)$ [**St**].

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