A PARAMETER ASIP FOR THE QUADRATIC FAMILY

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Abstract. Consider the quadratic family $T_a(x) = ax(1-x)$, for $x \in [0,1]$ and mixing Collet–Eckmann (CE) parameters $a \in (2,4)$. For bounded φ, set $\tilde\varphi_a := \varphi - \int \varphi\, d\mu_a$, with μ_a the unique acim of T_a, and put $(\sigma_a(\varphi))^2 := \int \tilde\varphi_a^2\, d\mu_a + 2\sum_{i>0} \int \tilde\varphi_a(\tilde\varphi_a \circ T_a^i)\, d\mu_a$. For any mixing Misiurewicz parameter a_*, we find a positive measure set Ω_* of mixing CE parameters, containing a_* as a Lebesgue density point, such that for any Hölder φ with $\sigma_{a_*}(\varphi) \neq 0$, there exists $\epsilon_\varphi > 0$ such that, for normalised Lebesgue measure on $\Omega_* \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$, the functions $\xi_i(a) = \tilde\varphi_a(T_a^{i+1}(1/2))/\sigma_a(\varphi)$ satisfy an almost sure invariance principle (ASIP) for any error exponent $\gamma > 2/5$. (In particular, the Birkhoff sums satisfy this ASIP.) Our argument goes along the lines of Schnellmann's proof for piecewise expanding maps. We need to introduce a variant of Benedicks–Carleson parameter exclusion and to exploit fractional response and uniform exponential decay of correlations from [BBS].

Contents

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1. Introduction

1.1. **Background and Motivation.** Let $(\Omega_*, m_*, \mathcal{F}_*)$ be a probability space. We say that a sequence of measurable functions $\xi_i \colon \Omega_* \to \mathbb{R}$, $i \geq 1$ satisfies the *almost sure invariance principle (ASIP) with error exponent* $\gamma < 1/2$ if there exist a probability space $(\Omega_W, m_W, \mathcal{F}_W)$ supporting a (centered) one-dimensional Brownian motion W and a sequence of measurable functions $\eta_i \colon \Omega_W \to \mathbb{R}$, $i \geq 1$, such that

i) The random variables $\{\xi_i\}_{i\geq 1}$ and $\{\eta_i\}_{i\geq 1}$ have the same[1] distribution.
ii) Almost surely, $\left|W(n) - \sum_{i=1}^n \eta_i\right| = O(n^\gamma)$ as $n \to \infty$.

Since a Brownian motion at integer times coincides with a sum of independent identically distributed (i.i.d.) Gaussian variables, the above definition can also be formulated as an almost sure approximation, with error $o(n^\gamma)$, by a sum of i.i.d. Gaussian variables.

It is a classical result (see [PS]) that if the $\{\xi_i\}$ satisfies the ASIP then it satisfies the law of the iterated logarithm (LIL), the central limit theorem (CLT) and the functional CLT: Letting $\sigma^2 > 0$ be the variance of the Brownian motion W (the expectation is zero by assumption), and denoting Lebesgue measure by m, the LIL says that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n \log\log n}} \sum_{i=1}^n \xi_i(a) = \sigma\,, \qquad \text{for } m_*\text{-almost every } a \in \Omega_*\,,$$

and the CLT (for the functional CLT, see [DLS, Lemma 5.1]) says that

$$\lim_{n\to\infty} m_*\left(\left\{ a \in \Omega_* \mid \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^n \xi_i(a) \leq y \right\}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-s^2/2}\, ds\,,\ \forall y \in \mathbb{R}\,.$$

We consider $I = [0,1]$ and the quadratic family

$$T_a(x) = ax(1-x)\,, \quad x \in I\,,\ a \in (2,4]\,.$$

Denote by $c = 1/2$ the critical point of T_a and set $c_j(a) = T_a^j(c)$ for $j \geq 1$.

If $\liminf_{n\to\infty} n^{-1} \log \partial_x(T_a^n)(T_a(c)) > 0$, we say that a is a Collet–Eckmann (CE) parameter. If a is CE, then T_a admits a unique absolutely continuous invariant probability measure (acim) $\mu_a = h_a dm$. Our goal is to find a positive Lebesgue measure set Ω_* of CE parameters with a Lebesgue density point $a_* \in \Omega_*$ such that for any Hölder continuous function $\varphi \colon I \to \mathbb{R}$ with $\sigma_{a_*}(\varphi) \neq 0$ (see (1.2)), there exists $\epsilon_\varphi > 0$ such that the ASIP holds for m_* the normalised Lebesgue measure on $\Omega_* \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$ and

$$\xi_j(a) := \varphi_a(c_{j+1}(a))\,, \qquad j \geq 0\,, \quad a \in \Omega_* \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]\,,$$

[1] By definition of the distribution of discrete-time real-valued stochastic processes, this means that for any $n \geq 1$ and any $\{y_i \in \mathbb{R} \mid 1 \leq i \leq n\}$, the joint probability that $\xi_i \leq y_i$ for all $1 \leq i \leq n$ coincides with the joint probability that $\eta_i \leq y_i$ for all $1 \leq i \leq n$.

where φ_a is a suitable normalisation of φ (see (1.6)). We follow the approach of Schnellmann [Sch], who developed this program for transversal families of piecewise expanding maps T_a, for which Ω_* can be taken to be an interval.

Our main motivation is to extend to the quadratic family the method developed by de Lima–Smania [DLS] in the setting of piecewise expanding maps, in order to study linear and fractional response. (This method requires a functional central limit theorem, see [DLS, Lemma 5.1].)

We say that T_a is mixing if it is topologically mixing on

$$K(a) := [c_2(a), c_1(a)] .$$

It will be convenient below to restrict to mixing maps T_a. Tiozzo recently showed [Ti, Cor 3.15] (his result holds in fact for more general unimodal maps) that T_a is (strongly) mixing for its unique measure of maximal entropy (MME) if its topological entropy is greater than $\log(2)/2$. If a is a CE parameter with strongly mixing MME, then T_a is topologically mixing on $K(a)$ since the measure of maximal entropy has[2] full support there. Since the topological entropy of T_4 is equal to $\log 2$, and the topological entropy of T_a is nondecreasing and continuous (in fact Hölder continuous [Gu]) in a, there exists $a_{\text{mix}} < 4$ such that for all $a \in (a_{\text{mix}}, 4] \cap CE$, the map T_a is topologically mixing on $K(a)$, and μ_a is strongly mixing, with support $K(a)$.

Melbourne and Nicol showed [MN] the ASIP in the phase space $x \in K(a)$, setting $\xi_i = T_a^i(x)$ for a fixed CE map T_a, using an induced uniformly expanding system (then [PS, Section 7] provides an ASIP which projects to the ASIP for the original CE map). However, to the best of our knowledge, the ASIP in the parameter a is still open.

In the parameter space, typicality (the law of large numbers, LLN) and the LIL are known: Avila–Moreira [AM2], showed that[3] for Lebesgue almost every CE map T_a the critical point is typical for its unique absolutely continuous invariant measure $\mu_a = h_a dm$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(c_i(a)) = \int_0^1 \varphi \, d\mu_a , \qquad \forall \varphi \in C^0 . \tag{1.1}$$

For Hölder continuous $\varphi\colon I \to \mathbb{R}$ and a topological mixing CE parameter a, define $\sigma_a(\varphi) \geq 0$ by

$$(\sigma_a(\varphi))^2 := \int_0^1 \left(\varphi - \int \varphi d\mu_a\right)^2 d\mu_a \tag{1.2}$$

$$+ 2 \sum_{i>0} \int_0^1 \left(\varphi - \int \varphi \, d\mu_a\right)\left(\varphi - \int \varphi \, d\mu_a\right) \circ T_a^i \, d\mu_a , \tag{1.3}$$

where the sum (1.3) is finite because topological mixing (i.e., the fact that the map is nonrenormalisable) implies [KN] exponential mixing for the acim and Hölder continuous observables.

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[2]Indeed, since T_a has no homtervals if $a \in CE$, it is conjugated to its piecewise linear model F_a by a homeomorphism which maps the MME of F_a to the MME of T_a, and the MME of F_a is absolutely continuous with a positive density on $[F_a^2(c), F_a(c)]$.

[3]Benedicks and Carleson established typicality in [BC1] for the Cantor set of CE parameters considered there.

In a work in progress, Gao and Shen [GS2] show that, for Lebesgue almost every a in the set of mixing CE parameters, for every Hölder observable φ, either $\sigma_a(\varphi) = 0$ or the critical point c of T_a satisfies the LIL for φ, i.e.,

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n\log\log n}} \sum_{i=1}^{n} \left(\varphi(T_a^i(c)) - \int \varphi \, d\mu_a \right) = \sigma_a(\varphi) \, .$$

1.2. **Statement of the ASIP (Theorem 1.1).** To state our main result, we need more notation and definitions. For $j \ge 0$ and $a \in (a_{\mathrm{mix}}, 4]$, set

$$x_j(a) = c_{j+1}(a) = T_a^{j+1}(c) \, , \qquad T_a'(x) = \partial_x T_a(x) \, , \qquad x_j'(a) = \partial_a x_j(a) \, .$$

The family T_a is *transversal* at a_* if (see [Ts1]) there exists $C \ge 1$ such that

$$\frac{1}{C} \le \left| \frac{x_j'(a_*)}{(T_{a_*}^j)'(c_1(a_*))} \right| \le C \, , \qquad \forall j \ge 1 \, . \tag{1.4}$$

By [Ts2, Theorem 3], all CE parameters are transversal. We refer to [Ts1, (NV_t)] for an equivalent condition expressed in terms of the postcritical orbit.

The map T_a is (H_a, κ_a)*-polynomially recurrent,* for $\kappa_a \ge 1$ and $H_a \ge 1$, if

$$|x_{j-1}(a) - c| = |T_a^j(c) - c| \ge \frac{1}{j^{\kappa_a}} \, , \qquad \forall j \ge H_a \, . \tag{1.5}$$

If $\inf_{j\ge1} |T_a^j(c) - c| > 0$ then a is called a *Misiurewicz parameter.* Misiurewicz parameters are CE and thus transversal. Avila and Moreira [AM1] showed that, for any $\kappa_0 > 1$, the set of parameters a which are (H_a, κ_0)-polynomially recurrent for some H_a has full measure in the set of CE parameters. The set of Misiurewicz parameters a is uncountable (it has full Hausdorff dimension [Za, Thm. 1.4] but zero Lebesgue measure).

Finally, we introduce the normalisation φ_a: Let φ be bounded such that $\sigma_a(\varphi) \ne 0$ for a mixing CE parameter a. Then the function

$$\varphi_a(x) := \frac{1}{\sigma_a(\varphi)} \left(\varphi(x) - \int_0^1 \varphi \, d\mu_a \right) \tag{1.6}$$

is well defined and satisfies

$$\sigma_a(\varphi_a) = 1 \qquad \text{and} \qquad \int \varphi_a \, d\mu_a = 0 \, . \tag{1.7}$$

Theorem 1.1 (Main Theorem: ASIP)**.** *For any Misiurewicz parameter $a_* \in (a_{\mathrm{mix}}, 4)$ there exists a positive Lebesgue measure set Ω_* of mixing polynomially recurrent parameters, containing a_* as a Lebesgue density point, such that for any Hölder continuous function φ with $\sigma_{a_*}(\varphi) \ne 0$, there exists*[4] *$\epsilon_\varphi > 0$ such that the functions*

$$\xi_n(a) := \varphi_a(x_n(a)) = \varphi_a(T_a^{n+1}(c)) \, , \qquad n \ge 1 \, , \tag{1.8}$$

satisfy the ASIP for normalised Lebesgue measure m_* on $\Omega_* \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$ and all error exponents $\gamma > 2/5$.

[4] The choice of ϵ_φ ensures in particular that $\sigma_a(\varphi) \ne 0$ if $\sigma_{a_*}(\varphi) \ne 0$.

The value $a_* = 4$ is not covered by our arguments for technical reasons, since c_1 and c_2 then lie on the boundary of I (see e.g. Footnote 29). It is possible (but a bit cumbersome) to handle (a one-sided neighbourhood of) this value by a change of coordinates as in [Ts1, Lemma 2.1].

We expect that the methods[5] of this paper can be extended to the case when the "root" a_* is mixing, but only Collet–Eckmann and polynomially recurrent (for large enough $\kappa_0 > 1$), instead of Misiurewicz. We restrict here to Misiurewicz parameters a_*, for the sake of simplicity. What is most desirable in view of our original motivation to extend the analysis of [DLS], is to obtain a "fatter" Cantor set Ω_* (as opposed to a fatter set of root points a_*): Indeed, this extension will probably require the ASIP on a set $\widetilde{\Omega}$ for which there exist $\beta > 1$ and a full measure subset $\widetilde{\Omega}_1 \subset \widetilde{\Omega}$ such that

$$\lim_{\epsilon \to 0} \frac{m([a-\epsilon, a+\epsilon] \setminus \widetilde{\Omega})}{\epsilon^\beta} = 0\,, \ \forall a \in \widetilde{\Omega}_1\,. \tag{1.9}$$

(See [BS2, (5), Prop. F], note that [BS2, Lemma E] even uses $\beta < 2$ close to 2.) Property (1.9) is known for all $\beta < 2$ for the sets $\widetilde{\Omega}_1 \subset \widetilde{\Omega}$ studied[6] by Tsujii [Ts1]. Although it is not stated in the literature, the Cantor set Ω_{BC} from the (exponential) Benedicks–Carleson construction at a Misiurewicz point a_* should[7] satisfy (1.9) at $a = a_*$ for some $\beta > 1$. For our Cantor set $\Omega_* \subset \Omega_{BC}$, we expect that for any $\kappa > 1$, taking κ_0 large enough in Proposition 2.2 the factor ϵ^β in (1.9) must be replaced by $\epsilon|\log \epsilon|^{-\kappa}$ (see (2.20)), which does not seem good enough. Attaining the goal of our original motivation may thus require establishing the ASIP on a Cantor set having larger density, and thus weakening the polynomial lower recurrence in the construction (see comments in the next paragraph). We view this as the most desirable improvement of our main theorem.

To clarify the role of Ω_*, it is useful to compare Schnellmann's proof with ours. In [Sch], Schnellmann studies suitable transversal one-parameter families of piecewise expanding interval maps and obtains a parameter ASIP on a set Ω_* which is just an interval $[0, \epsilon^\varphi]$ of parameters. Indeed, existence of an exponentially mixing acim enjoying fractional response (with uniform bounds) holds in an entire interval $[0, \epsilon^\varphi]$ in his setting [Sch, Prop. 4.3, Lemma4.5]. So $[0, \epsilon^\varphi]$ is the baseline parameter space for his analysis. Some parameters in this baseline cause difficulties ("exceptionally small sets"), but Schnellmann can get away with just *ignoring* them (taking advantage of the fact that their total measure is controlled [Sch, (III), Theorem 3.2, Lemma 4.1, proof of Lemmas 6.1–6.2]) instead of *excluding* them from the baseline. Our situation is different, since we need to *exclude* parameters which do not have an acim or for which exponential mixing or fractional

[5] For example, [A, Lemma 8.1] would replace [DMS, Lemma V.6.5] in the proof of Proposition 2.2.

[6] Beware that Tsujii's result cannot be used immediately. In particular, the main argument in the construction of the parameter set in Theorem 1 of *Pre-threshold fractional susceptibility function: holomorphy and response formula,* arxiv.org/2203.07942, is flawed.

[7] See (2.18), noting that $|\omega_0| = 2\epsilon$, and taking $j \geq N_0$ for $N_0 = O|\log \epsilon|$ (as is the case in the proof of Proposition 2.2).

response (with uniform bounds) does not hold: Our baseline set is a Cantor set, and the best we can do is to make it as fat as possible.

The polynomial recurrence (1.5) in our parameter exclusion (Proposition 2.2), which causes the "thinness" of Ω_*, is needed[8] to apply the results of [BBS] in Sections 2.4 and 2.5 (Propositions 2.5 and 2.6 on uniform decorrelation and fractional response, and its consequence, Lemma 2.8). Due to this we already *exclude* the parameters which could have exceptionally small image and we do not need to *ignore* them (Lemma 2.3, compare also the proof of [Sch, Lemma 6.1] with (4.13) below). In addition, we get an easy proof of the local distortion estimate (2.31). If the required consequences of [BBS] could be extended to sets of parameters which enjoy only exponential recurrence bounds, then we could use the (fatter) Benedicks–Carleson Cantor set Ω^{φ}_{BC} as a baseline instead of Ω_* (if necessary, the Benedicks–Carleson technique could be replaced by ideas from Tsujii [Ts1], Avila–Moreira [AM1] or Gao–Shen [GS1]). Next, one could try to *ignore* the parameters with exceptionally small images in Lemma 2.3. For (2.31), see also Footnote 24.

We also note for the record here that the characteristic function $1_{\widetilde{\Omega}}$ of a fat enough Cantor set $\widetilde{\Omega}$ belongs to a Sobolev space $H^s_q(I)$ with $s > 0$ (see [HM, Props 4.9 and 4.10]). Thus, working with a Cantor set of larger density may simplify some of our arguments (in the proof of Proposition 3.2, e.g.).

Finally, the results of this paper probably extend to more general families of smooth unimodal maps. In the present "proof of concept" work, we choose to restrict to the quadratic family.

1.3. **Structure of the Text.** Schnellmann pointed out [Sch, p. 370] that the "Markov partitions" given by the intervals in the celebrated Benedicks–Carleson [BC1, BC2] parameter exclusion construction would be the key to extend his result to nonuniformly expanding interval maps.

Our paper carries out this plan and is organised as follows: After recalling basic facts in Section 2.1, we adapt in Section 2.2 the Benedicks and Carleson procedure to construct, in a neighbourhood of a topologically mixing Misiurewicz point a_*, a sequence $\Omega_n \subset \Omega_{n-1}$ where Ω_n is a finite union of intervals in $\mathcal{P}_n$. At each step, some intervals in $\mathcal{P}_n$ are partitioned and the intervals which do not satisfy a time-n *polynomial* recurrence assumption are excluded. The remaining Cantor set $\Omega_*(a_*) = \cap_n \Omega_n$ is a positive Lebesgue measure set of parameters satisfying the Collet–Eckmann property, polynomial returns and distortion control, with uniform constants. (Our distortion bound (2.31) is new.) In addition, the construction ensures that there are no "exceptionally small" sets (Lemma 2.3). Applying results from [BBS], this ensures uniform exponential decay of correlations (Proposition 2.5) and fractional response (Proposition 2.6), from which we obtain regularity of the map $a \mapsto \sigma_a$ (Lemma 2.8).

Sections 3 and 4 contain the proof of the ASIP along the lines of [Sch]: First approximate the Birkhoff sum by a sum of blocks of polynomial size (Sections 4.1 and 4.2), then (Section 4.3) approximate these blocks by a martingale difference sequence Y_j and apply Skorokhod's representation

[8]See also (4.7) and (4.22), which may cause a different error exponent.

theorem linking a martingale with a Brownian motion (see [PS, Section 3]). The usual application of the approach of [PS, Chapter 7] in dynamics uses a strong independence condition (see [PS, 7.1.2]) which we do not have (the ξ_i's are not iterations of a fixed map and there is no[9] underlying invariant measure). We replace this strong independence condition by uniformity of constants in the exponential decay of correlations (given by [BBS]) which we translate into properties for the ξ_i by switching from parameter to phase space (see Proposition 3.2), giving estimates similar to those in [PS, Section 3].

For $\varpi \in (0,1)$, we shall denote by C^{ϖ} the set of ϖ-Hölder continuous functions $\varphi\colon I \to \mathbb{R}$, putting $\|\varphi\|_{\varpi} = \sup|\varphi| + H_{\varpi}(\varphi)$, with $H_{\varpi}(\varphi)$ the smallest H_{ϖ} such that $|\varphi(x) - \varphi(y)| \le H_{\varpi}|x-y|^{\varpi}$ for all x, y in I. The letter C is used throughout to represent a (large) uniform constant, which may vary from place to place.

2. Bounds for the Quadratic Family. The Cantor Set $\Omega_*(a_*)$

2.1. **Basic Properties.** Clearly, the maps

$$a \mapsto T_a'(x) = \partial_x T_a(x) = a(1-2x)\,, \qquad x \mapsto \partial_a T_a(x) = x(1-x)$$

are Lipschitz continuous uniformly in $x \in I$ and $a \in (a_{\mathrm{mix}}, 4]$, and in addition

$$\sup_{x \in I} |T_a'(x)| \le \Lambda := 4\,, \qquad \forall a \in (a_{\mathrm{mix}}, 4]\,. \tag{2.1}$$

Each T_a has two monotonicity intervals, with partition points 0, $c = 1/2$, and 1. The following easy lemma replaces[10] [Sch, (30)]:

Lemma 2.1. *There exists $C < \infty$ such that, for any $a_1, a_2 \in (2,4]$, we have*

$$|T^n_{a_1}(x) - T^n_{a_2}(x)| \le C\Lambda^n |a_1 - a_2|\,, \qquad \forall x \in I\,, \quad \forall n \ge 1\,. \tag{2.2}$$

Proof. Clearly, $|T_{a_1}(x) - T_{a_2}(x)| \le |a_1 - a_2|$. For $n \ge 2$, using the definition (2.1) of Λ, and setting $C = \sum_{j=0}^{\infty} \Lambda^{-j}$, we get

$$\begin{aligned}
&|T^n_{a_1}(x) - T^n_{a_2}(x)| \\
&\qquad \le |T_{a_1}(T^{n-1}_{a_1}(x)) - T_{a_2}(T^{n-1}_{a_1}(x))| + |T_{a_2}(T^{n-1}_{a_1}(x)) - T_{a_2}(T^{n-1}_{a_2}(x))| \\
&\qquad \le |a_1 - a_2| + \Lambda |T^{n-1}_{a_1}(x) - T^{n-1}_{a_2}(x)| \\
&\qquad \le |a_1 - a_2|(1+\Lambda) + \Lambda^2 |T^{n-2}_{a_1}(x) - T^{n-2}_{a_2}(x)| \le \dots \\
&\qquad \le |a_1 - a_2| \sum_{j=0}^{n-1} \Lambda^j \le C\Lambda^n |a_1 - a_2|\,.
\end{aligned}$$

□

[9] See the example (see [Ka], p. 646) discussed in [Sch]. Also, as pointed out by [Sch], it is not clear how to apply the spectral techniques of [Go] in our setting.

[10] We do not need as in [Sch, (30)] that x has the same combinatorics under T_{a_1} and T_{a_2} up to the $(n-1)$th iteration. We thus do not need any analogue of [Sch, Sublemma 5.4].

2.2. **A Polynomial Benedicks–Carleson Construction ($\Omega_*(a_*)$, $\mathcal{P}_n$).** For each $j \geq 0$, the function $x_j(a) = T_a^{j+1}(c)$ is a map from the parameter space $(a_{\text{mix}}, 4]$ to the phase space $I = [0, 1]$, with $x_j(a) \in K(a)$ for all a. The transversality condition (1.4) says that the derivatives of x_j and T_a^j are comparable at a_*, so that statistical properties (such as the ASIP) can be transferred from the maps $x \mapsto T_a^j(x)$ to the maps $a \mapsto x_j(a)$. To make this precise, we next construct a sequence of partitions in the parameter space. Our starting point is the following variant of the Benedicks and Carleson Cantor set[11] $\Omega_{BC} = \Omega_{BC}(a_*)$ (see [BC1, BC2]) associated to a Misiurewicz parameter a_* (which is automatically transversal):

Proposition 2.2 (The Cantor set $\Omega_* = \Omega_*(a_*, \kappa_0)$)**.** *Let $a_* \in (a_{\text{mix}}, 4]$ be a Misiurewicz parameter. There exist $\lambda_{CE} \in (1, \Lambda)$ and $C_0 \in (0, 1)$ such that, for any $d_1 \in (0, C_0 \log \lambda_{CE}/4)$ and $d_0 > 0$, there exists $\epsilon > 0$ such that, for any $\kappa_0 > 1/d_1$, for all large enough $N_0 \geq 1$ there exists a sequence $\mathcal{P}_j$ of finite sets of pairwise disjoint subintervals of*

$$\omega_0 := [a_* - \epsilon, a_* + \epsilon] \cap (a_{\text{mix}}, 4] ,$$

such that $\mathcal{P}_1 = \mathcal{P}_2 = \ldots = \mathcal{P}_{N_0}$ and, setting

$$\Omega_* = \Omega_*(a_*, \kappa_0) := \bigcap_{j \geq N_0} \Omega_j , \qquad \textit{with} \qquad \Omega_j := \bigcup_{\omega \in \mathcal{P}_j} \omega ,$$

we have $\Omega_{j+1} \subset \Omega_j$ for $j \geq N_0$, and[12]

$$\forall j \geq 1 , \quad \forall \omega \in \mathcal{P}_j , \quad \forall 0 \leq \ell < j , \quad \exists \omega' \in \mathcal{P}_\ell \textit{ such that } \omega \subset \omega' , \tag{2.3}$$

$$|x_j'(a)| > 0 , \quad |T_a^{j+1}(c) - c| > 0 , \quad \forall a \in \omega , \quad \forall \omega \in \mathcal{P}_j , \quad \forall j \geq 0 , \tag{2.4}$$

and there exists[13] *$C < \infty$ such that, for all $j \geq N_0$ and $\omega \in \mathcal{P}_j$,*

$$|(T_a^n)'(T_a(c))| \geq \lambda_{CE}^n , \qquad \forall N_0 \leq n \leq j , \quad \forall a \in \omega , \tag{2.5}$$

$$\frac{1}{C} \leq \left| \frac{x_n'(a)}{(T_a^n)'(T_a(c))} \right| \leq C , \qquad \forall N_0 \leq n \leq j , \quad \forall a \in \omega , \tag{2.6}$$

$$|\tilde{\omega}| \leq C \lambda_{CE}^{-n} |x_n(\tilde{\omega})| , \qquad \forall N_0 \leq n \leq j , \quad \forall \tilde{\omega} \subset \omega , \tag{2.7}$$

and, moreover,

$$|T_a^{n+1}(c) - c| > n^{-\kappa_0} , \qquad \forall N_0 \leq n \leq j , \quad \forall a \in \omega . \tag{2.8}$$

Finally, we have that $a_* \in \Omega_*$ is a Lebesgue density point of Ω_*, with

$$|\Omega_*| \geq (1 - d_0 \cdot e_j)|\Omega_{j-1}| , \qquad \forall j \geq N_0 , \quad \textit{where} \quad e_j := \sum_{n=j}^{\infty} n^{-d_1 \cdot \kappa_0} , \tag{2.9}$$

and we have the more precise (semi-local) bound

$$\sum_{\substack{\omega \in \mathcal{P}_\ell \\ \omega \subset \omega'}} |\omega \setminus (\omega \cap \Omega_*)| \leq d_0 \cdot e_{\ell - \ell'} |\omega'| , \quad \forall \omega' \in \mathcal{P}_{\ell'} , \forall \ell \geq \ell' \geq N_0 . \tag{2.10}$$

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[11] See (2.17) for the construction of Ω_{BC}.

[12] The first bound of (2.4) implies that $a \mapsto x_j(a) = T_a^{j+1}(c)$ is monotone on $\omega \in \mathcal{P}_j$.

[13] Note that (2.6) replaces [Sch, Lemma 2.4].

See Lemma 2.3 below regarding the absence of exceptionally small sets and Section 2.3 for a Hölder distortion property refining (2.16).

Clearly, (2.8) means that any $a \in \Omega_*$ is (N_0, κ_0)-polynomially recurrent.

The bound (2.9) implies that the Cantor set Ω_* has positive Lebesgue measure as soon as $d_1 \cdot \kappa_0 > 1$ (and N_0 is large enough). Proposition 2.2 holds for such κ_0, but we will need the stronger condition $d_1 \cdot \kappa_0 \ge 11/3$ to use (2.9) in the proof of Proposition 3.2 (and $d_1 \cdot \kappa_0 > 9/5$ for Lemma 4.2).

The local bound (2.10) is used in the proof of Lemma 4.1.

Proof of Proposition 2.2. Let $r_0 \ge 2$ be a large integer (to be chosen later, with $\epsilon \to 0$ as r_0 increases). For $r \ge r_0$, set $I_r = I_r^- \cup I_r^+$, where

$$I_r^+ = [c+e^{-r-1}, c+e^{-r}) \, , \ I_r^- = (c-e^{-r}, c-e^{-r-1}] \, , \ U_r = (c-e^{-r}, c+e^{-r}) \, ,$$

and cover each $I_r^\pm$ by r^2 pairwise disjoint intervals $I_{r,\ell}^\pm$ of equal size, each $I_{r,\ell}^\pm$ containing its boundary point closest to c. Let[14] $\beta_{BC} > \alpha_{BC} > 0$ where

$$e^{-n\alpha_{BC}} \le n^{-\kappa_0} \, , \ \forall n \ge N_0 \, ,$$

for N_0 a large integer to be chosen later.

For $a \in (a_{\mathrm{mix}}, 4]$, $\nu \ge 1$, and $r \ge r_0$ such that $T_a^\nu(c) \in I_r$, the *binding time* $p(a) = p(r, a, \nu)$ of U_r with $T_a^\nu(c)$ is the maximal $p \in \mathbb{Z}_+ \cup \{\infty\}$ such that

$$|T_a^j(x) - T_a^{j+\nu}(c)| \le e^{-j\beta_{BC}} \, , \qquad \forall 1 \le j \le p \, , \quad \forall x \in U_r \, .$$

The *first free return time* $\nu_1(a)$ of $a \in (a_{\mathrm{mix}}, 4]$ is the smallest integer $j \ge 1$ for which $T_a^j(c) \in U_{r_0}$. For an interval $\omega \subset (a_{\mathrm{mix}}, 4]$, the *first free return time* $\nu_1(\omega)$ is the smallest integer $j \ge 1$ for which there exists $a \in \omega$ with $T_a^j(c) \in U_{r_0}$. If there exists $r = r(\omega)$ such that $x_{\nu_1 - 1}(\omega) \subset I_r$ (recall that $T_a^{\nu_1}(c) = x_{\nu_1-1}(a)$), we define the *first binding time* of ω by $p_1(\omega) = \min_{a \in \omega} p(r, a, \nu_1(\omega))$. For $i \ge 2$, define inductively the ith *free return time* of (suitable) ω to be the largest integer $\nu_i(\omega) > \nu_{i-1}(\omega) + p_{i-1}(\omega) + 1$ such that

$$T_a^j(c) \cap U_{r_0} = \emptyset \, , \qquad \forall \nu_{i-1}(\omega) + p_{i-1}(\omega) + 1 \le j < \nu_i(\omega) \, , \ \forall a \in \omega \, ,$$

and, for $r(\omega)$ such that $x_{\nu_{i-1}-1}(\omega) \subset I_r$, set the ith *binding time* of ω to be

$$p_i(\omega) = \min_{a \in \omega} p(r, a, \nu_{i-1}(\omega)) \, .$$

(Similarly, define inductively for $i \ge 2$ and a such that $T_a^{\nu_{i-1}}(c) \in I_r$, the pointwise binding times $p_i(a)$ and free returns $\nu_i(a)$.) The iterates between $\nu_i(\omega)$ and $\nu_i(\omega) + p_i(\omega)$ form the ith bound period of ω, those between $\nu_{i-1}(\omega) + p_{i-1}(\omega) + 1$ and $\nu_i(\omega) - 1$ form its ith free period. Finally, if there exist $a \in \omega$ and $j \ge \nu_1(\omega)$ such that $T_a^j(c) \in U_{r_0}$, we say that j is a *return time* of ω. (Return times either are free returns $\nu_i(\omega)$ or they occur during the bound period.)

Note that for any fixed ϵ, setting $\omega_0 = [a_* - \epsilon, a_* + \epsilon]$, there exists N_ϵ such that $x_{N_\epsilon}(\omega_0)$ contains a neighbourhood of c (indeed, by transversality, for any $a \in \omega_0 \setminus \{a_*\}$, there exists $N(a)$ such that $T_{a_*}^{N(a)+1}(c)$ and $T_a^{N(a)+1}(c)$ lie on different sides of c). In particular, $\nu_1(\omega_0) < \infty$. Similarly, all $\nu_i(\omega_0)$ and $p_i(\omega_0)$ are finite.

[14]The constant α_{BC} is usually called α, but we shall need the letter α for another purpose in (2.30).

Let W_{a_*} be a neighbourhood of c disjoint from $\{T^n_{a_*}(c) \mid n \geq 1\}$. From now on, we only consider r_0 large enough such that $\overline{U}_{r_0-1} \subset W_{a_*}$. Set $W^+_{a_*,r_0} = W_{a_*} \cap [c+e^{-r_0}, 1]$ and $W^-_{a_*,r_0} = W_{a_*} \cap [0, c-e^{-r_0}]$. We claim that, for any fixed large r_0, we have that $x_{\nu_1(\omega_0)-1}(\omega_0)$ contains[15] $W^+_{a_*,r_0}$ or $W^-_{a_*,r_0}$ for all small enough ϵ. Indeed, $x_{\nu_1(\omega_0)-1}(\omega_0)$ is an interval intersecting U_{r_0}, and $x_{\nu_1(\omega_0)-1}(\omega_0)$ contains $T^{\nu_1(\omega)}_{a_*}(c) \notin W_{a_*}$.

For small $\epsilon > 0$ (to be chosen depending on r_0), the sequence $\mathcal{P}_j$ can now be defined[16] inductively: Start with the single interval $\mathcal{P}_0 = \mathcal{P}_1 = \ldots = \mathcal{P}_{N_0} = \{\omega_0\}$, for ϵ small enough such that $\nu_1(\omega_0) \geq N_0$ (note that $\nu_1(\omega_0)$ increases if r_0 increases or ϵ decreases).

For $j > N_0$, each $\omega \in \mathcal{P}_{j-1}$ is partitioned into finitely many (possibly just one) intervals, at least one of which will be included into an auxiliary partition $\mathcal{P}'_j$, as follows:

If j is not a free return[17] time of ω, we include ω in $\mathcal{P}'_j$. If j is a free return time of ω but $x_{j-1}(\omega)$ does not contain an interval $I^{\pm}_{r,\ell}$ (we call this an *inessential (free) return*), we also include ω in $\mathcal{P}'_j$.

Otherwise, j is a free return time of ω such that $x_{j-1}(\omega)$ contains at least one interval $I^{\pm}_{r,\ell}$. We call this an *essential (free) return.* In that case, we decompose $x_{j-1}(\omega)$ into the following intervals:

$$x_{j-1}(\omega) \setminus U_{r_0}\,, \qquad \{x_{j-1}(\omega) \cap I^{\pm}_{r,\ell} \mid r \geq r_0\,,\, 1 \leq \ell \leq r^2\}\,.$$

If $x_{j-1}(\omega) \setminus U_{r_0} \neq \emptyset$, but any of the (at most two) connected components of $x_{j-1}(\omega) \setminus U_{r_0}$ has size less than $e^{-r_0}(1-1/e)r_0^{-2} = |I^{\pm}_{r_0,\ell}|$, we join it to its neighbour $x_{j-1}(\omega) \cap I^{\pm}_{r_0,\ell} = I^{\pm}_{r_0,\ell}$. If a connected component of $x_{j-1}(\omega) \setminus U_{r_0}$ has size larger than $S := \sqrt{|U_{r_0}|}$, we subdivide it into pairwise disjoint intervals of lengths between $S/2$ and S. If $x_{j-1}(\omega) \cap I^{\pm}_{r,\ell} \neq \emptyset$, but $I^{\pm}_{r,\ell}$ is not contained in $x_{j-1}(\omega)$ (this can happen for at most two intervals $I^{\pm}_{r,\ell}$), we join $x_{j-1}(\omega) \cap I^{\pm}_{r,\ell}$ to its neighbour $x_{j-1}(\omega) \cap I^{\pm}_{r',\ell'} = I^{\pm}_{r',\ell'}$. Denote by $\{\hat\omega_{r,\ell} \mid r \geq r_0 - 1\}$ the partition of $x_{j-1}(\omega)$ thus obtained, where the index (r,ℓ) refers to the "host" interval $I_{r,\ell}$ contained in $\hat\omega_{r,\ell}$ if $r \geq r_0$, while $\hat\omega_{r_0-1,\ell} \subset I \setminus U_{r_0}$. Then we discard all intervals $\hat\omega_{r,\ell}$ for which

$$e^r \geq (j-1)^{\kappa_0}\,. \tag{2.11}$$

Mapping the remaining intervals via the inverse of the diffeomorphism (see [DMS, Prop. V.6.2]) x_{j-1} gives finitely many subintervals of ω which we include in $\mathcal{P}'_j$. Further intervals $\hat\omega_{r,\ell}$ need to be discarded from $\mathcal{P}'_j$, using a requirement denoted (FA_j) or (FA'_j) in [DMS, Section V.6], [Mo], which finally defines $\mathcal{P}_j$. For further use, we denote these remaining intervals by

$$\omega_{r,\ell} = x^{-1}_{j-1}(\hat\omega_{r,\ell})\,. \tag{2.12}$$

[15]This fact is used before [DMS, Lemma V.6.8]. (There, W_{a_*} is mistakenly mentioned instead of $W^{\pm}_{a_*,r_0}$. Our r_0 is denoted by Δ and our $x_n(a)$ is denoted $\xi_{n+1}(a)$ in [DMS].)

[16]We refer throughout to [DMS, Section V.6]. The original ideas and key estimates appeared previously in the work of Benedicks and Carleson [BC1, BC2]. See Footnote 18.

[17]That is, either j is not a return, or it is a return within the bound period.

It is well known[18] [BC1, BC2, DMS] that, if we replace the condition (2.11) (used to discard intervals) by the exponential condition

$$\omega \cap I^{\pm}_{r,\ell} \neq \emptyset \quad \text{and} \quad e^r \geq e^{\alpha_{BC}(j-1)}\,, \tag{2.13}$$

to construct sequences $\mathcal{P}_j^{\prime,BC}$ and $\mathcal{P}_j^{BC}$, then there exists $\lambda_{CE} > 1$ (called e^γ in [DMS, (V.6.4), Theorem V.6.2]) such that for any small enough $\beta_{BC} > \alpha_{BC} > 0$ there exist N_0' such that, if r_0 is large enough and $\epsilon > 0$ small enough, then the $\mathcal{P}_j^{BC}$ satisfy (2.3)–(2.7) ((2.5) is called (EX_j) in [DMS, Section V.6]) for some $C < \infty$, and the following condition (noted[19] (BA_j) in the literature) holds for all $j \geq N_0'$

$$2|T_a^{n+1}(c) - c| > e^{-n\alpha_{BC}}\,, \qquad \forall N_0' \leq n \leq j\,,\ \forall a \in \omega\ \ \forall \omega \in \mathcal{P}_j^{\prime,BC}\,. \tag{2.14}$$

Since λ_{CE} does not depend on α_{BC}, N_0, or N_0', we may assume that

$$14\alpha_{BC} < \log \lambda_{CE}$$

and we may replace N_0 by $\max\{N_0, N_0'\}$.

In particular [DMS, Prop. V.6.1, Lemma V.6.1 b), c)] give $\gamma_0 > 0$, $\lambda_{CE} = e^\gamma \in (1, e^{\gamma_0})$, and $C_0 > 0$ (independent of r_0 and ϵ) such that, if $a \in \Omega_n$ and $\nu_{\ell+1}(a) \leq n$, writing p_ℓ, ν_ℓ for $p_\ell(a)$, $\nu_\ell(a)$, we have

$$\begin{cases} |(T_a^{\nu_{\ell+1}-(\nu_\ell+p_\ell+1)})'(T_a^{\nu_\ell+p_\ell+1}(c))| \geq C_0 e^{\gamma_0(\nu_{\ell+1}-(\nu_\ell+p_\ell))} \\ |(T_a^{p_\ell+1})'(T_a^{\nu_\ell}(c))| \geq \lambda_{CE}^{p_\ell/4}\,. \end{cases} \tag{2.15}$$

To establish (2.5) (the bound below will also be used for (2.34)), one takes r_0 such that

$$r_0^2 C_0^2 \log \lambda_{CE} > |\log C_0|\,.$$

The key distortion bound [DMS, Prop. V.6.3] gives C such that

$$\left|\frac{x_j'(a_1)}{x_j'(a_2)}\right| \leq C\,, \qquad \forall N_0 \leq j \leq n\,, \quad \forall a_1, a_2 \in \omega\,, \tag{2.16}$$

whenever $n+1$ is a free return time of $\omega \in \mathcal{P}_n$ with $x_{n+1}(\omega) \subset U_{r_0/2}$. The bound (2.6) follows from [DMS, Prop. V.6.2 and Theorem V.6.2].

Let $\Omega_j' := \bigcup_{\omega \in \mathcal{P}_j'} \omega$, recall Ω_j, and define Ω_j^{BC} and $\Omega_j^{\prime,BC}$ accordingly, setting

$$\Omega_{BC} = \Omega_{BC}(a_*, \alpha_{BC}) = \cap_j \Omega_j^{BC}\,, \quad \text{so that} \quad \Omega_*(a_*) \subset \Omega_{BC}(a_*)\,. \tag{2.17}$$

It is easy to check that (2.11) implies (2.8) (for returns during a bound period, use that $\ell^{-\kappa_0} - e^{-\ell\beta_{BC}} \geq j^{-\kappa_0}$ for all $N_0 \leq \ell \leq j-1$, up to increasing N_0 again). Our choice of N_0 implies $\Omega_j \subset \Omega_j^{BC}$. Also, (2.6) with (2.4) imply that all points in Ω_* are transversal. Since (2.7) is an immediate consequence of (2.5)–(2.6), it only remains to establish that a_* is a Lebesgue density point in Ω_* (clearly, $a_* \in \Omega_*$) and that (2.9) and (2.10) hold.

[18] The original construction in [BC1, BC2] is for $a_* = 4$, see [Mo] for a self-contained account. It extends to Misiurewicz parameters: for CE parameters, the condition in [DMS, Theorem 6.1] is equivalent to (1.4), taking large enough k in the last line of [DMS, p. 406, Step 2].

[19] Strictly speaking, the condition (BA_j) does not involve the factor 2, and a condition (BA_j') requiring that for each $\omega \in \mathcal{P}_j^{\prime BC}$ there exists $a \in \omega$ with $|T_a^{n+1}(c) - c| > e^{-n\alpha_{BC}}$ for $N_0' \leq n \leq j$ is used in some lemmas. See [DMS, Section V.6, Step 5].

To show that a_* is a Lebesgue density point of Ω_*, we may follow[20] [DMS, Step 7 of the proof of Theorem V.6.1], replacing Ce^{-iC_0} there by $C'i^{-\kappa_0}$.

We next establish (2.9) and (2.10). For suitably small $\bar\eta > 0$, and for $J_0 \ge 1$ such that[21] $\prod_{j=J_0}^\infty (1-e^{-\bar\eta j}) > 3/4$, the parameter exclusion rule (2.13) gives $d_0' > 0$ (tending to zero with ϵ) such that ([DMS, Section V.6, Step 7], [Mo, §6])

$$(2.18)\qquad \begin{cases} |\omega \cap \Omega_j^{',BC}| \ge (1-d_0' e^{-j\bar\eta})|\omega|\,, & \forall \omega \in \mathcal{P}_{j-1}^{BC}\,,\ \forall j \ge J_0\,,\\ |\Omega_j^{BC}| \ge |\Omega_j^{',BC}| - e^{-j\bar\eta}|\omega_0|\,, & \forall j \ge J_0\,. \end{cases}$$

The above implies $|\Omega_j^{BC}| \ge (1-d_0'e^{-\bar\eta j})|\Omega_{j-1}^{BC}| - e^{-\bar\eta j}|\omega_0|$ for $j \ge J_0$, and, exploiting that $|\omega_0| = |\Omega_n^{BC}|$ for all $n \le N_0$ with $N_0 \ge J_0$, and using the definition of J_0, also that

$$|\Omega_j^{BC}| \ge \left(\prod_{n=J_0}^{j}(1-d_0'e^{-\bar\eta n}) - \sum_{n=J_0}^{j} e^{-\bar\eta n}\right)|\omega_0| \ge \frac{1}{2}|\omega_0|\,, \qquad \forall j \ge J_0\,.$$

(By taking larger J_0, i.e. smaller ϵ, we could replace $1/2$ by a number close to 1.) Thus, applying inductively

$$|\Omega_j^{BC}| \ge \big((1-d_0'e^{-\bar\eta j}) - 2e^{-\bar\eta j}\big)|\Omega_{j-1}^{BC}|\,, \qquad \forall j \ge J_0\,,$$

we find $\tilde\eta > 0$ such that for any $j \ge J_0$

$$(2.19)\quad |\Omega_{BC}| \ge \prod_{n=j}^{\infty}(1-(d_0'+2)e^{-\bar\eta n})|\Omega_{j-1}^{BC}| \ge (1-(\tilde d_0+2)e^{-\tilde\eta j})|\Omega_{j-1}^{BC}|\,.$$

Recall that we fixed $d_1 \in (0, \frac{C_0}{4}\log\lambda_{CE})$ (independently of κ_0). Let J_1 be such that $\prod_{j=J_1}^\infty (1-e^{-\bar\eta j} - j^{-2}) > 3/4$ and return to the sets Ω_j, Ω_j' constructed using the (polynomial) exclusion rule (2.11) for $\kappa_0 > 1/d_1$. We claim that for any $d_0 > 0$, if ϵ is small enough,

$$(2.20)\qquad \begin{cases} |\omega \cap \Omega_j'| \ge (1-d_0\cdot j^{-d_1\kappa_0})|\omega|\,, & \forall \omega \in \mathcal{P}_{j-1}\,,\ \forall j \ge J_1\,,\\ |\Omega_j| \ge |\Omega_j'| - e^{-j\bar\eta}|\omega_0|\,, & \forall j \ge J_1\,. \end{cases}$$

Before establishing this claim, we note that, mutatis mutandis, (2.20) combined with the arguments leading to (2.19) implies (2.9), while the more precise claim (2.10) follows from the refinement of (2.20) coming from the second statement of [DMS, Lemma V.6.9] (see the use of [Mo, Lemma 6.3] in [Mo, Lemma 6.4–Prop. 6.5]).

To show (2.20), we proceed in three steps, performing the necessary changes in the proof in [DMS, Section V.6]. Recall (2.12).

Firstly, up to taking larger N_0, the conclusion of [DMS, Lemma V.6.5] (which deals with (BA_j') for $\omega \in \mathcal{P}_{j-1}$ satisfying (BA_{j-1}') and (EX_{j-1}) and

[20]We mention a typo there: Although the constant $C = C(\epsilon)$ in the unnumbered equation on [DMS, p. 433] tends to zero as $\epsilon = |\omega_0|/2 \to 0$, the constant C_0 is (fortunately) uniformly bounded away from zero. See the proof of [DMS, Lemma V.6.5].

[21]Since $\bar\eta$ is independent of ϵ, r_0, N_0, we may take $N_0 \ge J_0$.

having a return at time j), if we replace the exponential rate (BA'_{j-1}) there by our polynomial rate (2.8), becomes

$$\frac{|\omega \setminus \bigcup_{r\geq\kappa_0\log j}\ \omega_{r,\ell}|}{|\omega|} \geq 1 - Cj^{-d_1\kappa_0}\,, \qquad \forall j \geq N_0\,. \tag{2.21}$$

To show this first claim, use that the constant $C_0 \in (0,1)$ (introduced above) is independent of κ_0 (because λ_{CE} does not depend on κ_0), and that [DMS, Lemma V.6.1] gives that the bound period p of a free return $\nu < j$ with

$$I_{r',\ell'} \subset x_\nu(\omega)\,, \quad \text{for} \quad r_0 \leq r' \leq \kappa_0 \log \nu \leq \kappa_0 \log j\,, \tag{2.22}$$

satisfies $p \geq C_0 r'$. Then, up to taking larger N_0, we can replace [DMS, V.(6.20)] in the proof of [DMS, Lemma V.6.5] by

$$|x_j(\omega)| \geq \lambda_{CE}^{p/4}\frac{e^{-r'}}{(r')^2} \geq \frac{e^{(-1+d_1)r'}}{(r')^2} \geq \frac{1}{j^{\kappa_0(1-d_1)}}\,, \qquad j \geq N_0\,, \tag{2.23}$$

where we used $d_1 \leq \frac{C_0}{4}\log\lambda_{CE}$ in the second inequality. We can thus replace the chain of inequalities after [DMS, V.(6.20)] (using the distortion bound (2.16) for $\tilde\omega \subset \omega$ the largest interval with $x_n(\tilde\omega) \subset U_{r_0/2}$, taking ϵ small enough and N_0 large enough such that (2.23) also holds for $\tilde\omega$) by

$$\frac{|\bigcup_{r\geq\kappa_0\log j}\ \omega_{r,\ell}|}{|\omega|} \leq \frac{|\bigcup_{r\geq\kappa_0\log j}\ \omega_{r,\ell}|}{|\tilde\omega|} \leq C\frac{1}{j^{\kappa_0}}\frac{1}{|x_j(\tilde\omega)|} \leq Cj^{-d_1\cdot\kappa_0}\,.$$

Secondly,[22] [DMS, Lemma V.6.6] (which deals with (FA_j)) uses (2.14) only via [DMS, Lemma V.6.3], while [DMS, Lemma V.6.3] still holds (with the same proof) if we replace (2.14) by our stronger assumption (2.8).

Thirdly, [DMS, Lemmas V.6.7–6.9] are unchanged, establishing (2.20). $\square$

Lemma 2.3 below is the analogue of [Sch, (III)']):

Lemma 2.3 (No Exceptionally Small Sets)**.** *For any $\kappa_1 > \kappa_0$ there exists $N_1 \geq N_0$ such that $|x_j(\omega)| > j^{-\kappa_1}$ for all $j \geq N_1$ and $\omega \in \mathcal{P}_j = \mathcal{P}_j(a_*, \kappa_0)$.*

Proof. We first show the lemma assuming that there exists $d_2 \in (0,1)$ such that for any $j \geq N_0$, and any $\omega \in \mathcal{P}_j$, we have

$$|x_j(\omega)| \geq \frac{d_2 e^{-r_0}(1-1/e)}{(\kappa_0\log j)^2 j^{\kappa_0}}\,, \tag{2.24}$$

with r_0 as in the proof of Proposition 2.2. Indeed (2.24) implies that

$$|x_j(\omega)| \geq \frac{d_2 e^{-r_0}(1-e^{-1})}{\kappa_0^2}\frac{1}{j^{\kappa_0}(\log j)^2}\,, \qquad \forall \omega \in \mathcal{P}_j\,, \forall j \geq N_0\,.$$

Clearly, there exists $N_1(\kappa_1) \geq N_0$ such that the right-hand side is larger than $j^{-\kappa_1}$ for all $j \geq N_1$.

To establish (2.24), we shall use (2.8). If $j+1$ is an essential free return time of ω, then taking r minimal such that $x_j(\omega)$ contains an interval $I^\pm_{r,\ell}$,

$$|x_j(\omega)| \geq |I^\pm_{r,\ell}| = e^{-r}\frac{1-1/e}{r^2} > \frac{j^{-\kappa_0}(1-1/e)}{(\kappa_0\log j)^2}\,. \tag{2.25}$$

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[22] We mention here a typo: [DMS, V.(6.24)] follows from [DMS, V.(6.22)] (and not [DMS, V.(6.20)] as stated there).

Otherwise, letting $j'+1 = \nu_{i'}(\omega) \geq \nu_1(\omega)$ be the largest essential free return time of ω such that $j'+1 < j+1$, we have $\omega \in \mathcal{P}_{j'}$ (since if $\tilde\omega \supset \omega$, $\tilde\omega \in \mathcal{P}_{j'}$, then $\tilde\omega$ is never cut between time j' and j), so that (2.25) implies

$$|x_{j'}(\omega)| > \frac{1-1/e}{(\kappa_0 \log j')^2 (j')^{\kappa_0}} > \frac{1-1/e}{(\kappa_0 \log j)^2 j^{\kappa_0}} .$$

We shall combine the above bound with [DMS, Lemma V.6.3, Props V.6.1–6.2] to handle the three cases left, namely: the time $j+1$ is an inessential free return of ω, the time $j+1$ is a return within a bound period of ω, and the intersection of $x_j(\omega)$ and U_{r_0} is empty.

If $j+1 = \nu_i(\omega)$ is an inessential free return then [DMS, V.(6.15) in Lemma V.6.3] gives, for $i' \leq i$ as defined above,

$$|x_j(\omega)| \geq 2^{i-i'} |x_{j'}(\omega)| > 2^{i-i'} \frac{1-1/e}{(\kappa_0 \log j)^2 j^{\kappa_0}} . \tag{2.26}$$

If $j+1$ is a return within the bound period of a previous free return $j''+1$ of ω, then using (2.25) for the bound period of an essential return, respectively (2.26) for the bound period of a nonessential return, and applying the first claim of [DMS, Lemma V.6.3], we find $d_2 \in (0,1)$ such that

$$|x_j(\omega)| \geq d_2 \lambda_{CE}^{j-j''} |x_{j''}(\omega)| > \frac{d_2(1-1/e)}{(\kappa_0 \log j)^2 j^{\kappa_0}} . \tag{2.27}$$

If $x_j(\omega) \cap U_{r_0} = \emptyset$ then [DMS, V.(6.2) in Prop. V.6.1 and Prop. V.6.2] and (2.25) give

$$|x_j(\omega)| \geq d_2 e^{-r_0} |x_{j'}(\omega)| > \frac{d_2 e^{-r_0}(1-1/e)}{(\kappa_0 \log j)^2 j^{\kappa_0}} . \tag{2.28}$$

We have shown (2.24) and thus Lemma 2.3. □

2.3. **A Hölder Local Distortion Estimate.** From now on, let $a_* \in (a_{\mathrm{mix}}, 4)$ be a Misiurewicz parameter, fix $\kappa_0 \geq 11/3d_1$, and let $\Omega_* = \Omega_*(a_*, \kappa_0) \subset \Omega_{BC} = \Omega_{BC}(a_*)$ be the positive measure Cantor set constructed in Section 2.2 via families $\mathcal{P}_j = \mathcal{P}_j(a_*, \kappa_0)$. The following[23] replaces [Sch, (33), (31)]. The bound (2.31) is new.

Lemma 2.4 (Hölder Distortion Bounds)**.** *There exists $C < \infty$ such that for all $n \geq N_0$ (with N_0 as in Proposition 2.2) and any $\omega \in \mathcal{P}_n = \mathcal{P}_n(a_*, \kappa_0)$*

$$\frac{1}{C} \leq \left| \frac{x_n'(a)/x_j'(a)}{(T_a^{n-j})'(x_j(a))} \right| \leq C , \qquad \forall 1 \leq j \leq n , \quad \forall a \in \omega . \tag{2.29}$$

In addition, there exist $C < \infty$ and $M_0 > \kappa_0$ such that, for all $n \geq N_0$, each $\tilde\omega \in \mathcal{P}_n = \mathcal{P}_n(a_*, \kappa_0)$, and every $\omega \subset \tilde\omega$ and $\alpha \in [0,1)$ satisfying

$$|x_n(\omega)| \leq n^{-M_0/(1-\alpha)} , \tag{2.30}$$

we have

$$\left| \frac{x_n'(a_1)}{x_n'(a_2)} \right| \leq 1 + C |x_n([a_1, a_2])|^{\alpha} , \qquad \forall a_1, a_2 \in \omega . \tag{2.31}$$

If $\alpha = 0$, and $n+1$ is a free return of $\omega \in \mathcal{P}_n$, the bound (2.31) is just (2.16). We shall require (2.31) for some $\alpha > 0$ in Corollary 3.4.

[23]For [Sch, (30)], see (2.2). We do not need [Sch, (32)].

Proof. The bound (2.29) is an immediate consequence of (2.6).

We first claim that[24] there exist C' and $\kappa_2 > 0$ such that for any n

$$\sum_{i=0}^{j-1} |x_i(\omega)| \le C' j^{\kappa_0+1+\kappa_2} |x_j(\omega)| \,, \quad \forall 1 \le j \le n \,, \forall \omega \subset \tilde\omega \in \mathcal{P}_n \,. \tag{2.32}$$

To start, there is C such that for any $0 \le i \le j \le n$, using (2.3) and (2.29) there exists $a = a(i,j,\omega) \in \omega$ such that, setting $X_{i,j} = x_j \circ x_i^{-1}$,

$$\frac{|x_i(\omega)|}{|x_j(\omega)|} = \frac{|x_i(\omega)|}{|X_{i,j}(x_i(\omega))|} = \frac{|x_i'(a)|}{|x_j'(a)|} \le \frac{C}{|(T_a^{j-i})'(T_a^{i+1}(c))|} \,. \tag{2.33}$$

(We used $X'_{i,j} = x'_j / x'_i$ and the mean value theorem in the second equality.) Next, let $s_j(a)$ be the largest ℓ with $\nu_\ell(a) \le j$, and put[25]

$$q_\ell(a) = \nu_{\ell+1}(a) - (\nu_\ell(a) + p_\ell(a) + 1) \,, \qquad \ell = 0, \dots, s_j(a) - 1 \,,$$

$$q_{s_j(a)}(a) = \max\{0, j - (\nu_{s_j(a)}(a) + p_{s_j(a)}(a) + 1)\} \,, \quad F_j(a) = \sum_{\ell=0}^{s_j(a)} q_\ell(a).$$

Set $p_\ell = p_\ell(a)$, $\nu_\ell = \nu_\ell(a)$, $q_\ell = q_\ell(a)$, and $s_j = s_j(a)$. Assume first that $i = 0$. Then, we have (see e.g. [DMS, V.(6.11)])

$$\begin{aligned} |(T_a^{j-i})'(T_a^{i+1}(c))| &= |(T_a^j)'(T_a(c))| \\ &= |(T_a^{\nu_1 - 1})'(T_a(c))| \cdot |(T_a^{j+1-\nu_{s_j}})'(T_a^{\nu_s}(c))| \\ &\quad \cdot \left(\prod_{\ell=1}^{s_j-1} |(T_a^{p_\ell+1})'(T_a^{\nu_\ell}(c))| |(T_a^{q_\ell})'(T_a^{\nu_\ell+p_\ell+1}(c))| \right) . \end{aligned}$$

Since a satisfies $(BA)_m$ and $(FA)_m$ for all $m \le n$, the bounds (2.15) give λ_{CE}, $\gamma_0 > 0$, and $C_0 > 0$ such that

$$\prod_{\ell=1}^{s_j-1} |(T_a^{p_\ell+1})'(T_a^{\nu_\ell}(c))| |(T_a^{q_\ell})'(T_a^{\nu_\ell+p_\ell+1}(c))| \ge \prod_{\ell=1}^{s_j-1} C_0 e^{\gamma_0 q_\ell} \lambda_{CE}^{p_\ell/4} \,.$$

Similarly, $|(T_a^{\nu_1-1})'(T_a(c))| > C_0 e^{\gamma_0 \nu_1}$. Next, if $j \le \nu_{s_j} + p_{s_j} + 1$, we have

$$|(T_a^{j+1-\nu_{s_j}})'(T_a^{\nu_{s_j}}(c))| \ge C_0^2 \lambda_{BC}^{j-\nu_{s_j}} j^{-\kappa_0} e^{-r_0} \,,$$

where we used (2.8) and [DMS, Lemma V.6.1.b, Prop. V.6.1]. If $j > \nu_{s_j} + p_{s_j} + 1$, we have, using [DMS, Lemma V.6.1.c, Prop. V.6.1]

$$|(T_a^{j+1-\nu_{s_j}})'(T_a^{\nu_s}(c))| \ge C_0^2 \lambda_{BC}^{p_{s_j}/4} e^{\gamma_0 (j - (\nu_{s_j} - p_{s_j} - 1))} e^{-r_0} \,,$$

Summarising,

$$|(T_a^j)'(T_a(c))| \ge \frac{C_0^{s_j+4}}{C} \lambda_{CE}^{(j-F_j(a))/4} e^{\gamma_0 F_j(a)} j^{-\kappa_0} \,.$$

[24] Our proof is inspired from that of [DMS, Theorem V.6.2]. This is suboptimal but enough for our purposes. Adapting instead [DMS, Lemma V.6.4] could enhance (2.31).

[25] The condition $(FA)_n$ implicitly used in Proposition 2.2 says that, for some fixed arbitrarily small $\tau > 0$, $F_\ell(a) \ge \ell(1-\tau)$ for $N_0 \le \ell \le n$. We shall not need this here.

Since $p_\ell \ge C_0 r_0$ (see after (2.22)), we have $j - F_j \ge j C_0 r_0$ while $s_j \le j/(C_0 r_0)$. We took r_0 large enough (see after (2.15)) such that

$$C_0^{s_j+4}\lambda_{CE}^{(j-F_j(a))/4} \ge 1\,. \tag{2.34}$$

Finally, using the trivial bound $e^{\gamma_0 F_j(a)} \ge 1$, we find

$$|(T_a^j)'(T_a(c))| \ge \frac{j^{-\kappa_0}}{C}\,.$$

If $i \ge 1$ and $\nu_{\ell_i}(a) + p_{\ell_i}(a) < i < \nu_{\ell_i+1}(a)$ for some $\ell_i \ge 1$, then we proceed as for $i = 0$, replacing $|(T_a^{\nu_1-1})'(T_a(c))|$ by $|(T_a^{\nu_{\ell_i+1}-i})'(T_a^{i+1}(c))|$, and setting $F_{i,j}(a) = \nu_{\ell_i+1}(a) - i + \sum_{\ell \ge \ell_i+1}^{s_j(a)} q_\ell(a)$. Then

$$|(T_a^{j-i})'(T_a^{i+1}(c))| \ge \frac{C_0^{s_j-s_i+4}}{C}\lambda_{CE}^{((j-i)-F_{i,j}(a))/4} e^{\gamma_0 F_{i,j}(a)} j^{-\kappa_0}\,.$$

We have $j - i - F_{i,j} \ge (j-i)C_0 r_0$ while $s_j - s_i \le (j-i)/(C_0 r_0)$, and we find, using $e^{\gamma_0 F_{i,j}(a)} \ge 1$ (we do not know or need $F_{i,j}(a) \ge (1-\tau)(j-i)$),

$$|(T_a^{j-i})'(T_a^{i+1}(c))| \ge \frac{j^{-\kappa_0}}{C}\,.$$

Otherwise, $\nu_{\ell_i}(a) \le i - 1 \le \nu_{\ell_i}(a) + p_{\ell_i}(a)$ for some $\ell_i \ge 1$. There may be (nonfree) returns during the ℓ_ith bound period. To bypass this difficulty, we exploit that the length of the ℓth bound period is of the order r if $x_{\nu_\ell}(a) \in I_r$ ([DMS, Lemma V.6.1a]). By (2.11), we have $r_{\ell_i} = O(\log(\nu_{\ell_i})) \le C\kappa_0 \log i$. Thus, the missing factor in the ℓth bound period is $\le \Lambda^{C\kappa_0 \log i} \le i^{\kappa_2}$, and

$$|(T_a^{j-i})'(T_a^{i+1}(c))| \ge \frac{j^{-\kappa_0}}{C i^{\kappa_2}}\,.$$

Summing over i, and recalling (2.33), this establishes (2.32).

Next, taking $a_1, a_2 \in \omega$, note that (2.7) (using the first bound of (2.4) if $i < N_0$) implies that for all $N_0 \le i \le n$, recalling $T_a'(x) = a(1-2x)$,

$$\begin{aligned}
&|T'_{a_1}(x_i(a_1)) - T'_{a_2}(x_i(a_2))| \\
&\qquad \le |T'_{a_1}(x_i(a_1)) - T'_{a_2}(x_i(a_1))| + |T'_{a_2}(x_i(a_1)) - T'_{a_2}(x_i(a_2))| \\
&\qquad \le |a_1 - a_2| + 2a_2|x_i(a_1) - x_i(a_2)| \le (C + 2a_2)|x_i(\omega)|\,.
\end{aligned} \tag{2.35}$$

(Note that (2.35) replaces [Sch, (36)].) We claim that there exists C'' with

$$\left|\frac{(T_{a_1}^j)'(x_0(a_1))}{(T_{a_2}^j)'(x_0(a_2))}\right| \le 1 + C'' e^{C''} j^{2\kappa_0+1+\kappa_2}|x_j(\omega)|\,, \forall 1 \le j \le n\,. \tag{2.36}$$

(The above replaces [Sch, (37)].) Indeed, using the classical bound

$$\prod_{i=0}^{j-1}(1+\upsilon_i) \le \exp\left(\sum_{i=0}^{j-1}\upsilon_i\right) \le 1 + e^{\sum_{i=0}^{j-1}\upsilon_i}\sum_{i=0}^{j-1}\upsilon_i\,, \quad \text{if all } \upsilon_i \ge 0\,,$$

we have, setting $C'' = 2C'C(C+2a_2)$,

$$\begin{aligned}\left|\frac{(T^j_{a_1})'(x_0(a_1))}{(T^j_{a_2})'(x_0(a_2))}\right| &= \left|\prod_{i=0}^{j-1}\frac{T'_{a_1}(x_i(a_1))}{T'_{a_2}(x_i(a_2))}\right| \\ &\le 1+e^{\sum_i\left|-1+\frac{T'_{a_1}(x_i(a_1))}{T'_{a_2}(x_i(a_2))}\right|}\cdot\sum_{i=0}^{j-1}\left|\frac{T'_{a_1}(x_i(a_1))}{T'_{a_2}(x_i(a_2))}-1\right| \\ &\le 1+e^{(C+2a_2)\sum_{i=0}^{j-1}C|x_i(\omega)|i^{\kappa_0}}\cdot(C+2a_2)\sum_{i=0}^{j-1}C|x_i(\omega)|i^{\kappa_0} \\ &\le 1+e^{C''j^{2\kappa_0+1+\kappa_2}|x_j(\omega)|}\cdot C''j^{2\kappa_0+1+\kappa_2}|x_j(\omega)|\,,\ \forall j\le n\,,\end{aligned}\tag{2.37}$$

where we used (2.35) and (2.8) (the first bound of (2.4) if $i < N_0$) in the second inequality, and (2.32) in the last inequality. Setting $M_0 := 4\kappa_0+3+2\kappa_2$, if (2.30) holds for ω, then (2.32) gives for all $N_0\le j\le n$

$$\begin{aligned}C''j^{2\kappa_0+1+\kappa_2}|x_j(\omega)| &\le C''j^{2\kappa_0+1+\kappa_2}C'j^{\kappa_0+1+\kappa_2}|x_n(\omega)| \\ &\le C'\frac{j^{3\kappa_0+3+2\kappa_2}}{n^{M_0/(1-\alpha)}}\le C'''\,.\end{aligned}$$

This proves (2.36). Similarly, $\left|\frac{(T^j_{a_2})'(x_0(a_2))}{(T^j_{a_1})'(x_0(a_1))}\right| \le 1+C''e^{C''}j^{2\kappa_0+1+\kappa_2}|x_j(\omega)|$. Therefore,

$$\left|\frac{1}{(T^j_{a_1})'(x_0(a_1))}-\frac{1}{(T^j_{a_2})'(x_0(a_2))}\right|\le C'''\frac{j^{2\kappa_0+1+\kappa_2}|x_j(\omega)|}{|(T^j_{a_1})'(x_0(a_1))|}\,.$$

We can then adapt the end of the proof of [Sch, (31)]: Comparing each term on the right-hand side of

$$\frac{x_n'(a)}{(T^n_a)'(x_0(a))}=x_0'(a)+\sum_{j=1}^n\frac{(\partial_aT_a)(x_{j-1}(a))}{(T^j_a)'(x_0(a))}\,,\ \forall a\in\tilde\omega\in\mathcal{P}_n\,,$$

for $a=a_1$ and $a=a_2$, we find, since $x_0'(a)=\partial_a c_1(a)=1/4$, and

$$|\partial_aT_a|_{a_1}(x_{j-1}(a_1))-\partial_aT_a|_{a_2}(x_{j-1}(a_2))|\le|x_{j-1}(a_1)-x_{j-1}(a_2)|\le|x_{j-1}(\omega)|\,,$$

recalling (2.5), and applying (2.32) and then (2.30) for $M_0=4\kappa_0+3+2\kappa_2$,

$$\begin{aligned}\left|\frac{x_n'(a_1)}{(T^n_{a_1})'(x_0(a_1))}\right| &\le \left|\frac{x_n'(a_2)}{(T^n_{a_2})'(x_0(a_2))}\right|+\hat C|a_1-a_2|+\hat C\sum_{j=1}^n\frac{j^{2\kappa_0+1+\kappa_2}}{\lambda^j_{CE}}|x_j(\omega)| \\ &\le \left|\frac{x_n'(a_2)}{(T^n_{a_2})'(x_0(a_2))}\right|+\bar CC'n^{4\kappa_0+3+2\kappa_2}|x_n(\omega)| \\ &\le \left|\frac{x_n'(a_2)}{(T^n_{a_2})'(x_0(a_2))}\right|+\tilde C|x_n(\omega)|^\alpha\,.\end{aligned}$$

Finally, we have, using (2.6) (which plays the role of [Sch, Lemma 2.4]),

$$\begin{aligned}\left|\frac{x_n'(a_1)}{x_n'(a_2)}\right| &\le \left|\frac{(T^n_{a_1})'(x_0(a_1))}{(T^n_{a_2})'(x_0(a_2))}\right|\left(1+\tilde C|x_n(\omega)|^\alpha\frac{|(T^n_{a_2})'(x_0(a_2))|}{|x_n'(a_2)|}\right) \\ &\le 1+C\tilde C|x_n(\omega)|^\alpha\,.\end{aligned}$$

□

2.4. **Uniform Decorrelation and Hölder Response.** The maps x_j are not the iterates of a fixed dynamical system admitting an invariant measure. To exploit statistical information on the iterates of the mixing CE map (T_{a_0}, μ_{a_0}), we will "switch locally" from x_j to $T^j_{a_0}$ (see Lemma 3.3), using that any $a \in \Omega_*$ satisfies[26] the following uniform decorrelation result for Hölder continuous observables. For $q > 1$ and $s \in [0, 1/q)$, we denote by $H^s_q(I) = F^s_{q,2}(I)$ the Sobolev space of functions of differentiability s and integrability q supported in I (see [RS]).

Proposition 2.5 (Uniform Decay of Correlations). *For any $s > 0$ and $q > 1$, there exist $C < \infty$ and $\rho^s_q < 1$ such that, for all $\varphi \in H^s_q(I)$, $\psi \in L^\infty(dm)$, $a \in \Omega_*(a_*, \kappa_0)$*

$$\left| \int_0^1 \varphi(\psi \circ T^n_a)\, dm - \int_0^1 \varphi\, dm \int_0^1 \psi\, d\mu_a \right| \le C \|\varphi\|_{H^s_q} \|\psi\|_{L^1(d\mu_a)} (\rho^s_q)^n\,, \forall n \ge 1\,.$$

For any $\varpi > 0$, there exist $C < \infty$ and $\rho_\varpi < 1$ such that, for all $\varphi \in C^\varpi$, $\psi \in L^\infty(dm)$, $a \in \Omega_*(a_*, \kappa_0)$

$$\left| \int_0^1 \varphi(\psi \circ T^n_a)\, d\mu_a - \int_0^1 \varphi\, d\mu_a \int_0^1 \psi\, d\mu_a \right| \le C \|\varphi\|_\varpi \|\psi\|_{L^1(d\mu_a)} (\rho_\varpi)^n\,, \forall n \ge 1\,.$$

We also use Hölder bounds on $a \mapsto \mu_a$ as a distribution (in Lemma 2.8):

Proposition 2.6 (Fractional Response). *For any $\Theta \in (0, 1/2)$, there exists C such that for all $\varphi \in C^{1/2}$*

$$\left| \int \varphi\, d\mu_a - \int \varphi\, d\mu_{a'} \right| \le C |a - a'|^\Theta \|\varphi\|_{1/2}\,, \quad \forall a, a' \in \Omega_*(a_*, \kappa_0)\,. \tag{2.38}$$

Our proof of Proposition 2.5 uses the following facts.

Sublemma 2.7. *For any $a \in \Omega_{BC}$, the density h_a of μ_a lies in $H^s_q(I)$ for all $s \in [0, 1/2)$ and $q \in (1, 2/(1+2s))$. In addition, for any (H_0, κ_0) polynomially recurrent a_*, there exists $C_{s,q,a_*} < \infty$ such that*

$$\sup_{a \in \Omega_* a_*, \kappa_0} \|h_a\|_{H^s_q(I)} \le C_{s,q,a_*}\,.$$

Proof. In the Misiurewicz case, the first claim is [Se, Theorem 10] , using Ruelle's [Ru, Theorem 9, Remark 16.a] decomposition of h_a into the sum of a C^1 function and an exponentially decaying sum of "spikes" $x \mapsto |x - c_k(a)|^{-1/2}$ and square root singularities $x \mapsto |x - c_k(a)|^{1/2}$. For a general $a \in \Omega_{BC}$, set $T^{-k}_{a,\varsigma} := (T^k_a|_{U_{k,a,\varsigma}})^{-1}$, for $k \ge 1$ and $\varsigma \in \pm$, where $U_{k,a,\varsigma}$ is the monotonicity interval of T^k_a containing c, located to the right of c for $\varsigma = +$, to the left of c for $\varsigma = -$. Then, since we assumed $\lambda_{CE} > e^{14\alpha_{BC}}$ in the proof of Proposition 2.2, use [BS1, Prop 2.7] that there exist a C^1 function $\psi_a \colon I \to \mathbb{R}_+$ and C^∞ functions $\Xi^k_{a,\pm} \colon [0,1] \to [0,1]$ supported in a neighbourhood of $c_k(a)$ in $T^k_a(U_{k,a,\pm})$, such that

$$h_a(x) = \psi_a(x) + \sum_{k=1}^\infty \sum_{\varsigma \in \{+,-\}} \chi_{k,a}(x) \frac{\Xi^k_{a,\varsigma}(T^{-k}_{a,\varsigma}(x)) \psi_a(T^{-k}_{a,\varsigma}(x))}{|(T^k_a)'(T^{-k}_{a,\varsigma}(x))|}\,, \tag{2.39}$$

[26] The factor $\|\varphi\|_{L_1(dm)}$ in the right-hand side of [Sch, Prop. 4.3] is replaced in Proposition 2.5 by $\|\varphi\|_{L^1(d\mu_a)} \le \|\varphi\|_{L^\infty(dm)}$. This does not impact [Sch, p. 36, use of Prop. 4.3].

where $\chi_{k,a}(x) = 1_{\pm x < \pm c_k(a)}$ if $\pm T_a^k$ has a local maximum at c. Setting $\Psi := \Xi_{a,\varsigma}^k \cdot \psi_a$, we find C^1 functions $\Psi_{k,\ell}$, for $\ell = 1, 2, 3$, with

$$\frac{\Psi(T_{a,\varsigma}^{-k}(x))}{|(T_a^k)'(T_{a,\varsigma}^{-k}(x))|} = \frac{\Psi_{k,1}(x)}{|x - c_k(a)|^{1/2}} + \Psi_{k,2}(x)|x - c_k(a)|^{1/2} + \Psi_{k,3}(x)\,, \tag{2.40}$$

for any $x \in \mathrm{supp}(\chi_{k,a})$. Finally, use [Se, Lemmas 11–12].

For the second claim, it is convenient to use an alternative decomposition of h_a. First recall that [BBS, Cor 1.6] gives a set Ω_{slow} of full measure in the set of mixing CE parameters such that, for any $\tilde a \in \Omega_{\text{slow}}$ and each $\kappa_0 > 1$, there exist $H_0 \ge 1$ and a set $\Delta_0(\tilde a, \kappa_0) \subset \Omega_{\text{slow}}$ of (H_0, κ_0)-polynomially recurrent (and thus transversal) parameters, with $\tilde a$ as a Lebesgue density point, such that Proposition 2.5 holds for all $a \in \Delta_0$. (It is unknown whether $\tilde a \in \Delta_0$.) The proof involves constructing a tower for each parameter in Δ_0. We claim that, up to reducing the value of ϵ in the proof of Proposition 2.2, we can replace $\tilde a$ by a_* and Δ_0 by $\Omega_*(a_*, \kappa_0)$. Indeed, Δ_0 was constructed in [BBS, Prop. 2.1], and it suffices to observe that the required uniformity in constants is satisfied by (2.5) and (2.8), while [BBS, (8) and (7)] are exactly [DMS, V.(6.1), V.(6.2) in Prop. V.6.1].

Let then

$$\Pi_a(\hat\psi)(x) = \sum_{j \ge 0, \varsigma \in \pm} \frac{\lambda^j}{|(T_a^j)'(T_{a,\varsigma}^{-j}(x))|} \psi_j(T_{a,\varsigma}^{-j}(x))\,,$$

(for a suitable $\lambda > 1$) be the projection from the tower with polynomial recurrence used in [BBS], and let $\hat{\mathcal{L}}_a$ be the lift $\mathcal{L}_a \Pi_a = \Pi_a \hat{\mathcal{L}}_a$ of the transfer operator $\mathcal{L}_a \varphi(x) = \sum_{T_a(y)=x} \varphi(y)/|T_a'(y)|$. Then[27] there exist $C < \infty$ and $\theta < 1$ such that, letting $\|\cdot\|_a'$ be the norm of the Sobolev space $\mathcal{B}_a^{W_1^1}$ of [BBS],

$$\|\hat{\mathcal{L}}_a^n(\hat\varphi) - \hat h_a \hat\nu(\hat\varphi)\|_a' \le C \|\hat\varphi\|_a' \theta^n\,, \qquad \forall \hat\varphi \in \mathcal{B}_a^{W_1^1}\,, \quad \forall a \in \Omega_{\tilde a}\,, \tag{2.41}$$

where $\hat h_a$ is the fixed point[28] of $\hat{\mathcal{L}}_a$ on $\mathcal{B}_a^{W_1^1}$ normalised by $\int \Pi_a \hat h_a \, dm = 1$, while $\hat\nu$, the nonnegative measure whose density with respect to Lebesgue in the level j of the tower is λ^j, is the fixed point of the dual of $\hat{\mathcal{L}}_a$ (see [BS1, (85)], note that $\nu(\hat h_a) = 1$ is automatic). Since $\Pi_a \hat h_a = h_a$ and the W_1^1 norm dominates any H_q^s norm on I if $s \in [0,1)$ and $1 < q < 1/s$ (by the Sobolev embedding, more precisely [RS, Chapter 2] the bounded inclusions $W_1^1 \subset W_1^\sigma = F_{1,1}^\sigma \subset F_{1,2}^\sigma \subset F_{q,2}^s = H_q^s$, if $\sigma = 1 + s - 1/q \in (0,1)$ and $q \in (1, \infty)$), the decomposition (2.40) combined with the uniform bound (2.41) (for $\hat\varphi$ vanishing on all levels ≥ 1 and constant on level zero of the tower, with $\hat\nu(\varphi) = 1$) gives the second claim of the sublemma, using again [Se, Lemmas 11–12]. $\square$

Proof of Proposition 2.5. Recall from the proof of Proposition 2.2 that we have $\lambda_{CE} > e^{14\alpha_{BC}}$. By mollification, it is enough to prove both bounds for C^1 functions φ. It is in fact enough to show the first bound for $\varphi \in C^1$: Indeed, again by mollification (see e.g. the proof of [Se, Lemma 14]), if the

[27] [BBS, (66)] gives uniform Lasota–Yorke estimates. [BBS, Lemma 3.8, Lemma 4.5, Lemma 4.6, Prop. 4.1] give the weak norm bounds needed for Keller–Liverani [KL].

[28] The fixed point property determines $\hat h_a$ by its value on the level zero of the tower.

first bound holds for $\varphi \in C^1$, then it holds for any $\varphi \in H^s_q(I)$ with $q > 1$ and $s > 0$. Therefore, since the density h_a of μ_a lies in $H^s_q(I)$ for all $s \in (0, 1/2)$ and $q \in (1, 2/(1 + 2s))$ by Sublemma 2.7 (with norm uniformly bounded in a), the second bound follows from the first bound for $\varphi \in C^1$ (using that C^1 functions are bounded multipliers on H^s_q).

Next, we observed in the proof of Sublemma 2.7 that we can replace the set called Δ_0 in [BBS, Cor. 1.6] by $\Omega_*(a_*, \kappa_0)$. The first bound for Lipschitz continuous φ thus follows from the second assertion of [BBS, Cor. 1.6], since $\Omega_* \subset [a_{\mathrm{mix}}, 4)$. Indeed, note first that a is topologically mixing if and only if its renormalisation period P_a is equal to one. Second, observe that the constant $C_{\varphi,\psi}$ in the second claim of [BBS, Cor. 1.6] can be replaced by $C\|\varphi\|_\varpi \|\psi\|_{L^1(d\mu_a)}$, for a constant C uniform in a in view of [BBS, Lemma 4.5, Lemma 4.6] and the principle of uniform boundedness. More precisely, using the notation from the proof of Sublemma 2.7, we have

$$\int (\psi \circ T_a^n)\varphi \, dm = \int \psi \Pi_a(\hat{\mathcal{L}}_a^n(\hat\varphi)) \, dm \qquad \text{if } \Pi_a(\hat\varphi) = \varphi \,,$$
$$\int (\psi \circ T_a^n)\varphi h_a \, dm = \int \psi \Pi_a(\hat{\mathcal{L}}_a^n(\hat\varphi_a)) \, dm \qquad \text{if } \Pi_a(\hat\varphi_a) = \varphi h_a \,.$$

Since[29] $\Pi_a \hat h_a = h_a$, any Lipschitz continuous φ can be written as $\Pi_a(\hat\varphi)$ (take $\hat\varphi_0 = \varphi$ on the level zero, and $\hat\varphi_j \equiv 0$ on levels $j \geq 1$) such that, on the one hand, $\|\hat\varphi\|'_a \leq C\|\varphi\|_1$ uniformly in a, and, on the other hand, $\hat\nu(\hat\varphi) = \int \varphi \, dm$, we conclude by applying (2.41) from the proof of Sublemma 2.7. □

Proof of Proposition 2.6. If $a = a_*$, the bound is an immediate consequence of the first claim of [BBS, Cor. 1.6], since we can replace the set denoted Δ_0 there by $\Omega_*(a_*, \kappa_0)$, as observed in the proof of Sublemma 2.7 and used in the proof of Proposition 2.5. If $a \neq a_*$, the uniformity of the constants given by Proposition 2.2 ensures that we may construct the reference tower in [BBS] at a (instead of a_*), viewing a' as a perturbation of a. □

2.5. Hölder Regularity of the Variance $\sigma_a(\varphi)$.

Propositions 2.5 and 2.6 will imply the following regularity of $a \mapsto \sigma_a(\varphi)$ on Ω_*.

Lemma 2.8 (Regularity of $\sigma_a(\varphi)$). *For any $\varpi \in (0, 1]$, there exist $\theta \in (0, \min\{1/2, \varpi\})$ and $C < \infty$ such that, for each $\varphi \in C^\varpi$ with $\sigma_{a_*}(\varphi) > 0$ there exists $\epsilon_\varphi > 0$ such that*

$$C_{\epsilon_\varphi}(\varphi) := \inf_{a \in \Omega_{a_*} \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]} \sigma_a(\varphi) > 0 \,,$$

and such that for all $a, a' \in \Omega_*(a_*, \kappa_0) \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$ we have

$$|\sigma_a(\varphi) - \sigma_{a'}(\varphi)| \leq \frac{C}{2C_\epsilon(\varphi)} \|\varphi\|_\varpi |a - a'|^\theta \,. \tag{2.42}$$

[29] The Banach space of [BBS] requires that the function on level zero of the tower be supported in $(0, 1)$, so this proof cannot cover the case $a = 4$.

Proof. Let $k_0 > 1$ be a large integer to be chosen at the end of the proof. By the second claim of Proposition 2.5, there exist $\rho = \rho_\varpi < 1$ and C_0 such that

$$\sum_{k>k_0} \left| \int \left(\varphi - \int \varphi \, d\mu_a \right) \cdot \left(\left(\varphi - \int \varphi \, d\mu_a \right) \circ T_a^k \right) d\mu_a \right| \\ \le C_0 \|\varphi\|_\varpi^2 \cdot \frac{\rho^{k_0}}{1-\rho} , \qquad \forall k_0 \ge 1 , \quad \forall a \in \Omega_{a_*} , \quad \forall \varphi \in C^\varpi .$$

Set $A_a = \int \varphi \, d\mu_a$. Since $\int \big((\varphi - A_a) \circ T_a^k\big)(\varphi - A_a) \, d\mu_a = \int (\varphi \circ T_a^k)\varphi \, d\mu_a - A_a^2$, we have

$$\begin{aligned} |\sigma_a(\varphi)^2 - \sigma_{a'}(\varphi)^2| \le{}& 2 \sum_{k=0}^{k_0-1} \left| \int \varphi(\varphi \circ T_a^k) \, d\mu_a - \int \varphi(\varphi \circ T_{a'}^k) \, d\mu_a \right| \\ &+ 2 \sum_{k=0}^{k_0-1} \left| \int \varphi(\varphi \circ T_{a'}^k) \, d\mu_a - \int \varphi(\varphi \circ T_{a'}^k) \, d\mu_{a'} \right| \\ &+ 2 \sum_{k=0}^{k_0-1} \left| \left(\int \varphi \, d\mu_a \right)^2 - \left(\int \varphi \, d\mu_{a'} \right)^2 \right| \\ &+ 4 C_0 \|\varphi\|_\varpi^2 \frac{\rho^{k_0}}{1-\rho} , \qquad \forall k_0 \ge 1 . \end{aligned}$$

Assume for a moment that $\varpi \ge 1/2$. The ϖ-Hölder constant of $\varphi(\varphi \circ T_{\bar a}^k)$ (for $\bar a = a$ or a') is bounded by $\Lambda^k \|\varphi\|_\varpi^2$. Thus, Proposition 2.6 gives for any $\Theta < 1/2$ a constant $C_1 = C_1(\Theta)$ such that for $a, a' \in \Omega_{a_*}$, and $\varphi \in C^\varpi$

$$\begin{aligned} |\sigma_a(\varphi)^2 - \sigma_{a'}(\varphi)^2| \le{}& k_0 C_1 \|\varphi\|_\varpi^2 \Lambda^{k_0} |a - a'|^\Theta + C_0 \|\varphi\|_\varpi^2 \frac{\rho^{k_0}}{1-\rho} \\ &+ 2 \sum_{k=0}^{k_0-1} \left| \int \varphi(\varphi \circ T_a^k) \, d\mu_a - \int \varphi(\varphi \circ T_{a'}^k) \, d\mu_a \right| , \qquad \forall k_0 \ge 1. \end{aligned}$$

Next, (2.2) gives that

$$\int |\varphi \circ T_a^k - \varphi \circ T_{a'}^k| \, d\mu_a \le \|\varphi\|_\varpi (C \Lambda^k |a - a'|)^\varpi .$$

Therefore, we find

$$\begin{aligned} \big|\sigma_a(\varphi)^2 - \sigma_{a'}(\varphi)^2\big| \le{}& k_0 C_1 \|\varphi\|_\varpi^2 \Lambda^{k_0} |a - a'|^\Theta + 4 C_0 \|\varphi\|_\varpi^2 \frac{\rho^{k_0}}{1-\rho} \\ &+ k_0 \|\varphi\|_\varpi (C \Lambda^{k_0} |a - a'|)^\varpi . \end{aligned} \tag{2.43}$$

We conclude the proof for $\varpi \ge 1/2$ by dividing (2.43) by $|a - a'|^\theta$, for small enough $\theta > 0$ and optimising in k_0, using also $(\sigma_a - \sigma_{a'})(\sigma_a + \sigma_{a'}) = \sigma_a^2 - \sigma_{a'}^2$.

If $\varpi \in (0, 1/2)$, mollification gives $\varphi_\upsilon \in C^{1/2}$ and C_4 such that

$$\|\varphi_\upsilon\|_{1/2} \le C_4 \upsilon^{\varpi - 1/2} \|\varphi\|_\varpi , \ \sup |(\varphi \circ T_{\bar a}^k)\varphi - (\varphi_\upsilon \circ T_{\bar a}^k)\varphi_\upsilon| \le C_4 \upsilon^\varpi \Lambda^k \|\varphi\|_\varpi ,$$

for all small $\upsilon > 0$, all $0 \le k \le k_0$, and all $\bar a \in \Omega_{a_*}$. To conclude, optimise in $\upsilon = |a - a'|^{\theta_0}$ for small $\theta_0 > 0$, taking θ smaller (in particular $\theta < \varpi \theta_0$). $\square$

3. Switching Locally from the Parameter to the Phase Space

Let a_*, $\mathcal{P}_j(a_*, \kappa_0)$, and $\Omega_* = \Omega_*(a_*, \kappa_0)$ be as in Proposition 2.2 for $\kappa_0 \geq 11/(3d_1)$, and fix $\varpi \in (0, 1)$. This section is devoted to Proposition 3.2, the main estimate (analogous to [Sch, Prop. 5.1]) towards a law of large numbers for the squares of the blocks which will be defined in Section 4 (see Lemma 4.2).

From now on, fix $\varpi \in (0, 1)$ and a ϖ-Hölder continuous function $\varphi \colon I \to \mathbb{R}$, recalling φ_a, $\sigma_a(\varphi)$ from (1.6), (1.3), and assume $\sigma_{a_*}(\varphi) > 0$. Lemma 2.8 gives $\epsilon_\varphi > 0$ such that

$$\sigma_a(\varphi) > 0\,, \quad \forall a \in \Omega_*^\varphi := \Omega_* \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]\,. \tag{3.1}$$

If $\epsilon_\varphi < \epsilon$, we replace Ω_* by Ω_*^φ by replacing ϵ in the proof of Proposition 2.2 with ϵ_φ. (This is harmless as it can only improve the constants.)

Remark 3.1 (θ-Hölder Whitney Extensions of φ_a and $\xi_n(a)$)**.** *By Proposition 2.6, the function $a \mapsto \int \varphi\, d\mu_a$ is Θ-Hölder continuous on Ω_* for any $\Theta < 1/2$. By Lemma 2.8, the function $a \mapsto \sigma_a(\varphi) \geq 0$ is θ-Hölder continuous on Ω_* for some $\theta < \min\{1/2, \varpi\}$, and uniformly bounded away from zero on Ω_*^φ. Taking $\Theta \geq \theta$, the map $a \mapsto \varphi_a(u) = (\varphi(u) - \int \varphi d\mu_a)/\sigma_a$ is θ-Hölder continuous on Ω_*^φ uniformly in $u \in I$. By the Whitney extension theorem, we extend each map $a \mapsto \varphi_a(u)$ to a θ-Hölder continuous map on $[a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$, uniformly in $u \in I$. In addition, there exists $\widetilde{C} < \infty$ such that*

$$\|\varphi_a\|_\infty \leq \|\varphi_a\|_\varpi \leq \widetilde{C}\|\varphi\|_\varpi\,, \qquad \forall a \in [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]\,. \tag{3.2}$$

Then, using (2.2)*, we may extend each map $a \mapsto \xi_n(a) = \varphi_a(T_a^{n+1}(c))$ to a θ-Hölder continuous map on $[a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$, with θ-Hölder constant bounded by $C\Lambda^{\theta(n+1)}$: Indeed, recalling $x_n(a) = T_a^{n+1}(c)$, just decompose*

$$\xi_n(a) - \xi_n(a') = \varphi_a(x_n(a)) - \varphi_{a'}(x_n(a)) + \varphi_{a'}(T_a^{n+1}(c)) - \varphi_{a'}(T_{a'}^{n+1}(c))\,. \tag{3.3}$$

Fix $\alpha \in (0, 1)$ such that (in view of the use of (2.30) in Corollary 3.4)

$$\frac{M_0}{1-\alpha} \leq \frac{3}{\alpha}\,. \tag{3.4}$$

Fix $q > 1$ and $0 < s < \min\{\varpi, 1/q\}$, and let

$$\lambda_0 = \min(\lambda_{CE}^\theta, \rho^{-1/2}) > 1\,, \tag{3.5}$$

where $\lambda_{CE} > 1$ is given by (2.5), while $\rho = \max\{\rho_q^s, \rho_\varpi\} < 1$ is given by Proposition 2.5, and $\theta \in (0, \min\{1/2, \varpi\})$ is given by Lemma 2.8. Finally, recalling Λ from (2.1), let $\eta \in (0, 1/2)$ be so small that

$$\Big(\frac{2\Lambda}{\lambda_{CE}}\Big)^\eta \leq \lambda_0 < \frac{\lambda_{CE}^\varpi}{\Lambda^{\eta\varpi}}\,. \tag{3.6}$$

Define the expectation $E(\psi)$ of $\psi \in L^\infty(\Omega_*^\varphi)$ by[30]

$$E(\psi) := \frac{1}{m(\Omega_*^\varphi)} \int_{\Omega_*^\varphi} \psi\, dm\,. \tag{3.7}$$

The following result is the key estimate on $\xi_j(a) = \varphi_a(T_a^{j+1}(c))$.

[30] We restrict to the Cantor set Ω_*^φ here and thus in (3.8). The bound (2.9) is used in the proof of (3.8) (but not for (3.9), Lemma 3.3, or Corollary 3.4).

Proposition 3.2. *There exist $C_\varphi < \infty$ and $K < \infty$ such that*

$$\left| E\left(\sum_{j=k}^{k+n-1} \xi_j\right)^2 - n\right| \le C_\varphi\,, \quad \forall k \ge \max\{2K, [2/\eta]\}\,,\ \forall 1 \le n \le \eta k/2\,, \tag{3.8}$$

and, setting[31] $v(k) = [k - k^{1/4}]$, *for every nontrivial interval* $\omega \subset \tilde\omega \in \mathcal{P}_{v(k)}$ *with* $\omega \cap \Omega_*^\varphi \neq \emptyset$ *and* $\lambda_0^{-k^{1/4}} \le |x_{v(k)}(\omega)| \le v(k)^{-3/\alpha}$, *we have*

$$\left| \frac{1}{|\omega|} \int_\omega \left(\sum_{j=k}^{k+n-1} \xi_j\right)^2 dm - n\right| \le C_\varphi\,, \quad \forall k \ge [2/\eta]\,,\ \forall 1 \le n \le \eta k/2\,, \tag{3.9}$$

and, for any sequence Ψ_k *with* $C_\Psi := \sup_k k^{-8/3} \sup |\Psi_k| < \infty$, *we have*

$$\left| E(\Psi_k) - \frac{1}{|\Omega^{\mathcal{Q}}_{*,v(k)}|} \int_{\Omega^{\mathcal{Q}}_{*,v(k)}} \Psi_k \, dm\right| \le C_\Psi C_\varphi\,,\ \forall k \ge [2/\eta]\,, \tag{3.10}$$

where

$$\mathcal{Q}_{*,v(k)} := \{\omega \in \mathcal{Q}_{v(k)} \mid \omega \cap \Omega_*^\varphi \neq \emptyset\}\,, \qquad \Omega^{\mathcal{Q}}_{*,v(k)} = \cup_{\omega \in \mathcal{Q}_{*,v(k)}} \omega\,, \tag{3.11}$$

for any refinement $\mathcal{Q}_{v(k)}$ *of* $\mathcal{P}_{v(k)}$ *such*[32] *that* $\lambda_0^{-k^{1/4}} \le |x_v(\omega)| \le v^{-3/\alpha}$ *for all* $\omega \in \mathcal{Q}_{v(k)}$.

Proposition 3.2 is proved in Section 3.1. Like for its analogue [Sch, Prop. 5.1], the first step will be to show the local estimate (3.9) using Lemma 3.3 through its Corollary 3.4 (the analogues of [Sch, Lemma 5.3, Cor. 5.5]).

Lemma 3.3 (Switching Locally from Parameter to Phase Space)**.** *Fix* $\ell_0 \in \{1, 2, 3, 4\}$. *There exists* $C < \infty$ *such that we have, for any integers*

$$n \le n_i \le n + \eta n\,, \qquad 1 \le i \le \ell_0\,,$$

for every $\tilde\omega \in \mathcal{P}_n$ *and each nontrivial interval* $\omega \subset \tilde\omega$ *with* $\omega \cap \Omega_*^\varphi \neq \emptyset$,

$$\begin{aligned}\int_{x_n(\omega)} \left| \prod_{\ell=1}^{\ell_0} \xi_{n_\ell}(x_n|_\omega^{-1}(y)) - \prod_{\ell=1}^{\ell_0} \varphi_{a_0}(T_{a_0}^{n_\ell - n}(y))\right| dy& \\ \le C\lambda_0^{-n}|x_n(\omega)|\,, \qquad \forall a_0 \in \omega \cap \Omega_*^\varphi\,.&\end{aligned} \tag{3.12}$$

Corollary 3.4. *There exists* $C_3 > 1$ *such that, for* ℓ_0, n, $n_1, \ldots, n_{\ell_0}$, *and* ω *as in Lemma 3.3, if, in addition,* $|x_n(\omega)| \le n^{-3/\alpha}$, *then for any* $a_0 \in \omega \cap \Omega_*^\varphi$,

$$\begin{aligned}\left| \frac{1}{|\omega|} \int_\omega \prod_{\ell=1}^{\ell_0} \xi_{n_\ell}(a)\, da - \frac{1}{|x_n(\omega)|} \int_{x_n(\omega)} \prod_{\ell=1}^{\ell_0} \varphi_{a_0}(T_{a_0}^{n_\ell - n}(y))\, dy\right|& \\ \le C_3(|x_n(\omega)|^\alpha + \lambda_0^{-n})\,.&\end{aligned}$$

[31]The stretched exponent 1/4 for $v(k)$ and the lower bound can be replaced by any number in $(0, 1)$, without changing the statements, up to adjusting intermediate constants.

[32]We have $\lambda_0^{-k^{1/4}} \le |x_v(\omega)|$ for all $\omega \in \mathcal{P}_{v(k)}$ by (3.21).

Proof. Since (3.4) implies (2.30) for ω, the change of variables $y = x_n(a)$ on ω, combined with the distortion estimate (2.31), gives

$$\begin{aligned}
&\left|\frac{1}{|\omega|}\int_\omega \prod_{\ell=1}^{\ell_0} \xi_{n_\ell}(a)\,da - \frac{1}{|x_n(\omega)|}\int_{x_n(\omega)} \prod_{\ell=1}^{\ell_0} \xi_{n_\ell}(x_n|_\omega^{-1}(y))\,dy\right| \\
&= \frac{1}{|x_n(\omega)|}\left|\int_{x_n(\omega)} \prod_{\ell=1}^{\ell_0} \xi_{n_\ell}(x_n|_\omega^{-1}(y))\left(\frac{1}{|x_n'(x_n|_\omega^{-1}y)|}\frac{|x_n(\omega)|}{|\omega|} - 1\right)dy\right| \qquad (3.13)\\
&\le C\frac{|x_n(\omega)|^\alpha}{|x_n(\omega)|}\int_{x_n(\omega)} \prod_{\ell=1}^{\ell_0} |\xi_{n_\ell}(x_n|_\omega^{-1}(y))|\,dy\,.
\end{aligned}$$

Since $\sup_k \|\xi_k\|_{L^\infty} < \infty$, the claim then follows from Lemma 3.3. □

Proof of Lemma 3.3. For $a_0 \in \omega$ as in the statement, the functions

$$\tilde\varphi_\ell(y) = \tilde\varphi_{\ell,a_0}(y) = \varphi_{a_0}(T_{a_0}^{n_\ell - n}(y))\,, \qquad \tilde\xi_\ell(y) = \tilde\xi_{\ell,\omega}(y) = \xi_{n_\ell}(x_n|_\omega^{-1}(y))$$

with

$$\xi_{n_\ell}(x_n|_\omega^{-1}(y)) = \varphi_{x_n|_\omega^{-1}(y)}(x_{n_\ell}(x_n|_\omega^{-1}(y)) = \varphi_{x_n|_\omega^{-1}(y)}(T^{n_\ell+1}_{x_n|_\omega^{-1}(y)}(c))$$

are bounded on $x_n(\omega \cap \Omega_*^\varphi)$. Decomposing

$$\begin{aligned}
|\tilde\xi_1\tilde\xi_2\tilde\xi_3\tilde\xi_4 - \tilde\varphi_1\tilde\varphi_2\tilde\varphi_3\tilde\varphi_4| &\le |(\tilde\xi_1 - \tilde\varphi_1)\tilde\xi_2\tilde\xi_3\tilde\xi_4| + |\tilde\varphi_1(\tilde\xi_2 - \tilde\varphi_2)\tilde\xi_3\tilde\xi_4| \\
&\quad + |\tilde\varphi_1\tilde\varphi_2(\tilde\xi_3 - \tilde\varphi_3)\tilde\xi_4| + |\tilde\varphi_1\tilde\varphi_2\tilde\varphi_3(\tilde\xi_4 - \tilde\varphi_4)|\,, \qquad (3.14)
\end{aligned}$$

it is enough to find a uniform constant $\bar C > 1$ such that

$$\frac{1}{|x_n(\omega)|}\int_{x_n(\omega)} |\tilde\xi_{\ell,\omega} - \tilde\varphi_{\ell,a_0}|\,dy \le \bar C\lambda_0^{-n}\,, \qquad \forall a_0 \in \omega \cap \Omega_*^\varphi\,,\ 1 \le \ell \le \ell_0\,.$$

We will do so by showing the pointwise estimate

$$|\tilde\xi_{\ell,\omega}(y) - \tilde\varphi_{\ell,a_0}(y)| \le \bar C\lambda_0^{-n}\,, \qquad \forall y \in x_n(\omega)\,,\ \forall a_0 \in \omega \cap \Omega_*^\varphi\,,\ 1 \le \ell \le \ell_0\,.$$

For $a = x_n|_\omega^{-1}(y)$, we decompose

$$\begin{aligned}
\tilde\xi_{\ell,\omega}(y) - \tilde\varphi_{\ell,a_0}(y) &= \xi_{n_\ell}(a) - \varphi_{a_0}(T_{a_0}^{n_\ell-n}(x_n(a))) \\
&= \varphi_a(x_{n_\ell}(a)) - \varphi_{a_0}(T_{a_0}^{n_\ell-n}(x_n(a))) \\
&= \varphi_a(x_{n_\ell}(a)) - \varphi_{a_0}(x_{n_\ell}(a)) + \varphi_{a_0}(x_{n_\ell}(a)) - \varphi_{a_0}(T_{a_0}^{n_\ell-n}(x_n(a)))\,. \qquad (3.15)
\end{aligned}$$

Using Remark 3.1, there exists C, independent of n_ℓ, such that

$$|\varphi_a(x_{n_\ell}(a)) - \varphi_{a_0}(x_{n_\ell}(a))| \le C|\omega|^\theta\,, \qquad \forall \{a, a_0\} \subset \omega\,. \qquad (3.16)$$

Hence, using our choice (3.5) of λ_0, and since $|\omega| \le C\lambda_{CE}^{-n}$ by (2.7), we get

$$|\varphi_a(x_{n_\ell}(a)) - \varphi_{a_0}(x_{n_\ell}(a))| \le C|\omega|^\theta \le C\lambda_0^{-n}. \qquad (3.17)$$

For the last two terms in the right-hand side of (3.15) note that since $a = x_n|_\omega^{-1}(y)$ implies $x_{n_\ell}(a) = T_a^{n_\ell+1}(c) = T_a^{n_\ell-n}(T_a^{n+1}(c)) = T_a^{n_\ell-n}(y)$, we have, using (2.2),

$$\begin{aligned}
\left|x_{n_\ell}(x_n|_\omega^{-1}(y)) - T_{a_0}^{n_\ell-n}(y)\right| &= |T_a^{n_\ell-n}(y) - T_{a_0}^{n_\ell-n}(y)| \qquad (3.18)\\
&\le C\Lambda^{n_\ell-n}|a - a_0| \le C\Lambda^{n_\ell-n}|\omega|\,, \qquad \forall y \in x_n(\omega)\,.
\end{aligned}$$

Then, since $n_\ell - n \le \eta n$, our choice of λ_0, η, with (3.2) at $a = a_0$ give[33]

$$
\begin{aligned}
(3.19)\qquad &|\varphi_{a_0}(x_{n_\ell}(a)) - \varphi_{a_0}(T_{a_0}^{n_\ell - n}(x_n(a)))| \\
&\qquad = \left|\varphi_{a_0}(x_{n_\ell}(x_n|_\omega^{-1}(y))) - \varphi_{a_0}(T_{a_0}^{n_\ell - n}(y))\right| \\
&\qquad \le C\tilde{C}\Lambda^{(n_\ell - n)\varpi}|\omega|^\varpi \le C\tilde{C}\Lambda^{\varpi\eta n}|\omega|^\varpi \le C\tilde{C}\lambda_0^{-n}\,,
\end{aligned}
$$

using again in the last inequality that $|\omega| \le C\lambda_{CE}^{-n}$ from (2.7). We conclude by combining (3.17) and (3.19) into (3.15). $\square$

3.1. **Proof of Proposition 3.2.** We first show (3.9). Let $\omega \subset \tilde{\omega} \in \mathcal{P}_{[k-k^{1/4}]}$, with $k \ge 2n/\eta$, be as in the assertion. Writing

$$
\int_\omega \Big(\sum_{j=k}^{k+n-1} \xi_j\Big)^2 dm = \sum_{j=k}^{k+n-1}\Big(\int_\omega \xi_j^2\, dm + 2\sum_{\ell=j+1}^{k+n-1}\int_\omega \xi_j\xi_\ell\, dm\Big)\,,
$$

it is sufficient to show that

$$
(3.20)\qquad \sum_{j=k}^{k+n-1}\left|1 - \frac{1}{|\omega|}\int_\omega\Big(\xi_j^2 + 2\sum_{\ell=j+1}^{k+n-1}\xi_j\xi_\ell\Big)dm\right| = O(1)\,.
$$

Fix $a_0 \in \omega \cap \Omega_*^\varphi$. By Corollary 3.4 for $\ell_0 = 2$, we have, for $k \le j \le k+n-1$,

$$
\begin{aligned}
&\frac{1}{|\omega|}\int_\omega\Big(\xi_j^2 + 2\sum_{\ell=j+1}^{k+n-1}\xi_j\xi_\ell\Big)dm \\
&\quad = \frac{1}{|x_v(\omega)|}\int_{x_v(\omega)}\Big(\varphi_{a_0}^2\circ T_{a_0}^{j-v} + 2\sum_{\ell=j+1}^{k+n-1}\varphi_{a_0}\circ T_{a_0}^{j-v}\varphi_{a_0}\circ T_{a_0}^{\ell-v}\Big)dm \\
&\qquad\qquad + O\big((k+n-j)(\lambda_0^{-(k-k^{1/4})} + |x_v(\omega)|^\alpha)\big)\,,
\end{aligned}
$$

(recall $v = [k - k^{1/4}]$). Since $0 < s < 1/q < 1$ we have that $1_{x_v(\omega)} \in H_q^s$, uniformly in v and ω (see [St]), so the first claim of Proposition 2.5 gives

$$
\begin{aligned}
&\int_{x_v(\omega)}(\varphi_{a_0}\circ T_{a_0}^{j-v})(\varphi_{a_0}\circ T_{a_0}^{\ell-v})\,dm \\
&\qquad = |x_v(\omega)|\int \varphi_{a_0}\cdot(\varphi_{a_0}\circ T_{a_0}^{\ell-j})\,d\mu_{a_0} + O(\rho^{j-v})\,,\quad \forall \ell \ge j\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{1}{|\omega|}\int_\omega\Big(\xi_j^2 + 2\sum_{\ell=j+1}^{k+n-1}\xi_j\xi_\ell\Big)dm &= \int\Big(\varphi_{a_0}^2 + 2\sum_{\ell=k+1}^{k+n-1}\varphi_{a_0}\cdot(\varphi_{a_0}\circ T_{a_0}^{\ell-j})\Big)d\mu_{a_0} \\
&\quad + O\big((k+n-j)(\lambda_0^{-(k-k^{1/4})} + |x_v(\omega)|^\alpha + \rho^{j-v}|x_v(\omega)|^{-1})\big)\,.
\end{aligned}
$$

By (1.3) and (1.7), we have

$$
1 = \int \varphi_{a_0}^2\,d\mu_{a_0} + 2\sum_{i=1}^{\infty}\int \varphi_{a_0}\cdot\varphi_{a_0}\circ T_{a_0}^i\,d\mu_{a_0}\,.
$$

[33] We do not need the analogue of Sublemma 5.4 from [Sch] here.

Therefore, the second claim of Proposition 2.5 gives

$$\int \Big(\varphi_{a_0}^2 + 2 \sum_{\ell=j+1}^{k+n-1} \varphi_{a_0} \cdot (\varphi_{a_0} \circ T_{a_0}^{\ell-j}) \Big) \, d\mu_{a_0} = 1 + O(\rho^{k+n-j}) \, .$$

Hence, we find, for $k \le j \le k+n-1$ and $v = [k - k^{1/4}]$,

$$\begin{aligned} &\left| 1 - \frac{1}{|\omega|} \int_\omega \Big(\xi_j^2 + 2 \sum_{\ell=j+1}^{k+n-1} \xi_j \xi_\ell \Big) \right| \\ &\qquad \le C(k+n-j) \big(\lambda_0^{-(k-k^{1/4})} + |x_v(\omega)|^\alpha + \rho^{j-v} |x_v(\omega)|^{-1} \big) + C\rho^{k+n-j} \, . \end{aligned}$$

To proceed we shall use several times that

$$\sup_n \sup_k \sum_{j=k}^{k+n-1} \frac{1}{(k+n-j)^2} \le \sup_n \sum_{\ell=1}^{n} \frac{1}{\ell^2} < \infty \, .$$

Clearly, $\rho^{k+n-j} \le \frac{C}{(k+n-j)^2}$. For the term $(k+n-j)\rho^{j-v}|x_v(\omega)|^{-1}$, we use $|x_v(\omega)| \ge \lambda_0^{-k^{1/4}}$ and the definition (3.5) of λ_0 to get, since $k \ge 2n/\eta$,

$$\frac{\rho^{j-v}}{|x_v(\omega)|} \le \rho^{k^{1/4}} \lambda_0^{k^{1/4}} \le \lambda_0^{-k^{1/4}} \le \frac{C}{n^3} \le \frac{C}{(k+n-j)^3} \, , \quad k \le j \le k+n-1 \, .$$

The term $(k+n-j)\lambda_0^{-(k-k^{1/4})}$ is similar. Finally, $|x_v(\omega)| \le v^{-3/\alpha}$ gives

$$\sum_{j=k}^{k+n-1} (k+n-j)|x_v(\omega)|^\alpha \le \sum_{j=k}^{k+n-1} \frac{k+n-j}{n^3} \le n \frac{k+n-k}{n^3} = \frac{1}{n} \, .$$

This proves (3.20), and hence (3.9).

We will next deduce (3.8) and (3.10) from (3.9). Fix $\kappa_1 > \kappa_0$, let $N_1(\kappa_1) \ge N_0$ be given by Lemma 2.3, and let $K \ge N_1$ be such that $k^{\kappa_1} \le \lambda_0^{k^{1/4}}$ for all $k \ge K$. Then, if $v = v(k) \ge K$ (so that $k \ge K$), we have

$$|x_v(\tilde\omega)| > v^{-\kappa_1} = [k - k^{1/4}]^{-\kappa_1} > \lambda_0^{-k^{1/4}} \, , \quad \forall \tilde\omega \in \mathcal{P}_v \, . \tag{3.21}$$

Refining $\mathcal{P}_v$ to a partition $\mathcal{Q}_v$ such that

$$\lambda_0^{-k^{1/4}} \le |x_v(\omega)| \le v^{-3/\alpha} \, , \qquad \forall \omega \in \mathcal{Q}_v \, ,$$

we set $\Omega^{\mathcal{Q}}_{*,v}$ as in (3.11) and we decompose

$$\begin{aligned} |\Omega_*^\varphi| \cdot E \Big(\sum_{j=k}^{k+n-1} \xi_j \Big)^2 &= \int_{\Omega^{\mathcal{Q}}_{*,v}} \Big(\sum_{j=k}^{k+n-1} \xi_j \Big)^2 dm - \int_{\Omega^{\mathcal{Q}}_{*,v} \setminus \Omega_*^\varphi} \Big(\sum_{j=k}^{k+n-1} \xi_j \Big)^2 dm \\ &= \int_{\Omega_v} \Big(\sum_{j=k}^{k+n-1} \xi_j \Big)^2 dm - \int_{\Omega_v \setminus \Omega_*^\varphi} \Big(\sum_{j=k}^{k+n-1} \xi_j \Big)^2 dm \, . \end{aligned}$$

Then, using (2.9), $\sup_k \sup |\xi_k| < \infty$, $\kappa_0 \ge 3/d_1$, and $v(k) \ge k/2 \ge n/\eta$,

$$\begin{aligned} 0 \le \int_{\Omega^{\mathcal{Q}}_{*,v} \setminus \Omega_*^\varphi} \Big(\sum_{j=k}^{k+n-1} \xi_j \Big)^2 dm &\le \int_{\Omega_v \setminus \Omega_*^\varphi} \Big(\sum_{j=k}^{k+n-1} \xi_j \Big)^2 dm \\ &\le C n^2 e_v \le C n^2 n^{1 - d_1 \kappa_0} \le C \, , \end{aligned} \tag{3.22}$$

which shows that

$$\int_{\Omega_*^\varphi} \left(\sum_{j=k}^{k+n-1} \xi_j \right)^2 dm = \int_{\Omega_{*,v}^{\mathcal{Q}}} \left(\sum_{j=k}^{k+n-1} \xi_j \right)^2 dm + O(1)$$
$$= \int_{\Omega_v} \left(\sum_{j=k}^{k+n-1} \xi_j \right)^2 dm + O(1) .$$

By (3.9),

$$\int_\omega \left(\sum_{j=k}^{k+n-1} \xi_j \right)^2 dm = |\omega|(n + O(1)) , \qquad \forall \omega \in \mathcal{Q}_v^* . \tag{3.23}$$

Summing (3.23) over $\omega \in \mathcal{Q}_v^*$, we get that

$$\int_{\Omega_{*,v}^{\mathcal{Q}}} \left(\sum_{j=k}^{k+n-1} \xi_j \right)^2 dm = n + O(1) .$$

Finally, using again (2.9) to see

$$0 \le \frac{|\Omega_{*,v}^{\mathcal{Q}}|}{|\Omega_*^\varphi|} - 1 \le \frac{|\Omega_v|}{|\Omega_*^\varphi|} - 1 = O(e_v) = O(e_n) ,$$

we have established (3.8), and also (3.10) in the case $\Psi_k = (\sum_{j=k}^{k+n-1} \xi_j)^2$ (note that $|\Psi_k| \le Cn^2 \le Ck^2$). For more general Ψ_k, the same argument, using $d_1\kappa_0 \ge 11/3$ in (3.22), gives (3.10). This ends the proof of Proposition 3.2.

4. Proof of Theorem 1.1 via Skorokhod's Representation Theorem

We will rearrange the Birkhoff sum as a sum of blocks of polynomial size, approximate the blocks by a martingale, and finally apply Skorokhod's representation theorem to this martingale. The size for the jth block $\mathbb{I}_j$ is $j^{2/3}$, which will give the error exponent $\gamma > 2/5$ in our ASIP.[34]

4.1. **Blocks $\mathbb{I}_M$. Approximations χ_i and y_j.** Fix a_*, $\varpi \in (0,1)$, q, $s \in (0, \min\{\varpi, 1/q\})$, ρ, θ, λ_0, η, α, $\varphi \in C^\varpi$, $\Omega_*^\varphi = \Omega_* \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$ as in the beginning of Section 3. Set

$$\mathcal{P}_{*,k} := \{\omega \in \mathcal{P}_k \mid |\omega \cap \Omega_*^\varphi| > 0\} , \qquad \Omega_{*,k} := \bigcup_{\omega \in \mathcal{P}_{*,k}} \omega, \qquad k \ge 1 .$$

Fix $\gamma \in (2/5, 1/2)$ and[35] $\delta \in (0, \min\{1/5, 2(\gamma - 2/5)\})$. For $i \ge 1$, we shall approach $\xi_i \colon [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi] \to \mathbb{C}$ (see Remark 3.1) by the stepfunction

$$\chi_i \colon \Omega_{*,r_i} \to \mathbb{C} , \qquad \chi_i = E(\xi_i | \mathcal{F}_{r_i}) , \quad \text{where } r_i = i + [i^\delta] ,$$

[34] A block size $\#\mathbb{I}_j = j^b$ replaces 3/5 in (4.4) by $1/(1+b)$, so that the first constraint becomes $N^\gamma > N^{b/(1+b)}$, see (4.5). Our bounds (4.25)–(4.26) (with Gál–Koksma and $M(N) \sim N^{1/(1+b)}$) give $N^\gamma > N^{(b+2)/(4(b+1))}$. Hence, $b = 2/3$ is the optimum. In the iid case a block size $j^{1/2}$ gives $\gamma > 1/3$ ([PS, p. 25]), see also the beginning of [Sch, Sec. 6].

[35] See Lemma 4.4 for the condition $\delta < 2(\gamma - 2/5)$.

with $\mathcal{F}_k$ the σ-algebra generated by the intervals in $\mathcal{P}_{*,k}$. Conditional expectations are only defined almost everywhere, but we may set (see (3.7))

$$\chi_i|_\omega \equiv \frac{\int_{\omega\cap\Omega_*^\varphi} \xi_i \, dm}{|\omega \cap \Omega_*^\varphi|}\,, \qquad \forall \omega \in \mathcal{P}_{*,r_i}\,,\ \forall i \geq 1\,. \tag{4.1}$$

Thus, χ_i is defined everywhere on Ω_{*,r_i}, allowing pointwise claims about it.

Recalling e_ℓ from (2.9) and our assumption $\lambda_{CE} > e^{14\alpha_{BC}}$ in the proof of Proposition 2.2, we have the following basic lemma:

Lemma 4.1. *For any $\tilde\lambda_{CE} \in (e^{\alpha_{BC}}, \sqrt{\lambda_{CE}} \cdot e^{-\alpha_{BC}})$, there exists C such that*

$$|\xi_i(a) - \chi_i(a)| \leq C\tilde\lambda_{CE}^{-\theta i^\delta}\,, \qquad \forall i \geq 1\,,\ \forall a \in \Omega_{*,r_i}\,, \tag{4.2}$$

and[36] *for all $i \geq 1$, $j \geq 0$ and all $a \in \Omega_{*,r_i}$*

$$|E(\xi_{i+j}|\mathcal{F}_{r_i})(a)| = |E(\chi_{i+j}|\mathcal{F}_{r_i})(a)| \leq C \min(1, e_{[\eta(j-2i^\delta)]})\,. \tag{4.3}$$

Following [PS, Sec 3.3], [Sch, Sec 6.1], we define inductively consecutive blocks $\mathbb{I}_j$ of integers and associated functions y_j: Let $\mathbb{I}_1 = \{1\}$, and let $\mathbb{I}_j$ for $j \geq 2$ contain $[j^{2/3}]$ consecutive integers. The first blocks are below:

$$\underbrace{1,}_{\mathbb{I}_1}\ \underbrace{2,}_{\mathbb{I}_2}\ \underbrace{3,4,}_{\mathbb{I}_3}\underbrace{5,6,}_{\mathbb{I}_4}\underbrace{7,8,}_{\mathbb{I}_5}\underbrace{9,10,11,}_{\mathbb{I}_6}\underbrace{12,13,14,}_{\mathbb{I}_7}\underbrace{15,16,17,18,}_{\mathbb{I}_8}\underbrace{19,20,21,22,}_{\mathbb{I}_9}\ldots$$

Let $M = M(N)$ be uniquely defined by $N \in \mathbb{I}_M$. There exists C such that

$$C^{-1}N^{3/5} \leq M(N) \leq CN^{3/5}\,, \qquad \forall N \geq 1\,. \tag{4.4}$$

By (4.2) in Lemma 4.1, there is C such that, for all $i \geq 1$ and all $a \in \Omega_{*,r_i}$,

$$\left|\sum_{i=1}^N \xi_i(a) - \sum_{j=1}^{M(N)} \sum_{i\in\mathbb{I}_j} \chi_i(a)\right| \leq \sum_{i=1}^N |\xi_i(a) - \chi_i(a)| + C\#\mathbb{I}_M \leq CN^{2/5}\,, \tag{4.5}$$

for all $N \geq 1$. Hence, in order to prove Theorem 1.1, it is sufficient to consider

$$y_j \colon \Omega_{*,[Cr_j^{5/3}]} \to \mathbb{C}\,, \qquad y_j := \sum_{i\in\mathbb{I}_j} \chi_i\,, \qquad j \geq 1\,.$$

Proof of Lemma 4.1. By (4.1), since ξ_i is continuous (see Remark 3.1), for any $\omega \in \mathcal{P}_{*,r_i}$, there exists $a' \in \omega$ such that $\chi_i|_\omega = \xi_i(a')$. Revisiting the decomposition (3.3), and using (3.2) and the θ-Hölder continuity of $a \mapsto \varphi_a(u)$ (as for (3.16)), we find C such that for all $i \geq 1$ and $\omega \in \mathcal{P}_{*,r_i}$

$$|\xi_i(a) - \chi_i(a)| = |\xi_i(a) - \xi_i(a')| \leq C(|\omega|^\theta + |x_i(\omega)|^\varpi) \leq C|x_i(\omega)|^\theta\,,\ \forall a \in \omega\,,$$

where we used $\theta \leq \varpi$ and (2.7) in second inequality. This establishes (4.2), since for any $\bar\lambda_{CE} \in (e^{\alpha_{BC}}, \sqrt{\lambda_{CE}} \cdot e^{-\alpha_{BC}})$, there exists $\bar C$ such that

$$|x_i(\omega)| \leq \bar C \cdot \bar\lambda_{CE}^{-i^\delta} \cdot i^{\kappa_0}\,, \qquad \forall \omega \in \mathcal{P}_{*,r_i}\,,\ \forall i\,. \tag{4.6}$$

To show (4.6) first note, using (2.29), that there exists $a \in \omega$ such that

$$|x_i(\omega)| \leq C\frac{|x_{r_i}(\omega)|}{|(T_a^{i^\delta})'(x_i(a))|}\,.$$

[36] The constant C in (4.3) goes to infinity as $\delta \to 0$, i.e. if $\gamma \to 2/5$.

Then, if $a \in \Omega_*$, the polynomial recurrence (2.8) and standard arguments give

$$|(T_a^{i^\delta})'(x_i(a))| \geq C i^{-\kappa_0} \bar{\lambda}_{CE}^{-i^\delta} \tag{4.7}$$

(see e.g. [BS1, Prop. 3.7] in the exponentially recurrent case). If $a \notin \Omega_*$, we may use bounded distortion (2.31) ($\alpha = 0$ suffices here) since $|\omega \cap \Omega_*| > 0$.

The equality in (4.3) follows from the definition since $\mathcal{F}_{r_i} \subset \mathcal{F}_{r_{i+j}}$. Indeed, for $a \in \omega \in \mathcal{P}_{*,r_i}$,

$$\begin{aligned} |\omega \cap \Omega_*^\varphi| \cdot |E(\xi_{i+j}|\mathcal{F}_{r_i})(a)| &= \int_{\omega \cap \Omega_*^\varphi} \xi_{i+j}\, dm \\ &= \sum_{\substack{\omega' \in \mathcal{P}_{*,r_{i+j}} \\ \omega' \subset \omega}} |\omega' \cap \Omega_*^\varphi| \cdot \frac{\int_{\omega' \cap \Omega_*^\varphi} \xi_{i+j}\, dm}{|\omega' \cap \Omega_*^\varphi|} \\ &= \sum_{\substack{\omega' \in \mathcal{P}_{*,r_{i+j}} \\ \omega' \subset \omega}} |\omega' \cap \Omega_*^\varphi| \cdot \chi_{i+j}|_{\omega'} = \sum_{\substack{\omega' \in \mathcal{P}_{*,r_{i+j}} \\ \omega' \subset \omega}} \int_{\omega' \cap \Omega_*^\varphi} \chi_{i+j}\, dm\,. \end{aligned} \tag{4.8}$$

Since $\sup_k \|\xi_k\|_{L^\infty} < \infty$, we may and shall assume that $j \geq 2i^\delta$ to prove the upper bound in (4.3). For such j, recalling $\eta \in (0, 1/2)$ from (3.6), define

$$k = k(i,j) = \max\left\{ i + [i^\delta] + \eta(j - i^\delta), \left\lceil \frac{i+j}{1+\eta} \right\rceil \right\} \tag{4.9}$$

so that $k \leq i + j - \frac{\eta}{1+\eta}(j - i^\delta) \leq i + j$ and $i + j \leq k(1+\eta)$.

Since δ is fixed, we may and shall assume that i is large enough such that $k(i,j) \geq N_1$ (with N_1 from Lemma 2.3) and

$$\max\{\lambda_0^{-(j+i)/(1+\eta)}, \rho^{\eta j/3} \cdot (2j)^{(\kappa_0+1)/\delta}\} \leq e_{[\eta(j-i^\delta)]}\,. \tag{4.10}$$

Since $k(i,j) \geq r_i$, we have, similarly as for (4.8),

$$|E(\xi_{i+j}|\mathcal{F}_{r_i})(a)| = |E(E(\xi_{i+j}|\mathcal{F}_{k(i,j)})|\mathcal{F}_{r_i})(a)|\,, \qquad \forall a \in \tilde{\omega} \in \mathcal{P}_{*,r_i}\,.$$

We must analyse the above decomposition more closely than in the proof of [Sch, Lemma 6.1]: Let $a \in \tilde{\omega} \in \mathcal{P}_{*,r_i}$, then

$$\begin{aligned} |\tilde{\omega} \cap \Omega_*^\varphi| \cdot |E(E(\xi_{i+j}|\mathcal{F}_{k(i,j)})|\mathcal{F}_{r_i})(a)| &= \Bigg| \sum_{\substack{\omega \in \mathcal{P}_{*,k(i,j)} \\ \omega \subset \tilde{\omega}}} \frac{|\omega \cap \Omega_*^\varphi|}{|\omega \cap \Omega_*^\varphi|} \int_{\omega \cap \Omega_*^\varphi} \xi_{i+j}\, dm \Bigg| \\ \leq \Bigg| \sum_{\substack{\omega \in \mathcal{P}_{*,k(i,j)} \\ \omega \subset \tilde{\omega}}} \frac{|\omega|}{|\omega|} \int_\omega \xi_{i+j}\, dm \Bigg| &+ \sup_{\tilde{a}} \|\varphi_{\tilde{a}}\|_{L^\infty} \cdot \sum_{\substack{\omega \in \mathcal{P}_{*,k(i,j)} \\ \omega \subset \tilde{\omega}}} |\omega \setminus (\omega \cap \Omega_*^\varphi)|\,. \end{aligned} \tag{4.11}$$

Since $\tilde{\omega} \in \mathcal{P}_{*,r_i}$, the bound (2.10) implies

$$\begin{cases} \frac{\sum_{\omega \in \mathcal{P}_{*,k(i,j)},\, \omega \subset \tilde{\omega}} |\omega \setminus (\omega \cap \Omega_*)|}{|\tilde{\omega} \cap \Omega_*^\varphi|} \leq \frac{d_0 e_{k(i,j)-r_i} |\tilde{\omega}|}{(1 - d_0 e_{r_i})|\tilde{\omega}|} \leq C d_0 e_{[\eta(j-i^\delta)]}\,, \\ |\tilde{\omega}|/|\tilde{\omega} \cap \Omega_*^\varphi| \leq \frac{|\tilde{\omega}|}{(1 - d_0 e_{r_i})|\tilde{\omega}|} \leq C\,. \end{cases} \tag{4.12}$$

In view of (2.10), (4.12) and (4.11), it suffices to show

$$\frac{1}{|\omega|}\left|\int_\omega \xi_{i+j}\, dm\right| \le C \min(1, e_{[\eta(j-2i^\delta)]})\,, \qquad \forall \omega \in \mathcal{P}_{*,k(i,j)}\,.$$

Fix $\omega \in \mathcal{P}_{*,k(i,j)}$. First note that, by (2.31) for $\alpha = 0$,

$$\frac{1}{|\omega|}\left|\int_\omega \xi_{i+j}(a)\, da\right| \le \frac{C}{|x_k(\omega)|}\left|\int_{x_k(\omega)} \xi_{i+j}(x_k|_\omega^{-1}(y))\, dy\right|. \tag{4.13}$$

Then, on the one hand, Lemma 3.3 for $\ell_0 = 1$ gives $a_0 \in \omega \cap \Omega_*^\varphi$ such that

$$\begin{aligned}\frac{1}{|x_k(\omega)|}&\left|\int_{x_k(\omega)} (\xi_{i+j}(x_k|_\omega^{-1}(y)) - \varphi_{a_0}(T_{a_0}^{i+j-k}(y)))\, dy\right| \\ &\le C\lambda_0^{-k(i,j)} \le C\lambda_0^{-(i+j)/(1+\eta)}\,.\end{aligned} \tag{4.14}$$

On the other hand, recalling $0 < s < 1/q$, since $1_{x_k(\omega)} \in H_q^s$ (uniformly in k and ω), the first claim of Proposition 2.5, with $\int \varphi_{a_0} d\mu_{a_0} = 0$, gives[37]

$$\begin{aligned}\frac{1}{|x_k(\omega)|}\left|\int_{x_k(\omega)} \varphi_{a_0}(T_{a_0}^{i+j-k}(y))\, dy\right| &\le C \cdot k(i,j)^{(\kappa_0+1)} \rho^{i+j-k(i,j)} \\ &\le C \cdot (i+j)^{\kappa_0+1} \rho^{\eta(j-i^\delta)/(1+\eta)} \le C \cdot (2j)^{(\kappa_0+1)/\delta} \rho^{\eta j/(2+2\eta)}\,.\end{aligned} \tag{4.15}$$

(We used $|x_k(\omega)| > Ck^{-\kappa_0+1}$ from Lemma 2.3.) Putting together (4.13), (4.14), (4.15) and (4.10), we conclude the proof of (4.3). □

4.2. **Law of Large Numbers for y_j^2.** Recall that $\gamma \in (2/5, 1/2)$ is fixed. The main ingredient in the proof of Theorem 1.1 is the following analogue of [Sch, Lemma 6.2], itself inspired by [PS, Lemma 3.3.1]:

Lemma 4.2. *For m_*-a.e. $a \in \Omega_*^\varphi$, there exists $C(a)$ such that*

$$\left|N - \sum_{j=1}^{M(N)} y_j^2(a)\right| \le C(a) N^{2\gamma}\,, \qquad \forall N \ge 1\,. \tag{4.16}$$

The proof of Lemma 4.2 (which uses Proposition 3.2 and (4.2), but not (4.3)) is based on the following theorem ([GK], see also [PS, Theorem A.1]).

Theorem 4.3 (Gál–Koksma's Strong Law of Large Numbers)**.** *Let z_j, $j \ge 1$, be zero-mean random variables. Assume there exist $p \ge 1$ and $C < \infty$ with*

$$E\Big(\sum_{j=m+1}^{m+n} z_j\Big)^2 \le C((m+n)^p - m^p)\,, \quad \forall\, m \ge 0 \text{ and } n \ge 1\,.$$

Then for all $\iota > 0$, we have $\frac{1}{n^{p/2+\iota}} \sum_{j=1}^n z_j \to 0$ almost surely.

Proof of Lemma 4.2. Set $w_j = \sum_{i \in \mathbb{I}_j} \xi_i$. Since $y_j^2 - w_j^2 = (y_j + w_j)(y_j - w_j)$ and $|y_j + w_j| \le Cj^{2/3}$, the bound (4.2) gives C such that $|y_j^2 - w_j^2| \le Cj^{2/3} \tilde\lambda_{CE}^{-\theta j^\delta}$ for all $j \ge 1$ and $a \in \Omega_{*, r_{Cj^{5/3}}}$. Hence, $\sup_{a \in \Omega_*^\varphi} \sum_{j \ge 1} |y_j^2 - w_j^2|$ is finite, and it suffices to show (4.16) with y_j replaced by w_j.

[37]A factor $|x_k(\omega)|^{-1}$ was omitted when applying [Sch, Prop. 4.3] on p. 400 of [Sch]: We fix this by using our polynomial lower bound on $|x_k(\omega)|$ (considering two different values of δ should work for [Sch]).

By (3.8) we have $|E(w_j^2) - \#\mathbb{I}_j| \le C$, and, since $\sum_{j=1}^{M(N)} \#\mathbb{I}_j = N$, we get $\left|\sum_{j=1}^{M(N)} E(w_j^2) - N\right| \le CM(N)$. Therefore,

$$\left|N - \sum_{j=1}^{M(N)} w_j^2\right| \le CM(N) + \left|\sum_{j=1}^{M(N)} w_j^2 - E(w_j^2)\right| . \tag{4.17}$$

Assume there exists C such that

$$E\left(\sum_{j=m+1}^{m+n} w_j^2 - E(w_j^2)\right)^2 \le C((m+n)^{8/3} - m^{8/3}) , \quad \forall m \ge 0,\ n \ge 1 . \tag{4.18}$$

Then Theorem 4.3 (Gál–Koksma) applied to $\iota \in (0, 10(\gamma - 2/5)/3]$, $p = 8/3$, and the zero-mean random variables $z_j = w_j^2 - E(w_j^2)$, implies that

$$\sum_{j=1}^{M(N)} w_j^2 - E(w_j^2) = o(M^{\frac{4}{3}+\iota}) , \qquad \text{almost surely} .$$

Hence, (4.17) gives $\left|N - \sum_{j=1}^{M(N)} w_j^2(a)\right| \le C(a)N^{4/5+3\iota/5} \le C(a)N^{2\gamma}$, almost surely (recall $M(N) \sim N^{3/5}$ by (4.4)). It remains to prove (4.18).

By Jensen's inequality we have $(E(w_j^2))^2 \le E(w_j^4)$ and therefore

$$\begin{aligned} & E\left(\sum_{j=m+1}^{m+n} w_j^2 - E(w_j^2)\right)^2 \\ & \qquad \le 2 \sum_{j=m+1}^{m+n} \left(E(w_j^4) + \sum_{k=j+1}^{m+n} |E(w_j^2 w_k^2) - E(w_j^2)E(w_k^2)| \right) . \end{aligned} \tag{4.19}$$

We consider first $E(w_j^4)$. Fix $\upsilon \in (0, 1/6)$ and, for $j \ge 1$, let

$$S_j = \{ \vec{v} \in \mathbb{I}_j^4 \mid v_1 \le v_2 \le v_3 \le v_4 \text{ and } \max\{v_2 - v_1, v_4 - v_3\} \ge j^\upsilon \}.$$

Then, since $\#(\{ \vec{v} \in \mathbb{I}_j^4 \mid v_1 \le v_2 \le v_3 \le v_4\} \setminus S_j) \le (j^{2/3+\upsilon})^2 = j^{4/3+2\upsilon}$, we find

$$\begin{aligned} \int_{\Omega_*^\varphi} w_j(a)^4 \, da &= \sum_{\vec{v} \in \mathbb{I}_j} \left| \int_{\Omega_*^\varphi} \prod_{\ell=1}^4 \xi_{v_\ell}(a) \, da \right| \le C \sum_{\substack{\vec{v} \in \mathbb{I}_j^4 \\ v_1 \le \ldots \le v_4}} \left| \int_{\Omega_*^\varphi} \prod_{\ell=1}^4 \xi_{v_\ell}(a) \, da \right| \\ &\le C \sum_{\vec{v} \in S_j} \left| \int_{\Omega_*^\varphi} \prod_{\ell=1}^4 \xi_{v_\ell}(a) \, da \right| + C j^{4/3+2\upsilon} . \end{aligned} \tag{4.20}$$

Let $\vec{v} \in S_j$ be such that $v_4 - v_3 \ge j^\upsilon$. For $\omega \in \mathcal{P}_{v_3}$ such that $\omega \cap \Omega_*^\varphi \ne \emptyset$, the change of variable in equation (3.13), together with an easy variant of Lemma 3.3 deduced from (3.14), give $a_0 \in \omega \cap \Omega_*^\varphi$ such that

$$\begin{aligned} & \frac{1}{|\omega|} \left| \int_\omega \prod_{\ell=1}^4 \xi_{v_\ell}(a) \, da \right| \\ & \quad \le \frac{C}{|x_{v_3}(\omega)|} \left| \int_{x_{v_3}(\omega)} \left(\prod_{\ell=1}^3 \xi_{v_\ell}(x_{v_3}|_\omega^{-1}(y)) \right) \varphi_{a_0}(T_{a_0}^{v_4 - v_3}(y)) \, dy \right| + C\lambda_0^{-v_3} . \end{aligned}$$

For $y \in x_{v_3}(\omega)$, setting $a = x_{v_3}|_\omega^{-1}(y)$, and recalling Remark 3.1, we find

$$\begin{aligned}|\xi_{v_\ell}(x_{v_3}|_\omega^{-1}(y)) - \varphi_{a_0}(x_{v_\ell} \circ x_{v_3}|_\omega^{-1}(y))| \\ = |\varphi_a(x_{v_\ell} \circ x_{v_3}|_\omega^{-1}(y)) - \varphi_{a_0}(x_{v_\ell} \circ x_{v_3}|_\omega^{-1}(y))| \le C|\omega|^\theta ,\end{aligned}$$

for $\ell = 1, 2, 3$. Thus, (2.9) and (2.7) imply (using $\sup_k \|\xi_k\|_{L^\infty} < \infty$)

$$\left|\int_{\Omega_*^\varphi} \prod_{\ell=1}^4 \xi_{v_\ell}(a)\, da\right| \le Ce_{v_3} + \sum_{\omega \in \mathcal{P}_{*,v_3}} |\omega| \left[\frac{C\lambda_{CE}^{-v_3\theta}}{|x_{v_3}(\omega)|} + C\lambda_0^{-v_3}\right] \tag{4.21}$$

$$+ \sum_{\omega \in \mathcal{P}_{*,v_3}} |\omega| \frac{C}{|x_{v_3}(\omega)|} \left|\int_{x_{v_3}(\omega)} \Big(\prod_{\ell=1}^3 \varphi_{a_0}(x_{v_\ell} \circ (x_{v_3}|_\omega^{-1})(y))\Big) \varphi_{a_0}(T_{a_0}^{v_4 - v_3}(y))\, dy\right| .$$

We claim that, for $\ell = 1, 2, 3$, and for each $\omega \in \mathcal{P}_{*,v_3}$,

$$|\partial_y(x_{v_\ell} \circ (x_{v_3}|_\omega^{-1}))(y)| \le Cv_\ell^{\kappa_0} , \quad \forall y \in x_{v_3}(\omega) . \tag{4.22}$$

Indeed, by (2.29), there exists $a \in \omega$ such that

$$|\partial_y(x_{v_\ell} \circ (x_{v_3}|_\omega^{-1}))(y)| \le C|(T_a^{v_3 - v_\ell})'(T_a^{v_\ell+1}(c))|^{-1} .$$

Thus, if $a \in \Omega_*$, standard arguments (see e.g. [BS1, Prop. 3.7], using our polynomial recurrence (2.8)) give the claim. Otherwise, since $|\omega \cap \Omega_*| > 0$, we may use (2.31) as for (4.6).

Therefore, we find C such that for each v_3 and $\omega \in \mathcal{P}_{*,v_3}$,

$$\|1_{x_{v_3}(\omega)} \cdot \prod_{\ell=1}^3 (\varphi_{a_0} \circ x_{v_\ell} \circ x_{v_3}|_\omega^{-1})\|_{H_q^s} \le C(v_1 v_2)^{\varpi\kappa_0} \|\varphi_{a_0}\|_{C^\varpi}^2 \|\varphi_{a_0}\|_{H_q^s} .$$

Indeed, on the one hand, there exists C such that, for any C^2 map $\mathcal{T}$, we have

$$\|\varphi_{a_0} \circ \mathcal{T}\|_{C^\varpi} \le C \sup |\mathcal{T}'|^\varpi \|\varphi_{a_0}\|_{C^\varpi} .$$

On the other hand, since $0 < s < 1/q < 1$, the characteristic function of an interval is a bounded multiplier on $H_q^s(I)$ (uniformly in the size of the interval), and since $s < \varpi$, a function in C^ϖ is a bounded multiplier on $H_q^s(I)$ ([St, Th]).

Hence, by the first claim of Proposition 2.5 (with (3.2) and $\int \varphi_{a_0} d\mu_{a_0} = 0$), we have

$$\begin{aligned}\frac{|\int_{x_{v_3}(\omega)} (\prod_{\ell=1}^3 \varphi_{a_0}(x_{v_\ell} \circ x_{v_3}|_\omega^{-1})) \varphi_{a_0}(T_{a_0}^{v_4 - v_3})\, dy|}{|x_{v_3}(\omega)|} &\le C(v_1 v_2)^{\varpi\kappa_0} \frac{\rho^{v_4 - v_3}}{|x_{v_3}(\omega)|} \\ &\le Cj^{10\varpi\kappa_0/3} v_3^{\kappa_1} \rho^{j^v} .\end{aligned}$$

(We used Lemma 2.3 and that $v_\ell \in \mathbb{I}_j$, implies $v_\ell \le Cj^{5/3}$,). Next,

$$\left|\int_{\Omega_*^\varphi} \prod_{\ell=1}^4 \xi_{v_\ell}\, da\right| \le C(e_{[Cj^{5/3}]} + j^{10\varpi\kappa_0/3} j^{5\kappa_1/3} \rho^{j^v} + j^{5\kappa_1/3} \lambda_0^{j^{5/3}}) \le Ce_{[Cj^{5/3}]} ,$$

for all $\vec{v} \in S_j$ with $v_4 - v_3 \ge j^v$ (if j is large enough).

Let now $\vec{v} \in S_j$ with $v_2 - v_1 \ge j^v$. Then applying directly Lemma 3.3 with $\ell_0 = 4$, a similar reasoning gives $|\int_{\Omega_*^\varphi} \prod_{\ell=1}^4 \xi_{v_\ell}\, da| \le Ce_{[Cj^{5/3}]}$.

Finally, since $\#S_j \le \#\mathbb{I}_j^4 \le j^{8/3}$ and $e_j \le j^{-d_1\kappa_0+1}$ with $d_1\kappa_0 \ge 3 > 9/5$, the bound (4.20) gives C such that[38]

$$E(w_j^4) \le C(j^{8/3} e_{[Cj^{5/3}]} + j^{4/3+2v}) \le Cj^{4/3+2v}\,, \qquad \forall j \ge 1\,. \tag{4.23}$$

We next bound $|E(w_j^2 w_k^2) - E(w_j^2)E(w_k^2)|$ for $k \ge j+1$. If $k = j+1$, by Cauchy's inequality and (4.23),

$$E(w_j^2 w_{j+1}^2) \le \sqrt{E(w_j^4)E(w_{j+1}^4)} \le Cj^{5/3}\,.$$

By (3.8) we have $E(w_j^2)E(w_{j+1}^2) \le Cj^{4/3}$. Hence

$$|E(w_j^2 w_{j+1}^2) - E(w_j^2)E(w_{j+1}^2)| \le Cj^{5/3}\,. \tag{4.24}$$

Assume now that $k \ge j+2$. By construction, y_j is constant on elements of $\mathcal{P}_v$ if $v \ge r_{j_1} = j_1 + [j_1^\delta]$, where j_1 is the largest number in $\mathbb{I}_j$. Let

$$k_0 = k_0(k) := \min \mathbb{I}_k \ge \frac{k^{5/3}}{C}\,.$$

Then, for large enough j, using that $x \mapsto x - x^{1/4}$ is increasing for large x, we find

$$k_0 - k_0^{1/4} \ge j_1 + \#\mathbb{I}_{j+1} - (j_1 + \#\mathbb{I}_{j+1})^{1/4} \ge j_1 + 2j_1^{2/3} - 2j_1^{1/4} \ge j_1 + j_1^{1/4}\,.$$

Since $k \ge j+2$ and $\delta < \frac{1}{4}$, we have that y_j is constant on elements of $\mathcal{P}_v$ for

$$v = v(k_0) = [k_0 - k_0^{1/4}]\,.$$

Lemma 2.3 gives $|x_{v(k_0)}(\omega)| \ge \lambda_0^{-k_0{}^{1/4}}$ if $\omega \in \mathcal{P}_{v(k_0)}$. Thus, there exists a refinement $\mathcal{Q}_{v(k_0)}$ of $\mathcal{P}_{v(k_0)}$ such that,

$$\lambda_0^{-k_0{}^{1/4}} \le |x_{v(k_0)}(\omega)| \le v(k_0)^{-3/\alpha} = [k_0 - k_0^{1/4}]^{-3/\alpha}\,, \quad \forall \omega \in \mathcal{Q}_{v(k_0)}\,.$$

Therefore, for large enough k, the local bound (3.9) in Proposition 3.2 gives for all $\omega \in \mathcal{Q}_v$ with non-empty intersection with Ω_*^φ that

$$\left| \frac{1}{|\omega|} \int_\omega w_k^2\, dm - \#\mathbb{I}_k \right| \le C\,,$$

since $n = \#\mathbb{I}_k = [k^{2/3}] \le \eta k_0/2$. As in (3.11), we write $\mathcal{Q}_{*,v}$ for the set of $\omega \in \mathcal{Q}_v$ with nonempty intersection with Ω_*^φ, and $\Omega_{*,v}^{\mathcal{Q}} = \cup \mathcal{Q}_{*,v}$. Thus, using that y_j is constant on each $\omega \in \mathcal{Q}_v$ (since $\mathcal{Q}_v$ refines $\mathcal{P}_v$),

$$\begin{aligned}\int_{\Omega_{*,v}^{\mathcal{Q}}} (y_j^2 w_k^2)\, dm &= \sum_{\omega \in \mathcal{Q}_v^*} |\omega| \cdot y_j^2|_\omega \cdot \frac{1}{|\omega|} \int_\omega w_k^2\, dm \\ &\in \left[\int_{\Omega_{*,v}^{\mathcal{Q}}} y_j^2\, dm(\#\mathbb{I}_k - C), \int_{\Omega_{*,v}^{\mathcal{Q}}} y_j^2\, dm(\#\mathbb{I}_k + C) \right].\end{aligned}$$

Recall that $j \le k-2$. Since $|y_j^2| \le Cj^{4/3} \le Ck^{4/3}$, we get

$$\frac{1}{m(\Omega_{*,v}^{\mathcal{Q}})} \int_{\Omega_{*,v}^{\mathcal{Q}}} y_j^2\, dm = E(y_j^2) + O(1)\,,$$

[38] For the purposes of the present lemma, a version of (4.23) with $Cj^{5/3}$ in the right-hand side would suffice. The stronger statement is needed for (4.31).

by (3.10) applied to $\Psi_k = y_j^2$, and since $|y_j^2 w_k^2| \le Ck^{8/3}$, we have

$$\frac{1}{m(\Omega^{\mathcal{Q}}_{*,v})}\int_{\Omega^{\mathcal{Q}}_{*,v}} (y_j^2 w_k^2)\, dm = E(y_j^2 w_k^2) + O(1)\,,$$

by (3.10) applied to $\Psi_k = (y_j^2 w_k^2)$. That is,

$$|E(y_j^2 w_k^2) - \#\mathbb{I}_k E(y_j^2)| \le C(E(y_j^2)+1)\,.$$

Next, the global estimate (3.8) in Proposition 3.2 gives $|E(y_j^2)E(w_k^2) - \#\mathbb{I}_k E(y_j^2)| \le CE(y_j^2)$. Therefore[39]

$$|E(y_j^2 w_k^2) - E(y_j^2)E(w_k^2)| \le C(2E(y_j^2)+1)) \le C\#\mathbb{I}_j\,.$$

Hence, for large enough j and all $k \ge j+2$, since $\sup|w_j + y_j| \le C\#\mathbb{I}_j$,

$$\begin{aligned}
|E(w_j^2 w_k^2) - E(w_j^2)E(w_k^2)| &\le |E(y_j^2 w_k^2) - E(y_j^2)E(w_k^2)| \\
&\quad + |E(y_j^2 w_k^2) - E(w_j^2 w_k^2)| + |E(w_j^2)E(w_k^2) - E(y_j^2)E(w_k^2)| \\
&\le C\#\mathbb{I}_j + CE(w_k^2)\sup|w_j - y_j|\cdot \sup|w_j+y_j| \\
&\le Cj^{2/3} + Ck^{2/3}j^{2/3}\tilde\lambda_{CE}^{-\theta j^{5\delta/3}}\,. \qquad (4.25)
\end{aligned}$$

(We used (4.2) to get $\sup|w_j - y_j| \le C\#\mathbb{I}_j\tilde\lambda_{CE}^{-\theta j^{5\delta/3}}$.)

Finally, we plug (4.25), (4.24), (4.23) into (4.19), and get, since $2\upsilon < 1/3$,

$$\begin{aligned}
&E\Big(\sum_{j=m+1}^{m+n} w_j^2 - E(w_j^2)\Big)^2 \\
&\le C\sum_{k=m+3}^{m+n} k^{2/3}\sum_{j=m+1}^{\infty} j^{2/3}\tilde\lambda_{CE}^{-\theta j^{5\delta/3}} + C\sum_{j=m+1}^{m+n}\Big(j^{5/3} + \sum_{k=j+2}^{m+n} j^{2/3}\Big) \\
&\le C\Bigg((m+n)^{5/3} - m^{5/3} + \sum_{j=m+1}^{m+n}\Big(j^{5/3} + (m+n-j)j^{2/3}\Big)\Bigg)\,. \qquad (4.26)
\end{aligned}$$

This proves (4.18). □

4.3. **Martingale Differences Y_j. Skorokhod's Representation Theorem.** As in Schnellmann's adaptation of [PS, Section 3.4–3.5] in [Sch, Section 6.3], let $\mathcal{L}_j$ be the σ-algebra generated by $\{y_\ell\}_{1\le\ell\le j}$, and set

$$u_j = \sum_{k\ge 0} E(y_{j+k} \mid \mathcal{L}_{j-1})\,, \qquad Y_j = y_j + u_{j+1} - u_j\,, \qquad j \ge 2\,. \qquad (4.27)$$

Then $\{Y_j, \mathcal{L}_j\}$ is a martingale difference sequence. Using (4.3), we show that $\{Y_j\}$ inherits the law of large numbers established for $\{y_j\}$ in Lemma 4.2:

Lemma 4.4. *For m_*-a.e. $a \in \Omega_*^\varphi$, there exists $C(a)$ such that*

$$\Big|N - \sum_{j=1}^{M(N)} Y_j^2(a)\Big| \le C(a)N^{2\gamma}\,, \qquad \forall N \ge 1\,, \qquad (4.28)$$

[39] The expression $\#\mathbb{I}_j = j^b = j^{2/3}$ in the right-hand side already leads to $\gamma > 2/5$. See Footnote 34.

and

$$\left|\sum_{j=1}^{M(N)} E(Y_j^2 \mid \mathcal{L}_{j-1}) - Y_j^2(a)\right| \le C(a)N^{2\gamma}\,, \qquad \forall N \ge 1\,. \tag{4.29}$$

Proof. Recalling the σ-algebra $\mathcal{F}_{r_i}$ generated by the intervals in $\mathcal{P}_{r_i}$, we have $\mathcal{L}_{\ell-1} \subset \mathcal{F}_{r_{i(\ell)}}$, where $i(\ell) = \max\{i \in \mathbb{I}_{\ell-1}\} \le C\ell^{5/3}$ by (4.4). Then

$$u_\ell = \sum_{j\ge 1} E(E(\xi_{i(\ell)+j} \mid \mathcal{F}_{r_{i(\ell)}}) \mid \mathcal{L}_{\ell-1})\,.$$

Since $\sum_{j=1}^\infty e_j < \infty$, the bound (4.3) in Lemma 4.1 gives

$$|u_\ell(a)| \le \sum_{j\ge 1} C\min\{1, e_{[\eta(j-2i(\ell)^\delta))]}\} \le \frac{2C}{\eta} i(\ell)^\delta \le C\ell^{5\delta/3}\,. \tag{4.30}$$

Put $v_j = u_j - u_{j+1}$, so that $Y_j^2 = y_j^2 - 2y_jv_j + v_j^2$.

We claim that (4.28) follows if for a.e. $a \in \Omega_*^\varphi$, there exists C such that $\sum_{j=1}^{M(N)} v_j^2 \le CN^{4\gamma-1}$. Indeed, since $\gamma < 1/2$, Lemma 4.2 and Cauchy's inequality then give (using $\sum_{j=1}^{M(N)} y_j^2 \le CN$)

$$\begin{aligned}\left|N - \sum_{j=1}^{M(N)} Y_j^2\right| &= \left|N - \sum_{j=1}^{M(N)} (y_j^2 - 2y_jv_j + v_j^2)\right| \\ &\le \left|N - \sum_{j=1}^{M(N)} y_j^2\right| + \sum_{j=1}^{M(N)} v_j^2 + 2\sqrt{\sum_{j=1}^{M(N)} y_j^2 \sum_{j=1}^{M(N)} v_j^2} \\ &\le C(a)N^{2\gamma} + CN^{2\gamma} + C\sqrt{NN^{4\gamma-1}} \le C(a)N^{2\gamma}\,.\end{aligned}$$

But since we have $v_j^2 \le Cj^{10\delta/3}$ (by (4.30)), we find, using $\delta < 2(\gamma - 2/5)$,

$$\sum_{j=1}^{M(N)} v_j^2 \le CM^{1+10\delta/3} \le N^{3/5+2\delta} \le CN^{4\gamma-1}\,.$$

It remains to prove (4.29). Set $R_j = Y_j^2 - E(Y_j^2 \mid \mathcal{L}_{j-1})$ and observe that $\{R_j, \mathcal{L}_j\}$ is a martingale difference sequence. By Minkowski's inequality

$$E(R_j^2) \le \left(\sqrt{E(Y_j^4)} + \sqrt{E(E(Y_j^2 \mid \mathcal{L}_{j-1})^2)}\right)^2 \le \left(2\sqrt{E(Y_j^4)}\right)^2 = 4E(Y_j^4)\,.$$

Since $Y_j = y_j - v_j$, we have, again by Minkowski's inequality,

$$\begin{aligned}E(R_j^2) \le 4E(Y_j^4) &\le 4\Big((E(y_j^4))^{\frac14} + (E(v_j^4))^{\frac14}\Big)^4 \le C(E(y_j^4) + E(v_j^4)) \\ &\le C(E(w_j^4) + E(|w_j^4 - y_j^4|) + E(v_j^4))\,.\end{aligned}$$

Since $w_j^4 - y_j^4 = (w_j^2+y_j^2)(w_j+y_j)(w_j-y_j)$, we get from (4.2) that $E(|w_j^4-y_j^4|)$ is uniformly bounded. By (4.30), we have $|u_j| \le Cj^{5\delta/3}$. Hence, $|v_j| \le |u_j| + |u_{j-1}| \le Cj^{5\delta/3}$, and $E(v_j^4) \le Cj^{20\delta/3} \le Cj^{4/3}$, since $\delta < 1/5$. For arbitrary $\iota > 0$ the bound (4.23), gives C such that $E(w_j^4) \le C j^{4/3+\iota}$. Thus

$$\sum_{j\ge 1} \frac{E(R_j^2)}{j^{7/3+\iota}} < \infty \,, \tag{4.31}$$

and a martingale result (see [Ch]) implies that $\sum_{j\ge 1} R_j / j^{7/6+\iota}$ converges almost surely. For m_*-a.e. $a \in \Omega_*^\varphi$, Kronecker's Lemma gives $C(a)$ with

$$\sum_{j=1}^{M(N)} R_j \le C(a) M^{7/6+\iota} \le C(a) N^{21/30+\iota} \,,$$

using (4.4) in the last inequality. Since $21/30 < 2\gamma$ this establishes (4.29). □

We shall apply the following embedding result. (See [HH, Theorem A.1].)

Theorem 4.5 (Skorokhod's Representation Theorem). *For any zero-mean square-integrable martingale $\{\sum_{k=1}^j Y_k,\ \mathcal{L}_j \mid j \ge 1\}$, there exist a probability space supporting a (standard) Brownian motion W, and nonnegative variables $\{T_k\,,\ k \ge 1\}$, such that $\{\sum_{k=1}^j Y_k\}_{j\ge 1}$ and $\{W(\sum_{k=1}^j T_k)\}_{j\ge 1}$ have the same distribution, and, in addition, letting $\mathcal{G}_0$ be the trivial σ-algebra (the empty set and the entire space), and $\mathcal{G}_j$, for $j \ge 1$, be the σ-algebra generated by*

$$\{W(t) \mid 0 \le t \le \tau_j\}\,, \quad \textit{where } \tau_j := \sum_{k=1}^{j} T_k \,,$$

then τ_j is $\mathcal{G}_j$-measurable, while $E(T_1 \mid \mathcal{G}_0) = E(W(T_1)^2 \mid \mathcal{G}_0)$, and

$$E(T_j \mid \mathcal{G}_{j-1}) = E\big((W(\tau_j) - W(\tau_{j-1}))^2 \mid \mathcal{G}_{j-1}\big)\,, \quad \forall j \ge 2\,, \quad \textit{almost surely.}$$

By the last claim of Theorem 4.5 and properties of Brownian motion

$$E(T_j \mid \mathcal{G}_{j-1}) = E(W(T_j)^2 \mid \mathcal{G}_{j-1})\,, \qquad \forall j \ge 1\,, \tag{4.32}$$

almost surely. (Indeed, letting W_1 be an independent copy of W we have $W(\tau_j) = W_1(\tau_{j-1} + T_j) = W_1(\tau_{j-1}) + W(T_j)$ in distribution, so that $W(\tau_j) - W(\tau_{j-1}) = W(T_j)$ in distribution.)

We need one last lemma. Recall that $\gamma \in (2/5, 1/2)$ is fixed.

Lemma 4.6 (Strong Law of Large Numbers for the Sequence T_j). *For m_*-a.e. $a \in \Omega_*^\varphi$, there exists $C(a)$ such that*

$$\left| N - \sum_{j=1}^{M(N)} T_j \right| \le C(a) N^{2\gamma}\,, \qquad \forall N \ge 1\,. \tag{4.33}$$

Proof. To start, apply Theorem 4.5 to the martingale difference sequence Y_j from (4.27), with $\mathcal{L}_j$ generated by $\{y_\ell\}_{1\le \ell \le j}$. Let $\tilde Y_j = W(\tau_j) - W(\tau_{j-1})$, so that $W(\tau_j) = \sum_{k=1}^j \tilde Y_k$ and $\tilde Y_j = W(T_j)$. By (4.32), we have, almost surely,

$$N - \sum_{j=1}^{M(N)} T_j = \big[N - \sum_{j=1}^{M} \tilde Y_j^2\big] + \sum_{j=1}^{M} \big[\tilde{Y}_j^{\,2} - E(\tilde Y_j^2 \mid \mathcal{G}_{j-1})\big]$$
$$+ \sum_{j=1}^{M} \big[E(T_j \mid \mathcal{G}_{j-1}) - T_j\big]\,,\ \forall N \ge 1\,.$$

Then, since Y_j and $\tilde{Y}_j$ have the same distribution, the bound (4.28) in Lemma 4.4 gives $C(a)$ such that, for all $N \geq 1$, the first sum in the right-hand side above is not larger than $C(a)N^{2\gamma}$.

For the second sum in the right-hand side above, we use (4.29). Since conditional expectations can be expressed in terms of distributions, (4.29) is also valid with Y_j replaced by $\tilde{Y}_j$. Thus the second sum in the right-hand side is also bounded by $C(a)N^{2\gamma}$ for all $N \geq 1$.

Finally, let $R_j = E(T_j \mid \mathcal{G}_{j-1}) - T_j$. Then $\{R_j, \mathcal{G}_j\}$ is a martingale difference sequence by (4.32). As in the proof of (4.29), we can estimate $E(R_j^2) \leq 4E(W(T_j)^4)$, and thus there exists $C(a)$ such that, for all $N \geq 1$, we have $\sum_{j=1}^{M(N)} R_j \leq CN^{21/30+\iota} \leq C(a)N^{2\gamma}$ almost surely. □

Proof of Theorem 1.1. Just like Schnellmann, we follow the proof of [PS, Lemma 3.5.3], replacing their $1/2 - \alpha/2 + \gamma$ by γ, and replacing Lemma 3.5.1 there by our Lemma 4.6. We then obtain that, almost surely,

$$\left| \sum_{j=1}^{M(N)} Y_j - W(N) \right| = O(N^\gamma) .$$

Then, using (4.30) and (4.4), we find

$$\left| \sum_{j=1}^{M(N)} y_j - Y_j \right| = \left| \sum_{j=1}^{M(N)} (u_{j+1} - u_j) \right| = |u_{M(N)+1} - u_1| \leq CN^\delta . \tag{4.34}$$

Since $\delta < 2/5$, and recalling (4.5), this establishes Theorem 1.1. □

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