SPECTRA OF DIFFERENTIABLE HYPERBOLIC MAPS

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Abstract. These are notes for the course with the same title given by the first named author during the workshop "Resonances and periodic orbits: spectrum and zeta functions in quantum and classical chaos" at IHP, Paris, June 27-July 5, 2005. We refer to our joint paper ([4], arxiv.org, 20 pages) for a complete self-contained proof in a more general setting. Our goal here is to give a reader-friendly presentation of the key ideas in our work in the simplest possible setting.

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1. (Lecture I) Introduction

1.1. **Results.** Let X be a d-dimensional C^∞ compact Riemannian manifold, and let $\mathcal{T} : X \to X$ be a diffeomorphism, which is $C^r(X)$ for $r > 1$. [1] We assume here that $\mathcal{T}$ is Anosov (a weaker hyperbolicity assumption suffices, as will be clear from the proof, see also [4]). This means that there exist constants $C > 0$ and $0 < \lambda_s < 1 < \nu_u$, and an invariant decomposition $TM = E^u \oplus E^s$ of the tangent bundle, satisfying

$$\|D\mathcal{T}^m_x|_{E^s}\| \le C\lambda_s^m \text{ and } \|D\mathcal{T}^{-m}_x|_{E^u}\| \le C\nu_u^{-m}\,, \forall m \ge 0\,, \forall x \in X\,.$$

For a C^{r-1} function $g : X \to \mathbb{C}$, we define the Ruelle transfer operator

$$\mathcal{L}_{\mathcal{T},g} u(x) = g(x) \cdot u \circ T(x)\,.$$

(For a – very short – moment we can view $\mathcal{L}_{\mathcal{T},g}$ as acting on continuous functions on X.) For example, letting ω be the Riemannian volume form on X, and letting $|\det DT|$ be the Jacobian of T (the function given by $T^*\omega = |\det D\mathcal{T}| \cdot \omega$), we may consider the pull-back operator

$$T^* u := \mathcal{L}_{\mathcal{T},1} u = u \circ \mathcal{T}\,, \tag{1}$$

and the Perron-Frobenius operator

$$\mathcal{P}u := \mathcal{L}_{\mathcal{T}^{-1}, |\det D\mathcal{T}^{-1}|} u = |\det D\mathcal{T}^{-1}| \cdot u \circ \mathcal{T}^{-1}\,. \tag{2}$$

These operators are adjoint to each other in the sense that

$$\int_X \mathcal{T}^* u \cdot v \, d\omega = \int_X u \cdot \mathcal{P} v \, d\omega\,. \tag{3}$$

This course is about the spectrum of the operator $\mathcal{L}_{T,g}$ on suitable Banach spaces of distributions on X.

More precisely, our goal is to find large enough spaces (they should contain at least all C^r functions) on which $\mathcal{L}_{T,g}$ is bounded and has small essential spectral radius. Recall the following definition:

Definition (Essential spectral radius)**.** The essential spectral radius $r_{ess}(\mathcal{L}|_\mathcal{B})$ of a bounded operator $\mathcal{L}$ on a Banach space $\mathcal{B}$ is the infimum of the real numbers $\rho > 0$ so that, outside of the disc of radius ρ, the spectrum of $\mathcal{L}$ on $\mathcal{B}$ consists of isolated eigenvalues of finite multiplicity.

The following basic fact will be at the very center of our proof. (It is behind most techniques to estimate the essential spectral radius: Lasota-Yorke or Doeblin-Fortet bounds, Hennion's theorem, the Nussbaum formula, etc, see e.g. the course [11] given by F. Naud in the same workshop at IHP.)

[1] For non-integer $s > 0$, a function on X is of class C^s if all its partial derivatives of order $[s]$ are $(s - [s])$-Hölder.

Exercise 1.1. If we can decompose $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_0$ where $\mathcal{L}_1$ is compact on $\mathcal{B}$ and $\mathcal{L}_0$ is bounded on $\mathcal{B}$, then the essential spectral radius of $\mathcal{L}|_\mathcal{B}$ is not larger than the spectral radius of $\mathcal{L}_0|_\mathcal{B}$. (See [6].)

For $m \ge 1$, we write $g^{(m)}(x) = \prod_{k=0}^{m-1} g(\mathcal{T}^k(x))$, and we set

$$R(\mathcal{T}, g) = \lim_{m\to\infty} (\sup |g^{(m)}(x)|)^{1/m}$$

(the limit is well-defined and equal to the infimum, by a standard subbadditivity argument). We shall give the key steps in the proof of our following result [4]:

Theorem 1.2. *[Essential spectral radius] Let $\mathcal{T} : X \to X$ be a C^r Anosov diffeomorphism, and let g be a C^{r-1} function, with $r > 1$. For real numbers $q < 0 < p$ with $p - q < r - 1$, there is a Banach space $C^{p,q}_*(\mathcal{T})$ of distributions on X, containing all C^s functions with $s > p$, on which $\mathcal{L}_{\mathcal{T},g}$ extends boundedly and so that*

$$r_{ess}(\mathcal{L}_{\mathcal{T},g}|_{C^{p,q}_*(\mathcal{T})}) \le R(\mathcal{T}, g) \max\{\lambda_s^p, \nu_u^q\} \, .$$

In particular the pull-back operator satisfies for all $q < 0 < p$ with $p - q < r - 1$

$$r_{ess}(\mathcal{T}^*|_{C^{p,q}_*(\mathcal{T})}) \le \max\{\lambda_s^p, \nu_u^q\} < 1 \, .$$

Note that [4] contains a more precise result: $R(\mathcal{T}, g) \max\{\lambda_s^p, \nu_u^q\}$ is replaced there by a weighted local hyperbolicity exponent (this makes the final part of the proof in [4] a little messy, requiring the use of fine partitions of unity) in a more general framework (hyperbolic diffeomorphisms instead of Anosov diffeomorphisms). In [4], we also study a scale of anisotropic Sobolev spaces. For a historical discussion and references to previous works ([5], [3], etc.) in particular the important paper of Gouëzel and Liverani [7], we refer to the introduction of [4]. We would also like to attract the reader's attention to the work of Avila–Gouëzel–Tsujii [1]: although the spaces in [1] are formally related to the Banach spaces of Gouëzel and Liverani [7], the key idea leading to the construction of the Banach spaces in [4], and discussed in the present notes, was in fact implicitly present in [1].

Remark 1.3 (Motivation and decay of correlations). Once we have the estimates in Theorem 1.2, it is not difficult to see that the spectral radius of the pull-back operator $\mathcal{T}^*$ on $C^{p,q}_*(\mathcal{T})$ is equal to one. (The constant function is a fixed function.) If $\mathcal{T}$ is topologically transitive (and thus topologically mixing) in addition, then 1 is the unique eigenvalue on the unit circle, it is a simple eigenvalue, and the fixed vector gives of the dual operator to $\mathcal{T}^*$ rise to the SRB measure μ: This corresponds to exponential decay of correlations for C^p observables and μ. (See Blank–Keller–Liverani [5, §3.2] for example.) Luckily, this is one of the last lectures of the workshop and no further motivation is needed!

Remark 1.4 (Spectral stability). It is not difficult to see that there is $\epsilon > 0$ so that if $\widetilde{\mathcal{T}}$ and $\widetilde{g}$, respectively, are ϵ-close to $\mathcal{T}$ and g, respectively, in the C^r, resp. C^{r-1}, topology, then the associated operator $\mathcal{L}_{\widetilde{\mathcal{T}},\widetilde{g}}$ has the same spectral properties than $\mathcal{L}_{\mathcal{T},g}$ on *the same Banach spaces*. Spectral stability can then be proved, as it has been done in [5] or [7] for the norms defined there.

We end this subsection by making a connection with dynamical determinants: Kitaev [9] proved that the following "dynamical Fredholm determinant"

$$d(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\mathcal{T}^n(x)=x} \frac{1}{|\det(D\mathcal{T}^n(x) - \mathrm{Id})|} \tag{4}$$

extends to a holomorphic function in the disc $\{z \,:\, |z|\cdot\max\{\lambda_s^p, \nu_u^q\} < 1\}$ for all $q < 0 < p$ with $p - q < r - 1$. Kitaev's bounds are slightly more general, but no spectral interpretation of his result is given in [9], although $d(z)$ is formally the Fredholm determinant of $\mathcal{L}$. Combining the results of [4] (or [7]) with those of Liverani [10] gives a connection between the eigenvalues of the transfer operator and the zeroes of a dynamical determinant, in a disk which is smaller, however, than the one indicated by Kitaev's results.

1.2. **Organisation of the course and toolbox.** In the rest of the first lecture (Section 2), we consider a much simpler situation: C^r locally expanding maps. In this case, a relevant Banach space is $C^{r-1}(X)$, and the analogue of Theorem 1.2 together with the connection with the dynamical determinants is well-known (see [12], [13]). We revisit these known results in Fourier coordinates: this allows us to present the skeleton of our argument (which reduces to careful applications of integration by parts!). In the second lecture (Section 3) we shall give the definition of the Banach space $C_*^{p,q}(\mathcal{T})$ of distributions suitable for the hyperbolic case, and adapt the argument from Section 2 to deal with hyperbolic data.

We end this introduction by presenting the main tools in the proof, in addition to Exercise 1.1, and the fact that

$$\int_{\mathbb{R}^d} (1 + \|\xi\|)^{-s} d\xi < \infty$$

if and only if $s > d$. They are quite elementary:

Ascoli-Arzelà's theorem. A reference is e.g. [6]. The Ascoli-Arzelà theorem is used to prove the compact embedding statements (Propositions 2.6 and 3.3). We refer to [4], and do not give the details here.

Young's inequality. For all $1 < t \le \infty$, we have the following inequality for the L^t norm of a convolution:

$$\|v * u\|_{L^t} \le \|v\|_{L^1} \|u\|_{L^t} . \tag{5}$$

In this note we shall only need the (trivial) case $t = \infty$. The general case is useful in [4] to consider Sobolev spaces.

Integration by parts. By "integration by parts on w," we will mean application, for $f \in C^2(\mathbb{R}^d)$ with $\sum_{j=1}^d (\partial_j f(w))^2 \neq 0$ and a compactly supported $g \in C^1(\mathbb{R}^d)$, of the formula

$$\int e^{if(w)} g(w) dw = -\sum_{k=1}^d \int i(\partial_k f(w)) e^{if(w)} \cdot \frac{i(\partial_k f(w)) \cdot g(w)}{\sum_{j=1}^d (\partial_j f(w))^2} dw \tag{6}$$

$$= i \cdot \int e^{if(w)} \cdot \sum_{k=1}^d \partial_k \left(\frac{\partial_k f(w) \cdot g(w)}{\sum_{j=1}^d (\partial_j f(w))^2} \right) dw , \tag{7}$$

where $w = (w_k)_{k=1}^d \in \mathbb{R}^d$, and ∂_k denotes partial differentiation with respect to w_k. (Note that if f is C^r we can only integrate by parts $[r]-1$ times in the above sense, even if g is C^r and compactly supported.)

Regularised integration by parts If $f \in C^{1+\delta}(\mathbb{R}^d)$ and $g \in C_0^\delta(\mathbb{R}^d)$, for $\delta \in (0,1)$, and $\sum_{j=1}^d (\partial_j f)^2 \neq 0$ on $\mathrm{supp}(g)$, we shall consider the following "regularised integration by parts:" [2] Set, for $k = 1, \dots, d$

$$h_k := \frac{i(\partial_k f(w)) \cdot g(w)}{\sum_{j=1}^d (\partial_j f(w))^2}.$$

Each h_k belongs to $C_0^\delta(\mathbb{R}^d)$. Let $h_{k,\epsilon}$, for small $\epsilon > 0$, be the convolution of h_k with $\epsilon^{-d} \upsilon(x/\epsilon)$, where the C^∞ function $\upsilon : \mathbb{R}^d \to \mathbb{R}_+$ is supported in the unit ball and satisfies $\int \upsilon(x) dx = 1$. There is C, independent of f and g, so that for each small $\epsilon > 0$ and all k,

$$\|\partial_k h_{k,\epsilon}\|_{L^\infty} \le C \|h_k\|_{C^\delta} \epsilon^{\delta - 1}, \quad \|h_k - h_{k,\epsilon}\|_{L^\infty} \le C \|h_k\|_{C^\delta} \epsilon^\delta .$$

[2] We thank S. Gouëzel for suggesting this.

Finally, for every real number $\Lambda \geq 1$

(8)

$$\begin{aligned}\int e^{i\Lambda f(w)} g(w) dw &= -\sum_{k=1}^{d} \int i\partial_k f(w) e^{i\Lambda f(w)} \cdot h_k(w) dw \\ &= \int \frac{e^{i\Lambda f(w)}}{\Lambda} \cdot \sum_{k=1}^{d} \partial_k h_{k,\epsilon}(w) dw \\ &\quad - \sum_{k=1}^{d} \int i\partial_k f(w) e^{i\Lambda f(w)} \cdot (h_k(w) - h_{k,\epsilon}(w)) dw.\end{aligned}$$

2. (Lecture I) Locally expanding endomorphisms

2.1. **The result for locally expanding maps (Theorem 2.8).** Let $\mathcal{T} : X \to X$ be C^r for $r > 1$ and X a d-dimensional compact manifold. In this section, we assume that $\mathcal{T}$ is a locally expanding map, i.e., there are $C > 0$ and $\lambda_s < 1$ so that for each x, all $m \geq 1$ and all $v \in T_x X$, we have $|D_x \mathcal{T}^m v| \geq C\lambda_s^{-m}|v|$. The function g is assumed to be C^r. We study the operator

$$\mathcal{L}_{\mathcal{T}^{-1},g} u(x) := \sum_{y:\mathcal{T}(y)=x} g(y)u(y)\,.$$

(This is the transfer operator associated to the branches of $\mathcal{T}^{-1}$, which contract by at least λ_s.) Note that

$$R(\mathcal{T}^{-1}, g) := \lim_{m\to\infty} \big(\sup_x \sum_{y:\mathcal{T}^m(y)=x} |g^{(m)}(x)|\big)^{1/m}$$

is the spectral radius of $\mathcal{L}_{\mathcal{T}^{-1},g}$ acting on continuous functions. [3]

For $p > 0$, recall that the C^p norm of $u \in C^\infty(\mathbb{R}^d)$ is $\|u\|_{C^p} =$

$$\max\left\{ \max_{|\alpha|\leq[p]} \sup_{x\in\mathbb{R}^d} |\partial^\alpha u(x)|, \max_{|\alpha|=[p]} \sup_{x\in\mathbb{R}^d} \sup_{y\in\mathbb{R}^d/\{0\}} \frac{|\partial^\alpha u(x+y) - \partial^\alpha u(x)|}{|y|^{p-[p]}} \right\},$$

where $\partial^\alpha u$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+$ denotes the partial derivative $\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} u$, and $|\alpha| = \sum_{j=1}^d \alpha_j$. For $u \in C^p(X)$ (in the sense of the footnote 1), the above norm can be used in charts to define a norm $\|u\|_{C^p(X)}$.

We shall prove the following result:

[3]If $g = |\det D\mathcal{T}|^{-1}$ then it is well-known that $R(\mathcal{T}^{-1}, g) = 1$.

Theorem 2.1. *[Essential spectral radius for expanding maps] Let $\mathcal{T}$ be C^r and expanding, and let g be C^r for $r > 1$. For any noninteger $0 < p < r$, let $C^p_*(X)$ be the closure of $C^\infty(X)$ for the C^p norm. Then the operator $\mathcal{L}_{\mathcal{T}^{-1},g}$ is bounded on $C^p_*(X)$ and*

$$r_{ess}(\mathcal{L}_{\mathcal{T}^{-1},g}|_{C^p_*(X)}) \le R(\mathcal{T}^{-1}, g) \cdot \lambda_s^p \,.$$

The main interest of the proof given in Lecture I is that it can be generalised to the hyperbolic case.

Exercise 2.2. Prove that for any noninteger $p > 0$ we have $C^p_*(X) \subset C^p(X)$, and that the inclusion is strict.

If p is an integer, the proof below gives the same bounds for a Zygmund space $C^p_*(X)$.

Ruelle [12] proved the statement of Theorem 2.1 for $C^p(X)$ instead of $C^p_*(X)$. (It is in fact possible to modify the definitions in Subsection 2.2, to get a new proof of Ruelle's result. This modification is cumbersome when dealing with distributions in the later sections, and we do not present it here.)

2.2. **Local definition of Hölder norms in Fourier coordinates.** We present here the "dyadic decomposition" approach to compactly supported Hölder functions in $\mathbb{R}^d$ (for $d \ge 1$). Fix a C^∞ function $\chi : \mathbb{R}_+ \to [0, 1]$ with

$$\chi(s) = 1, \quad \text{for } s \le 1, \qquad \chi(s) = 0, \quad \text{for } s \ge 2.$$

Define $\psi_n : \mathbb{R}^d \to [0, 1]$ for $n \in \mathbb{Z}_+$, by $\psi_0(\xi) = \chi(\|\xi\|)$, and

$$\psi_n(\xi) = \chi(2^{-n}\|\xi\|) - \chi(2^{-n+1}\|\xi\|)\,, \quad n \ge 1\,.$$

We have $1 = \sum_{n=0}^{\infty} \psi_n(\xi)$, and $\mathrm{supp}(\psi_n) \subset \{\xi \mid 2^{n-1} \le \|\xi\| \le 2^{n+1}\}$ for $n \ge 1$. Also $\psi_n(\xi) = \psi_1(2^{-n+1}\xi)$ for $n \ge 1$. Thus, for every multi-index α, there exists a constant C_α such that $\|\partial^\alpha \psi_n\|_{L^\infty} \le C_\alpha 2^{-n|\alpha|}$ for all $n \ge 0$, and the inverse Fourier transform of ψ_n,

$$\widehat{\psi}_n(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi} \psi_n(\xi) d\xi\,, \quad x \in \mathbb{R}^d\,,$$

decays rapidly in the sense of Schwartz. Furthermore we have $\widehat{\psi}_n(x) = 2^{d(n-1)}\widehat{\psi}_1(2^{n-1}x)$ for $n \ge 1$ and all x, and

$$\sup_n \int_{\mathbb{R}^d} |\widehat{\psi}_n(x)| dx < \infty\,. \tag{9}$$

Exercise 2.3. Prove the above claims on ψ_n and $\widehat{\psi}_n$.

Fix a compact subset $K \subset \mathbb{R}^d$ with non-empty interior and let $C^\infty(K)$ be the space of complex-valued C^∞ functions on $\mathbb{R}^d$ supported on K. Decompose each $u \in C^\infty(K)$ as $u = \sum_{n\ge 0} u_n$, by defining for $n \in \mathbb{Z}_+$ and $x \in \mathbb{R}^d$

$$(10) \qquad u_n(x) = \psi_n(D)u(x) := (2\pi)^{-d} \int_K \int_{\mathbb{R}^d} e^{i(x-y)\xi} \psi_n(\xi) u(y)\, dy d\xi\,.$$

Note that u_n is not necessarily supported in K, although it satisfies good decay properties when $|x| \to \infty$: we say that the operator $\psi_n(D)$ is not a "local" operator, but it is "pseudo-local." (See [4]. The pseudo-local estimates there are useful e.g. to show the compactness Proposition 3.3.)

Remark 2.4. The notation $a(D)$ for the operator sending a compactly supported $u \in C^\infty(\mathbb{R}^d)$ to

$$a(D)u(x) := (2\pi)^{-d} \int_K \int_{\mathbb{R}^d} e^{i(x-y)\xi} a(\xi) u(y)\, dy d\xi = (\widehat{a} * u)(x)\,,$$

associated to $a \in C^\infty(\mathbb{R}^d)$ so that $\partial^\alpha a(\xi) \le C_\alpha(a) \|\xi\|^{-|\alpha|}$ for each multi-index α, stands for the "pseudo-differential operator associated to the symbol" a. We shall not need any knowledge about pseudodifferential operators, and shall not require symbols depending on both x and ξ.

Definition (Little hölder space C^p_*)**.** For a real number $p > 0$, define on $C^\infty(K)$ the norm

$$\|u\|_{C^p_*} = \sup_{n \ge 0} 2^{pn} \|u_n\|_{L^\infty(\mathbb{R}^d)}\,.$$

The space $C^p_*(K)$ is the completion of $C^\infty(K)$ with respect to $\|\cdot\|_{C^p_*}$.

Remark 2.5. It is known that if p is not an integer then the norm $\|u\|_{C^p_*}$ is equivalent to the C^p norm. (See [14, Appendix A].)

We shall not give a proof of the following, very standard, result (the proof is based on the Ascoli-Arzelà lemma; see Proposition 3.3 for an anisotropic analogue):

Proposition 2.6. *[Compact embeddings] If* $0 < p' < p$ *the inclusion* $C^p_*(K) \subset C^{p'}_*(K)$ *is compact.*

2.3. **Compact approximation for local maps.** Let $r > 1$. Let $K, K' \subset \mathbb{R}^d$ be compact subsets with non-empty interiors, and take a compact neighbourhood W of K. Let $T : W \to K'$ be a C^r diffeomorphism onto its image (the reader should think of T as being a local inverse branch of an expanding map $\mathcal{T}$, in charts). Let $g : \mathbb{R}^d \to \mathbb{C}$

be a C^{r-1} function such that $\mathrm{supp}(g) \subset K$. In this section we study a local transfer operator:

$$L : C^{r-1}(K') \to C^{r-1}(K), \qquad Lu(x) = g(x) \cdot u \circ T(x).$$

We define a "weakest contraction" exponent

$$\|T\|_+ = \sup_{x \in K} \sup_{\xi \neq 0} \frac{\|DT_x^{tr}(\xi)\|}{\|\xi\|} \, .$$

Remark 2.7. Strictly speaking, $\|T\|_+$ is of course the "strongest expansion" of T, but if $\|T\|_+ < 1$ then $\|T\|_+$ deserves to be called the "weakest contraction." Although we do not assume that $\|T\|_+ < 1$ in this section, the result below is most interesting if $\|T\|_+ < 1$.

The following result is the key to the proof of Theorem 1.2:

Theorem 2.8. *For any real number $p > 0$ such that $p < r - 1$, there is a constant C, so that for each C^r map T and every g in $C^{r-1}(K)$ there is a compact operator $L_1 : C^p_*(K') \to C^p_*(K)$ such that for any $u \in C^p_*(K')$*

$$\|Lu - L_1 u\|_{C^p_*(K)} \leq C\|g\|_{L^\infty} \cdot \|T\|_+^p \|u\|_{C^p_*(K')} \, .$$

If $g \in C^r(K)$ then the condition on p may be relaxed to $0 < p < r$.

It is essential that the constant C in the statement of the theorem does not depend on T and g.

We sketch how to deduce Theorem 1.2 from Theorem 2.8: fix $\epsilon > 0$ and take an iterate $\mathcal{T}^m$ so that $\|D\mathcal{T}_x^m v\| \geq (\lambda_s + \epsilon)^{-m}|v|$ for all x and v. Considering suitable charts K_i and a suitable partition of unity on X, each inverse branch T_i of $\mathcal{T}^m$ in charts satisfies $\|T_i\|_+ \leq (\lambda_s + \epsilon)^m$. Summing over the branches gives Theorem 1.2. It is crucial that the constant C in Theorem 2.8 is independent of T and thus of the iterate m. We use also that $C^p_*(X)$ is embedded in the direct sum of the local $C^p_*(K_i)$ spaces. (See Section 3.3 for the analogous argument in the hyperbolic case.)

Proof of Theorem 2.8. We need a couple more notations. Recall the function χ from Section 2.2. Define $\tilde{\psi}_\ell : \mathbb{R}^d \to [0, 1]$ by

$$\tilde{\psi}_\ell(\xi) = \begin{cases} \chi(2^{-\ell-1}\|\xi\|) - \chi(2^{-\ell+2}\|\xi\|), & \text{if } \ell \geq 1, \\ \chi(2^{-1}\|\xi\|), & \text{if } \ell = 0. \end{cases}$$

Note that $\tilde{\psi}_\ell(\xi) = 1$ if $\xi \in \mathrm{supp}(\psi_\ell)$.

We write[4]

- $\ell \hookrightarrow n$ if $2^n \le \|T\|_+ 2^{\ell+4}$,
- $\ell \not\hookrightarrow n$ otherwise.

By the definition of $\not\hookrightarrow$ there exists an integer $N(T) > 0$ such that

$$\inf_x d(\mathrm{supp}(\psi_n), DT_x^{tr}(\mathrm{supp}(\tilde\psi_\ell))) \ge 2^{\max\{n,\ell\}-N(T)} \quad \text{if } \ell \not\hookrightarrow n. \tag{11}$$

Noting that $L(f)$ is well-defined if $f \in C^\infty(\mathbb{R}^d)$ because g is supported in K, we may define L_1 and L_0 by $L_j u = \sum_n (L_j u)_{(n)}$ with

$$(L_0 u)_{(n)} = \sum_{\ell : \ell \hookrightarrow n} \psi_n(D)(L\, u_\ell)\,,$$

and

$$(L_1 u)_{(n)} = \sum_{\ell : \ell \not\hookrightarrow n} \psi_n(D)(L\, \tilde\psi_\ell(D) u_\ell)\,.$$

Since $\tilde\psi_\ell(D) u_\ell = u_\ell$, we have $L_0 + L_1 = L$. By Proposition 2.6, it is enough to show that there is C, which does not depend on T and g, so that

$$\|L_0 u\|_{C^p_*(K)} \le C \|T\|_+^p \|g\|_{L^\infty} \|u\|_{C^p_*(K')}\,,$$

and that for each $0 < p' < p$ there is $C(T, g)$ so that

$$\|L_1 u\|_{C^p_*(K)} < C(T, g) \|u\|_{C^{p'}_*(K')}\,. \tag{12}$$

(Note that if a Banach space $\mathcal{B}_1'$ is compactly included in a Banach space $\mathcal{B}_0'$, then any bounded linear operator from $\mathcal{B}_0'$ to $\mathcal{B}_1$ is compact when restricted to $\mathcal{B}_1'$, using that the composition of a compact operator followed by a bounded operator is compact.)

Notice that there is C (independent of T and g) so that

$$\sum_{\ell : \ell \hookrightarrow n} 2^{pn - p\ell} \le 2^{4p} \|T\|_+^p \sum_{j=0}^{\infty} 2^{-j} \le C \|T\|_+^p,\ \forall n\,. \tag{13}$$

[4]By definition, if $\ell \not\hookrightarrow n$ then $n > \ell - n(T)$ for some $n(T)$ depending only on T. This feature will not be present in the hyperbolic case.

The bound for L_0 is then easy:

$$\begin{aligned}\|L_0 u\|_{C^p_*} &= \sup_m 2^{pm} \|\psi_m(D)\big(\sum_n (L_0 u)_{(n)}\big)\|_{L^\infty(\mathbb{R}^d)} \\ &\le \sup_m 2^{pm} \sum_{|n-m|<5} \sum_{\ell:\ell\hookrightarrow n} \|\psi_n(D)(Lu_\ell)\|_{L^\infty(\mathbb{R}^d)} \\ &\le C\|g\|_{L^\infty} \sup_m 2^{pm} \sum_{|n-m|<5} \sum_{\ell:\ell\hookrightarrow n} \|u_\ell\|_{L^\infty(\mathbb{R}^d)} \\ &\le C\|g\|_{L^\infty} \sup_m \sum_{|n-m|<5} \big(\sum_{\ell:\ell\hookrightarrow n} 2^{pn-p\ell}\big)\|u\|_{C^p_*} \\ &\le C\|g\|_{L^\infty}\|T\|_+^p \|u\|_{C^p_*}\end{aligned}$$

(We used $\psi_n(D)f = \widehat{\psi}_n * f$ which implies $\|\psi_n(D)f\|_{L^\infty} \le C\|f\|_{L^\infty}$ for all n, by Young's inequality (5) for L^∞, and

$$\sup_{\mathbb{R}^d} |g(x) f_n(T(x))| \le \sup_K |g(x)| \sup_{T(K)} |f_n(x)| \,.)$$

We will have to work a little harder for L_1.

Assume first that $p \le r-1$. Then it is enough to prove that for each $f \in C^\infty(\mathbb{R}^d)$ with rapid decay, and all n

(14)
$$\|\psi_n(D)(L(\tilde\psi_\ell(D)f))\|_{L^\infty} \le C(T,g) 2^{-(r-1)\max\{n,\ell\}} \|f\|_{L^\infty} \text{ if } \ell \not\hookrightarrow n.$$

Indeed, the above bound implies that

(15)
$$\|L_1 u\|_{C^p_*(K)} \le C(T,g)\cdot \sup_n \left(\sum_{\ell:\ell\not\hookrightarrow n} 2^{pn-p'\ell-(r-1)\max\{n,\ell\}}\right) \|u\|_{C^{p'}_*(K')} \,,$$

and the conditions $p \le r-1$ and $p' > 0$ ensure that the supremum over n of the sum over ℓ such that $\ell \not\hookrightarrow n$ above is finite (recall the footnote 4).

To show (14), we note that

$$(\psi_n(D) L\, \tilde\psi_\ell(D) f)(x) = (2\pi)^{-2d} \int_{\mathbb{R}^d} V_n^\ell(x,y)\cdot f\circ T(y) |\det DT(y)| dy \,,$$

where we have extended T to a C^r diffeomorphism of $\mathbb{R}^d$, linear outside of a neighbourhood of W, with $\sup_{\mathbb{R}^d} |\det DT| \le 2 \sup_W |\det DT|$ and

(16)
$$V_n^\ell(x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} e^{i(x-w)\xi + i(T(w)-T(y))\eta} g(w)\psi_n(\xi)\tilde\psi_\ell(\eta) dw d\xi d\eta \,.$$

Since $\|f \circ T \cdot |\det DT|\|_{L^\infty(\mathbb{R}^d)} \le C(T)\|f\|_{L^\infty(\mathbb{R}^d)}$, the inequality (14) follows if we show that there exists $C(T,g)$ such that for all $\ell \not\leftrightarrow n$ the operator norm of the integral operator

$$H_n^\ell : f \mapsto \int_{\mathbb{R}^d} V_n^\ell(x,y) f(y) dy$$

acting on $L^\infty(\mathbb{R}^d)$ is bounded by $C(T,g) \cdot 2^{-(r-1)\max\{n,\ell\}}$.

Define the integrable function $b : \mathbb{R}^d \to \mathbb{R}_+$ by

$$b(x) = 1 \quad \text{if } \|x\| \le 1, \qquad b(x) = \|x\|^{-d-1} \quad \text{if } \|x\| > 1. \tag{17}$$

The required estimate on H_n^ℓ follows if we show

$$|V_n^\ell(x,y)| \le C(T,g) 2^{-(r-1)\max\{n,\ell\}} \cdot 2^{d\min\{n,\ell\}} b(2^{\min\{n,\ell\}}(x-y)), \tag{18}$$

for some $C(T,g) > 0$ and all $\ell \not\leftrightarrow n$. Indeed, as the right hand side of (18) is written as a function of $x-y$, say $B(x-y)$, we have, by Young's inequality (5) in $L^\infty(\mathbb{R}^d)$,

$$\begin{aligned} \|H_n^\ell f\|_{L^\infty} &\le \|B * f\|_{L^\infty} \le \|B\|_{L^1}\|f\|_{L^\infty} \\ &\le C(T,g) 2^{-(r-1)\max\{n,\ell\}} \cdot \|b\|_{L^1} \cdot \|f\|_{L^\infty}. \end{aligned}$$

(Note that by Young's inequality (5) for $L^t(\mathbb{R}^d)$ with $1 < t < \infty$, the operator H_n^ℓ acting on each $L^t(\mathbb{R}^d)$ is also bounded by $C(T,g) \cdot 2^{-(r-1)\max\{n,\ell\}}$. This is useful to control the essential spectral radius on Sobolev spaces, see [4].)

We now prove (18).

If $r \ge 2$ (otherwise we do nothing at this stage), integrating (16) by parts $[r]-1$ times on w (recall (6)), we obtain

$$V_n^\ell(x,y) = \int e^{i(x-w)\xi + i(T(w)-T(y))\eta} F(\xi,\eta,w)\psi_n(\xi)\tilde{\psi}_\ell(\eta) dw d\xi d\eta, \tag{19}$$

where $F(\xi,\eta,w)$ is a $C^{r-[r]}$ function in w which is C^∞ in the variables ξ and η. The following exercise is an important (but straightforward) step in the proof:

Exercise 2.9. Using (11), check that if $\psi_n(\xi) \cdot \tilde{\psi}_\ell(\eta) \ne 0$ then

$$\|F(\xi,\eta,\cdot)\|_{C^{r-[r]}} \le C(T,g) 2^{-([r]-1)\max\{n,\ell\}}. \tag{20}$$

The estimate (20) looks promising, but applying it naively is not enough: since we are integrating over ξ in the support of ψ_n and over η in the support of $\tilde{\psi}_\ell$, we would get an additional factor $2^{dn+d\ell}$. In order to get rid of this factor, we shall use another exercise:

Exercise 2.10. Using (11), show that if $\psi_n(\xi)\cdot\tilde\psi_\ell(\eta)\neq 0$, then for all multi-indices α and β

$$\|\partial_\xi^\alpha\partial_\eta^\beta F(\xi,\eta,\cdot)\|_{C^{r-[r]}}\le C_{\alpha,\beta}(T,g)2^{-n|\alpha|-\ell|\beta|-([r]-1)\max\{n,\ell\}}\,. \tag{21}$$

Assume first that r is an integer (then, $r=[r]\ge 2$). Put

$$G_{n,\ell}(\xi,\eta,w)=F(\xi,\eta,w)\psi_n(\xi)\tilde\psi_\ell(\eta)$$

and consider the scaling $\widetilde G_{n,\ell}(\xi,\eta,w)=G_{n,\ell}(2^n\xi,2^\ell\eta,w)$.

The estimate (21) implies that for all α and β

$$\|\partial_\xi^\alpha\partial_\eta^\beta\widetilde G_{n,\ell}(\xi,\eta,\cdot)\|_{C^{r-[r]}}\le C_{\alpha,\beta}(T,g)2^{-([r]-1)\max\{n,\ell\}}\,,\forall\xi,\eta,n,\ell\,. \tag{22}$$

Then, denoting by $\mathcal F$ the inverse Fourier transform with respect to the variable (ξ,η), and setting $W_n^\ell(u,v,w):=$

$$(\mathcal F\widetilde G_{n,\ell})(u,v,w)=(2\pi)^{-2d}\int_{\mathbb R^d}\int_{\mathbb R^d}e^{iu\xi}e^{iv\eta}\widetilde G_{n,\ell}(\xi,\eta,w)\,d\xi d\eta\,, \tag{23}$$

the bounds (22) imply that for any nonnegative integers γ and γ'

$$\big\|\|u\|^\gamma\|v\|^{\gamma'}W_n^\ell(u,v,\cdot)\big\|_{L^\infty}\le\widetilde C_{\gamma,\gamma'}(T,g)2^{-([r]-1)\max\{n,\ell\}}\,,\forall u,v,n,\ell\,. \tag{24}$$

(Just note that the integrand in (23) is supported in $\max\{\|\xi\|,\|\eta\|\}\le 2$, and integrate by parts with respect to ξ and η as many times as desired.) Applying (24) to γ, γ' in $\{0,d+1\}$, we get $C(T,g)$ so that for each $w\in K$, and all n, ℓ, u, v

$$|W_n^\ell(u,v,w)|\le C(T,g)2^{-([r]-1)\max\{n,\ell\}}b(u)b(v)\,. \tag{25}$$

(For $w\notin K$ we have $W_n^\ell(u,v,w)=0$ for all u, v, n, ℓ.) Therefore, since $(\mathcal FG_{n,\ell})(u,v,w)=2^{dn+d\ell}W_n^\ell(2^nu,2^\ell v,w)$, we get by definition,

$$\begin{aligned}|V_n^\ell(x,y)|&\le\int_K|(\mathcal FG_{n,\ell})(x-w,T(w)-T(y),w)|\,dw\\ &\le C\int_K 2^{dn+d\ell}|W_n^\ell(2^n(x-w),2^\ell(T(w)-T(y)),w)|\,dw\\ &\le C(T,g)2^{-([r]-1)\max\{n,\ell\}+dn+d\ell}\int_K b(2^n(x-w))b(2^\ell(T(w)-T(y))dw\,.\end{aligned}$$

Next, using $u=2^n(x-w)$, note $w_u=x-2^{-n}u$, and write

$$\begin{aligned}&\int_K 2^{dn+d\ell}b(2^n(x-w))b(2^\ell(T(w)-T(y)))\,dw\\ &=\int_{\mathbb R^d}2^{d\ell}b(u)b(2^\ell(T(w_u)-T(y)))\,du\,.\end{aligned} \tag{26}$$

Since $\ell \le n + N(T)$ (see footnote 4), we get by using

$$\int b(u) b(2^\ell (T(w_u) - T(y)))\, du \le \int b(u)\, du < \infty\,, \tag{27}$$

that $|V_n^\ell(x,y)| \le C(T,g) 2^{d \min\{n,\ell\} - ([r]-1)\max\{n,\ell\}}$.

If $\|x-y\| > 2^{-\min\{n,\ell\}}$, we can improve the estimate: let $q_0 \le \min\{\ell, n\}$ be the integer so that $\|x-y\| \in [2^{-q_0}, 2^{-q_0+1})$. At least one of the following conditions holds:

$$\|u\| = 2^n \|x - w_u\| \ge 2^{C(T)+n-q_0} \ge 2^{C(T)+\ell-q_0}$$
$$2^\ell \|T(w_u) - T(y)\| > 2^{M(T)} \|w_u - y\| > 2^{C(T)+\ell-q_0}\,.$$

If the first condition holds replacing $\int_{\mathbb{R}^d} b(u)\, du$ by $\int_{\|u\| \ge 2^{C(T)+\ell-q_0}} b(u)\, du$ in (27), we get the claim $|V_n^\ell(x,y)| \le 2^{-\min\{n,\ell\} - ([r]-1)} 2^{(d+1)q_0}$. In the second case, replace the right-hand-side of (27) by

$$\int_{\|T(w_u)-T(y)\| \ge 2^{C(T)-q_0}} b(2^\ell (T(w_u) - T(y)))\, du\,,$$

where $w_u = x - 2^{-n}u$, and find $|V_n^\ell(x,y)| \le 2^{-\min\{n,\ell\} - ([r]-1)} 2^{(d+1)q_0}$.

If $r > 1$ is not an integer, we start from (19) and rewrite $V_n^\ell(x,y)$ as

$$\int e^{i\Lambda(x-w)(\xi/\Lambda) + i\Lambda(T(w)-T(y))(\eta/\Lambda)} F(\xi,\eta,w) \psi_n(\xi) \tilde\psi_\ell(\eta) dw d\xi d\eta, \tag{28}$$

for $\Lambda = 2^{\max\{\ell,n\}}$. Recalling (8), we apply to (28) one regularised integration by parts for $\delta = r - [r]$ (noting that T is $C^{1+\delta}$). We get two terms $F_{1,\epsilon}(\xi,\eta,w)$ and $F_{2,\epsilon}(\xi,\eta,w)$. Choosing $\epsilon = \Lambda^{-1}$, we may apply the above procedure to each of them.

The case when g is C^r and $r-1 < p \le r$ is done in Appendix A. □

3. (Lecture II) The Anosov case

3.1. **Local definition of the anisotropic norms.** In this subsection we define the anisotropic norms in a compact domain of $\mathbb{R}^d$. (The Banach space in Theorem 1.2 will be constructed by patching together such local spaces in coordinate charts.) Let $\mathbf{C}_+$ and $\mathbf{C}_-$ be closed cones in $\mathbb{R}^d$ with nonempty interiors, such that $\mathbf{C}_+ \cap \mathbf{C}_- = \{0\}$. Let then $\varphi_+, \varphi_- : \mathbf{S}^{d-1} \to [0,1]$ be C^∞ functions on the unit sphere $\mathbf{S}^{d-1}$ in $\mathbb{R}^d$ satisfying

$$\varphi_+(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \mathbf{C}_+, \\ 0, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \mathbf{C}_-, \end{cases} \qquad \varphi_-(\xi) = 1 - \varphi_+(\xi)\,. \tag{29}$$

(What the reader can have in mind is that $\mathbf{C}_+$ is a cone containing a stable bundle and $\mathbf{C}_-$ a cone containing an unstable bundle. The justification for the choice of $\pm$ is given in Remark 3.2 below.)

Except in Exercise 3.6 below, we shall work in this subsection with a fixed pair of cones $\mathbf{C}_\pm$ and fixed functions $\varphi_\pm$, they will not appear in the notation for the sake of simplicity. Recall ψ_n and χ from Subsection 2.2. For $n \in \mathbb{Z}_+$ and $\sigma \in \{+, -\}$, we define

$$\psi_{n,\sigma}(\xi) = \begin{cases} \psi_n(\xi)\varphi_\sigma(\xi/\|\xi\|), & \text{if } n \geq 1, \\ \chi(\|\xi\|)/2, & \text{if } n = 0. \end{cases}$$

Exercise 3.1. Prove that the $\psi_{n,\sigma}$ enjoy similar properties as those of the ψ_n, in particular the L^1-norm of the rapidly decaying function $\widehat{\psi}_{n,\sigma}$ is bounded uniformly in n.

Fix $K \subset \mathbb{R}^d$ compact and with nonempty interior. For $u \in C^\infty(K)$, define for each $n \in \mathbb{Z}_+$, $\sigma \in \{+, -\}$, and $x \in \mathbb{R}^d$:

$$u_{n,\sigma}(x) = (\psi_{n,\sigma}(D)u)(x) = (\widehat{\psi}_{n,\sigma} * u)(x)\,.$$

Since $1 = \sum_{n=0}^\infty \sum_{\sigma=\pm} \psi_{n,\sigma}(\xi)$, we have $u = \sum_{n\geq 0} \sum_{\sigma=\pm} u_{n,\sigma}$.

Definition (Anisotropic hölder spaces)**.** Let $\mathbf{C}_\pm$ and $\varphi_\pm$ be fixed, as above. Let p and q be arbitrary real numbers. Define the anisotropic hölder norm $\|u\|_{C_*^{p,q}}$ for $u \in C^\infty(K)$, by

$$\|u\|_{C_*^{p,q}} = \max\left\{ \sup_{n\geq 0} 2^{pn}\|u_{n,+}\|_{L^\infty},\ \sup_{n\geq 0} 2^{qn}\|u_{n,-}\|_{L^\infty} \right\}. \tag{30}$$

Let $C_*^{p,q}(K)$ be the completion of $C^\infty(K)$ for the norm $\|\cdot\|_{C_*^{p,q}}$.

Remark 3.2. In our application, $p > 0$, and $q < 0$. Recalling Section 2 for contracting branches and $p > 0$, it is then natural that $\mathbf{C}_+$ and φ_+ be associated to a contracting (i.e., stable) cone for the dynamics and $\mathbf{C}_-$ and φ_- be associated to an expanding (i.e., unstable) cone. Elements of $C_*^{p,q}(K)$ are distributions which are at least p-smooth in the directions in $\mathbf{C}_+$ and at most q-"rough" in the directions of $\mathbf{C}_-$.

We shall not give a proof of the following result, referring instead to [4]. (The proof is based on the Ascoli-Arzelà theorem.)

Proposition 3.3. *[Compact embeddings] If $p' < p$ and $q' < q$, the inclusion $C_*^{p,q}(K) \subset C_*^{p',q'}(K)$ is compact.*

3.2. **Compact approximation for local hyperbolic maps.** Let $r > 1$. Let $K, K' \subset \mathbb{R}^d$ be compact subsets with non-empty interiors, and take a compact neighborhood W of K. Let $T : W \to K'$ be a C^r diffeomorphism onto its image. Let $g : \mathbb{R}^d \to \mathbb{C}$ be a C^{r-1} function such that $\mathrm{supp}(g) \subset K$. In this section we study the transfer operator

$$L : C^{r-1}(K') \to C^{r-1}(K), \qquad Lu(x) = g(x) \cdot u \circ T(x)\,.$$

For a pair of cones $\mathbf{C}_\pm$ as in Subsection 3.1, we make the following *cone-hyperbolicity* assumption on T:

$$DT_x^{tr}(\mathbb{R}^d \setminus \mathrm{interior}\,(\mathbf{C}_+)) \subset \mathrm{interior}\,(\mathbf{C}_-) \cup \{0\} \quad \text{for all } x \in W, \tag{31}$$

where DT_x^{tr} denotes the transpose of the derivative of T at x. (The above condition is sufficient in the neighbourhood of a hyperbolic fixed point. More generally, it will be useful to allow more flexibility and to work with two pairs of cones. See Exercise 3.6 below.)

Put

$$\|T\|_+ = \sup_x \sup_{0 \neq DT_x^{tr}(\xi) \notin \mathbf{C}_-} \frac{\|DT_x^{tr}(\xi)\|}{\|\xi\|}$$
$$\text{(the “weakest contraction (outside of } (DT_x^{tr})^{-1}\mathbf{C}_-)\text{”)}\,,$$

$$\|T\|_- = \inf_x \inf_{0 \neq \xi \notin \mathbf{C}_+} \frac{\|DT_x^{tr}(\xi)\|}{\|\xi\|}$$
$$\text{(the “weakest expansion (outside of } \mathbf{C}_+)\text{”)}\,.$$

(The terminology “weakest contraction/expansion” is to be understood in the sense of Remark 2.7.)

The following result is the key to the proof of Theorem 1.2 (see also Exercise 3.6):

Theorem 3.4. *[Estimates for local cone-hyperbolic maps] For any $q < 0 < p$ such that $p - q < r - 1$, there exists a constant C so that for each C^r diffeomorphism T and each C^{r-1} function g as above (assuming in particular (31)), there is a compact operator $L_1 : C_*^{p,q}(K') \to C_*^{p,q}(K)$, such that for any $u \in C_*^{p,q}(K')$*

$$\|Lu - L_1 u\|_{C_*^{p,q}(K)} \le C\|g\|_{L^\infty} \cdot \max\{\|T\|_+^p, \|T\|_-^q\}\|u\|_{C_*^{p,q}(K')}\,.$$

It is essential that the constant C does not depend on T and g.

Proof of Theorem 3.4. We need more notation. By (31) there exist a closed cone $\widetilde{\mathbf{C}}_+$ contained in the interior of $\mathbf{C}_+$ such that for all $x \in W$

$$DT_x^{tr}(\mathbb{R}^d \setminus \mathrm{interior}(\widetilde{\mathbf{C}}_+)) \subset \mathrm{interior}(\mathbf{C}_-) \cup \{0\}\,. \tag{32}$$

Fix also a closed cone $\widetilde{\mathbf{C}}_-$ contained in the interior of $\mathbf{C}_-$ and let $\tilde\varphi_\pm : \mathbf{S}^{d-1} \to [0,1]$ be C^∞ functions satisfying

$$\tilde\varphi_-(\xi) = \begin{cases} 0, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \widetilde{\mathbf{C}}_+, \\ 1, & \text{if } \xi \notin \mathbf{S}^{d-1} \cap \mathbf{C}_+, \end{cases} \qquad \tilde\varphi_+(\xi) = \begin{cases} 1, & \text{if } \xi \notin \mathbf{S}^{d-1} \cap \mathbf{C}_-, \\ 0, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \widetilde{\mathbf{C}}_-. \end{cases}$$

Recalling $\tilde\psi_\ell$ from the beginning of the proof of Theorem 2.8, define for $\sigma \in \{+,-\}$

$$\tilde\psi_{\ell,\sigma}(\xi) = \begin{cases} \tilde\psi_\ell(\xi)\tilde\varphi_\sigma(\xi/\|\xi\|), & \text{if } \ell \ge 1, \\ \chi(2^{-1}\|\xi\|), & \text{if } \ell = 0. \end{cases}$$

Note that $\tilde\psi_{\ell,\tau}(\xi) = 1$ if $\xi \in \operatorname{supp}(\psi_{\ell,\tau})$.

Up to slightly changing the cones $\widetilde{\mathbf{C}}_-$, we can guarantee that

$$\inf_{x\in K} \inf_{0\ne\xi\notin\widetilde{\mathbf{C}}_+} \frac{\|DT_x^{tr}(\xi)\|}{\|\xi\|} \ge \|T\|_-/2\,, \tag{33}$$

$$\sup_{x\in K} \sup_{0\ne DT_x^{tr}(\xi)\notin\widetilde{\mathbf{C}}_-} \frac{\|DT_x^{tr}(\xi)\|}{\|\xi\|} \le 2\|T\|_+\,. \tag{34}$$

We write $(\ell,\tau) \hookrightarrow (n,\sigma)$ if either

- $(\tau,\sigma) = (+,+)$ and $2^n \le 2^{\ell+5}\|T\|_+$, or
- $(\tau,\sigma) = (-,-)$ and $2^{\ell-5}\|T\|_- \le 2^n$, or
- $(\tau,\sigma) = (+,-)$ and $2^n \ge 2^5\|T\|_-$ or $2^\ell \ge 2^5\|T\|_+$.

We write $(\ell,\tau) \not\hookrightarrow (n,\sigma)$ otherwise.

Exercise 3.5. Let $\check{\mathbf{C}}_\pm$ be two closed cones with disjoint interiors, so that $\check{\mathbf{C}}_+ \cap \check{\mathbf{C}}_- = \{0\}$, and with

$$\text{closure}\,(\mathbb{R}^d \setminus \mathbf{C}_+) \subset \text{interior}\,(\check{\mathbf{C}}_-) \cup \{0\}\,.$$

Let $\check\varphi_\pm : \mathbf{S}^{d-1} \to [0,1]$, and $\check\psi_{\ell,\sigma} : \mathbb{R}^d \to [0,1]$, for $\sigma \in \{+,-\}$ and $\ell \in \mathbb{Z}_+$, be functions defined just like $\varphi_\pm$ and $\psi_{\ell,\sigma}$, but replacing the cones $\mathbf{C}_\pm$ by $\check{\mathbf{C}}_\pm$.

Using (32) and (33), check that there exists an integer $N(T) > 0$ such that for all $x \in \operatorname{supp}(g)$

$$d(\operatorname{supp}(\check\psi_{n,\sigma}), DT_x^{tr}(\operatorname{supp}(\tilde\psi_{\ell,\tau}))) \ge 2^{\max\{n,\ell\}-N(T)} \quad \text{if } (\ell,\tau) \not\hookrightarrow (n,\sigma). \tag{35}$$

Hint: For $(\tau,\sigma) = (-,+)$, use (32). See [4] for further details.

Note that (35) is *exactly* the same lower bound as (11).

Define L_1 and L_0 by $L_j u = \sum_{n,\sigma} (L_j u)_{(n,\sigma)}$ with

$$(L_0 u)_{(n,\sigma)} = \sum_{(\ell,\tau):(\ell,\tau)\hookrightarrow(n,\sigma)} \check\psi_{n,\sigma}(D)(L\, u_{\ell,\tau})\,,$$

and

$$(L_1 u)_{(n,\sigma)} = \sum_{(\ell,\tau):(\ell,\tau)\not\hookrightarrow(n,\sigma)} \check\psi_{n,\sigma}(D)(L\,\tilde\psi_{\ell,\tau}(D)u_{\ell,\tau})\,.$$

Since $\tilde\psi_{\ell,\tau}(D)u_{\ell,\tau} = u_{\ell,\tau}$, we have $L_0 + L_1 = L$. Note also that by definition of the cones $\check{\mathbf{C}}_\pm$, if $|n-m| > 5$ or $\upsilon = +$ and $\sigma = -$ then for $i = 0$ and $i = 1$:

$$\psi_{m,\upsilon}(D)(L_i u)_{(n,\sigma)} = 0\,. \tag{36}$$

By Proposition 3.3, it is enough to show that there is C, which does not depend on T and g, so that

$$\|L_0 u\|_{C^{p,q}_*(K)} < C\max\{\|T\|_+^p, \|T\|_-^q\}\|g\|_{L^\infty}\|u\|_{C^{p,q}_*(K')}\,,$$

and that for each $0 < p' < p$ and $q' < q$ so that $p - q' < r-1$, there is $C(T,g)$ so that

$$\|L_1 u\|_{C^{p,q}_*(K)} < C(T,g)\|u\|_{C^{p',q'}_*(K')}\,.$$

The bound for L_0 is easy, like in the proof of Theorem 2.8: Notice that there is C so that, setting $c(+) = p$, $c(-) = q$,

$$\sum_{(\ell,\tau):(\ell,\tau)\hookrightarrow(n,\sigma)} 2^{c(\sigma)n - c(\tau)\ell} \le C\max\{\|T\|_+^p, \|T\|_-^q\},\ \forall (n,\sigma)\,, \tag{37}$$

and recall that $\sup_{(n,\sigma)}\int|\widehat{\psi}_{n,\sigma}(x)|dx < \infty$.

Consider next L_1. It is enough to prove that if $(\ell,\tau)\not\hookrightarrow(n,\sigma)$ then

$$\|\check\psi_{n,\sigma}(D)(L\,\tilde\psi_{\ell,\tau}(D)f)\|_{L^\infty} \le C(T,g)2^{-(r-1)\max\{n,\ell\}}\|f\|_{L^\infty}\,. \tag{38}$$

Indeed, setting $c'(+) = p'$, and $c'(-) = q'$, (38) and (36)imply that

$$\begin{aligned}
&\|L_1 u\|_{C^{p,q}_*(K)}\\
&\le \sup_{(m,\upsilon)}\sum_{(n,\sigma)}\sum_{(\ell,\tau):(\ell,\tau)\not\hookrightarrow(n,\sigma)} 2^{c(\upsilon)m}\|\psi_{m,\upsilon}(D)\check\psi_{n,\sigma}(D)(L\,\tilde\psi_{\ell,\tau}(D)u_{\ell,\tau})\|_{L^\infty}\\
&\le C(T,g)\cdot\sup_{(n,\sigma)}\left(\sum_{(\ell,\tau):(\ell,\tau)\not\hookrightarrow(n,\sigma)} 2^{c(\sigma)n - c'(\tau)\ell - (r-1)\max\{n,\ell\}}\right)\|u\|_{C^{p',q'}_*(K')}\,.
\end{aligned}$$

Then, since $p \le r-1$, $p - q' < r-1$, and thus $-q < r-1$, we see from the definition of $\not\hookrightarrow$ that

$$\sup_{(n,\sigma)}\left(\sum_{(\ell,\tau):(\ell,\tau)\not\hookrightarrow(n,\sigma)} 2^{c(\sigma)n - c'(\tau)\ell - (r-1)\max\{n,\ell\}}\right) < \infty\,. \tag{39}$$

(Note that $p - q \le r-1$ is not enough to guarantee the above bound because of the case $(\tau,\sigma) = (-,+)$.)

To show (38), extend T to $\mathbb{R}^d$ as in the proof of Theorem 2.8, and rewrite

$$(\psi_{n,\sigma}(D)(L\,\tilde\psi_{\ell,\tau}(D)f)(x) = (2\pi)^{-2d}\int V^{\ell,\tau}_{n,\sigma}(x,y)\cdot f\circ T(y)|\det DT(y)|dy\,,$$

where

$$V^{\ell,\tau}_{n,\sigma}(x,y) = \int e^{i(x-w)\xi+i(T(w)-T(y))\eta}g(w)\psi_{n,\sigma}(\xi)\tilde\psi_{\ell,\tau}(\eta)dwd\xi d\eta\,. \tag{40}$$

Recall b from (17). If we show

$$|V^{\ell,\tau}_{n,\sigma}(x,y)| \le C(T,g)2^{-(r-1)\max\{n,\ell\}}\cdot 2^{d\min\{n,\ell\}}b(2^{\min\{n,\ell\}}(x-y))\,, \tag{41}$$

for some $C(T,g)>0$ and all $(\ell,\tau)\not\hookrightarrow(n,\sigma)$ then (38) follows from Young's inequality, as in the expanding case from Section 2.3.

Finally, the proof of (41) is *exactly* the same as the proof of (18), up to using the change of variable $v=2^\ell(T(w)-T(y))$ instead of $u=2^n(x-w)$ in (26) if $\ell>n$. □

Exercise 3.6. Consider now two pairs of cones $\mathbf{C}_\pm$ and $\mathbf{C}'_\pm$, and construct, for each p and q, two spaces $C^{p,q}_*(K)$ and $(C'_*)^{p,q}(K')$ (by choosing $\varphi_\pm$ and $\varphi'_\pm$ as above). Introduce a more general condition for T:

$$DT^{tr}_x(\mathbb{R}^d\setminus\text{interior}\,(\mathbf{C}'_+))\subset\text{interior}\,(\mathbf{C}_-)\cup\{0\}\quad\text{for all } x\in W. \tag{42}$$

Put

$$\|T\|_+ = \sup_{x\in K}\sup_{0\neq DT^{tr}_x(\xi)\notin\mathbf{C}_-}\frac{\|DT^{tr}_x(\xi)\|}{\|\xi\|}\quad(\text{the “weakest contraction”})\,,$$

$$\|T\|_- = \inf_{x\in K}\inf_{0\neq\xi\notin\mathbf{C}'_+}\frac{\|DT^{tr}_x(\xi)\|}{\|\xi\|}\quad(\text{the “weakest expansion”})\,.$$

Check that a small modification of the proof of Theorem 3.4 gives:

Theorem 3.7. *For any $q<0<p$ such that $p-q<r-1$ there exist a constant C so that for each C^r diffeomorphism T and C^{r-1} function g, assuming (42), there exists a compact operator $L_1:(C')^{p,q}_*(K')\to C^{p,q}_*(K)$ such that for any $u\in(C'_*)^{p,q}(K')$*

$$\|Lu-L_1u\|_{C^{p,q}_*(K)}\le C\|g\|_{L^\infty}\cdot\max\{\|T\|^p_+,\|T\|^q_-\}\|u\|_{(C'_*)^{p,q}(K')}\,.$$

(See [4]. Note that the above result may be applied to T the identity map, i.e. L the operator of multiplication by a function g, up to taking suitable pairs of cones.)

3.3. **Transfer operators for Anosov diffeomorphisms.** In this section we prove Theorem 1.2 by reducing to the model of Subsections 3.1 and 3.2.

Proof of Theorem 1.2. We first define the space $C^{p,q}_*(\mathcal{T})$ by using local charts to patch the anisotropic spaces from Subsection 3.1. Fix a finite system of C^∞ local charts $\{(V_j, \kappa_j)\}_{j=1}^J$ that cover X, and a finite system of pairs of closed cones [5] $\{(\mathbf{C}_{j,+}, \mathbf{C}_{j,-})\}_{j=1}^J$ in $\mathbb{R}^d$ with the properties that for all $1 \le j, k \le J$:

(a) The closure of $\kappa_j(V_j)$ is a compact subset of $\mathbb{R}^d$.
(b) $\mathbf{C}_{j,+} \cap \mathbf{C}_{j,-} = \{0\}$.
(c) If $x \in V_j$, the cones $(D\kappa_j)^*(\mathbf{C}_{j,+})$ and $(D\kappa_j)^*(\mathbf{C}_{j,-})$ in the cotangent space contain the normal subspaces of $E^s(x)$ and $E^u(x)$, respectively.
(d) If $\mathcal{T}^{-1}(V_k) \cap V_j \neq \emptyset$, setting $U_{jk} = \kappa_j(\mathcal{T}^{-1}(V_k) \cap V_j)$, the map in charts $T_{jk} := \kappa_k \circ \mathcal{T} \circ \kappa_j^{-1} : U_{jk} \to \mathbb{R}^d$ enjoys the cone-hyperbolicity condition:

$$D\mathcal{T}^{tr}_{jk,x}(\mathbb{R}^d \setminus \text{interior } (\mathbf{C}_{k,+})) \subset \text{interior}(\mathbf{C}_{j,-}) \cup \{0\}, \quad \forall x \in U_{jk}. \tag{43}$$

The fact that such systems of cones exist is standard for Anosov maps, see e.g. [8].

Choose C^∞ functions $\varphi_j^+, \varphi_j^- : \mathbf{S}^{d-1} \to [0,1]$ for $1 \le j \le J$ which satisfy (29) with $\mathbf{C}_\pm = \mathbf{C}_{j,\pm}$, as in Section 3.1. This defines for each j a local space denoted $C^{p,q,j}_*$. Choose finally a C^∞ partition of the unity $\{\phi_j\}$ subordinate to the covering $\{V_j\}_{j=1}^J$, that is, the support of each $\phi_j : X \to [0,1]$ is contained in V_j, and we have $\sum_{j=1}^J \phi_j \equiv 1$ on X.

Definition. We define the Banach spaces $C^{p,q}_*(\mathcal{T})$ to be the completion of $C^\infty(X)$ for the norm

$$\|u\|_{C^{p,q}_*(\mathcal{T})} := \max_{1 \le j \le J} \|(\phi_j \cdot u) \circ \kappa_j^{-1}\|_{C^{p,q,j}_*} .$$

By definition, $C^{p,q}_*(\mathcal{T})$ contains $C^s(X)$ for $s > p$. If $0 \le p' < p$ and $q' < q$, Lemma 3.3 and a finite diagonal argument over $\{1, \dots, J\}$, imply that the inclusion $C^{p,q}_*(\mathcal{T}) \subset C^{p',q'}_*(\mathcal{T})$ is compact.

For $m \ge 1$ and j, k so that

$$V_{m,jk} := \mathcal{T}^{-m}(V_k) \cap V_j \neq \emptyset ,$$

we may consider the map in charts

$$T^m_{jk} = \kappa_k \circ \mathcal{T}^m \circ \kappa_j^{-1} : \kappa_j(V_{m,jk}) \to \mathbb{R}^d .$$

[5] We regard $\mathbf{C}_{j,\pm}$ as constant cone fields in the *cotangent* bundle $T^*\mathbb{R}^d$.

Note that (43) implies that

$$(DT^m_{jk,x})^{tr}(\mathbb{R}^d \setminus \text{interior }(\mathbf{C}_{k,+})) \subset \text{interior}(\mathbf{C}_{j,-}) \cup \{0\}\,,\ \ \forall x \in \kappa_j(V_{m,jk})\,.$$

Set

$$R_m = \max_{j,k} \sup_{x \in \kappa_j(V_{m,jk})} |g^{(m)} \circ \kappa_j^{-1}(x)| \cdot \max\{\|T^m_{jk}\|_+^p, \|T^m_{jk}\|_-^q\}\,,$$

where

$$\|T^m_{jk}\|_+ = \sup_{x \in \kappa_j(V_{m,jk})} \sup \left\{ \frac{\|(DT^m_{jk})^{tr}_x(\xi)\|}{\|\xi\|}\ ;\ 0 \neq (DT^m_{jk})^{tr}_x(\xi) \notin \mathbf{C}_{j,-} \right\}\,,$$

and

$$\|T^m_{jk}\|_- = \inf_{x \in \kappa_j(V_{m,jk})} \inf \left\{ \frac{\|(DT^m_{jk})^{tr}_x(\xi)\|}{\|\xi\|}\ ;\ 0 \neq \xi \notin \mathbf{C}_{k,+} \right\}\,.$$

A standard argument in uniformly hyperbolic dynamics gives

$$\lim_{m\to\infty} (\|T^m_{jk}\|_+)^{1/m} \le \lambda_s\,,$$

and

$$\lim_{m\to\infty} (\|T^m_{jk}\|_-)^{1/m} \ge \nu_u\,.$$

Therefore

$$\limsup_{m\to\infty} (R_m)^{1/m} \le R(\mathcal{T}, g) \max\{\lambda_s^p, \nu_u^q\}\,. \tag{44}$$

Since $p - q < r - 1$, we can apply Theorem 3.7 to obtain for each m

$$\|g^{(m)} \circ \kappa_j^{-1} \cdot u_{jk} \circ T^m_{jk} - L^{(m)}_{1,jk} u_{jk}\|_{C^{p,q,j}_*} \le C R_m \cdot \|u_{jk}\|_{C^{p,q,k}_*}\,.$$

with $L^{(m)}_{1,jk} : C^{p',q',k}_* \to C^{p,q,j}_*$ compact. This implies the claimed upper bound for the essential spectral radius of $\mathcal{L}_{\mathcal{T},g}$. □

Remark 3.8. Though it is not explicit in our notation, choosing a different system of local charts, a different partition of unity, or a different set of cones or functions $\varphi_\pm$, does not a priori give rise to equivalent norms. This is a little unpleasant, but does not cause problems.

Appendix A. Theorem 2.8 when both T and g are C^r

Proof. We only need to adapt the estimate (12) on L_1 to the case when g is C^r and $r - 1 < p \le r$, for $r > 1$, for some $0 < p' < p$. Recall V^ℓ_n from (16) and b from (17). We shall show

$$|V^\ell_n(x,y)| \le C(T,g) 2^{-r\max\{n,\ell\}} \cdot 2^{(d+1)\min\{n,\ell\}} b(2^{\min\{n,\ell\}}(x-y)), \tag{45}$$

for some $C(T,g) > 0$ and all $\ell \not\hookrightarrow n$.

Exercise A.1. Show that (45) combined with

$$\sup_n \left(\sum_{\ell:\ell \not\to n} 2^{pn-p'\ell+\min\{n,\ell\}-r\max\{n,\ell\}} \right) < \infty \,,$$

gives the claim. (Recall footnote 4 and take $p' > p$ very close to p.)

Define for each y a C^r function:

$$A_y(w) = T(w) - T(y) - DT(y)(w-y)\,.$$

We may rewrite (16) as

$$V_n^\ell(x,y) = \int e^{i(x-w)\xi + iDT(y)(w-y)\eta}\big(e^{iA_y(w)\eta} g(w)\big)\psi_n(\xi)\tilde\psi_\ell(\eta)dwd\xi d\eta\,.$$

Integrating (16) by parts once on w, we obtain

$$V_n^\ell(x,y) = \int e^{i(x-w)\xi + i(T(w)-T(y))\eta}\check F(\xi,\eta,w)\psi_n(\xi)\tilde\psi_\ell(\eta)dwd\xi d\eta\,, \tag{46}$$

where $\check F(\xi,\eta,w)$ is a C^{r-1} function in w which is C^∞ in the variables ξ and η. (We used properties of the derivative of an exponential to "reconstruct" $e^{i(T(w)-T(y))\eta}$.) Then, integrate (46) $[r]-1$ times by parts on w, giving

$$V_n^\ell(x,y) = \int e^{i(x-w)\xi + i(T(w)-T(y))\eta}\widetilde F(\xi,\eta,w)\psi_n(\xi)\tilde\psi_\ell(\eta)dwd\xi d\eta\,, \tag{47}$$

with $\widetilde F(\xi,\eta,w)$ a $C^{r-[r]}$ function in w which is C^∞ in the variables ξ and η. By (11), if $\psi_n(\xi)\cdot\tilde\psi_\ell(\eta)\neq 0$, then we have for all α and β

$$\|\partial_\xi^\alpha\partial_\eta^\beta \widetilde F\|_{C^{r-[r]}} \le C_{\alpha,\beta}(T,g)2^\ell 2^{-n|\alpha|-\ell|\beta|-[r]\max\{n,\ell\}}\,. \tag{48}$$

(The price we have to pay for the first integration by parts is the factor 2^ℓ. What we gained is $2^{-[r]\max\{n,\ell\}}$, with $[r]$ instead of $[r]-1$.) Then (48) implies (45), just like in Section 2.3 (recall that $\ell \le n$).

□

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