Spectrum and Statistical Properties of Chaotic Dynamics

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Abstract. We present new developments on the statistical properties of chaotic dynamical systems. We concentrate on the existence of an ergodic physical (SRB) invariant measure and its mixing properties, in particular decay of its correlation functions for smooth observables. In many cases, there is a connection (via the spectrum of a Ruelle–Perron–Frobenius transfer operator) with the analytic properties of a weighted dynamical zeta function, weighted dynamical Lefschetz function, or dynamical Ruelle–Fredholm determinant, built using the periodic orbit structure of the map.
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1. Introduction

We are concerned with the orbits of smooth iterated maps $f : M \to M$ on a (say, compact) Riemannian manifold, in situations where the dynamics is unpredictable because of sensitive dependence to initial conditions (related to positive Lyapunov exponents). We seek invariant ergodic Borel measures μ which describe the asymptotic behaviour of most or many points. Here, "most or many" is in the sense of the normalised Lebesgue measure.

To fix ideas, assume that f is a C^2 diffeomorphism. If μ is a *probability* measure, a desirable property is that for each $\varphi \in L^1(d\mu)$ there is a positive *Lebesgue* measure set of initial conditions x, with time averages $\frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k(x)$ converging to the space average $\int \varphi \, d\mu$. Even if μ is *ergodic,* this is more than what the Birkhoff theorem gives, and we call such an invariant (ergodic) probability measure a *physical measure.* Requiring absolute continuity with respect to Lebesgue is too strong, and a more appropriate sufficient condition to be a physical measure is often the *SRB* (Sinai–Ruelle–Bowen) property that the conditional measures along the unstable manifolds given by Pesin theory are absolutely continuous with respect to the measure induced by Lebesgue. (See, e.g., the efficient survey [104].)

If μ is a *mixing* physical or SRB probability measure, one would like to know how fast the *correlation function*

$$\rho_{\varphi\psi}(n) = \int \varphi \circ f^n \, \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \tag{1}$$

decays to zero for φ and ψ smooth (Hölder) *test functions* (or *observables*). Also, if μ is not volume itself, it is relevant to ask whether $(f^n)_*(\text{Lebesgue})$ converges to μ, and about the speed of this *convergence to equilibrium.* Such quantitative information on the *decay of correlations* may be used to prove a *central limit theorem* for smooth observables. (See [68, 97, 102] and references therein.)

Similar questions exist for continuous-time dynamics, but we shall completely leave aside Dolgopyat's [41, 42, 43] beautiful recent results on the speed of decay of correlations of Axiom A flows. (See [46] for an inspiring account. An alternative strategy is examined in [71].) Many of the techniques applied below to smooth attractors extend to repellors and saddles, or systems with holes, or other interesting (Gibbs or equilibrium) invariant measures, such as the measure of maximal entropy, but we do not delve into these topics. We do not venture into noncompact situations (including spatially extended systems such as chaotic weakly coupled map lattices, or symbolic situations with countably many states). Neither do we enter the very rich theory of hyperbolic and parabolic behaviour for holomorphic (polynomial, rational, etc.) dynamics. We completely neglect (Poincaré maps of) billiards, referring to [102, 33, 34] for recent results. We abstain from discussing the efficient numerical computation by Dellnitz et al. [39] of correlation spectrum via Ulam matrices. Finally, we do not take a "generic" approach to smooth dynamics on compact manifolds, referring to Palis [77] for a perhaps more holistic *Weltanschauung.*

Our aim is to give an overview of the constellation of recent and ongoing work on the decay of correlations associated to SRB invariant measures and the connection with *dynamical zeta functions or determinants.* In *Section 2* we recall "classical" results (a very subjective notion, appearing to coincide with "before I got my doctoral degree"), mentioning more modern proofs. In *Section 3,* we present some progress made during the last decade. The *final section* contains comments and questions. We refer, e.g., to the monographs [12, 97], or the surveys [104, 11, 10] for technical definitions and additional references. To the more physically inclined readers, we recommend Cvitanović's monograph [37] for many concrete applications (the Helium atom!) and bold insights on the periodic orbit structure of physically relevant chaotic dynamics.

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2. Classical results

2.1. Exponential mixing for uniformly hyperbolic systems

A diffeomorphism $f : M \to M$ is called *Axiom A* ([90], see also, e.g., [78]) if the periodic points are dense in the nonwandering set and the tangent bundle splits as $TM = E^u \oplus E^s$ with $\|Df^{-n}|_{E^u}\| \le C\lambda^{-n}$ and $\|Df^n|_{E^s}\| \le C\lambda^{-n}$, for some $\lambda > 1$, $C > 0$, and all $n \in \mathbf{Z}_+$. If $\Lambda_j \subset M$ is a (transitive) *attracting basic* set for a $C^{1+\epsilon}$ *Axiom A* diffeomorphism f, then by results of Sinai and Ruelle there is a unique ergodic physical probability invariant measure μ_j on Λ_j, which is also an *SRB measure.* If $f^{p_j}|_{\Lambda_{jk}}$ is topologically mixing, with $p_j \geq 1$ and $\Lambda_j = \sqcup_{k=1}^{p_j} \Lambda_{jk}$ from Smale's spectral decomposition, then work of Ruelle and Bowen shows that the *correlation function* for $(f^{p_j}|_{\Lambda_{jk}}, \mu_j)$ and Hölder observables *decays exponentially fast.* The first proofs used finite Markov partitions and a transfer operator associated to a one-sided (expanding) symbolic dynamics model. (This specific transfer operator is positive, and the relevant theorem deserves to be called a *Ruelle–Perron–Frobenius theorem.*) A more recent approach, exploiting Hilbert (projective) metrics to analyse the action of the transfer operator on Birkhoff cones of functions on M, was introduced by Liverani [67] and extended by Viana [97]. Yet another strategy, using a tower model (more precisely, estimating the asymptotics of return times to a suitable hyperbolic Cantor subset of Λ), was implemented by Young in two guises. She [102] first appealed to the spectral gap of a transfer operator on the tower, and then [103] adapted a probabilistic coupling method to the tower with recurrence bounds. The probabilistic coupling was recently carried out by Bressaud and Liverani [23] directly on the manifold (see also [44]).

Similar results (with simpler proofs) hold for $C^{1+\epsilon}$ (locally) *uniformly expanding* endomorphisms f. (I.e., assuming that $\|Df^n_x\| \geq C\lambda^n$ for some $C > 0$, $\lambda > 1$, all $x \in M$, and all $n \in \mathbf{Z}_+$.) In this case the physical probability measure is absolutely continuous with respect to Lebesgue, and the transfer operator $\mathcal{L}$ is defined on Hölder functions $\varphi : M \to \mathbf{C}$ by

$$\mathcal{L}\varphi(x) = \sum_{f(y)=x} \frac{\varphi(y)}{|\det Df(y)|} \, .$$

2.2. Exponential mixing for one-dimensional piecewise expanding maps

Here, $M = I \subset \mathbf{R}$ is just a compact interval, and f is assumed to be piecewise C^2 (or C^1, with inverse derivative of bounded variation). The easiest case is when there are finitely many intervals of monotonicity, i.e., finitely many singularities. Usually there does *not* exist a finite Markov partition, however: This is the main difference with previous settings. Assuming additionally that $|f'| > 1$ on the intervals of monotonicity, Lasota and Yorke showed that there are *finitely many absolutely continuous and ergodic* invariant probability measures μ_j (which are SRB and physical measures). Hofbauer's spectral decomposition gives $I = \bigsqcup_{\ell=1}^m W_\ell \sqcup \bigsqcup_{j=1}^k I_j$ with the W_ℓ wandering, $f|_{I_j}$ topologically transitive, and $I_j = \sqcup_{k=1}^{p_j} I_{jk}$ with each $(f^{p_j}|_{I_{jk}}, \mu_j)$ mixing. Using a spectral gap approach, Hofbauer and Keller [53] proved that the associated correlation functions *decay exponentially* fast for observables of bounded variation. More recently, the Birkhoff cone method was applied by Liverani to show the same result [69]. Finally, both the spectral gap and the probabilistic coupling methods on Young's tower [102, 103] also yield exponential decay of correlations. (See also the results from [16] inspired by the Milnor–Thurston theorem in Subsection 3.7.)

2.3. Dynamical zeta functions and dynamical Fredholm determinants

General references for dynamical zeta functions include the book [78] and the survey [10]. The *weighted dynamical zeta function* of a map $f : M \to M$ and a weight $g : M \to \mathbf{C}$ is formally defined by setting $g^{(n)}(x) = \prod_{k=0}^{n-1} g(f^k(x))$ and

$$\zeta_g(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix} f^n} g^{(n)}(x) \,. \tag{2}$$

Let f be a (transitive) $C^{1+\epsilon}$ Axiom A diffeomorphism, or expanding endomorphism, on a compact manifold M, and let $g : M \to \mathbf{C}$ be Hölder. Combining results of Pollicott [80], Ruelle [84], and Haydn [51], one gets that $\zeta_g(z)$ is analytic in the disc of radius $\exp(-P(\log|g|))$, where $P(\cdot)$ denotes topological pressure. It admits a meromorphic extension to the disc of radius $\theta^{-1/2}\exp(-P(\log|g|))$, where $0 < \theta < 1$ is related to the Hölder exponent of the invariant laminations and to the hyperbolicity factor $\lambda > 1$ of f. If g is positive, the poles of $\zeta_g(z)$ in this disc are in bijection with the poles of the Fourier transform of the correlation function (1) of the equilibrium measure μ of $\log g$ and Hölder φ, ψ in a strip.

In the case of interval dynamics, Keller and I [14] showed that the dynamical zeta function of a piecewise monotone interval map f with a generating partition, and a continuous weight g of bounded variation is analytic in the disc of radius $\exp(-P(\log|g|))$ and admits a meromorphic extension to a disc of inverse radius $\limsup_{n\to\infty} \sup |g^{(n)}|^{1/n}$. If g is positive, the poles of $\zeta_g(z)$ in this disc are related to the poles of the Fourier transform of the correlation function of the equilibrium measure of $\log g$ and observables of bounded variation.

The proof of both results is by studying the spectrum of a linear transfer operator $\mathcal{L}_g$ which (just as $\mathcal{L}$ in Subsection 2.1) is *not compact* but whose essential spectral radius can be estimated. For interval maps we simply have $\mathcal{L}_g\varphi(x) = \sum_{f(y)=x} g(y)\varphi(y)$. The poles of the zeta function are related to part of the discrete spectrum of the transfer operator and are sometimes called (Ruelle) *resonances,* I also like the phrase *correlation spectrum.*

The setting relevant to this note is $g = |\mathrm{det} Df|_{E^u}|$, respectively $g = 1/|f'|$. *Restricting* f to an attracting basic set Λ, respectively to a *transitive* subinterval Λ, the corresponding equilibrium measure is the physical (and SRB) ergodic measure. The associated zeta function $\zeta_g(z)$ (counting weighted periodic orbits in Λ) is meromorphic in a disc of radius > 1 with a simple pole at 1. If $f|_\Lambda$ is topologically mixing, then there are no other poles on the unit circle. The exponential decay of correlations for Hölder observables on a mixing attracting set Λ is thus reflected in the fact that the simple pole at 1 is the only singularity of the zeta function. More generally, the analytic properties of the full zeta function $\zeta_g(z)$ reflect the topological spectral decomposition: It is meromorphic in a disc of radius > 1, with a pole at $z = 1$ having the number of attracting transitive components (including the trivial sinks in the Axiom A case, on which $\mathrm{det} Df|_{E^u} = 1$) as multiplicity, the other poles of modulus one being roots of unity associated to the orders of mixing of the transitive components.

The *dynamical (Ruelle-Fredholm) determinant* of a differentiable map $f : M \to M$ and a weight $g : M \to \mathbf{C}$ is defined formally by

$$d_g(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix} f^n} \frac{g^{(n)}(x)}{|\det(1 - D_x f^{-n})|} . \tag{3}$$

In analytic expanding settings, the transfer operator $\mathcal{L}_g$ associated to the contracting inverse branches of f is compact. Applying Grothendieck's theory of Fredholm determinants for nuclear operators, Ruelle proved [83] that if g is real analytic and f is a real analytic expanding endomorphism, or if f a real analytic Anosov diffeomorphism *with real analytic stable and unstable foliations* (a very nongeneric assumption), then $d_g(z)$ is an entire function of finite order, whose zeroes are related to eigenvalues of a transfer operator. This result also holds in *piecewise* expanding, piecewise analytic situations, *when the partition is Markov,* e.g., if f is the Gauss map and $g = 1/|f'|$ [72]. (Dynamical Fredholm determinants and transfer operators are also useful in quantum chaos, at the triple intersection of number theory, geometry, and group theory. Mayer [73] wrote very readable discussion on the Selberg zeta function and transfer operators applied to quantum chaos on the modular surface. See also [32] and references therein.)

Ruelle [86] later extended the Grothendieck-Fredholm theory to nonanalytic situations, showing in particular that if f is a C^r and λ-expanding endomorphism ($\lambda > 1$) on a compact manifold, and g is a C^r function (with $r \geq 1$), then, setting $g = 1/|\det Df|$ to fix ideas, $d_g(z)$ is an analytic function in the disc of radius λ^r, where its zeroes are inverse eigenvalues of the (noncompact) transfer operator $\mathcal{L}_g$.

3. Recent developments

3.1. Low-dimensional nonuniformly hyperbolic dynamics: Unimodal maps, Hénon and Hénon-like maps; Multidimensional nonuniform hyperbolicity

In dimension one, Keller–Nowicki [63] and Young [101] independently proved that a class of (nonrenormalisable) *good unimodal maps* containing the logistic map $x \mapsto a - x^2$ (for a positive measure-set of *good* parameters a) enjoys *exponential decay of correlations* for observables of bounded variation and their unique absolutely continuous and ergodic (thus, SRB and physical) invariant probability measure. Existence of this measure (and positive Lebesgue measure of the good parameter set) had been obtained by Jakobson. Exponential decay also holds for Hölder test functions [103].

Keller and Nowicki [63] also showed that the zeta function $\zeta_{1/|f'|}(z)$ (2) of such a good unimodal map is meromorphic in a disc of radius > 1 with a simple pole at $z = 1$ as only singularity. Nowicki and Sands [76] list conditions (in particular the Collet-Eckmann condition $|(f^n)'(f(0))| \geq C\lambda^n$, for $n \in \mathbf{Z}_+$, with $\lambda > 1$) equivalent to exponential decay of correlations for the absolutely continuous invariant measure *(a.c.i.m).*

Bruin proved that some unimodal maps with weaker (inverse-summable) growth of the postcritical derivative also possess an ergodic absolutely continuous invariant probability measure. Using Young's probabilistic coupling machinery [103], Bruin, Luzzatto, and Strien [24] estimated the rates of decay of correlations in such nonexponential situations.

Twenty years after Hénon studied numerically the surface diffeomorphism

$$(x, y) \mapsto (1 - ax^2 + y, bx)\,, \tag{4}$$

for $a = 1.4$ and $b = 0.3$, Benedicks and Young [21] showed that the correlations functions of the SRB measure of the *Hénon maps with good parameters* (a, b) decayed exponentially (for Hölder observables). In between, a positive two-dimensional Lebesgue set of good parameters had been exhibited by Benedicks and Carleson, who aimed at a positive Lyapunov exponent, and the SRB measure had been constructed by Benedicks and Young [20] for good parameters. Existence of an SRB measure and exponential decay of its correlations has recently been obtained by Wang and Young [99] for a more general class of two-dimensional attractors (*(generalised) solenoidal attractors)* "close" to a one-dimensional map with nonuniform hyperbolicity and arising from generic homoclinic bifurcations.

Viana [96] showed that an *open set of nonuniformly expanding endomorphisms* of the cylinder close to

$$(x, \theta) \mapsto (a - x^2 + \delta \sin\theta, D\theta (\mathrm{mod}\ 1))\,, \qquad \mathrm{D} \in \mathbf{Z}_+\,, \quad \mathrm{D} \gg 16\,, \tag{5}$$

(with a a good parameter for the logistic map, and $\delta > 0$ small) possess *two* positive Lyapunov exponents. Alves [2] proved that they admit an absolutely continuous (physical, SRB) invariant probability measure which is mixing. (Viana [96] also studied the Lyapunov exponents of higher-dimensional invertible systems with multidimensional nonuniform hyperbolicity, replacing in particular $a - x^2$ by a Hénon map and $D\theta$ by a hyperbolic solenoid. To our knowledge the existence of an SRB measure has not been obtained yet for these systems.)

3.2. Piecewise expanding maps in arbitrary dimensions

Let f be an endomorphism of a compact manifold M partitioned into subsets on which the restriction of f is a $C^{1+\epsilon}$ uniformly expanding map. If the partition is not Markov (i.e., the image of each piece is not exactly a union of pieces), new problems appear in higher dimensions due to the accumulation boundary discontinuities under iterations. (See the introduction of [27] for references to piecewise C^r – arbitrary r – pathological surface transformations due to Tsujii, and piecewise affine pathological examples of Buzzi.) Another, more technical, difficulty consists in finding a convenient higher-dimensional version of bounded variation.

For higher-dimensional piecewise expanding maps, the existence of absolutely continuous invariant probability measure had only been proved in specific cases such as β-transformations, certain piecewise C^2 surface transformations (studied by Keller [60] over 20 years ago), or piecewise analytic and expanding surface endomorphisms (independently studied by Tsujii [94] and Buzzi [27], who also obtained exponential decay of correlations for Lipschitz observables). Around 1998, *three similar conditions* expressing that "the expansion must be stronger than the asymptotic growth of singularity accumulation" and *guaranteeing existence of an a.c.i. (probability) m.* were discovered independently (each of the three conditions is generic, i.e., open and dense in a natural class of piecewise expanding maps):

Saussol [91] introduced a weak sufficient (local) condition guaranteeing existence of the a.c.i. probability and its exponential mixing for Hölder observables. In dimension $N \geq 2$, when there are finitely many open domains of injectivity and their boundaries are piecewise C^1 codimension-one embedded compact submanifolds, the sufficient condition reduces to

$$s(f)^{\epsilon} + \frac{4s(f)}{1-s(f)} Y(f) \frac{\gamma_{N-1}}{\gamma_N} < 1\,,$$

where $\gamma_N = \pi^{N/2}/(N/2)!$ is the volume of the N-dimensional ball in $\mathbf{R}^N$, $s(f) < 1$ is the largest contraction coefficient of the inverse branches, $0 < \epsilon \leq 1$ is the Hölder exponent of the Jacobian of the inverse branches, and $Y(f) \geq 1$ is the maximal number of boundaries of the partition having a common intersection point. Sausol's result holds more generally, for countable dynamical partitions, with possibly fractal boundary we refer to [91] for the (technical) statement of the general sufficient condition (G).

Simultaneously, Buzzi [25], introduced another (local) condition sufficient to guarantee existence of a finite a.c.i.m.: Letting $\mathcal{Z}$ denote the finite partition into bounded open subsets of $\mathbf{R}^N$ on which $f : X \to X$ is uniformly expanding and $C^{1+\epsilon}$ (the approach works under weaker smoothness assumptions on the Jacobian), and writing $P(A, f)$ for the *topological pressure* of a subset $A \subset X$ and the weight $-\log|\det Df|$, the assumption is that the pressure of the boundary is smaller than the total pressure:

$$P(\partial\mathcal{Z}, f) < P(X, f)\,. \tag{6}$$

Assuming additionally that the discontinuities are in general position, in the sense that for each $N+1$-tuple $(Z_{k_i}, i = 1, \ldots, N+1)$ with $Z_{k_i} \in \mathcal{Z}$ the intersection $\cap_{i=1}^{N+1}(f|_{Z_{k_{i-1}}} \circ \ldots \circ f|_{Z_{k_0}})^{-1}\partial Z_{k_i}$ is empty (this "transversality condition" in fact implies (6)), Buzzi and Maume [29] then proved exponential decay of the corresponding correlations when f is $C^{1+\epsilon}$ and slower rates of mixing when the Jacobian is allowed a (summable) modulus of continuity weaker than Hölder. The papers [25] and [29] also give results for other equilibrium states.

Cowieson [36] proved existence of an absolutely continuous invariant measure (and stochastic stability) for piecewise expanding, piecewise C^2 maps $f : M \to M$ satisfying another transversality condition. Restricting to a finite partition $\mathcal{Z}$ into C^2 submanifolds with corners his main assumption is (see [60] for surface maps)

$$\exists n \geq 1 \text{ with } \lambda(f^{n+1}) > F(\mathcal{Z}_n)\,,$$

where $\lambda(f) > 1$ is the minimal local expansion coefficient, $\mathcal{Z}_n$ is the partition into domains of injectivity for f^n, and $F(\Pi)$ is the *cut index* of a partition Π, i.e., the maximum number of boundary pieces of the partition (which are not on the boundary of M) with a common intersection.

Buzzi and Keller [28] successfully carried through the analysis of the dynamical zeta function $\zeta_{1/|\det Df|}(z)$ (2) when f is a piecewise *affine and expanding surface* transformation (Jérôme Buzzi informed us that this can be extended to arbitrary dimensions by using Tsujii's [95] work). They prove that the zeta function is analytic in the open unit disc and meromorphic in a disc of larger radius, where its poles are the inverse eigenvalues of the associated transfer operator acting on functions of higher-dimensional bounded variation. In particular, f is topologically mixing if and only if the zeta function has a simple pole at 1 as only singularity in the larger disc. See also [75] and references therein.

3.3. Partially hyperbolic diffeomorphisms

Partially hyperbolic diffeomorphisms naturally generalise uniformly hyperbolic diffeomorphisms (see e.g. Sections 7–8 of [98] for an overview, including references to fundamental work by Brin–Pesin and Pesin–Sinai, as well as links with the theory of robust transitivity). The tangent bundle TM is assumed to split as either $E^c \oplus E^u$, with E^u uniformly expanding and dominating the expansion in E^c (the central bundle), or $E^s \oplus E^c$, with E^s uniformly contracting and dominating E^c. This includes time-one maps of hyperbolic flows, but also some maps without zero Lyapunov exponents. There does not seem to be a general description of the statistical properties of C^2 partially hyperbolic diffeomorphisms. If the central direction is *mostly contracting,* Bonatti–Viana [22] showed that there are finitely many SRB measures, and Castro [31] and Dolgopyat [44] independently showed, under mild additional technical assumptions, that correlation functions (for Hölder observables) decay *exponentially fast.* When the central direction is mostly expanding, Alves–Bonatti–Viana [3] proved that there exist SRB measures (and finitely many such measures if the Lyapunov exponents are bounded away from zero).

Skew-product constructions yielding partially hyperbolic diffeomorphisms are given by compact group extensions over a mixing uniformly hyperbolic system. The SRB measure is available for free, and in the volume preserving case mixing follows from work of Brin. Dolgopyat [45] combined key results of Brin and Katok on partially hyperbolic skew products with information gained from his previous study of hyperbolic flows [41, 42, 43] to prove that a diophantine condition on the Brin group implies rapid (i.e., faster than any polynomial) decay of correlations for C^∞ observables. The speed of mixing is not necessarily exponential for Hölder observables. These examples may not carry any periodic orbits.

3.4. Almost hyperbolic maps (neutral periodic points)

Consider the following "almost expanding" one-dimensional $C^{1+\epsilon}$ interval map: Assume that $I = [0,1]$ is partitioned into two subintervals which are mapped injectively by f onto I, that $f' \geq \lambda > 1$ on the right interval, and that $f' > 1$ on the left interval, except at $0 = f(0)$. Suppose also that f possesses an expansion

$$f(x) = x + x^{1+\gamma}(1 + u(x)),$$

with $\gamma > 0$ and $u(x) \to 0$ as $x \to 0$. (Additional technical assumptions are in fact useful.) A piecewise affine model for such maps was introduced by Gaspard-Wang [50] and has been much studied (see Isola [57, 58], also for references to work of Prellberg, Mori, Lopes, and others on intermittent maps).

Returning to f itself, it has been known for quite a while (see Pianigiani [79], Thaler [92], and for recent extensions to the non-Markov case, Zweimüller [106]) that if $\gamma \geq 1$ then f does *not have* an absolutely continuous invariant probability measure. However there always exists an *infinite (σ-finite) absolutely continuous measure,* about whose density much is known. This is perhaps the simplest (piecewise) smooth example where the SRB measure μ is not a probability. The physical measure is just the Dirac mass at 0, which is also an equilibrium measure for $-\log|f'|$. More generally, if μ is not a probability but only (conservative, invariant and) σ-finite, the Birkhoff ergodic theorem is not relevant. One finds inspiration in the Hopf ergodic theorem instead [1], seeking an *(anomalous) scaling rate,* i.e., a sequence $a_n \to \infty$, with $a_n = o(n)$, and $\frac{1}{a_n}\sum_{k=0}^{n-1} \varphi \circ f^k(x) \to \int \varphi \, d\mu$, for all $\varphi \in L^1(d\mu)$ and a positive Lebesgue measure set of x. Few rigorous results are available in general although the scaling rates a_n have been established for the one-dimensional maps discussed here [30, 35, 93].

If f is additionally assumed to be piecewise real analytic, Rugh [89] exploited a *regularisation* idea to combine information from the Grothendieck-Fredholm theory and Fatou coordinates. (Leaving out the neutral periodic point somehow mirrors the algebraic identity for a dynamical determinant: $d(z) = (z - \mathcal{M} - \mathcal{N})^{-1} = (z-\mathcal{M})^{-1}(1-\mathcal{D}(z))^{-1}$, for $\mathcal{D}(z) = \mathcal{N}(z-\mathcal{M})^{-1}$ with $\mathcal{M}$ a bounded transfer operator associated to the neutral branch, and $\mathcal{N}$ a nuclear transfer operator associated to the expanding branch.) He unveiled the analytic structure of the dynamical determinant $d_{1/|f'|}(z)$, as well as the spectrum of the transfer operator $\mathcal{L}_{1/|f'|}$ acting on a suitable space of singular functions (there is no spectral gap).

For $0 < \gamma < 1$, there exists an absolutely continuous invariant probability μ, which is mixing. The speed of mixing is *polynomial.* Almost optimal upper bounds were obtained by Liverani–Saussol–Vaienti [70]. The exact upper bound

$$|\int \varphi \circ f^n \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu| \leq \frac{C_{\varphi\psi}}{n^{1/\gamma - 1}} \, ,$$

(for Hölder continuous test functions φ, ψ) was obtained by Young [103] and Hu [54] who used different methods (Hu showed it is optimal).

Higher-dimensional “almost expanding” or “almost hyperbolic” models have been studied in recent years. Assuming the existence of two invariant (expanding and contracting) cone fields, except on a finite set of neutral periodic points, Hu [55, 56] found necessary conditions for the existence of finite or σ-finite SRB measures, as well as polynomial estimates for the rate of decay of correlations when the SRB measure is a probability.

One of the motivations to investigate such maps comes from hydrodynamics, we recommend the paper on intermittency by Pomeau-Manneville [81]. Expanding maps with neutral fixed points also appear naturally when studying the Selberg zeta function associated to arbitrary cofinite Fuchsian groups or simply investigating the asymptotic growth of closed geodesics on manifolds with cusps [7]. Almost expanding maps also occur in number theory when evaluating the quality of diophantine approximations of various continued fraction algorithms (from the Farey map to more exotic multidimensional continued fractions, see e.g. [105]).

3.5. Small random perturbations and stochastic stability

It is natural to ask whether the statistical properties of a deterministic dynamical system are stable under perturbation by small random noise. (More generally, one may investigate random compositions of dynamical systems within a given class, see Kifer's survey [64] and references therein). We do not attempt to give general formal definitions here, and restrict to *independent identically distributed* (i.i.d.) noise where a Markov chain model χ^n is relevant: Assume we are given a probability space (Ω_0, ν), and consider the Bernoulli measure $\mathbf{P}_+(d\omega) = \prod_{i \in \mathbf{Z}_+} \nu(\omega_i)$ on $\Omega_+ = \Omega_0^{\mathbf{Z}_+}$, and for $\omega \in \Omega_+$ a "dynamical system valued random-variable" f_ω which only depends on ω_0. The transition probabilities are given by Prob $(\chi^{n+1} \in E \mid \chi^n = \{x\}) = \int \mathbf{1}_E(f_\omega(x))\, d\nu(\omega_0)$.

Letting σ_+ be the one-sided shift, we write $f^n_\omega = f_{\sigma_+^{n-1}\omega} \circ \cdots \circ f_\omega$. Mild assumptions suffice to ensure that the invariant measure for the Markov chain is the weak limit of the Birkhoff averages of dirac masses

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x)}, \quad n \to \infty,$$

for Lebesgue almost all initial x and $\mathbf{P}_+$-almost all random itineraries ω.

Young [100] showed weak convergence of the unique invariant probability of the Markov chain associated to small i.i.d. perturbations of transitive Axiom A attractors towards the SRB measure.

For the *Markov chain* associated to small i.i.d. perturbations of *smooth uniformly expanding* or one-dimensional mixing piecewise uniformly expanding very strong properties hold (Baladi–Young [18]): L^1 stability of the density of absolutely continuous invariant probability measures, and stability of the correlation spectrum, in particular the exponential rates of decay of the (averaged) correlation function of the Markov chain. Cowieson [36] extended the L^1 stability to higher-dimensional piecewise expanding dynamics. See also [62] for recent improvements of relevant spectral stability results by Keller and Liverani.

For *nonuniformly expanding dynamics,* such as good unimodal maps, results almost as strong, including L^1 stability of the density and some spectral stability, were obtained by Baladi–Viana [17]. (Weaker stability had been previously proved by Katok–Kifer, Collet, and Benedicks–Young in similar settings.) Benedicks and Viana [98] proved stochastic stability (weak convergence of the Markov chain invariant measure towards the SRB measure) of good Hénon maps.

Instead of considering the averaged behaviour given by a Markov chain, one may investigate *almost sure* random behaviour, i.e., consider the *two-sided*

Bernoulli space $(\mathbf{P}, \Omega)$ and study invariant measures of the form $\mu_\omega(dx)P(d\omega)$ of the random skew product $(x, \omega) \mapsto (f_\omega(x), \sigma\omega)$, where σ is the two-sided shift. In good situations one expects the disintegrations $\mu_\omega = (f_{\sigma^{-1}\omega})_* \mu_{\sigma^{-1}\omega}$ to be absolutely continuous or SRB quasi-invariant measures. It is also natural to conjecture

$$\lim_{n\to\infty} (f^n_{\sigma^{-n}\omega})_*(\text{Lebesgue}) = \mu_\omega \, , \tag{7}$$

for almost all ω. In uniformly expanding cases, Baladi–Kondah–Schmitt (see [12]) proved that the convergence in (7) takes place exponentially fast uniformly for all ω, that the μ_ω have Hölder densities which all converge to the invariant density as the noise level goes to zero and that the corresponding "random" correlations for Hölder observables decay exponentially (with uniform rate in the noise level).

In a Lasota-Yorke random context (one-dimensional and some particular multidimensional, piecewise monotone non Markov, expanding in average), Buzzi [26], who had proved existence of quasi-invariant absolutely continuous μ_ω, obtained almost sure exponential speed of convergence in (7), and exponential decay of the random correlation functions for Lipschitz observables, sometimes in the absence of exponential decay of averaged correlations. (His setting consists in random compositions of maps which are not necessarily close to a given map, and the result does not have the flavour of stochastic stability.)

Investigating the almost sure (as opposed to the averaged, Markov chain) behaviour of small random perturbations of good unimodal maps involves difficulties similar to those apearing when dealing with the deterministic skew product (5). With Benedicks and Maume-Deschamps [13], we consider a good unimodal map f (satisfying Benedicks–Carleson type assumptions), and for small $\epsilon > 0$ a probability ν_ϵ on $\Omega_0 = [-\epsilon, \epsilon]$, setting $f_\omega(x) = f(x) + \omega_0$. Assuming e.g. that ν_ϵ is absolutely continuous with a density bounded uniformly in ϵ, we show that the convergence to equilibrium (7) almost surely takes place at least stretched exponentially fast:

Theorem 3.1. (Almost sure rates of mixing) *For $\mathbf{P} = \prod_{\mathbf{Z}} \nu_\epsilon$ almost all ω, the quasi-invariant measure $\mu_\omega(dx)$ is absolutely continuous. For each small enough $\epsilon > 0$ there are $C > 1$ and a random variable C_ω with $\mathbf{P}(\omega | C_\omega > K) \le \frac{C}{K^u}$ (for some $u > 1$, independent of ϵ), and such that for $\mathbf{P}$-almost all ω*

$$|(f^n_{\sigma^{-n}\omega})_*(\text{Lebesgue}) - \mu_\omega| \le C_\omega \exp(-n^{1/16}/C) \, , \forall n \in \mathbf{Z}_+ \, . \tag{8}$$

No lower bounds are known, but analogous almost sure upper bounds hold for the *random correlations* of the μ_ω's and Lipschitz observables. The above theorem is *not* about stochastic stability: Our estimates on C_ω and C blow up as $\epsilon \to 0$.

See Section 2.7 in [12] for relations between the Lyapunov exponents of the random transfer operators, the decay of correlations, and random zeta functions.

3.6. Dynamical Fredholm determinants for hyperbolic dynamics

What makes transfer operators "nice" (compact, or at least quasi compact) is that they are basically compositions by contractions, on smooth function spaces. Until the Ph.D. thesis of Rugh (see [88] and references therein) dynamical zeta functions for smooth hyperbolic dynamics were studied mainly through an expanding system (with contracting inverse branches) obtained by quotienting out along stable manifolds. This is the reason behind the very strong assumption of analyticity of the foliations required in Ruelle's [83] theorem from Subsection 2.3. In a two-dimensional analytic hyperbolic setting, Rugh introduced *pinning coordinates,* i.e., for each $n \in \mathbf{Z}_+$ two parametrised real analytic contractions φ_x^n and ψ_z^n, so that

$$f^n(x, \varphi_x^n(z)) = (\psi_z^n(x), z) \,. \tag{9}$$

Showing nuclearity of an associated transfer operator, he proved that if f is an *analytic Axiom A* diffeomorphism on a compact *surface,* the determinant

$$d_{SRB}(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix} f^n} \frac{1}{|\det(D_x f^n - 1)|} \tag{10}$$

is an entire function. (The dynamical *foliations* of f *need not be smoother* than Hölder.) Despite our optimistic notation, the relation of the zeroes of the determinant $d_{SRB}(z)$ with the correlation of the SRB measure has not been completely elucidated (see [87], [49]). Introducing more conceptual tools, Fried [48] proved that $d_{SRB}(z)$ is meromorphic in the whole complex plane in any dimensions. This applies to flows, proving a conjecture of Smale [90].

In a remarkable paper which deserves more attention, Kitaev [66] performed the breakthrough step from analyticity to *finite differentiability,* showing that if f is a C^r *Anosov* diffeomorphism on a compact manifold, with hyperbolicity factor $\lambda > 1$, then $d_{SRB}(z)$ defines an analytic function in the disc of radius $\lambda^{r/2}$. (The "loss" of half of the differentiability mirrors a similar phenomenon in symbolic dynamics, see [78].) Fried [49] has recently announced a spectral interpretation of the zeroes of $d_{SRB}(z)$ in this framework.

3.7. Sharp determinants and dynamical Lefschetz functions in dimension one

For continuous interval maps, assuming only existence of a finite partition into monotonicity intervals, Milnor and Thurston [74] defined a *negative zeta function* using the (finite) set $\mathrm{Fix}^- f^n$ of points with $f^n(x) = x$ such that f^n is decreasing in a neighbourhood of x, and putting $\zeta_-(z) = \exp \sum_{n \in \mathbf{Z}_+} \frac{z^n}{n} 2\#\mathrm{Fix}^- f^n$. They showed that $\zeta_-(z)$ is the determinant of a finite matrix (with coefficients power series), the *kneading matrix* $D(z)$. (In fact, as was explained to us long ago by Jack Milnor, $\zeta_-(z)$ is essentially a Lefschetz zeta function.)

The desire, on the one hand to introduce weights of bounded variation (such as $g = 1/|f'|$) in the negative zeta function, and on the other to understand the link between the zeta function, the kneading matrix, and the transfer operator, led to a series of papers by Ruelle and/or myself ([16] and references therein). In the weighted case, a kneading operator $\mathcal{D}_g(z) = z\mathcal{N}_g(1 - z\mathcal{L}_g)^{-1} S$ (which has a bounded kernel) replaces the kneading matrix: The understanding emerged that the mechanism behind the Milnor–Thurston result is (again, see Subsection 3.4) *regularisation.* A weighted *sharp zeta function* was introduced (where sums of integrals $\int \mathrm{sgn}(f_{\mathrm{br}}^{-1} x - x)/2 \, dg(x)$, with sgn the sign function, and

f_{br}^{-1} an inverse branch, replace the usual sums over fixed points). The equality $\zeta_g^{\#}(z) = \det(1+\mathcal{D}_g(z))$ was obtained and used to relate the poles of $\zeta_g^{\#}(z)$ and the discrete eigenvalues of the transfer operator $\mathcal{L}_g$. These results use technical assumptions, in particular the weight g should vanish on the boundary, but this can be weakened [82]. Just like the Milnor–Thurston theorem, they are true *also in the presence of uncountably many periodic orbits.* Integrating by parts, if the periodic points are isolated, the negative zeta function and the sharp zeta function can be viewed as counting all periodic orbits with *Lefschetz signs.* In the present non necessarily Markov context even the unweighted (Milnor-Thurston) Lefschetz zeta function is not always rational. Allowing weights is perhaps a way of revamping what Smale [90] called the "false" zeta function.

We refer to the survey [10] or the original papers for technical details. See also [15] for a partial extension of this analysis to one-dimensional complex dynamics and Baillif's article [8] for kneading determinants for tree maps. Alves and Sousa-Ramos [4, 5] have yet another, more functorial, approach to the unweighted Milnor–Thurston theorem for both interval and tree endomorphisms.

4. Closing remarks

4.1. Low dimensions

Let us say that the *SRB zeta function of f has a gap* if there exist $\tau > 0$, and $r > 1$ so that for $t = 0$ and $t = \tau$ the zeta function

$$\zeta^{(t)}(z) = \exp \sum_{n\in\mathbf{Z}_+} \frac{z^n}{n} \sum_{x\in \mathrm{Fix} f^n} \frac{1}{|(f^n)'(x)|^{1-t}} \tag{11}$$

is meromorphic in the disc of radius r, with a simple pole at $q(t)$ as a unique singularity, and $q(0) \geq 1$. Let f be a nonrenormalisable unimodal interval map with negative Schwarzian deriative and at least two periodic orbits. Keller [61] proved last year that the SRB zeta function of f has a gap *if and only if* it admits an absolutely continuous invariant probability measure with exponential decay of correlations for Hölder observables. (The new implication is the "only if", this uses Nowicki–Sands [76] and Keller–Nowicki [63].)

The first question which comes to mind is whether one can weaken the SRB-gap property by taking $\tau = 0$ in Keller's theorem. Also, one wonders whether the topological spectral decomposition of multimodal maps with negative Schwarzian (assuming perhaps a Collet–Eckmann property to simplify) is readable in the multiplicity of the pole of the SRB (or unweighted) zeta function at 1 (or $\exp(\mathrm{h_{top}})$).

Recalling the results of Subsection 3.2 in particular [28], one can ask whether the SRB zeta function of a piecewise C^2 expanding map f (replacing $|f'|$ by $|\det Df|$ in (11) and assuming one of the three conditions in Subsection 3.2) has a gap (in the strong or weakened sense) if and only if f has an absolutely continuous invariant probability measure with correlations decaying exponentially fast (for Hölder observables). Is the presence of the gap generic?

Equivalence of the existence of a gap for the SRB zeta function and exponential mixing of an SRB measure is also open to our knowledge for the Hénon family (or more generally solenoidal attractors *à la* Wang-Young), or Viana's maps (5). (In view of the results by [13] mentioned above, one can expect the correlations of (5) to decay at least stretched exponentially.)

Away from the purely exponential setting, there are still few rigorous results relating the (nonpolar) singularity spectrum of dynamical zeta functions and the statistical properties of chaotic dynamics. For example, what can be said about the dynamical zeta function (11) associated to the multimodal maps studied by Bruin–Luzzatto–van Strien [24] which enjoy polynomial or stretched exponential mixing? (Véronique Maume suggested to investigate the abstract model from [103].) In the presence of neutral fixed points, numerical and heuristical work of Dahlqvist [38], and results of Isola [58] for countable Markov chains are inspiring rigorous enquiries in smooth settings such as the interval maps with neutral fixed points and the almost Anosov systems from Subsection 3.4. One would also like to know whether the analytic properties of dynamical zeta functions for analytic intermittent maps discovered by Rugh [89] may be linked to ergodic properties of an absolutely continuous invariant measure, also when it is only sigma-finite.

4.2. Higher dimensions: plethora and penury

The recent increase of our understanding of weighted dynamical zeta functions leaves me with the feeling that we have not pushed the theory to its limits yet. Can one generalise Keller's theorem and find a dynamical zeta function which describes statistical properties of (generic?) C^2 or piecewise C^2 dynamics on a compact manifold, going beyond hyperbolicity and assuming, e.g., existence of finitely many SRB measures without zero Lyapounov exponents (with sufficiently fast decay of correlations for Hölder observables)? If we seek this "mother of all" dynamical zeta functions, we must handle both overabundance and scantiness of periodic orbits.

We may be overwhelmed by a *plethora* of closed orbits even if we follow Artin–Mazur and count only *isolated* periodic points. Artin and Mazur [6] had showed that for a dense subset of C^r diffeomorphisms on a compact manifold the corresponding growth was at most exponential. However, recent work of Kaloshin (see [59] and references therein) reveals that superexponential growth of isolated periodic orbits occurs on a residual set of C^r diffeomorphisms, $r \geq 2$. Dima Dolgopyat recently pointed out to us that by combining Kaloshin's results with Castro's [31] one can construct a four-dimensional partially hyperbolic skew-product with superexponential growth of periodic orbits but having a single mixing SRB measure with exponential decay of correlations.

If zero Lyapunov exponents are present, we have to face *scarcity* even in a three-dimensional uniformly hyperbolic context: Just take the time-t map of a geodesic flow on a surface of constant negative curvature, with t not a common divisor of the orbit lengths, to find a volume preserving diffeomorphism with exponentially decaying correlations for Hölder observables, but not a single closed orbit. Other systems admitting an SRB measure with rapid mixing properties but not a single closed orbit are compact group extensions of hyperbolic systems as studied by Dolgopyat [45]. (Each of these discrete-time dynamical system without closed orbits has a single zero Lyapunov exponent and is imbedded in a hyperbolic flow with the "right" periodic orbit structure, the zeta function of which has the expected analytic properties [41, 42, 43]. Fayad's [47] volume-preserving polynomially mixing flow on the two-torus only has a single fixed point, but it has zero entropy, it is not known if the decay is summable.)

Focusing mainly on the first of these two difficulties, let us briefly explore what tools are available, seeking insights from the low-dimensional results. In our opinion, a dynamical zeta function or determinant is interesting only if its poles or zeroes reflect topological or ergodic information: The topological entropy (or more generally the topological pressure), e.g., does not coincide in general with asymptotic (weighted) growth of periodic orbits. (See [19] and [52] for a characterisation in terms of (n, α)- separated sets of ϵ-pseudo periodic orbits.) We have by now (re)learned sophisticated ways of counting the periodic orbits. This can mean introducing weights: The Jacobian along unstable directions, if one is concerned with SRB measures, but also analyticity-improving Jacobians in the denominators, as in the Ruelle-Fredholm determinants (3), useful in presence of strong smoothness. It can also signify rehabilitating Lefschetz-type signs: We saw in Subsection 3.7 that generalised Lefschetz signs attenuate superfluous growth for isolated periods and are a way to cleverly ignore the nonisolated orbits. Higher-dimensional version of the sharp traces of [16] could perhaps be devised (involving perhaps some kind of Leray-Schauder-Lefschetz index for compact continua of fixed points [40], as pointed out to us by André de Carvalho).

At this time we do not have candidates for a suitable class of smooth dynamics and its adapted dynamical zeta function. Let us finish by mentioning exciting ongoing work: In the case of a smooth diffeomorphism f satisfying a weak transversality assumption ensuring that periodic orbits are isolated, and a smooth weight g, Baillif [9] proves, using ideas of Kitaev [65], a higher-dimensional version of the Milnor–Thurston [74] theorem which should lead to a spectral interpretation of the zeroes of the following weighted Lefschetz zeta function ($L(\psi, x)$ denotes the usual Lefschetz index):

$$\exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix} f^n} g^{(n)}(x) L(f_x^{-n}, x) \, .$$

References

[1] J. Aaronson, *An Introduction to Infinite Ergodic Theory,* Mathematical Surveys and Monographs, Amer. Math. Soc., (Providence, RI), **1997.**

[2] J.F. Alves, *SRB measures for nonhyperbolic systems with multidimensional expansion,* Annales scient. Ecole normale sup. (4), **33** (2000), 1–32.

[3] J.F. Alves, C. Bonatti, and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding,* Preprint (1999), to appear Invent. Math.

[4] J.F. Alves and J. Sousa Ramos, *Kneading theory: a functorial approach,* Comm. Math. Phys. **204** (1999), 89–114.

[5] J.F. Alves and J. Sousa Ramos, *Zeta function of a tree map,* Preprint (1999).

[6] M. Artin and B. Mazur, *On periodic points,* Ann. of Math. (2), **21** (1965), 82–99.

[7] M. Babillot and M. Peigné, *Homologie des géodésiques fermées sur des variétés hyperboliques avec bouts cuspidaux,* Ann. scient. Ecole normale sup. (4), **33** (2000), 81–120.

[8] M. Baillif, *Dynamical zeta functions for tree maps,* Nonlinearity, **12** (1999), 1511–1529.

[9] M. Baillif, *Sharp determinants and transfer operators in higher dimensions,* in preparation.

[10] V. Baladi, *Periodic orbits and dynamical spectra,* Ergodic Theory Dynam. Systems, **18** (1998), 255–292.

[11] V. Baladi, *Decay of correlations,* Preprint (1999), to appear Proceedings of the AMS Summer Institute on Smooth ergodic theory and applications, held in Seattle, 1999.

[12] V. Baladi, *Positive Transfer Operators and Decay of Correlations,* (2000) World Scientific (Advanced Series in Nonlinear Dynamics), Singapore, to appear.

[13] V. Baladi, M. Benedicks, and V. Maume, *Almost sure rates of mixing for i.i.d. unimodal maps,* Preprint (1999).

[14] V. Baladi and G. Keller, *Zeta functions and transfer operators for piecewise monotone transformations,* Comm. Math. Phys., **127** (1990), 459–479.

[15] V. Baladi, A. Kitaev, D. Ruelle, and S. Semmes, *Sharp determinants and kneading operators for holomorphic maps,* Proc. Steklov Inst. Math., **216** (1997) 186–228.

[16] V. Baladi and D. Ruelle, *Sharp determinants,* Invent. Math., **123** (1996), 553–574.

[17] V. Baladi and M. Viana, *Strong stochastic stability and rate of mixing for unimodal maps,* Annales scient. Ecole normale sup. (4), **29** (1996), 483–517.

[18] V. Baladi and L.-S. Young, *On the spectra of randomly perturbed expanding maps,* Comm. Math. Phys., **156** (1993), 355–385 (see also Erratum, Comm. Math. Phys., **166** (1994), 219–220).

[19] M. Barge and R. Swanson, *Pseudo-orbits and topological entropy,* Proc. Amer. Math. Soc., **109** (1990), 559-566.

[20] M. Benedicks and L.-S. Young, *Sinai-Bowen-Ruelle measures for certain Hénon maps,* Invent. Math., **112** (1993), 541–576.

[21] M. Benedicks and L.-S. Young, *Markov extensions and decay of correlations for certain Hénon maps,* Géométrie complexe et systèmes dynamiques, Proceedings of the Colloque en l'honneur d'Adrien Douady held in Orsay, June 1995, Astérisque **261** (Paris), **2000,** 13–56.

[22] C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting,* Preprint (1998), to appear Israel J. Math.

[23] X. Bressaud and C. Liverani, *Anosov diffeomorphisms and coupling,* Preprint (2000).

[24] H. Bruin, S. Luzzatto, and S. van Strien, *Decay of correlations in one-dimensional dynamics,* Preprint (1999).

[25] J. Buzzi, *A.c.i.m.s as equilibrium states for piecewise invertible dynamical systems,* Preprint (1998).

[26] J. Buzzi, *Exponential decay of correlations for random Lasota-Yorke maps,* Comm. Math. Phys., **208** (1999), 25–54.

[27] J. Buzzi, *Piecewise expanding and* $\mathbf{R}$*-analytic mappings of the plane,* Preprint (1998), to appear Ergodic Theory Dynam. Systems.

[28] J. Buzzi and G. Keller, *Zeta functions and transfer operators for multidimensional piecewise affine and expanding maps,* Preprint, to appear Ergodic Theory Dynam. Systems (1999).

[29] J. Buzzi and V. Maume-Deschamps, *Decay of correlations for piecewise expanding maps in higher dimensions,* Preprint (1999).

[30] M. Campanino and S. Isola, *Infinite invariant measures for non-uniformly expanding transformations on* $[0,1]$*: weak law of large numbers with anomalous scaling,* Forum. Math., **8** (1996), 71–92.

[31] A. Castro, *Backward inducing and exponential decay of correlations for partially hyperbolic attractors with mostly contracting direction,* Ph.D. thesis, IMPA (1998).

[32] C.-H. Chang, *The transfer operator from statistical mechanics to quantum chaos,* Physics Bimonth., **21** (1999), 496–509.

[33] N. Chernov, *Statistical properties of piecewise smooth hyperbolic systems in high dimensions,* Discrete Contin. Dynam. Systems, **5** (1999), 425–448.

[34] N. Chernov, *Decay of correlations and dispersing billiards,* J. Statist. Phys., **94** (1999), 513–556.

[35] P. Collet and P. Ferrero, *Some limit ratio theorem related to a real endomorphism in case of a neutral fixed point,* Ann. Inst. H. Poincaré. Phys. Théor., **52** (1990), 283–301.

[36] W. Cowieson, *Piecewise smooth expanding maps in* $\mathbf{R}^d$*,* Ph.D. thesis UCLA (1999).

[37] P. Cvitanović, *Classical and Quantum Chaos: A Cyclist Treatise,* `http://www.nbi.dk/ChaosBook`.

[38] P. Dahlqvist, *The roles of singularities in chaotic spectroscopy,* Chaos Solitons Fractals, **8** (1997), 1011–1029.

[39] M. Dellnitz and O, Junge, *On the approximation of complicated dynamical behaviour,* SIAM J. Numer. Anal., (1999) **36,** 491–515.

[40] A. Dold, *Fixed point index and fixed point theorem for Euclidean neighbourhood retracts,* Topology, **4** (1965), 1–8.

[41] D. Dolgopyat, *On decay of correlations in Anosov flows,* Ann. of Math. (2), **147** (1998), 325–355.

[42] D. Dolgopyat, *Prevalence of rapid mixing in hyperbolic flows,* Ergodic Theory Dynam. Systems, **5** (1998), 1097–1114.

[43] D. Dolgopyat, *Prevalence of rapid mixing-II: topological prevalence,* Preprint (1998), to appear Ergodic Theory Dynam. Systems.

[44] D. Dolgopyat, *On dynamics of mostly contracting diffeomorphisms,* Preprint (1998), to appear Comm. Math. Phys.

[45] D. Dolgopyat, *On mixing properties of compact group extensions of hyperbolic systems,* Preprint (2000).

[46] D. Dolgopyat and M. Pollicott, Addendum to *Periodic orbits and dynamical spectra,* Ergodic Theory Dynam. Systems, **18** (1998), 293–301.

[47] B. Fayad, *Polynomial decay of correlations for a class of smooth flows on the two torus,* Preprint (2000).

[48] D. Fried, *Meromorphic zeta functions for analytic flows,* Comm. Math. Phys., **174** (1995), 161–190.

[49] D. Fried, *The zeroes of dynamical zeta functions,* IHES talk (1999), in preparation.

[50] P. Gaspard and X.-J. Wang, *Sporadicity: between periodic and chaotic dynamical behaviors,* Proc. Nat. Acad. Sci. USA, **85** (1988), 4591–4595.

[51] N.T.A. Haydn, *Meromorphic extension of the zeta function for Axiom A flows,* Ergodic Theory Dynam. Systems, **10** (1990), 347–360.

[52] He, Lianfa, *Pseudo-orbits and topological pressure,* Chinese J. Contemp. Math., **17** (1996), 405–414.

[53] F. Hofbauer and G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations,* Math. Z., **180** (1982), 119–140.

[54] H. Hu, *Decay of correlations for piecewise smooth maps with indifferent fixed points,* Preprint (1998).

[55] H. Hu, *Conditions for the existence of SBR measures for "almost Anosov" diffeomorphisms,* Preprint (1998), to appear Trans. Amer. Math. Soc.

[56] H. Hu, *Equilibrium states with polynomial decay of correlations,* Preprint (1999).

[57] S. Isola, *On the rate of convergence to equilibrium for countable ergodic Markov chains,* Preprint (1997).

[58] S. Isola, *Renewal sequences and intermittency,* J. Stat. Phys., **97** (1999), 263–280.

[59] V.Y. Kaloshin, *An extension of the Artin-Mazur theorem,* Ann. of Math. (2) **150** (1999), 729–741.

[60] G. Keller, *Ergodicité et mesures invariantes pour les transformations dilatantes par morceaux d'une région bornée du plan,* C.R. Acad. Sci. Paris Sér. I Math., **289,** (1979), 625–627.

[61] G. Keller, *Dynamical zeta functions for S-unimodal maps,* Preprint (1999).

[62] G. Keller and C. Liverani, *Stability of the spectrum for transfer operators,* Ann. Scuola Norm. Sup. Pisa. Cl. Sc., **28** (1999), 141–152.

[63] G. Keller and T. Nowicki, *Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps,* Comm. Math. Phys., **149** (1992), 31–69.

[64] Yu. Kifer, *Random dynamics and its applications,* Proceedings of the International Congress of Mathematicians held in Berlin, August 1998, Doc. Math., Extra Vol. II (electronic), (Bielefeld), **1998,** 809–818.

[65] A. Kitaev, *Kneading functions for higher dimensions,* Personal communication (1995).

[66] A. Kitaev, *Fredholm determinants for hyperbolic diffeomorphisms of finite smoothness,* Nonlinearity, **12** (1999) 141–179 (see also Corrigendum, 1717–1719).

[67] C. Liverani, *Decay of correlations,* Ann. of Math. (2), **142** (1995), 239–301.

[68] C. Liverani, *Central limit theorem for deterministic systems,* Proceedings of the International Conference on Dynamical Systems held in Montevideo, 1995, Addison Wesley Longman, (Harlow), **1996,** 56–75.

[69] C. Liverani, *Decay of correlations for piecewise expanding maps,* J. Stat. Phys., **78** (1995), 1111–1129.

[70] C. Liverani, B. Saussol, and S. Vaienti, *A probabilistic approach to intermittency,* Ergodic Theory Dynam. Systems, **19** (1999), 671–686.

[71] C. Liverani and D. Urbach, *A functional analytic approach to decay of correlations in Anosov flows,* in preparation.

[72] D.H. Mayer, *Continued fractions and related transformations,* Proceedings of the conference on Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces held in Trieste, April 1989, Oxford University Press (New York), **1991,** 175–222.

[73] D.H. Mayer, *Thermodynamic formalism and quantum mechanics on the modular surface,* From Phase Transitions to Chaos, World Sci. Publishing (River Edge, NJ), **1992,** 521–529.

[74] J. Milnor and W. Thurston, *On iterated maps of the interval,* Proceedings of the conference on Dynamical Systems held at College Park, MD, 1986–87, Springer Lect. Notes in Math. **1342,** (Berlin-New York), **1988,** 465–563.

[75] M. Mori, *Fredholm determinant for higher-dimensional piecewise linear transformations,* Japan. J. Math. (N.S.), **25** (1999), 317–342.

[76] T. Nowicki and D. Sands, *Non-uniform hyperbolicity and universal bounds for S-unimodal maps,* Invent. Math., **132** (1998), 633–680.

[77] J. Palis, *A global view of dynamics and a conjecture on the denseness of finitude of attractors,* Géométrie complexe et systèmes dynamiques, Proceedings of the Colloque en l'honneur d'Adrien Douady held in Orsay, June 1995, Astérisque **261** (Paris) **2000,** 335–348.

[78] W. Parry and M. Pollicott, *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics,* Astérisque, **187–188,** Soc. Math. de France (Paris) **1990.**

[79] M. Pianigiani, *First return map and invariant measure,* Israel J. Math., **35** (1980), 32–48.

[80] M. Pollicott, *On the rate of mixing of Axiom A flows,* Invent. Math., **81** (1985), 413–426.

[81] Y. Pomeau and P. Manneville, *Intermittent transition to turbulence in dissipative dynamical systems,* Comm. Math. Phys., **74** (1980), 189–197.

[82] L. Rey-Bellet, unpublished notes on sharp determinants (1995).

[83] D. Ruelle, *Zeta functions for expanding maps and Anosov flows,* Invent. Math., **34** (1976), 231–242.

[84] D. Ruelle, *One-dimensional Gibbs states and Axiom A diffeomorphisms,* J. Diff. Geom., **25** (1987), 117–137.

[85] D. Ruelle, *The thermodynamic formalism for expanding maps,* Comm. Math. Phys., **125** (1989), 239–262.

[86] D. Ruelle, *An extension of the theory of Fredholm determinants,* Inst. Hautes Etudes Sci. Publ. Math., **72** (1990), 175–193.

[87] H.H. Rugh, *Fredholm determinants for real-analytic hyperbolic diffeomorphisms of surfaces,* Proceedings of the XIth International Congress of Mathematical Physics held in Paris, 1994, Internat. Press, (Cambridge, MA), **1995,** 297–303.

[88] H.H. Rugh, *Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems,* Ergodic Theory Dynam. Systems, **16** (1996), 805–819.

[89] H.H. Rugh, *Intermittency and regularized Fredholm determinants,* Invent. Math., **135** (1999), 1–25.

[90] S. Smale, *Differentiable dynamical systems,* Bull. A.M.S., **73** (1967), 747–817.

[91] B. Saussol, *Absolutely continuous invariant measures for multidimensional expanding maps,* Preprint (1997), to appear Israel J. Math.

[92] M. Thaler, *Transformations on* $[0,1]$ *with infinite invariant measures,* Israel J. Math., **46** (1983), 67–96.

[93] M. Thaler, *A limit theorem for the Perron–Frobenius operator of transformations on* $[0,1]$ *with indifferent fixed points,* Israel J. Math., **91** (1995), 111–127.

[94] M. Tsujii, *Absolutely continuous invariant measures for piecewise real-analytic expanding maps on the plane,* Preprint (1998), to appear Comm. Math. Phys.

[95] M. Tsujii, *Absolutely continuous invariant measures for expanding piecewise linear maps,* Preprint (1999).

[96] M. Viana, *Multidimensional nonhyperbolic attractors,* Inst. Hautes Études Sci. Publ. Math., **85** (1997), 63–96.

[97] M. Viana, *Stochastic Dynamics of Deterministic Systems,* Col. Bras. de Matemática (Rio de Janeiro) **1997.**

[98] M. Viana, *Dynamics: a probabilistic and geometric perspective,* Proceedings of the International Congress of Mathematicians held in Berlin, August 18–27 1998, Doc. Math., Extra Vol. I (electronic), (Bielefeld), **1998,** 557–578.

[99] Q. Wang and L.-S. Young, *Analysis of a class of strange attractors,* Preprint (1999).

[100] L.-S. Young, *Stochastic stability of hyperbolic attractors,* Ergodic Theory Dynam. Systems, **6** (1986), 311–319.

[101] L.-S. Young, *Decay of correlations for certain quadratic maps,* Comm. Math. Phys., **146** (1992), 123–138.

[102] L.-S. Young, *Statistical properties of systems with some hyperbolicity including certain billiards,* Ann. of Math. (2), **147** (1998), 585–650.

[103] L.-S. Young, *Recurrence times and rates of mixing,* Israel. J. Math., **110** (1999), 153–188.

[104] L.-S. Young, *Ergodic theory of chaotic dynamical systems,* Proceedings of the XIIth Congress of Mathematical Physics held in Brisbane, 1997, Internat. Press (Cambridge, MA) **1999,** 131–143.

[105] M. Yuri, *Thermodynamic formalism for certain nonhyperbolic maps,* Ergodic Theory Dynam. Systems, **19** (1999), 1365–1378.

[106] R. Zweimüller, *Ergodic structure and invariant densities of non-Markovian interval maps with indifferent fixed points,* Nonlinearity, **11** (1998), 1263–1276.

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