Almost sure rates of mixing for i.i.d. unimodal maps

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Final version, Summer 2000

Abstract. It has been known since the pioneering work of Jakobson and subsequent work by Benedicks-Carleson and others that a positive measure set of quadratic maps admit an absolutely continuous invariant measure. Young and Keller-Nowicki proved exponential decay of its correlation functions. Benedicks-Young [BeY] and Baladi-Viana [BV] studied stability of the density and exponential rate of decay of the Markov chain associated to i.i.d. small perturbations. The almost sure statistical properties of the sample stationary measures of i.i.d. itineraries are more difficult to estimate than the "averaged statistics." Adapting to random systems, on the one hand partitions associated to hyperbolic times due to Alves [A], and on the other a probabilistic coupling method introduced by Young [Yo2] to study rates of mixing, we prove stretched exponential upper bounds for the almost sure rates of mixing.

Résumé. On sait depuis les travaux de Jakobson d'abord, de Benedicks-Carleson et d'autres auteurs ensuite, que les transformations quadratiques de l'intervalle admettent une mesure de probabilité invariante absolument continue pour un ensemble de paramètres de mesure de Lebesgue positive. Young et Keller-Nowicki ont montré la décroissance exponentielle des fonctions de corrélation correspondantes. Benedicks-Young [BeY] et Baladi-Viana [BV] ont étudié la stabilité de la densité invariante et du taux de décroissance exponentiel de la chaîne de Markov associée à de petites perturbations i.i.d. Les propriétés statistiques presque sûres des mesures échantillonnées des itinéraires aléatoires sont plus difficile à estimer que les "statistiques moyennisées." En adaptant aux systèmes aléatoires, d'une part une partition associée aux temps hyperboliques d'Alves [A], d'autre part une méthode de couplage probabiliste introduite par Young [Yo2] pour étudier la vitesse de mélange, nous obtenons des bornes supérieures sous-exponentielles pour la vitesse de mélange presque sûre.

1991 *Mathematics Subject Classification.* 60G10 58F11; 28D20 58F30.

This work was made possible by support from the PRODYN Programme of the European Science Foundation. V.B. was partially supported by the Fonds National Suisse de la Recherche Scientifique and is thankful to the Göran Gustafsson foundation, KTH, Stockholm, for hospitality during part of the preparation of this work. M.B. was partially supported by the Swedish Natural Science Research Council. We are grateful to J. Alves for remarks on a preliminary version of the manuscript and to D. Sands for enlightening comments, which prompted us to amend three arguments. Finally, we thank M. Babillot for useful references, and the referee for encouraging us to reorganise the exposition.

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1. Introduction

An important class of discrete-time deterministic dynamical systems (given by a transformation f on a Riemann manifold) are those which are both "chaotic" (i.e., satisfy some sensitiveness of initial conditions property) and statistically predictable, i.e., there is (an ergodic) invariant measure μ so that, for each integrable observable φ, Lebesgue almost every point x_0 has a time average converging to the space average:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x_0)) = \int \varphi \, d\mu \, . \tag{1.1}$$

A stronger ergodic property is mixing. If μ is mixing, we have *convergence to equilibrium:*

$$\lim_{n\to\infty} \int \varphi \circ f^n \, d\mathrm{Leb} = \int \varphi \, d\mu \, , \tag{1.2}$$

and more generally, for any pair of square integrable observables we have *decay of the operational correlations:*

$$\lim_{n\to\infty} \left(\int \varphi \circ f^n \, \psi \, d\mathrm{Leb} - \int \varphi \, d\mu \int \psi \, d\mathrm{Leb} \right) = 0 \, . \tag{1.3}$$

(Essentially equivalently, the classical correlations tend to zero, which is the same as $\lim_{n\to\infty} \int (\varphi \circ f^n) \, \psi \, d\mu = \int \varphi \, d\mu \int \psi \, d\mu$. The proofs below apply to both notions, and we concentrate on the operational correlations, more accessible experimentally.) When (1.2–1.3) hold, a natural question is: "how fast does the convergence take place?" Such quantified information on *rates of mixing* may sometimes be obtained for smooth enough observables, and often yields a central limit theorem for them. See e.g. [Yo1] and references therein for a discussion of this class of problems and some specific examples of uniformly and nonuniformly hyperbolic dynamical systems where the rate of mixing is exponential. One of these examples is the quadratic family $x \mapsto a - x^2$ on the interval for "good" (so called Collet–Eckmann or Benedicks–Carleson) values of the parameter a, or more generally unimodal maps satisfying certain axioms.

Our present object of study is small random perturbations of dynamical systems. Since our results are for independent identically distributed perturbations of good unimodal maps, we can be a little more specific without being too technical: let $I \subset \mathbb{R}$ be a compact interval and let $f : I \to I$ be a smooth transformation with $f(I)$ a subset of the interior of I. For small $\epsilon > 0$, let ν_ϵ be a probability measure on $[-\epsilon, \epsilon]$. We may consider two models for the random compositions of $f + \omega_0$ with ω_0 selected in $[-\epsilon, \epsilon]$ following the law ν_ϵ:

Markov chain. In words, we are averaging over all possible random realisations. Because of the i.i.d. setting, this can be done by averaging at each time-step. More formally, this means considering the Markov chain $\{X_n\}_{n=1}^{\infty}$ with transition probabilities (here, $x \in I$ and $E \subset I$ with characteristic function χ_E)

$$\mathrm{Prob}\,(X_{n+1} \in E \mid X_n = x) = \int_{-\epsilon}^{\epsilon} \chi_E(f(x) + \omega_0)\, d\nu_\epsilon(\omega_0)\,. \tag{1.4}$$

Under rather weak assumptions, it is possible to show that the Markov chain admits a *unique* invariant probability measure, i.e., a measure μ_ϵ on I with

$$\mu_\epsilon(E) = \int_{-\epsilon}^{\epsilon} \int \chi_E(f(x) + \omega_0)\, d\nu_\epsilon(\omega_0)\, d\mu_\epsilon(x).$$

Writing $f_\omega(x) = f(x) + \omega_0$, and by induction $f_\omega^n(x) = f_{\sigma\omega}^{n-1} \circ f_\omega(x)$, one defines operational correlation functions

$$\int \varphi \circ f_\omega^n\, \psi\, d\mathrm{Leb} \prod_{i=0}^{n-1} d\nu_\epsilon(\omega_i) - \int \varphi\, d\mu_\epsilon \int \psi\, d\mathrm{Leb}\,. \tag{1.5}$$

for the Markov chain. It is of obvious interest to study *stochastic stability,* i.e., whether $\mu_\epsilon \to \mu$ (at which speed? in which topology?) and whether the rate of decay of correlations is stable as $\epsilon \to 0$.

Random skew product. Alternatively, we may wish to state "almost sure" results. Formally, we consider the skew product $T : I \times \Omega \to I \times \Omega$, with $\Omega = [-\epsilon, \epsilon]^{\mathbb{Z}}$,

$$T(x, \omega) = (f_\omega(x), \sigma(\omega))\,, \text{ where } (\sigma\omega)_k = \omega_{k+1}\,. \tag{1.6}$$

The natural objects of study are the invariant probability measures for T of the form $\mu_\omega \mathrm{Leb}(dx)\, P(d\omega)$ with $P = \nu_\epsilon^{\mathbb{Z}}$, in particular those for which almost each μ_ω is absolutely continuous with respect to Lebesgue measure. In the present i.i.d. setting such a family of absolutely continuous *sample stationary measures* $\mu_\omega = h_\omega \mathrm{Leb}$ (which satisfy $(f_\omega)_* \mu_\omega = \mu_{\sigma\omega}$) may be obtained by disintegrating a natural extension of $\mu_\epsilon \times \nu_\epsilon^{\mathbb{Z}_+}$ (with $\mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n \geq 0\}$). It is natural in this context to consider both the *future* ("aiming at a moving target"), and the *past rates* of convergence to equilibrium:

$$R_\omega^{(f)}(n) = |(f_\omega^n)_*(\mathrm{Leb}) - \mu_{\sigma^n\omega}| \quad \text{and} \quad R_\omega^{(p)}(n) = |(f_{\sigma^{-n}\omega}^n)_*(\mathrm{Leb}) - \mu_\omega|\,, \tag{1.7}$$

where $|\cdot|$ denotes the total mass of a signed measure. We may also consider the "future" and "past" random operational correlations:

$$\begin{aligned} C_{\omega,\varphi,\psi}^{(f)}(n) &= \left| \int (\varphi \circ f_\omega^n)\psi\, d\mathrm{Leb} - \int \varphi\, d\mu_{\sigma^n\omega} \int \psi\, d\mathrm{Leb} \right| \\ C_{\omega,\varphi,\psi}^{(p)}(n) &= \left| \int (\varphi \circ f_{\sigma^{-n}\omega}^n)\psi\, d\mathrm{Leb} - \int \varphi\, d\mu_\omega \int \psi\, d\mathrm{Leb} \right|. \end{aligned} \tag{1.8}$$

The goal here is to obtain for P-almost all ω, upper bounds of the type $C_\omega \cdot \rho(n)$ or $C_{\varphi,\psi} C_\omega \cdot \rho(n)$ on the expressions (1.7) respectively (1.8), where $\rho(n) \to 0$ at a certain rate, independently of ω. (In general it is not immediate to obtain bounds on the future random correlation functions from estimates on the past random correlation functions, and vice versa.) Asymptotic bounds on

$$P(\{\omega \mid C_\omega > n\}) \tag{1.9}$$

are also desirable. The stochastic stability questions mentioned in the framework of the Markov chain may also be asked here.

Obviously, controlling (1.5) is not enough to estimate (1.8). In the other direction, averaging estimates (1.8) yield corresponding bounds for (1.5) whenever the control in (1.9) is enough to guarantee that $C_\omega \in L^1(P)$. (In fact, some additional information is needed – and often available – to estimate expressions of the type $\int \phi_1(\sigma^n \omega)\phi_2(\omega)\, dP_\epsilon(\omega) - \int \phi_1(\omega)\, dP_\epsilon(\omega) \int \phi_2(\omega)\, dP_\epsilon(\omega)$.) Also, it may be argued that a control of "almost all random itineraries" with information of the type (1.9) is more relevant to an actual physical experiment (e.g.) than bounds for the the averages (1.5). After all, only finitely many experiments may be actually realised!

Before we state our main new results, let us recall previously known facts. For smooth expanding (in any dimension) or piecewise smooth and piecewise expanding one-dimensional maps, the Markov chain was studied by Baladi–Young [BaY] who proved exponential decay of correlations and strong stochastic stability. Baladi–Viana [BV] then extended these results to a positive measure set of nonuniformly expanding unimodal maps, for which Benedicks–Young [BeY] had previously obtained a weaker form of stochastic stability. (We also refer to results of Katok-Kifer [KaK] for more general perturbations, but under a Misiurewicz assumption, as well as to work of Collet [Co1].)

Let us now discuss random skew products for which a large body of literature is available (in particular by Kifer, and the school of L. Arnold in Bremen), we restrict to results related to the physical measures of small random perturbations of strongly mixing discrete-time dynamics. Bogenschütz [Bo] and Baladi et al. [BKS] studied random correlations for smooth expanding dynamics, proving exponential decay of future and past correlations together with a strong form of stochastic stability (this was done by using a very naive idea: all transfer operators in play map a given function cone strictly inside itself). We mention also the work of Khanin–Kifer [KhK] who were interested in more general equilibrium states for random compositions of maps expanding in average (they studied neither stability nor rates of mixing). More recently, Buzzi [Bu1, Bu2] considered random compositions of piecewise monotone interval maps (not necessarily close to a fixed map) having some expansion in average property. He showed existence of absolutely continuous sample stationary measures and exponential decay of both future and past correlations, using a probabilistic approach.

Informal statement of results

Starting from a "good" unimodal map f (our assumptions are stated in an axiomatic way, see (H1)–(H4) in Section 2, they apply to a positive measure set of parameters of the quadratic family) and an atomless probability measure ν_ϵ on $[-\epsilon, \epsilon]$ (the precise assumption is given in (2.1)), we consider for small enough ϵ the i.i.d. compositions of $f + \omega_0$. We show that for almost every $\omega \in \Omega$:

(1) There is a unique family of absolutely continuous sample stationary measures $h_{\sigma^n \omega}\,\mathrm{Leb}$ for $n \in \mathbb{Z}$.
(2) We have *stretched exponential decay* for the rates of mixing. More precisely, there are $0 < u < 1$, $v > 1$, $C(\epsilon) > 1$, and a random variable C_ω with $P(\{\omega \mid C_\omega > n\}) \le \frac{C(\epsilon)}{n^v}$ such that for all Lipschitz test functions φ, ψ, there is $C(\varphi, \psi)$, depending only on their Lipschitz constants so that with $R^{(f)}_\omega(n)$ $R^{(p)}_\omega(n)$, $C^{(f)}_{\omega,\varphi,\psi}(n)$, $C^{(p)}_{\omega,\varphi,\psi}(n)$ as in (1.7), (1.8) we have

$$\max\left(R^{(f)}_\omega(n), R^{(p)}_\omega(n), C^{(f)}_{\omega,\varphi,\psi}(n), C^{(p)}_{\omega,\varphi,\psi}(n)\right) \le C(\epsilon) C_\omega\, C(\varphi, \psi)\; e^{-n^u}\;, \forall n \in \mathbb{Z}_+ .$$

In fact, we can prove the bounds for the universal exponent $u = 1/16$ if we allow a factor $C(\epsilon) \ge 1$ as follows:

$$\max\left(R^{(f)}_\omega(n), R^{(p)}_\omega(n), C^{(f)}_{\omega,\varphi,\psi}(n), C^{(p)}_{\omega,\varphi,\psi}(n)\right) \le C(\epsilon) C_\omega\, C(\varphi, \psi)\; e^{-n^{1/16}/C(\epsilon)} .$$

We believe that this is the first time that estimates have been obtained for the almost sure rates of mixing in a concrete nonuniformly hyperbolic dynamical setting. We hope that they may be used to prove a random central limit theorem (see Kifer [Ki]).

Since the bound on $C^{(f,p)}_{\omega,\varphi,\psi}$ is integrable, averaging our results on the random correlations gives that the Markov chain correlation decays faster than $C(\epsilon) e^{-n^u}$ for some $0 < u < 1$, a result not as good as the exponential decay obtained in [BV]. Note also that our upper bounds for the various constants $C_\omega(\epsilon)$, $C(\epsilon)$ blow up when $\epsilon \to 0$. (In particular, we do *not* address in the present paper the question of stochastic stability.) In view also of the fact that the transition from exponential (Lemma 7.8) to stretched exponential bounds occurs rather late in the proof (it is a consequence of the waiting times interfering with the combinatorial bounds e.g. in the proof of Proposition 8.3, see below for more details), it is not clear whether the subexponentiality is an artifact of our proof.

One of the advantages of this work as contrasted to the previous studies ([BV, BaY, BeY], etc.) of the Markov chain approach is that it is naturally suited to extensions to the non-autonomous case. More precisely, instead of assuming full i.i.d., that is $P = \nu_\epsilon^{\mathbb{Z}}$, we could suppose that (σ, P) is "strongly" mixing, and try to implement a variant of the geometric construction of Viana [V] to replace e.g. Lemma 7.4.

The basic idea in our proof is to construct a random version of the towers of Young [Yo2], showing that the coupling method she introduced can be randomised. The first difficulty here is to modify the standard partition (see e.g. [Yo1]) and obtain good estimates on points with large return times. Here, a beautiful idea due to Alves [A] was instrumental. He studied (maps close to) a deterministic skew product $T(x, \theta) = (a - x^2 + \epsilon\theta, D\theta \bmod 1)$ where $D \gg 1$ gives a "strongly mixing" deterministic dynamical system on the circle. In order to construct absolutely continuous invariant measure for T on the cylinder, Alves introduced good partitions into rectangles, involving a crucial notion of "hyperbolic times" (an abstraction of the escape times relevant for unimodal or Hénon maps, which was later applied by Alves-Bonatti-Viana and Castro to analyze partially hyperbolic systems). He also exploited bounds on "exceptional sets" previously obtained by Viana [V], who was the first to study this skew product model and proved that it possesses two positive Lyapunov exponents. Although we consider a slightly more general framework than the Misiurewicz setting in [A] and [V], many properties become easier to prove in our i.i.d. setting (see Lemma 7.4, where we obtain exponential estimates, to be contrasted with the stretched exponential bounds in [V]). The key observation then is that the bounds obtained on the set of ω such that a given x behaves well by following [A], [V] are uniform in x, so that a careful application of Fubini's theorem allows us to exchange x and ω (up to a zero-measure residue of bad ω:s which may be excluded). On the other hand, we are forced to introduce "waiting times" (see Lemma 7.7) which render the coupling argument more intricate and make us lose exponential bounds. (This loss occurs first in Proposition 8.3, when we ensure that the return map to the basis is Markov. See in particular the choice $k = \sqrt{\ell}$ which is optimal because we use the rough estimate (8.7).)

Finally, one surprising fact was that an estimate of Young (see the "choice of n_0" in [Lemma 1, Yo2]) which was a trivial consequence of the mixing property of the measure, becomes more troublesome in the random case. To deal with this, we bootstrap from the mixing property of the Markov chain on the tower (which follows from mixing of the random skew product in Section 4) applied in (yet) another large deviation argument (Sections 5–6) within the coupling estimates.

Sketch of contents

The article is organised as follows. In Section 2 we give precise statements of our hypotheses and results, including an application to random countable Markov chains.

Sections 3 to 6 constitute Part I of the proof, which is an abstract random analogue of the coupling method on towers in [Yo2]: In Section 3, we formulate a set of axioms for random towers endowed with a reference measure, and show that they ensure the existence of absolutely continuous sample stationary measures. Section 4 is devoted to general remarks on random mixing and random exactness, followed by a proof that the skew product associated to the dynamics on random towers is exact (and thus mixing) if the greatest common denominator of the return times to the basis is equal to one. These remarks are used in a large deviations argument in Section 5, where the coupling method of [Yo2] is implemented on the towers from Section 3 to study the rate of decay of the "future" correlation function. Finally, in Section 6 we further adapt the coupling method to study the "past" correlations.

In Part II, we show that our random unimodal maps fit in the abstract framework of Sections 3–6. Section 7 is devoted to constructing random partitions of the interval, and estimating random return times to a well-chosen subinterval (adapting the hyperbolic times techniques in [A], and the bounds in [V]), after suitable "waiting times." Section 8 is centered around Proposition 8.3 which gives upper bounds on the random recurrence asymptotics. These bounds enable us to construct random towers satisfying the axioms in Section 3 and to complete the proof of our main theorem in Section 9.

To keep the length of this article within reasonable bounds, we put the emphasis on those of our arguments which are new or differ nontrivially from previous ones, giving precise references to published computations (in particular in [A, BeY, V, Yo2]).

2. Setting and statement of results

Let $I = [L, R]$ be a compact interval containing 0 in its interior and $f : I \to I$ be a C^2 unimodal map (i.e., f is increasing on $[L, 0]$, decreasing on $[0, R]$) satisfying $f''(0) \neq 0$, $\sup_I |f'| < 8$, and,

(H1) There are $0 < \alpha < 1$ and $1 < \lambda \leq 4$ with $200\alpha < (\log \lambda)^2$ so that
(i) $|(f^n)'(f(0))| \geq \lambda^n$ for all $n \in \mathbb{Z}_+$.
(ii) $|f^n(0)| \geq e^{-\alpha n}$, for all $n \in \mathbb{Z}_+$.

(H2) For each small enough $\delta > 0$, there is $M = M(\delta) \in \mathbb{Z}_+$ for which
(i) If $x, \ldots, f^{M-1}(x) \notin (-\delta, \delta)$ then $|(f^M)'(x)| \geq \lambda^M$;
(ii) For each n, if $x, \ldots, f^{n-1}(x) \notin (-\delta, \delta)$ and $f^n(x) \in (-\delta, \delta)$, then $|(f^n)'(x)| \geq \lambda^n$.

(H3) $f(I)$ is a subset of the interior of I.

(H4) f is topologically mixing on $[f^2(0), f(0)]$.

Examples of unimodal maps satisfying (H1)–(H4) are quadratic maps $a - x^2$ for a positive measure set of parameters a. (See e.g. [BV] for notations similar to those of the present paper; the estimate $200\alpha < (\log \lambda)^2$ used here in Lemmas 7.1–7.4 corresponds in [BV] to $e^{2\alpha} < \lambda$.) Condition (H2) is in fact implied by the existence of $\delta > 0$ and $M \in \mathbb{Z}_+$ such that (H2)(i)–(ii) hold. See the remark in Section 7.A.

Fixing $\epsilon_0 > 0$ small enough to guarantee $f(x) \pm \epsilon_0 \in I$ for all $x \in I$, we assume that we are given a constant $C > 0$ and for each $0 < \epsilon < \epsilon_0$ a probability measure ν_ϵ on $[-\epsilon, \epsilon]$ such that for any subinterval $J \subset [-\epsilon, \epsilon]$,

$$\nu_\epsilon(J) \leq \frac{C|J|}{\epsilon}. \tag{2.1}$$

(This is used in Lemma 7.4.) Assumption (2.1) may be relaxed, but we do not pursue this aim here. It cannot be completely suppressed since there are open intervals of parameters corresponding to periodic attractors arbitrarily close to a. Assumption (2.1) holds if ν_ϵ has a density with respect to Lebesgue which is bounded above by C/ϵ. It does *not* imply that 0 belongs to the support of ν_ϵ.

For fixed $\epsilon > 0$, we write $\Omega = \Omega_\epsilon = [-\epsilon, \epsilon]^{\mathbb{Z}}$, $\sigma : \Omega \to \Omega$ for the shift to the left, and $P = P_\epsilon = \nu_\epsilon^{\mathbb{Z}}$. Our aim is to study the random compositions of maps $f_\omega(x) = f(x) + \omega_0$ with $\omega \in \Omega$ following the law P. For $n \geq 1$ we write $f^n_\omega(x) = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x)$. Denoting Lebesgue measure on I by Leb, and $|\mu|$ for the total mass of a signed measure, our first main result is stretched exponential bounds for the speed of approach to equilibrium (as usual, the test functions φ and ψ can be assumed to be Hölder instead of Lipschitz):

Main Theorem. (Stretched exponential mixing for i.i.d. unimodals). *If ϵ is small enough (depending on f) then for P_ϵ-almost each $\omega \in \Omega_\epsilon$ there is a sample stationary measure $h_\omega \,\mathrm{Leb}$. There exist $C(\epsilon) \ge 1$ and, for almost every $\omega \in \Omega_\epsilon$, $C^{(1)}_\omega = C^{(1)}_\omega(\epsilon) > 0$ such that for each Lipschitz function $\varphi : I \to \mathbb{C}$, and all $n \ge 1$,*

$$\left|(f^n_{\sigma^{-n}\omega})_*(\varphi \,\mathrm{Leb}) - (h_\omega \,\mathrm{Leb})\right| \le C^{(1)}_\omega \,\mathrm{Lip}\,\varphi \, e^{-(n^{1/16}/C(\epsilon))} \,. \tag{2.2}$$

Additionally, for almost every $\omega \in \Omega$, there are $C^{(2)}_\omega > 0$, $C^{(3)}_\omega > 0$ such that for each Lipschitz function $\psi : I \to \mathbb{C}$ and every bounded function $\varphi : I \to \mathbb{C}$, the "past" and "future" random correlation function satisfy for all $n \ge 1$

$$\left|\int \varphi \circ f^n_{\sigma^{-n}\omega}\, \psi \, d\mathrm{Leb} - \int \varphi h_\omega \, d\mathrm{Leb} \int \psi \, d\mathrm{Leb}\right| \le C^{(2)}_\omega \,\sup|\varphi| \,\mathrm{Lip}\,\psi e^{-(n^{1/16}/C(\epsilon))} \,, \tag{2.3}$$

and

$$\left|\int \varphi \circ f^n_{\omega}\, \psi \, d\mathrm{Leb} - \int \varphi h_{\sigma^n\omega} \, d\mathrm{Leb} \int \psi \, d\mathrm{Leb}\right| \le C^{(3)}_\omega \,\sup|\varphi| \,\mathrm{Lip}\,\psi e^{-(n^{1/16}/C(\epsilon))} \,. \tag{2.4}$$

There are $C(\epsilon)$ and $v > 1$ so that the maximum $C_\omega = \max(C^{(1)}_\omega, C^{(2)}_\omega, C^{(3)}_\omega)$ satisfies

$$P(\{\omega \in \Omega_\epsilon \mid C_\omega > n\}) \le \frac{C(\epsilon)}{n^v} \,, \ \textit{for all } n \ge 1 \,. \tag{2.5}$$

Finally, there is $0 < u < 1/16$ so that the factor $e^{-(n^{1/16}/C(\epsilon))}$ in (2.2–2.4) may be replaced by e^{-n^u}.

Remarks.

(1) Our proof gives the same upper estimates for the "classical" correlations.
(2) See e.g. [BKS] for the operational significance of, and experimental access to, the rates in (2.2–2.4).
(3) By the work of Bahnmüller [Ba] (who extended previous work of Ledrappier and Young [LY] to noninvertible situations), the Pesin formula holds for the random skew product invariant measure $h_\omega \,\mathrm{Leb}\, P$.
(4) If (H4) does not hold, a result of Blokh-Lyubich [BL] says that f is renormalisable, i.e., that there is a cycle of intervals $\{I_i\}_{i=0}^m$, $f : I_i \to I_{i+1}$, $I_m = I_0$, where $\{I_i\}$ have disjoint interiors. This is reflected in the greatest common denominator $\mathcal{G}$ of return times satisfying $\mathcal{G} \ne 1$, also for the random towers (see the axiom (A.VI) in Section 3.B and (8.10)). Our proof yields stretched exponential decay of correlation and speed of mixing for the $\mathcal{G}$th iterate $f^{\mathcal{G}}_\omega$ of the random system.

A simplification of the results in Sections 3–6 yields a result on random countable Markov chains with estimates on the recurrence times (after waiting times) which we were unable to locate in the literature. The setting is the following: Let $\sigma : \Omega \to \Omega$, with $\Omega = \prod_{\mathbb{Z}} E$, where (E, ν) is a probability space, be a two-sided Bernoulli shift (preserving the probability measure $P = \prod_{\mathbb{Z}} \nu$). Let $X_\omega^{(n)}$ be a random Markov chain for (σ, Ω) on the countable state space $\mathbb{Z}_+$ given by the random transition probabilities

$$\text{Prob}\left(X_\omega^{(n+1)} = j \mid X_\omega^{(n)} = i\right) = p_{ij,\sigma^n\omega}\,, \forall n \in \mathbb{Z}_+\,.$$

(In particular, for almost all ω and all i, $\Sigma_{j=1}^\infty p_{ij,\omega} = 1$.) We say that a family of probability measures $\{\mu_\omega\}$ on $\mathbb{Z}_+$ are sample stationary probability measures if $\sum_i \mu_\omega(i) p_{ij,\omega} = \mu_{\sigma\omega}(j)$ for all $j \in \mathbb{Z}_+$ and almost all ω. In this case, we write for $n \in \mathbb{Z}_+$, $\omega \in \Omega$, and $\varphi, \psi \in \ell^\infty(\mathbb{Z}_+)$

$$\mathbb{E}[\varphi(X_\omega^{(n)})] = \mathbb{E}_{\mu_\omega}[\varphi(X_\omega^{(n)})] = \sum_{(j_0,\cdots,j_n)\in\mathbb{Z}_+^{n+1}} \varphi(j_n) \left(\prod_{k=0}^{n-1} p_{j_k j_{k+1},\sigma^k\omega}\right) \mu_\omega(\{j_0\})\,,$$

$$\mathbb{E}[\varphi(X_\omega^{(n)})\psi(X_{\sigma^{-n}\omega}^{(0)})] = \sum_{(j_0,\cdots,j_n)\in\mathbb{Z}_+^{n+1}} \varphi(j_n)\, \psi(j_0) \left(\prod_{k=0}^{n-1} p_{j_k j_{k+1},\sigma^k\omega}\right) \mu_\omega(\{j_0\})\,.$$

We shall assume that the Markov chain has the following *irreducible tower structure* on a subset Λ of $\mathbb{Z}_+$: On the one hand, for all $j \in \mathbb{Z}_+$ there are $i \in \Lambda$ and $k \in \mathbb{Z}_+$ with Prob $(X_\omega^{(0)} = i; X_\omega^{(k)} = j) > 0$, and on the other hand for all $i \in \Lambda$ and P-almost every $\omega \in \Omega$ there is a (first return time) $R_\omega(i) \geq 1$ so that

$$\begin{aligned}
&\text{Prob}\,(X_\omega^{(k)} \notin \Lambda\,, \forall k = 1, \ldots, R_\omega(i) - 1 \mid X_\omega^{(0)} = i) = 1\,,\\
&\text{Prob}\,(X_\omega^{R_\omega(i)} = j \mid X_\omega^{(0)} = i) > 0 \quad \text{if and only if } j \in \Lambda\,.
\end{aligned}$$

In particular, there are three ways of expressing $R_\omega(i)$:

$$R_\omega(i) = \begin{cases} 1 + \max_{\ell\geq 0}\{\text{ Prob }(X_\omega^{(k)} \notin \Lambda\,, \forall k = 1, \ldots, \ell \mid X_\omega^{(0)} = i) = 1\}\,,\\ \min_{\ell \geq 1}\{\text{ Prob }(X_\omega^{(k)} \notin \Lambda\,, \forall k = 1, \ldots, \ell \mid X_\omega^{(0)} = i) = 0\}\,,\\ \sum_{\ell\geq 1} \text{ Prob }(X_\omega^{(k)} \notin \Lambda\,, \forall k = 1, \ldots, \ell \mid X_\omega^{(0)} = i)\,. \end{cases}$$

Also, $\sum_{j\in\Lambda}$ Prob $(X_\omega^{R_\omega(i)} = j \mid X_\omega^{(0)} = i) = 1$ for all $i \in \Lambda$. We say that the chain is *aperiodic* if the return times satisfy (A.VI) in Section 3.A below.

Main Corollary (Application to i.i.d. countable Markov chains). *Let $X_\omega^{(n)}$ be a random Markov chain for (σ, Ω) on $\mathbb{Z}_+$, with irreducible tower structure on Λ. Assume that there is a (reference) probability measure m on Λ, with $m_i = m(\{i\}) > 0$ for all $i \in \Lambda$, so that for P-almost every $\omega \in \Omega$*

$$\begin{aligned}
&\sum_{\ell=1}^\infty \sum_{i\in\Lambda} m_i \text{ Prob }(X_{\sigma^{-\ell}\omega}^{(k)} \notin \Lambda\,, \forall k = 1, \ldots, \ell \mid X_{\sigma^{-\ell}\omega}^{(0)} = i) < \infty\\
&\lim_{\ell_0\to\infty} \sum_{\ell\geq\ell_0} m(i \in \Lambda \mid R_{\sigma^{\ell_0-\ell}\omega}(i) = \ell) = 0\,.
\end{aligned}$$

Then there is a unique family $\{\mu_\omega\}$ of sample stationary probability measures on $\mathbb{Z}_+$.

Assume further that the chain is aperiodic and that there are $0 < u', v' \le 1$ and $n_1 : \Omega \to \mathbb{Z}_+$ with

$$P(\{\omega \mid n_1(\omega) > n\}) < e^{-n^{v'}}$$

such that for P-almost every $\omega \in \Omega$

$$\sum_{i \in \Lambda} m_i \text{ Prob } (X_\omega^{(k)} \notin \Lambda \,, \forall k = 1, \ldots, \ell \mid X_\omega^{(0)} = i) < e^{-\ell^{u'}} \,, \forall \ell \ge n_1(\omega) \,. \tag{2.6}$$

Then, there are $0 < u < u'$ and $C_\omega^{(4)} \ge 1$ such that for each φ and ψ in $\ell^\infty(\mathbb{Z}_+)$, the "past random correlations" satisfy

$$\left| \mathbb{E}\big[\varphi(X_{\sigma^{-n}\omega}^{(n)})\psi(X_{\sigma^{-n}\omega}^{(0)})\big] - \mathbb{E}[\varphi(X_\omega^{(0)})]\mathbb{E}[\psi(X_{\sigma^{-n}\omega}^{(0)})] \right| \le C_\omega^{(4)} \sup|\psi| \sup|\varphi| e^{-n^u} \,. \tag{2.7}$$

Finally, there are $v > 1$, $C > 1$ so that

$$P(\{\omega \in \Omega \mid C_\omega^{(4)} > n\}) \le \frac{C}{n^v} \,.$$

Remarks.

(1) Obviously one may formulate the main corollary for future correlations, approach to equilibrium, etc. The main corollary can be also expressed as a result on speed of convergence to the maximal eigenvector of random products of stochastic matrices having a "tower structure." The cumbersome exercise is left to the reader. We refer to the papers of Hennion [He] and the book of Bougerol-Lacroix [BoL, especially Chapter A.III] for references on the classical work of Furstenberg, Kesten, Guivarc'h, Ledrappier, and others, on applications of the Oseledec theorem yielding exponential bounds for the speed of convergence to the maximal eigenvector of random products of *finite* stochastic matrices, under assumptions guaranteeing that the maximal Lyapunov exponent is simple.

(2) Adapting Sections 5 and 6 similarly as the corresponding proofs of Theorem 2(II) of [Yo2], we may also obtain exponential (respectively polynomial) estimates in (2.7) if we change the assumptions accordingly.

Open questions.

(1) As mentioned in the introduction, by adapting Kifer's methods in [Ki], we expect that it is possible to prove a random central limit theorem in the setting of the present paper.

(2) We also pointed out already that it is of obvious interest (and, in view of [V], probably feasible while retaining stretched exponential bounds) to generalise our i.i.d. setting to weaker forms of mixing. One could also attempt to study non-additive perturbations.

(3) We have restricted ourselves to perturbations of exponentially mixing maps. It would be interesting to see if our approach can be extended to unimodal maps with slower rates of mixing. See the recent study by Bruin, Luzzatto, and van Strien [BLS], based on Young's coupling argument [Yo2].

3. Random towers with waiting times

3.A Abstract setting

Suppose we are given a finite measure space (Λ, m) and for each small $\epsilon > 0$ a two-sided Bernoulli shift $\sigma : \Omega_\epsilon \to \Omega_\epsilon$ where $\Omega_\epsilon = \prod_{\mathbb{Z}} E_\epsilon$, with $(E_\epsilon, \nu_\epsilon)$ a probability space, and $P_\epsilon = \prod_{\mathbb{Z}} \nu_\epsilon$ is a σ-invariant probability measure. (We suppress ϵ from the notation whenever possible.)

For each ϵ and P-almost each ω, suppose there are a countable (modulo m) partition $\Lambda = \cup_j \Lambda_j(\omega)$ and a "return-time" function $R_\omega : \Lambda \to \mathbb{Z}_+ \cup \{\infty\}$, constant on each $\Lambda_j(\omega)$. Define the random towers for each ϵ and almost every ω by

$$\Delta_\omega = \left\{ (x, \ell) \in \Lambda \times \mathbb{Z}_+ \mid x \in \bigcup_j \Lambda_j(\sigma^{-\ell}\omega) \,, \ell \in \mathbb{Z}_+ \,, 0 \le \ell \le R_{\sigma^{-\ell}\omega}(x) - 1 \right\}.$$

(I.e., layer $R_\omega(x) - 1$ disjoint copies of $\Lambda_j(\omega)$ in Pisa tower fashion.) Denote by $\Delta_{\omega,\ell}$ the ℓth level of the tower $\{(x,\ell) \in \Delta_\omega\}$. We sometimes slightly abuse notation and identify $\Delta_{\omega,\ell}$ with $\{x \in \Lambda \mid R_{\sigma^{-\ell}\omega}(x) > \ell\} = \{x \mid (x,\ell) \in \Delta_\omega\}$; in particular $\Delta_{\omega,0} = \Lambda$ for all ω. Δ will denote the family $\{\Delta_\omega\}_{\omega\in\Omega}$.

Assume that for each ϵ and almost every ω we are given a return map $f^R_\omega : \Lambda \to \Lambda$ The dynamics $F_\omega : \Delta_\omega \to \Delta_{\sigma\omega}$ consists in hopping from one tower to the next above $(x, 0)$, stopping at level $R_\omega(x) - 1$ if $R_\omega(x) < \infty$, and falling down to the zeroth level of $\Delta_{\sigma^{R_\omega(x)}\omega}$ using the return map. In other words, we set

$$F_\omega(x, \ell) = \begin{cases} (x, \ell + 1), & \text{if } \ell + 1 < R_{\sigma^{-\ell}\omega}(x) \,, \\ (f^R_{\sigma^{-\ell}\omega}(x), 0), & \text{if } \ell + 1 = R_{\sigma^{-\ell}\omega}(x) \,. \end{cases}$$

(In particular, $F^{R_\omega}_\omega|_{\Delta_{\omega,0}} = f^R_\omega|_\Lambda$.)

For each ℓ we consider the countable partition $\mathcal{Z}_{\omega,\ell}$ of $\Delta_{\omega,\ell}$ induced by $\bigcup_j \Lambda_j(\sigma^{-\ell}\omega)$

$$\Delta_{\omega,\ell} = \bigcup_{j \text{ s.t. } R_\omega|_{\Lambda_j(\sigma^{-\ell}\omega)} \ge \ell+1} \Lambda_j(\sigma^{-\ell}\omega) \,,$$

we also let $\mathcal{Z}_\omega$, $\mathcal{Z}$ be the corresponding partitions of Δ_ω, respectively Δ.

Without risk of confusion, denote by m the lift of the reference measure m on Δ_ω (suppressing the dependence on ω from the notation). Observe that $\sup_\omega m(\Delta_\omega)$ is not necessarily finite (this plays a role e.g. in the proof of Proposition 5.6, (5.5–5.6)).

Next, we extend R_ω to Δ_ω (keeping the same notation without risk of confusion) by setting $R_\omega(x, \ell) = R_{\sigma^{-\ell}\omega}(x, 0) - \ell$. (I.e., $R_\omega(x, \ell)$ is the first positive integer n for which $F^n_\omega(x, \ell) \in \Delta_{\sigma^n\omega,0}$.)

Finally, for almost each ω, we introduce a *separation time* $s_\omega : \Delta_\omega \times \Delta_\omega \to \mathbb{Z}_+ \cup \{\infty\}$

$$s_\omega(x, y) = \min \left\{ n \ge 0 \mid F^n_\omega(x) \text{ and } F^n_\omega(y) \text{ lie in distinct elements of } \mathcal{Z} \right\}.$$

3.B Axioms

We shall assume that our random towers satisfy the following axioms:

(A.I) [**Return and separation times**] $R_\omega : \Delta_\omega \to \mathbb{Z}_+$ is constant on each interval of the partition $\mathcal{Z}_\omega$; with $R_\omega \geq p_0(\epsilon)$. If (x,ℓ) and (y,ℓ) are both in the same interval of the partition $\mathcal{Z}_\omega$, then $s_\omega((x,0),(y,0)) \geq \ell$. For any $(x,0),(y,0)$ in the same interval of $\mathcal{Z}_\omega$,

$$s_\omega((x,0),(y,0)) = R_\omega(x,0) + s_{\sigma^{R_\omega}(\omega)}\big(F^{R_\omega(x,0)}(x,0), F^{R_\omega(y,0)}(y,0)\big)\,.$$

(A.II) [**Markov property**] For each element J_ω of $\mathcal{Z}_\omega$, the map $F_\omega^{R_\omega}|_{J_\omega} : J_\omega \to \Lambda$ is a bijection.

(A.III) [**Weak forward expansion**] The partition $\mathcal{Z}_\omega$ is generating for F_ω in the sense that the diameters of the partitions $\bigvee_{j=0}^n F_{\sigma^{-j}\omega}^{-j} \mathcal{Z}_\omega$ tend to zero as $n \to \infty$.

(A.IV) [**Bounded distortion**] There are $C(\epsilon) > 1$ and $0 < \beta < 1$ (β is independent of ϵ) such that for all ω and each element J_ω of $\mathcal{Z}_\omega$, the map $F_\omega^{R_\omega}|_{J_\omega}$ and its inverse are nonsingular with respect to Lebesgue measure, and, writing $JF_\omega^{R_\omega} > 0$ for the corresponding jacobian, we have for each $x, y \in J_\omega$, writing s for $s_{\sigma^{R_\omega(x)}(\omega)}$,

$$\left| \frac{JF_\omega^{R_\omega}(x)}{JF_\omega^{R_\omega}(y)} - 1 \right| \leq C(\epsilon)\, \beta^{s(F_\omega^{R_\omega}(x,0), F_\omega^{R_\omega}(y,0))}\,. \tag{3.1}$$

(A.V) [**Return times asymptotics**] There are constants $C_1(\epsilon) > C_2(\epsilon) > 1$, a full measure subset $\Omega_1 \subset \Omega$ and random variables n_1 on Ω_1, so that for each $\omega \in \Omega_1$:

$$\begin{cases} m(\{x \in \Lambda \mid R_\omega(x) > n\}) \leq Ce^{-(n^{\frac14}/C_1(\epsilon))}, & \forall n \geq n_1(\omega)\,, \\ P(\{n_1(\omega) > n\}) \ \leq \ Ce^{-(n^{\frac14}/C_2(\epsilon))}\,, \forall n\,. \end{cases} \tag{3.2}$$

For the existence, respectively mixing, of finite invariant sample stationary measures the following two "summability" assumptions suffice: For almost all ω

$$m(\Delta_\omega) = \sum_{\ell \in \mathbb{Z}_+} m(\{R_{\sigma^{-\ell}\omega} > \ell\}) < \infty\,. \tag{3.3}$$

$$\text{For almost all } \omega\,, \lim_{\ell_0 \to \infty} \sum_{\ell \geq \ell_0} m(\Delta_{\sigma^{\ell_0}\omega,\ell}) = 0\,. \tag{3.4}$$

(A.VI) [**Gcd(Return times)=1 (mixing)**] There are $N_0(\epsilon) \geq 1$ and $\{t_i \in \mathbb{Z}_+, i = 1, \ldots N_0\}$ with g.c.d. $\{t_i\} = 1$ such that for all $\omega \in \Omega_2$, all $n \in \mathbb{Z}$ all $1 \leq i \leq N_0$ we have $m(\{x \in \Lambda \mid R_\omega(x) = t_i\}) > 0$.

Remark 3.1. Axiom (A.V, (3.2)) implies that for each fixed $\widehat{N} \in \mathbb{Z}_+$, there are constants $1 \le C_2(\epsilon) < C_1(\epsilon)$, a full measure subset $\Omega_3 \subset \Omega$, and $n_3 : \Omega_3 \to \mathbb{Z}_+$ so that

$$\begin{cases} n_1(\sigma^{-\ell}\omega) \le \ell \quad \text{and} \quad n_1(\sigma^{\widehat{N}+\ell}\omega) \le \ell \,, \quad \forall \ell \ge n_3(\omega)\,, \\ m(\{x \in \Delta_\omega \mid R_\omega(x) > n\}) \le Ce^{-(n^{\frac{1}{4}}/C_1(\epsilon))} m(\Delta_\omega), \quad \forall n \ge n_3(\omega)\,, \\ m(\Delta_\omega) = \sum_{\ell \in \mathbb{Z}_+} m(\{R_{\sigma^{-\ell}\omega} > \ell\}) \le n_3(\omega) + C(\epsilon) \\ P(\{n_3(\omega) > n\}) \;\le\; Ce^{-(n^{\frac{1}{4}}/C_2(\epsilon))}\,, \forall n\,. \end{cases} \tag{3.5}$$

In particular, (3.2) implies (3.3) and (3.4). (For (3.4), use that if $\ell_0 \ge n_3(\omega) \ge n_2(\omega)$ then $n_1(\sigma^{\widehat{N}+\ell_0}\omega) \le \ell_0$ by definition of n_2, so that $\sum_{\ell \ge \ell_0} m(\Delta_{\sigma^{\widehat{N}+\ell_0}\omega, \ell}) \le C(\epsilon)e^{-\ell_0^w}$ for some $w > 0$ by (3.2)).

To show (3.5), we first prove that there are $\Omega_2 \subset \Omega_1$, of full measure, and a random variable $n_2 \ge n_1$ on Ω_2, so that

$$\begin{cases} n_1(\sigma^{-\ell}\omega) \le \ell \text{ and } n_1(\sigma^{\widehat{N}+\ell}\omega) \le \ell \,, \quad \forall \ell \ge n_2(\omega)\,, \\ P\big(\{n_2(\omega) > n\}\big) \le Ce^{-(n^{\frac{1}{4}}/C_2(\epsilon))}\,, \forall n\,. \end{cases} \tag{3.6}$$

Indeed, just set $n_2(\omega) = \inf\{\ell \ge n_1(\omega) \mid \forall n \ge \ell \,, n_1(\sigma^{-n}\omega) \le n \text{ and } n_1(\sigma^{\widehat{N}+n}\omega) \le n\}$, and use that

$$\begin{aligned} P\big(\{n_2(\omega) > \ell\}\big) &\le \sum_{n \ge \ell} P\big(\{n_1(\sigma^{-n}\omega) > n\}\big) + \sum_{n \ge \ell} P\big(\{n_1(\sigma^{\widehat{N}+n}\omega) > n\}\big) \\ &\le \sum_{n \ge \ell} P\big(\{n_1(\omega) > n\}\big) + \sum_{n \ge \ell} P\big(\{n_1(\omega) > n\}\big)\,. \end{aligned}$$

Now, if $\omega \in \Omega_2$

$$\mathrm{Leb}(\Delta_\omega) = \sum_{\ell \in \mathbb{Z}_+} \mathrm{Leb}(\{R_{\sigma^{-\ell}\omega} > \ell\}) \le n_2(\omega) + C(\epsilon) \sum_{\ell > n_2(\omega)} e^{-(\ell^{\frac{1}{4}}/C_1(\epsilon))} < \infty\,. \tag{3.7}$$

This proves our claim, up to taking slightly larger constants $1 \le C_2(\epsilon) < C_1(\epsilon)$: just let

$$n_3(\omega) = \min\{m \ge n_2(\omega) \mid \forall n \ge m \text{ and } \forall \ell \ge 0\,, n_2(\sigma^{-\ell}\omega) \le n + \ell\}\,. \quad \square$$

3.C Dynamical Lipschitz and bounded random function spaces

Consider the following "dynamical Lipschitz" space of densities on Δ (with $\beta < 1$ as in (A.IV)):

$$\begin{aligned} \mathcal{F}_\beta^+ = \{\varphi_\omega : \Delta_\omega \to \mathbb{C} \mid \exists C_\varphi > 0\,, \quad \forall J_\omega \in \mathcal{Z}_\omega\,, \text{ either } \varphi_\omega|_{J_\omega} \equiv 0\,, \\ \text{or } \varphi_\omega|_{J_\omega} > 0 \text{ and } \left| \log \frac{\varphi_\omega(x)}{\varphi_\omega(y)} \right| \le C_\varphi \beta^{s_\omega(x,y)}\,, \forall x, y \in J_\omega\}\,, \end{aligned}$$

For a random variable $\mathcal{K}_\omega : \Omega \to \mathbb{R}_+$ with $\inf_\Omega \mathcal{K}_\omega > 0$ and

$$P\big(\{\omega \mid \mathcal{K}_\omega > n\}\big) \leq P\big(\{\omega \mid n_3(\omega) > n/3\}\big) \leq C(\epsilon) e^{-(n^{\frac{1}{4}}/C_2(\epsilon))}, \tag{3.8}$$

we introduce on the one hand a space of random Lipschitz functions:

$$\mathcal{F}_\beta^{\mathcal{K}_\omega} = \{\varphi_\omega : \Delta_\omega \to \mathbb{C} \mid \exists C_\varphi > 0\,, \\ |\varphi_\omega(x) - \varphi_\omega(y)| \leq C_\varphi \mathcal{K}_\omega \beta^{s_\omega(x,y)}\,, |\,\varphi_\omega(x)\,| \leq C_\varphi \mathcal{K}_\omega\,, \forall x, y \in \Delta_\omega\}\,,$$

and on the other, a space of random bounded functions:

$$\mathcal{L}_\infty^{\mathcal{K}_\omega} = \{\varphi_\omega : \Delta_\omega \to \mathbb{C} \mid \exists C'_\varphi > 0\,, \sup_{x\in\Delta_\omega} |\varphi_\omega(x)| \leq C'_\varphi\, \mathcal{K}_\omega\}\,.$$

Note for further use (in Section 5) that (3.8) together with (A.V) give that $\mathcal{L}_\infty^{\mathcal{K}_\omega}$, and thus $\mathcal{F}_\beta^{\mathcal{K}_\omega}$, is a subset of $L^2(\Delta, m)$.

Slightly abusing language (see Lemma 9.1) we refer to the smallest possible C_φ or C'_φ as the *Lipschitz constant,* or *supremum,* of φ in $\mathcal{F}_\beta^+$ or $\mathcal{F}_\beta^{\mathcal{K}_\omega}$, respectively $\mathcal{L}_\infty^{\mathcal{K}_\omega}$. Clearly, $\mathcal{F}_\beta^{\mathcal{K}_\omega}$ and $\mathcal{L}_\infty^{\mathcal{K}_\omega}$ with the norms $\|\varphi\|_{\mathcal{F}} = C_\varphi$ respectively $\|\varphi\|_{\mathcal{L}_\infty} = C'_\varphi$ are Banach spaces.

3.D Constructing the absolutely continuous stationary measures

We may now show how to obtain sample stationary measures absolutely continuous with respect to the reference measure m:

Theorem 3.2. (Sample measure). *Let $\{F_\omega : \Delta_\omega \to \Delta_{\sigma\omega}\}$ satisfy axioms (A.I)–(A.IV) together with the summability condition* (3.3) *in (A.V). Then there is for almost each $\omega \in \Omega$ an absolutely continuous sample stationary probability measure $\mu_\omega = h_\omega\, m$ on Δ_ω, i.e. so that $(F_\omega)_*(\mu_\omega) = \mu_{\sigma\omega}$.*

Additionally, $\{h_\omega\} \in \mathcal{F}_\beta^+$. If (3.2) *holds then there is a random variable $\mathcal{K}_\omega$ satisfying* (3.8) *so that both h_ω and $1/h_\omega$ belong to $\mathcal{F}_\beta^{\mathcal{K}_\omega} \subset \mathcal{L}_\infty^{\mathcal{K}_\omega}$.*

From now on, $\mathcal{K}_\omega$ will refer to the random variable from Theorem 3.2.

Proof of Theorem 3.2. Let $F_\omega^R : \Delta_{\omega,0} \to \bigcup_{n\geq p_0} \Delta_{\sigma^n\omega,0}$ denote the return map $F_\omega^R(x,0)$. If the meaning is clear, we just write F^R.

For any $E \subset \Lambda$ (recall $\Delta_{\sigma^{-n}\omega,0} = \Lambda$ for all ω and n)

$$\big[(F^R)^{-1}\big]_\omega(E) = \big\{(x,n) \in \Delta_{\sigma^{-n}\omega,0} \times \mathbb{Z}_+ \mid R_{\sigma^{-n}\omega}(x) = n \text{ and } F_{\sigma^{-n}\omega}^R(x,0) \in E\big\}\,.$$

We define $[(F^R)^{-j}]_\omega(E)$ by induction, and for probability measures $\{\nu_{\sigma^{-n}\omega} \mid n \in \mathbb{Z}_+\}$ on $\bigsqcup_{n\in\mathbb{Z}_+} \Delta_{\sigma^{-n}\omega,0}$, we set $([(F^R)^j]_\omega)_*\nu(E) = \sum_n \nu_{\sigma^{-n}\omega}([(F^R)^{-j}]_\omega(E) \cap \Delta_{\sigma^{-n}\omega,0})$.

Let m_0 be the probability measure $m|_{\Delta_{\omega,0}}/m(\Delta_{\omega,0})$ on $\Delta_{\omega,0} = \Lambda$.

Now, let $A \subset \bigsqcup_{n\in\mathbb{Z}_+} \Delta_{\sigma^{-n}\omega,0}$ with $A \in \bigvee_{\ell=0}^{j-1} [(F^R)^{-\ell}]_\omega \mathcal{Z}_\omega$ and set

$$\phi_{j,A} = \frac{d}{dm_0}\Big(\Big[\big(F^R\big)^j\Big]_\omega\Big)_*(m_0 \mid A)\,.$$

For $x, y \in \Delta_{\omega,0}$, letting $x', y' \in A$ be such that $x' \in [(F^R)^{-j}]_\omega(x)$, $y' \in [(F^R)^{-j}]_\omega(y)$, and setting n to be so that $x', y' \in \Delta_{\sigma^{-n}\omega,0}$, we find by (3.1) in (A.IV) that for a suitable sequence $0 \le n_\ell \le n$,

$$\begin{aligned}\log \frac{\phi_{j,A}(y)}{\phi_{j,A}(x)} = \log \frac{(J(F^R_{\sigma^{-n}\omega})^j)(x')}{(J(F^R_{\sigma^{-n}\omega})^j)(y')} &= \sum_{\ell=0}^{j-1} \log \frac{JF^R_{\sigma^{-n_\ell}\omega}\big((F^R_{\sigma^{-n_{\ell-1}}\omega})^\ell(x')\big)}{JF^R_{\sigma^{-n_\ell}\omega}\big((F^R_{\sigma^{-n_{\ell-1}}\omega})^\ell(y')\big)} \\ &\le \sum_{\ell=0}^{j-1} C(\epsilon)\beta^{s_\omega(x,y)+(j-\ell)-1} \le C(\epsilon)\beta^{s_\omega(x,y)}\,,\end{aligned} \tag{3.9}$$

which is uniform in j, A, and ω. As a consequence, for each ω the sequence

$$\phi_{n,\omega} = \frac{1}{n}\sum_{j=0}^{n-1}\Big(\Big[\big(F^R\big)^j\Big]_\omega\Big)_*(m_0)$$

is relatively compact in $L^\infty(\Lambda, m)$ (see [Yo2, Proof of Theorem 1]). We set $\hat\nu_\omega$ to be a measure whose density is a accumulation point of the $\phi_{n,\omega}$. (By construction, this density is bounded from above and from below on Λ and it belongs to $\mathcal{F}^+_\beta(\Delta)$.)

Then, we saturate (see e.g. the proof of Theorem 1 in [Yo1] or [Yo2]) to construct a measure on $\bigcup_{\ell\in\mathbb{Z}} \Delta_{\sigma^\ell\omega}$:

$$\hat\mu_\omega = \sum_{\ell=0}^{\infty}(F^\ell_{\sigma^{-\ell}\omega})_*(\hat\nu_{\sigma^{-\ell}\omega} \mid R_{\sigma^{-\ell}\omega} > \ell)\,.$$

Assumption (3.3) in (A.V) implies

$$\hat\mu_\omega(\Delta_\omega) \le C\sum_{\ell=0}^{\infty} m(\{R_{\sigma^{-\ell}\omega} > \ell\}) < \infty\,,$$

In particular, $\hat\mu_\omega$ can be normalised to get an absolutely continuous probability measure μ_ω. Its density satisfies the conditions needed to be in $\mathcal{F}^+_\beta$ (which only involve ratios).

The upper and lower bounds for the density of $\hat\nu_\omega$ and its Lipschitz constant translate into bounds for that of $\hat\mu_{\sigma^{-n}\omega}$, depending on ω through $n_3(\sigma^{-n}\omega)$, and we get the final claim in the theorem by setting $\mathcal{K}_\omega$ to be the maximum of the upper bounds for h_ω and its Lipschitz constant, and the corresponding bounds for $1/h_\omega$. $\square$

4. Mixing for the skew product: random exactness

In the previous section, we constructed an absolutely continuous sample stationary measure for the random tower map $\{F_\omega : \Delta_\omega \longrightarrow \Delta_{\sigma\omega}\}$. We shall now study its mixing properties, after introducing further notation.

The *random skew product* is the fibered map $F = (F_\omega)_{\omega\in\Omega}$ on Δ. Let $\mathcal{B}_\omega$ be the Borel σ-algebra of Δ_ω, and let $\mathcal{B}$ be the family of σ-algebras $\mathcal{B}_\omega$. In Theorem 3.2 we constructed absolutely continuous fibered invariant measures $(\mu_\omega)_{\omega\in\Omega}$. Let μ be the corresponding invariant measure for the random skew product: $\mu(A) = \displaystyle\int_\Omega \mu_\omega(A_\omega)$, for $A \in \mathcal{B}$. Let $L^2(\mu)$ be the Hilbert space of $\phi = (\phi_\omega : \Delta_\omega \longrightarrow \mathbb{C})_{\omega\in\Omega}$ such that $\phi_\omega \in L^2(\mathcal{B}_\omega,\ \mu_\omega)$ for almost all ω, and $\int_\Omega \int_{\Delta_\omega} |\phi_\omega|^2 d\mu_\omega dP(\omega) < \infty$.

For $n \in \mathbb{Z}_+$ we denote by $F^{-n}(\mathcal{B})$ the family $([F_{\sigma^{n-1}\omega} \circ \cdots \circ F_\omega]^{-1}(\mathcal{B}_{\sigma^n\omega}))_{\omega\in\Omega}$ and by F_ω^n the compositions $F_{\sigma^{n-1}\omega} \circ \cdots \circ F_\omega$.

We recall definitions which are standard for deterministic dynamics:

Definitions (Random exactness, mixing).
(1) The random skew product $(F, \mu) = (F_\omega, \mu_\omega)_{\omega\in\Omega}$ is exact if each $B \in \mathcal{B}$ which belongs to all $F^{-n}\mathcal{B}$, $n \in \mathbb{Z}_+$ is trivial. (I.e., for almost all ω, either $\mu_\omega(B_\omega) = 0$ or $\mu_\omega(B_\omega) = 1$.)
(2) The random skew product (F, μ) is mixing if for all φ and ψ in $L^2(\mu)$,

$$\lim_{n\to\infty} \left| \int_\Omega \int_{\Delta_\omega} \varphi_{\sigma^n\omega} \circ F_\omega^n \cdot \psi_\omega \, d\mu_\omega dP(\omega) - \int_\Omega \int_{\Delta_\omega} \varphi_\omega \, d\mu_\omega dP(\omega) \int_\Omega \int_{\Delta_\omega} \psi_\omega \, d\mu_\omega dP(\omega) \right| = 0 \,.$$

Remark. In our particular case of random towers, instead of a random dynamical system, we may consider a skew-product map F acting on $\Lambda \times \mathbb{Z}_+ \times \Omega$, endowed with the invariant measure $\mu = \mu_\omega \times P$, where μ_ω has support on $\Delta_\omega \subset \Lambda \times \mathbb{Z}_+ \times \{\omega\}$. Then the definition reduces to the usual notions of exactness and mixing.
The following proposition may be proved as in the deterministic case (see e.g. [PY]):

Proposition 4.1. *If F is exact then it is mixing.*

The following result is less standard. We shall not need it (our main theorem says much more), but we include it for completeness:

Lemma 4.2 (Forward fibered mixing). *Assume that the random skew product (F, μ) is exact. Then for all φ such that*

$$\sup_\omega \int |\varphi_\omega|^2 d\mu_\omega < \infty \,,$$

and all ψ in $L^2(\mu)$, we have for almost all $\omega \in \Omega$:

$$\lim_{n\to\infty} \left| \int_{\Delta_\omega} \varphi_{\sigma^n\omega} \circ F_\omega^n \cdot \psi_\omega \, d\mu_\omega - \int_{\Delta_{\sigma^n\omega}} \varphi_{\sigma^n\omega} \, d\mu_{\sigma^n\omega} \int_{\Delta_\omega} \psi_\omega \, d\mu_\omega \right| = 0 \,.$$

Proof of Lemma 4.2. This goes along the lines of the classical proof of Proposition 4.1 (see [PY]). Indeed, exactness implies that for almost all ω,

$$L^2(\mathcal{B}_\omega,\ \mu_\omega) \supset L^2(F_\omega^{-1}\mathcal{B}_{\sigma\omega}) \supset \cdots \supset L^2(F_\omega^{-n}\mathcal{B}_{\sigma^n\omega}) \supset \cdots \supset \mathbb{C}.$$

Choose $\{k_\omega^\alpha\,, \alpha \in \mathbb{Z}_+\}$ an orthonormal basis of $L^2(\mathcal{B}_\omega) \ominus L^2(F_\omega^{-1}\mathcal{B}_{\sigma\omega})$, then $\{k_{\sigma\omega}^\alpha \circ F_\omega\,, \alpha \in \mathbb{Z}_+\}$ is an orthonormal basis of $L^2(F_\omega^{-1}\mathcal{B}_{\sigma\omega}) \ominus L^2(F_\omega^{-2}\mathcal{B}_{\sigma^2\omega})$, and $\{k_{\sigma^j\omega}^\alpha \circ F_\omega^j\,, \alpha \in \mathbb{Z}_+\,, j \in \mathbb{Z}_+\}$ is an orthonormal basis of $L^2(\mathcal{B}_\omega) \ominus \mathbb{C}$. Writing $\varphi_{\sigma^n\omega}$ and ψ_ω in these bases, we get:

$$\left| \int\limits_{\Delta_\omega} \varphi_{\sigma^n\omega} \circ F_\omega^n \; \psi_\omega \, d\mu_\omega - \int\limits_{\Delta_{\sigma^n\omega}} \varphi_{\sigma^n\omega} \, d\mu_{\sigma^n\omega} \int\limits_{\Delta_\omega} \psi_\omega \, d\mu_\omega \right|$$
$$\leq \int\limits_{\Delta_{\sigma^n\omega}} |\varphi_{\sigma^n\omega}|^2 \, d\mu_{\sigma^n\omega} \sum_{j,\ \alpha} \left| \int k_{\sigma^{n+j}\omega}^\alpha \circ F_\omega^{n+j} \cdot \psi_\omega \, d\mu_\omega \right|^2 \xrightarrow{n\to\infty} 0. \quad \square$$

Proposition 4.3 (Exactness of random map). *Let (F, μ) satisfy (A.I)–(A.IV) and the summability conditions* (3.3, 3.4) *from (A.V), with μ from Theorem 3.2. If (A.VI) holds then (F, μ) is exact and thus mixing.*

Proof of Proposition 4.3. We first give a proof assuming the following property, stronger than (A.VI), and which will hold in the application to random unimodal maps. Suppose that there is $\widehat{N}(\epsilon)$ so that for almost all ω and each $r \geq \widehat{N}(\epsilon)$ we have $m\{x \in \Lambda \mid R_\omega(x) = r\} > 0$ (take $\widehat{N}$ in (3.2) to be this integer). First we prove: if, for every $\kappa > 0$ and almost all ω, there exists an integer $t(\kappa, \omega)$ such that $m(F_\omega^t(\Delta_{\omega,0})) > 1 - \kappa$, then F is exact.

We adapt Young's proof ([Yo2, Theorem 1 (iii)]) to our random setting. Let $A \in \bigcap_n F^{-n}\mathcal{B}$. Fixing ω such that $\mu_\omega(A_\omega) > 0$, we are going to prove that for any $\kappa > 0$, $\mu_\omega(A_\omega) > 1 - \kappa$. Let $t(\kappa/2, \omega)$ be given by hypothesis. For each $n \in \mathbb{Z}_+$ we have $A_\omega = (F_\omega^{n+t})^{-1}(B_{\sigma^{n+t}\omega})$ and

$$\mu_\omega(A_\omega) = \mu_{\sigma^{n+t}\omega}(B_{\sigma^{n+t}\omega}) = \mu_{\sigma^{n+t}\omega}(F_{\sigma^n\omega}^t \circ F_\omega^n(A_\omega))\,.$$

Now, the non singularity of $F_{\sigma^n\omega}^t$, the absolute continuity of $\mu_{\sigma^{n+t}\omega}$ with respect to m on $\Delta_{\sigma^{n+t}\omega}$, and the definition of t imply the existence of $\upsilon(\kappa, \omega, t, n) > 0$ such that

$$m(\Delta_{\sigma^n\omega,0} \setminus D_{\sigma^n\omega}) < \upsilon \Rightarrow \mu_{\sigma^{n+t}\omega}(F_{\sigma^n\omega}^t D_{\sigma^n\omega}) > 1 - \kappa\,.$$

Thus, if we can find $n \in \mathbb{Z}_+$ such that $m(\Delta_{\sigma^n\omega,0} \setminus F_\omega^n(A_\omega)) < \upsilon$, then we shall conclude that $\mu_\omega(A_\omega) > 1 - \kappa$. Let us prove the existence of such an integer n.

Since we assumed that $\mu_\omega(A_\omega) > 0$, we may choose $\xi_\omega \in \Xi_n(\sigma^n\omega)$ with $F_\omega^n(\xi_\omega) = \Delta_{\sigma^n\omega,0}$ and $m(A_\omega \cap \xi_\omega) \,/\, m(\xi_\omega) > 1 - \upsilon/2$. If n is large enough we may assume that $s_\omega|_{\xi_\omega}$ is large enough so that $C(\epsilon)\beta^{s_\omega} < \upsilon$. Then, the bounded distortion estimate (3.9) gives

$$\frac{m\big(F_\omega^n(A_\omega \cap \xi_\omega)\big)}{m(\Delta_{\sigma^n\omega,0})} > (1 - \upsilon)\frac{m(A_\omega \cap \zeta_\omega)}{m(\zeta_\omega)} > 1 - 2\upsilon.$$

Finally, we prove that for any $\kappa > 0$ and all $\omega \in \Omega_2$, there exists an integer $t(\kappa, \omega)$ such that $m(F_\omega^t(\Delta_{\omega,0})) > 1 - \kappa$, following ideas from Markov chains. By the hypothesis that all return times after $\widehat{N}$ occur, $m(\Delta_{\omega,0} \cap F_\omega^{-t}(\Delta_{\sigma^t\omega,0})) > 0$ for all $t \geq \widehat{N}$. Let $\ell_0 \geq \max[\widehat{N}, n_3(\omega)]$, the tower structure gives

$$F_\omega^{\widehat{N}+\ell_0}(\Delta_{\omega,0}) \supset \bigcup_{\ell \leq \ell_0} \Delta_{\sigma^{\widehat{N}+\ell_0}\omega,\ell}$$

Now, for fixed ϵ and almost all ω we can make $\sum_{\ell \geq \ell_0} m(\Delta_{\sigma^{\widehat{N}+\ell_0}\omega,\ell})$ arbitrarily small by taking large enough ℓ_0 by (3.4).

If we replace the assumption that all return times $\geq \widehat{N}$ occur with positive probability by the weaker "g.c.d.=1" assumption, we may use the following argument: Define

$$U = \Big\{ t \in \mathbb{Z}_+ \mid \forall\, \omega \in \Omega_2\,, m\big(\Delta_{\omega,0} \cap (F_\omega^t)^{-1}(\Delta_{\sigma^t\omega,0})\big) > 0 \Big\}\,.$$

The Markov property (A.II) of the tower gives that U is stable under addition, and it follows from the assumption in (A.VI) that g.c.d. $U = 1$. Then, Lemma A.3 in Seneta [S] gives that U contains all but a finite number of positive integers, so that there exists t_0 such that for all $t \geq t_0$ and all ω

$$m(\Delta_{\omega,0} \cap F_\omega^{-t}(\Delta_{\sigma^t\omega,0})) > 0\,.$$

Replacing $\widehat{N}$ by t_0 in the previous paragraph completes the argument. $\square$

5. Random coupling argument, "future" correlations

5.A Large deviations and joint returns to the basis.

Let (F, μ) satisfy (A.I)–(A.IV) from Section 3, and conditions (3.3, 3.4) from (A.V), with μ from Theorem 3.2. Adapting Young's definitions ([Yo2, §3.3]) to our random setting, we introduce *stopping times* τ_i^ω and a *joint return time* T_ω on $\Delta_\omega \times \Delta_\omega$ for P-almost each ω and $x, x' \in \Delta \times \Delta$, as follows. Set

$$\begin{aligned}
\tau_1^\omega(x, x') &= \inf\{n \geq \ell_0 \mid F_\omega^n(x) \in \Delta_{\sigma^n\omega,0}\}\,,\\
\tau_2^\omega(x, x') &= \inf\{n \geq \ell_0 + \tau_1^\omega(x, x') \mid F_\omega^n(x') \in \Delta_{\sigma^n\omega,0}\}\,,\\
\tau_3^\omega(x, x') &= \inf\{n \geq \ell_0 + \tau_2^\omega(x, x') \mid F_\omega^n(x) \in \Delta_{\sigma^n\omega,0}\}\,,
\end{aligned}$$

and so on, with the action alternating between x and x'. Define then $T_\omega(x, x')$ to be the smallest integer $n \geq \ell_0$ such that $(F_\omega^n(x), F_\omega^n(x'))$ belongs to $\Delta_{\sigma^n\omega,0} \times \Delta_{\sigma^n\omega,0}$.

For fixed ω and $j \in \mathbb{Z}_+$, consider also the partition $\widetilde{\Xi}_j^\omega$ of $\Delta_\omega \times \Delta_\omega$ into maximal subsets on which the $\tau_i^\omega(x, x')$ are constant for $0 \leq i \leq j$.

In order to make use of the random mixing properties, for $\ell \in \mathbb{Z}_+$, consider the random variable:

$$V_\omega^\ell = m(\Delta_{\omega,0} \cap F_\omega^{-\ell}(\Delta_{\omega,0})) = \int (\chi_{\Delta_{\sigma^{\ell-1}\omega,0}} \circ F_\omega^\ell) \cdot \chi_{\Delta_{\omega,0}}\, dm.$$

Recall that μ_ϵ is the invariant measure for the Markov chain and $\Delta_{\omega,0} = \Lambda$ for all ω. For small $\gamma > 0$, to be chosen later, since F is mixing by Propositions 4.1 and 4.3, there exists ℓ_0 such that for all $\ell \geq \ell_0$, the expectation of V_ω^ℓ satisfies

$$\left| \int_\Omega V_\omega^\ell \, dP(\omega) - m(\Lambda) \cdot \mu_\epsilon(\Lambda) \right| < \gamma. \tag{5.1}$$

(In order to deduce (5.1) from mixing of F, we also used that $h_\omega^{-1} \cdot \chi_{\Delta_{\omega,0}}$ belongs to $L^2(\mu)$. This follows from Theorem 3.2.)

For any $q \in \mathbb{Z}_+$ and each fixed sequence of integers $\tau_0 = 0 < \tau_1 < \cdots < \tau_q$ such that $\tau_{i+1} - \tau_i \geq \ell_0$, define:

$$S_q^{\{\tau_i\}}(\omega) = \sum_{i=1}^q V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i - \tau_{i-1}}.$$

Lemma 5.1 (Large deviations for $S_q^{\{\tau_i\}}$). *There exist $\rho > 0$ and $0 < \kappa < 1$ such that for each q and all $\tau_0 = 0 < \tau_1 < \cdots < \tau_q$ such that $\tau_{i+1} - \tau_i \geq \ell_0$,*

$$P(\{S_q^{\{\tau_i\}}(\omega) < q\rho\}) \leq \kappa^q. \tag{5.2}$$

Proof of Lemma 5.1. The random variable V_ω^ℓ depends only on $\omega_0, \ldots, \omega_{\ell-1}$, so V_ω^ℓ and $V_{\sigma^j\omega}^k$ are independent provided $j \geq \ell$. In particular, $S_q^{\{\tau_i\}}$ is a sum of independent random variables.

For any $\upsilon > 0$ and $t > 0$,

$$\begin{aligned}
P(\{S_q^{\{\tau_i\}}(\omega) < t\}) &\leq \int \exp[\upsilon(t - S_q^{\{\tau_i\}}(\omega))]\, dP(\omega) \\
&\leq e^{\upsilon \cdot t} \int \exp[-\upsilon S_q^{\{\tau_i\}}(\omega)]\, dP(\omega) \\
&\leq e^{\upsilon \cdot t} \prod_{i=0}^{q-1} \int \exp[-\upsilon V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i - \tau_{i-1}}]\, dP(\omega) \quad \text{(by independence)}.
\end{aligned}$$

We have $0 \leq V_\omega^\ell \leq m(\Lambda)$ and, by (5.1), $\int V_\omega^\ell \, dP(\omega) \geq m(\Lambda) \cdot \mu_\epsilon(\Lambda) - \gamma$, provided $\ell \geq \ell_0$. Now, since $0 \leq \upsilon V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i - \tau_{i-1}} \ll 1$,

$$\int \exp[-\upsilon V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i - \tau_{i-1}}]\, dP(\omega) \leq 1 - \upsilon \Big[m(\Lambda)\mu_\epsilon(\Lambda) - \gamma - m(\Lambda)^2 \frac{\upsilon}{2} \Big] =: a(\upsilon, \gamma).$$

Choose $\upsilon < 2\mu_\epsilon(\Lambda)$ and then $\gamma > 0$ small enough so that $0 < a(\upsilon, \gamma) < 1$. We get $P(\{S_q^{\{\tau_i\}}(\omega) < q\rho\}) \le (e^{\upsilon\rho} \cdot a(\upsilon, \gamma))^q \le \kappa^q$, for some $0 < \kappa < 1$ by choosing $0 < \rho < \frac{1}{\upsilon} \log(1/a(\upsilon, \gamma))$. $\square$

We shall now use Lemma 5.1 to perform yet another parameter exclusion which will be useful later on to estimate the joint return time on $\Delta \times \Delta$. First observe that the lemma may be reformulated as follows: For each q, and every fixed sequence of integers $\tau_0 = 0 < \tau_1 < \cdots < \tau_q$ such that $\tau_{i+1} - \tau_i \ge \ell_0$, there is a set $M_q^{\{\tau_i\}} \subset \Omega$ with $P(M_q^{\{\tau_i\}}) \le \kappa^q$ and such that if $\omega \notin M_q^{\{\tau_i\}}$ then $S_q^{\{\tau_i\}}(\omega) \ge q \cdot \rho$. Next define

$$M'_q = \{(\omega, x, x') \in \bigcup_{\omega \in \Omega} (\{\omega\} \times \Delta_\omega \times \Delta_\omega) \mid \omega \in M_q^{\{\tau_i^\omega(x,x')\}}\} .$$

Corollary 5.2. *Assume additionally* (3.2) *and let $\mathcal{K}_\omega$ be given by Theorem 3.2. There is $0 < \kappa < 1$ such that for each large enough q the set $\widetilde{M}_q \subset \Omega$ defined by*

$$\widetilde{M}_q = \{\omega \in \Omega \mid \int_{\Delta_\omega \times \Delta_\omega} \chi_{M'_q}(\omega, x, x')\, \mathcal{K}_\omega^2\, (dm \times dm)(x, x') > \kappa^{q/2}\} \tag{5.3}$$

has P-measure smaller than $\kappa^{q/4}$. Furthermore, there is a random variable n_4 defined on a full measure set $\Omega_4 \subset \Omega$ and such that

$$\begin{cases} n \ge n_4(\omega) \Longrightarrow \omega \notin \widetilde{M}_n\,, \\ P(\{n_4(\omega) > n\}) \le C\kappa^{n/2}\,. \end{cases} \tag{5.4}$$

Proof of Corollary 5.2. The first claim is a Fubini argument. Indeed, if $\widetilde{M}_q$ had P-measure greater than $\kappa^{q/4}$, then

$$\int_{\widetilde{M}_q} \int_{\Delta_\omega \times \Delta_\omega} \chi_{M'_q}(\omega, x, x')\, \mathcal{K}_\omega^2\, dm \times dm(x, x')\ dP(\omega) > \kappa^{q/2} \times \kappa^{q/4}.$$

However, denoting by $\mathbb{P}$ the finite measure on $\cup_{\omega \in \Omega}(\{\omega\} \times \Delta_\omega \times \Delta_\omega)$ defined by:

$$\mathbb{P}(A) = \int_\Omega \int_{\Delta_\omega \times \Delta_\omega} \chi_A(\omega, x, x')\, \mathcal{K}_\omega^2\, dm \times dm(x, x')\, dP(\omega)\,,$$

using (3.3) and (3.8), we find for large enough q

$$\begin{aligned}
\mathbb{P}(M'_q) &= \int \chi_{M'_q}(\omega, x, x')\, \mathcal{K}_\omega^2\, dm \times dm(x, x')\, dP(\omega) \\
&= \sum_{\tau_1 < \cdots < \tau_q} \mathbb{P}(M'_q \cap \{\tau_i^\omega(x, x') = \tau_i\,, i = 1, \ldots, q\}) \\
&\le \kappa^q \cdot \sup_{\substack{\omega \in \Omega_2 \\ (\mathcal{K}_\omega^2 \cdot m(\Delta_\omega)^2) \le \kappa^{-q/8}}} \sum_{\tau_1 < \cdots < \tau_q} \mathcal{K}_\omega^2 \cdot m \times m((\Delta_\omega \times \Delta_\omega) \cap \{\tau_i^\omega(x, x') = \tau_i\,, i\}) \\
&\qquad\qquad + P(\{\mathcal{K}_\omega^2 m(\Delta_\omega)^2 > \kappa^{-q/8}\}) \\
&\le \kappa^{3q/4}\,,
\end{aligned}$$

a contradiction. Setting $\Omega_4 = \{\omega \mid \exists\, n_4(\omega)$ so that $\forall n \geq n_4(\omega),\ \omega \notin \widetilde{M}_n\}$, a large deviations argument as in Lemma 5.1 together with the first claim of the corollary gives the second claim. $\square$

5.B Estimates on stopping times and joint return times.

From now to the end of Section 6, the notations λ, λ', $\tilde{\lambda}$ will be used to denote probability measures, absolutely continuous with respect to m on Δ or $m \times m$ on $\Delta \times \Delta$. There should be no confusion with the constant λ from (H1)–(H2) which will not reappear until Section 7. We shall write $\Pi_\omega : \Delta_\omega \times \Delta_\omega \to \Delta_\omega$ for the projection on the first factor. Before proving the main estimate of this section (Proposition 5.6), we state two lemmas which are randomised versions of Lemmas 1 and 2 in [Yo2].

Lemma 5.3 (Lower bound for $P(\{T_\omega = \tau_i\})$). *Let λ, λ' be absolutely continuous probability measures on $\{\Delta_\omega\}$, with densities φ, φ' in $\mathcal{F}_\beta^+$. If $\Gamma \in \widetilde{\Xi}_i^\omega$ is such that $T_\omega|\Gamma > \tau_{i-1}$, then, letting $V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i - \tau_{i-1}}$ be associated to the $\tau_j(\Gamma)$,*

$$(\lambda \times \lambda')(\{T_\omega > \tau_i\}|\Gamma) \leq 1 - V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i-\tau_{i-1}}/C_{\lambda,\lambda'}(\epsilon)\,,$$

where $C_{\lambda,\lambda'}(\epsilon) > 1$ depends on the Lipschitz constants of φ and φ'. This dependence may be removed if we consider $i \geq i_0(\lambda, \lambda')$.

Lemma 5.4 (Relating stopping times and return times). *Let λ, λ' be absolutely continuous probability measures on $\{\Delta_\omega\}$, with densities φ, φ' in $\mathcal{F}_\beta^+$. For each $\Gamma \in \widetilde{\Xi}_i^\omega$, we have*

$$\begin{aligned}(\lambda \times \lambda')_\omega(\{\tau_{i+1} - \tau_i > \ell_0 + n\} \mid \Gamma)& \\ \leq C_{\lambda,\lambda'}(\epsilon) m(\{R_{\sigma^{\tau_i+\ell_0}\omega} > n\}) &\cdot m(\Delta_{\sigma^{\tau_i+\ell_0}\omega}),\end{aligned}$$

where $C_{\lambda,\lambda'}(\epsilon)$ depends on the Lipschitz constants of φ, φ'. This dependence may be removed if we consider $i \geq i_0(\lambda, \lambda')$.

The proofs of Lemmas 5.3 and 5.4 are based on the following sublemma, which is a randomised version of Sublemmas 1 and 2 in [Yo2] (recall that the bounded distortion inequality (A.IV) is uniform in ω).

Sublemma 5.5 (Consequences of bounded distorsion).
(1) There is M_0 such that for all $n \in \mathbb{Z}_+$, and $\omega \in \Omega$,

$$\frac{d(F_\omega^n)_*(m)}{dm} \leq M_0\, m(\Delta_\omega).$$

(2) Let λ be a family of absolutely continuous probability measures on $\{\Delta_\omega\}$, with densities φ in $\mathcal{F}_\beta^+$. There is $C_\lambda(\epsilon) > 1$ so that for each $\omega \in \Omega$, every $k \in \mathbb{Z}_+$, letting $\Gamma \in \mathcal{Z}_\omega^{k-1}$ be such that $F_\omega^k(\Gamma) = \Delta_{\sigma^k\omega,0}$, and setting $\nu_{\sigma^k\omega} = (F_\omega^k)_*(\lambda_\omega|\Gamma)$, then for all x, $y \in \Delta_{\sigma^k\omega,0}$

$$\left|\frac{d\nu_{\sigma^k\omega}}{dm}(x) \Big/ \frac{d\nu_{\sigma^k\omega}}{dm}(y) - 1\right| \leq C_\lambda(\epsilon).$$

The dependence of $C_\lambda(\epsilon)$ on λ may be removed if the number of $i \le k$ such that $F^i_\omega(\Gamma) \subset \Delta_{\sigma^i\omega,0}$ is greater than some $j_0 = j_0(\lambda)$.

Proof of Sublemma 5.5. The proof of (1) follows verbatim the proof of Sublemma 1 in [Yo2] (making use of (3.3)), we omit it.

We sketch how to prove (2). Let x_0 and $y_0 \in \Gamma$ be such that $F^k_\omega(x_0) = x$ and $F^k_\omega(y_0) = y$. It is not difficult to check that

$$\left| \frac{\varphi_\omega(x_0)}{JF^k_\omega(x_0)} \Big/ \frac{\varphi_\omega(y_0)}{JF^k_\omega(y_0)} - 1 \right| \le (1 + C_\varphi \beta^k)\, C(\epsilon) + C_\varphi \beta^k,$$

where $C(\epsilon)$ only depends on the constants from (A.IV). $\square$

Proof of Lemma 5.3. Assume for definiteness that i is even. For $\Gamma \in \tilde{\Xi}_i$, let $\tilde{\lambda}_\omega = \lambda_\omega \times \lambda'_\omega$, so $\Pi_{\omega *}(\tilde{\lambda}_\omega|\Gamma) = C \cdot \lambda_\omega|\Pi_\omega(\Gamma)$. Let $\nu_{\sigma^{\tau_{i-1}}\omega} = F^{\tau_{i-1}}_\omega{}_*(\lambda_\omega|\Pi_\omega(\Gamma))$, we have:

$$(\lambda \times \lambda')_\omega(\{T_\omega = \tau_i\}|\Gamma) = \frac{1}{\nu_{\sigma^{\tau_{i-1}}\omega}(\Delta_{\sigma^{\tau_{i-1}}\omega,0})} \cdot \nu_{\sigma^{\tau_{i-1}}\omega}(\Delta_{\sigma^{\tau_{i-1}}\omega,0} \cap F^{-(\tau_i - \tau_{i-1})}_{\sigma^{\tau_{i-1}}\omega} \Delta_{\sigma^{\tau_i}\omega,0}),$$

Sublemma 5.5 (2) applies to ν and the result follows from the definition of $V^{\tau_i-\tau_{i-1}}_{\sigma^{\tau_{i-1}}\omega}$. $\square$

We omit the proof of Lemma 5.4 which is based on Sublemma 5.5 (1) and (2).

The main estimate of this subsection follows (see Proposition 5.7 for its relevance):

Proposition 5.6 (Joint return time asymptotics). *If (F, Δ) satisfies (A.I)–(A.VI) (in particular, all conditions in (A.V)), then there exist $\widehat{C}_2(\epsilon) < \widehat{C}_1(\epsilon)$, a subset $\Omega_5 \subset \Omega_3 \cap \Omega_4$ of full measure, and a random variable $n_5 \ge \max(n_3, n_4)$ on Ω_5 so that*

$$P(\{n_5(\omega) > n\}) \le Ce^{-(n^{\frac{1}{8}}/\widehat{C}_2(\epsilon))}$$

and such that for every pair λ, λ' of absolutely continuous probability measures on $\{\Delta_\omega\}$ having densities φ, φ' in $\mathcal{F}^+_\beta \cap \mathcal{L}^{\mathcal{K}_\omega}_\infty$, there is $C_{\lambda,\lambda'}(\epsilon)$, so that for each $\omega \in \Omega_5$ and all $n > n_5(\omega)$

$$(\lambda \times \lambda')_\omega(\{T_\omega > n\}) \le C_{\lambda,\lambda'}(\epsilon) e^{-(n^{\frac{1}{8}}/\widehat{C}_1(\epsilon))}\,.$$

Moreover, $C_{\lambda,\lambda'}$ depends on λ and λ' only through the Lipschitz constants of φ and φ'.

Proof of Proposition 5.6. We use the notation $\tilde{\lambda} = \lambda \times \lambda'$. For $0 < v < 1/4$ to be fixed later, we have, just like (8.2):

$$\begin{aligned}\tilde{\lambda}(\{T_\omega > n\}) &= \sum_{i \le n^v} \tilde{\lambda}(\{T_\omega > n\} \cap \{\tau^\omega_{i-1} \le n < \tau^\omega_i\}) + \tilde{\lambda}(\{T_\omega > n\} \cap \{\tau^\omega_{[n^v]} \le n\}) \\ &=: (\mathrm{I}) \ + \ (\mathrm{II}).\end{aligned}$$

The key remark to estimate (I) and (II) is that for a fixed $\omega \in \Omega$, the points (x, x') of each element of $\tilde{\Xi}^{(\omega)}_q$ are either all good or all bad for the condition $S^{\{\tau^\omega_i(x,x')\}}_q(\omega) > q\rho$.

Moreover, $V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i-\tau_{i-1}}$ depends only on τ_j for $1 \le j \le i$. For ω and $i \le q$, we say that an element $\Gamma \in \widetilde{\Xi}_i^{(\omega)}$ is *q-bad* if it only contains points such that $S_q^{\{\tau_i^\omega(x,x')\}} \le q\rho$. The other $\Gamma \in \widetilde{\Xi}_i^{(\omega)}$ are called *q-good*.

Fixing $\omega \in \Omega_4 \cap \Omega_3$, we omit the dependence of $\tilde{\lambda}$, T, and τ_i on ω from the notation.

Let us focus first on the term (II). Since the densities of λ and λ' are in $\mathcal{L}_\infty^{\mathcal{K}_\omega}$, for n such that $n^v \ge n_4(\omega)$, Corollary 5.2 gives

$$(\mathrm{II}) = \tilde{\lambda}(\{T_\omega > n\} \cap \{\tau_{[n^v]} \le n\}) \ \le\ C_{\varphi,\varphi'}\, \kappa^{[n^v/2]} + \sum_{\substack{\Gamma \in \widetilde{\Xi}^\omega_{[n^v]} \\ \Gamma\ [n^v]\text{-good}}} \tilde{\lambda}(\{T_\omega > \tau_{[n^v]}\} \cap \Gamma).$$

Now, denoting by Γ_i the element of $\widetilde{\Xi}_i^\omega$ containing $\Gamma \in \widetilde{\Xi}^\omega_{[n^v]}$ for $i \le [n^v]$, we may decompose

$$\tilde{\lambda}(\{T > \tau_{[n^v]}\} \cap \Gamma) = \tilde{\lambda}(\{T > \tau_2\} \cap \Gamma_2) \prod_{i=3}^{[n^v]} \frac{\tilde{\lambda}(\{T > \tau_i\} \cap \Gamma_i)}{\tilde{\lambda}(\{T > \tau_{i-1}\} \cap \Gamma_{i-1})}\,.$$

Therefore for each $[n^v]$-good Γ, using $V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i-\tau_{i-1}}$ associated to the corresponding stopping times, we obtain from Lemma 5.3,

$$\begin{aligned}
&\tilde{\lambda}(\{T > \tau_{[n^v]}\} \cap \Gamma) \\
&\quad = \tilde{\lambda}(\Gamma_2)\ \tilde{\lambda}(\{T > \tau_2\} | \Gamma_2) \prod_{i=3}^{[n^v]} \tilde{\lambda}(\{T > \tau_i\} | \{T > \tau_{i-1}\} \cap \Gamma_i) \ \prod_{i=3}^{[n^v]} \frac{\tilde{\lambda}(\{T > \tau_{i-1}\} \cap \Gamma_i)}{\tilde{\lambda}(\{T > \tau_{i-1}\} \cap \Gamma_{i-1})} \\
&\quad \le \tilde{\lambda}(\Gamma_2) \prod_{i=2}^{[n^v]} (1 - V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i-\tau_{i-1}}/C_{\lambda,\lambda'}) \ \prod_{i=3}^{[n^v]} \frac{\tilde{\lambda}(\{T > \tau_{i-1}\} \cap \Gamma_i)}{\tilde{\lambda}(\{T > \tau_{i-1}\} \cap \Gamma_{i-1})}.
\end{aligned}$$

Hence (making use of the consequences of $i \ge i_0(\lambda,\lambda')$ in Lemma 5.3),

$$\begin{aligned}
&\sum_{\substack{\Gamma \in \widetilde{\Xi}^\omega_{[n^v]} \\ \Gamma\ [n^v]\text{-good}}} \tilde{\lambda}(\{T > \tau^\omega_{[n^v]}\} \cap \Gamma) \\
&\qquad\qquad \le \sum_{\substack{\Gamma_2 \subset \Gamma \\ \text{good}}} \tilde{\lambda}(\Gamma_2) \sum_{\substack{\Gamma_3 \subset \Gamma_2 \\ \text{good}}} \frac{\tilde{\lambda}(\{T > \tau_2\} \cap \Gamma_3)}{\tilde{\lambda}(\{T > \tau_2\} \cap \Gamma_2)} \times \dots \\
&\qquad\qquad\qquad \times \sum_{\substack{\Gamma \subset \Gamma_{[n^v]-1} \\ \text{good}}} \frac{\tilde{\lambda}(\{T > \tau_{[n^v]-1}\} \cap \Gamma)}{\tilde{\lambda}(\{T > \tau_{[n^v]-1}\} \cap \Gamma_{[n^v]-1})} \times \prod_{i=2}^{[n^v]} (1 - V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i-\tau_{i-1}}/C_{\lambda,\lambda'}) \\
&\qquad\qquad \le e^{-[n^v]\rho/C}\,,
\end{aligned}$$

where we used $\omega \notin \widetilde{M}_{[n^v]}$ and also the fact that

$$\sum_{\substack{\Gamma_2\subset\Gamma\\ \text{good}}} \tilde\lambda(\Gamma_2) \sum_{\substack{\Gamma_3\subset\Gamma_2\\ \text{good}}} \frac{\tilde\lambda(\{T>\tau_2\}\cap\Gamma_3)}{\tilde\lambda(\{T>\tau_2\}\cap\Gamma_2)} \cdots \sum_{\substack{\Gamma\subset\Gamma_{[n^v]-1}\\ \text{good}}} \frac{\tilde\lambda(\{T>\tau_{[n^v]-1}\}\cap\Gamma)}{\tilde\lambda(\{T>\tau_{[(n^v]-1}\}\cap\Gamma_{[n^v]-1})} \le 1\,.$$

Finally, we get (II) $\le C_{\lambda,\lambda'}\kappa^{[n^v/2]} + e^{-[n^v]\rho/C}$.

Let us turn our attention to the term (I). Fix $0\le i\le n^v$ and decompose

$$\begin{aligned}&\tilde\lambda(\{T>n;\ \tau_{i-1}\le n<\tau_i\})\\ &\quad= \sum_{\substack{(k_1\dots k_{i-1})\\ \sum k_j\le n}} \tilde\lambda\Big(\{\tau_i-\tau_{i-1}>n-\sum_{j=1}^{i-1}k_j\,;\ \tau_j-\tau_{j-1}=k_j\ ,j=1,\dots,i-1\}\Big)\end{aligned}$$

Fixing $k_1,\dots,k_{i-1}$, conditioning, using Lemma 5.4 and the asymptotics (A.V) on the return times, we get if $n>\sum k_j + n_3(\sigma^{\tau_{i-1}+\ell_0}\omega)+\ell_0$:

$$\begin{aligned}&\tilde\lambda(\{\tau_i-\tau_{i-1}>n-\sum_{j=1}^{i-1}k_j;\tau_j-\tau_{j-1}=k_j\ ,j=1,\dots,i-1\})\\ &\quad\le \prod_{j=1}^{i-1} Cm(\Delta_{\sigma^{\tau_j+\ell_0}\omega}) \prod_{\substack{j=1,\dots i-1\\ k_j>n_3(\sigma^{\tau_{j-1}+\ell_0}\omega)+\ell_0}} e^{-[k_j-\ell_0]^{1/4}/C_1}\\ &\qquad\qquad \cdot Ce^{-\left(\left[n-\sum k_j-\ell_0\right]^{1/4}/C_1(\epsilon)\right)}\\ &\quad\le \left(\prod_{j=0}^{i-1} n_3(\sigma^{\tau_j+\ell_0}\omega)\right) e^{-n^{1/4}/C_1(\epsilon)}\left(C(\epsilon)e^{\ell_0^{1/4}/C_1(\epsilon)}\right)^i\\ &\qquad\qquad \cdot e^{\left(\sum_j\left[k_j|k_j\le\ell_0+n_3(\sigma^{\tau_{j-1}+\ell_0}\omega)\right]^{1/4}/C_1(\epsilon)\right)}\,.\end{aligned}\tag{5.5}$$

Now, since $P(\{n_3(\omega)>n\})\le Ce^{-(n^{1/4}/C_2(\epsilon))}$, conditioning with respect to elements of the partition $\widetilde{\Xi}_{[n^v]}$ (see the similar argument appearing between (8.7) and (8.8) in the proof of Proposition 8.3 below for details), we get for $0<\hat\rho<1$ a subset $\Omega_5\subset\Omega_\cap\Omega_3$ of full measure with the following property: For $\omega\in\Omega_5$, there exists $n_5(\omega)\ge\max(n_4(\omega),n_3(\omega))$ (with the bounds stated in Proposition 5.6) such that $\forall n\ge n_5(\omega)$, the $\tilde\lambda$-measure of the cylinders in (I) which violate the condition

$$\sum_{j=0}^{i}(n_3(\sigma^{\tau_j+\ell_0}\omega))^{1/4}\le\hat\rho n^{1/4} \text{ and } \prod_{j=0}^{i-1}n_3(\sigma^{\tau_j+\ell_0}\omega)\le e^{n^v\log(\hat\rho n)}\,,\quad \forall i\le n^v \tag{5.6}$$

is less than $e^{-(n^{1/4-v}/C(\epsilon))}$.

Next, summing (5.5) over the k_j such that $n > \sum k_j + n_3(\sigma^{\tau_{i-1}+\ell_0}\omega) + \ell_0$, the contribution of those cylinders which satisfy (5.6) is not larger than

$$Ce^{n^w} e^{n^v \log(n)\hat{\rho}} \; e^{-(n^{\frac{1}{4}}/C_1)} \left(C \; e^{\ell_0^{1/4}}\right)^{n^v} \; e^{n^v \ell_0^{1/4}} e^{\hat{\rho} n^{1/4}} \; \leq C(\epsilon) e^{-(n^{1/4}/\widehat{C}_1(\epsilon))} \,,$$

where the factor e^{n^w} with $v < w < 1/4$ comes from the different choices for $(k_1, \cdots, k_i)$.

It only remains to consider the sum over terms with $n \leq \sum k_j + n_3(\sigma^{\tau_{i-1}+\ell_0}\omega) + \ell_0$ which may be estimated by

$$\left(\prod_{j=0}^{i-1} n_3(\sigma^{\tau_j+\ell_0}\omega)\right) \left(Ce^{\ell_0^{1/4}}\right)^i e^{-(\sum k_j^{\frac{1}{4}}/C_1)} e^{\left(\sum k_j^{\frac{1}{4}} | k_j \leq \ell_0 + n_3(\sigma^{\tau_{j-1}+\ell_0}\omega)\right)/C}$$

$$\leq \left(\prod_{j=0}^{i-1} n_3(\sigma^{\tau_j+\ell_0}\omega)\right) \left(Ce^{\ell_0^{1/4}}\right)^i e^{-(n^{1/4}/C_1)} e^{\left(\sum_{j=0}^{i} n_3^{\frac{1}{4}}(\sigma^{\tau_j+\ell_0}\omega)\right)/C} .$$

So, if $n \geq n_5(\omega)$ the contribution to the sum over those terms of the cylinders satisfying (5.6) is not larger than $Ce^{-(n^{1/4}/\widehat{C}_1(\epsilon))}$. Finally, we get that (I) is less than $C\big(e^{-(n^{1/4}/\widehat{C}_1(\epsilon))} + e^{-(n^{1/4-v}/\widehat{C}_1(\epsilon))}\big)$. Combining this with the estimate on (II) ends the proof of Proposition 5.6 with upper bound $\max\big(e^{-(n^v/\widehat{C}_1(\epsilon)}, e^{-(n^{1/4-v}/\widehat{C}_1(\epsilon))}\big)$. The optimal choice is $v = 1/4 - v$, i.e., $v = 1/8$. $\square$

5.C Random coupling: matching $(F_\omega^n)_*(\lambda_\omega)$ with $(F_\omega^n)_*(\lambda'_\omega)$.

Let λ, λ' be absolutely continuous probability measures on $\{\Delta_\omega\}$ with densities φ, φ' in $\mathcal{F}_\beta^+ \cap \mathcal{L}_\infty^{\mathcal{K}_\omega}$. In this subsection, we shall match $(F_\omega^n)_*(\lambda_\omega)$ with $(F_\omega^n)_*(\lambda'_\omega)$. We just summarise the strategy, since the computations follow straightforwardly along the lines of [Yo2, § 3.4]. The relevant dynamical system is $\widehat{F_\omega} = (F_\omega \times F_\omega)^{T_\omega}$ which maps $\Delta_\omega \times \Delta_\omega$ into $\Delta_{T_\omega} \times \Delta_{T_\omega}$. The "matching" is done using a sequence of *(joint) stopping times* which are the successive entrance times into $\Delta_{\cdot,0} \times \Delta_{\cdot,0}$:

$$T_{1,\omega} = T_\omega, \quad T_{n,\omega} = T_{n-1,\omega} + T_{\sigma^{T_{n-1}}\omega} \circ \widehat{F}^{n-1}.$$

Denote by $\hat{\Xi}_i^\omega$ the largest partition of $\Delta_\omega \times \Delta_\omega$ on which $T_{1,\omega}, \cdots, T_{i,\omega}$ are constant.

Proposition 5.7 (Matching, joint return times, joint stopping times). *Let λ, λ' be absolutely continuous probability measures on $\{\Delta_\omega\}$ with densities φ, φ' in $\mathcal{F}_\beta^+ \cap \mathcal{L}_\infty^{\mathcal{K}_\omega}$, and let $i_1(\varphi, \varphi')$ be such that $\max(C_\varphi, C_{\varphi'})\beta^{i_1} < C$. There exists $0 < \theta < 1$ such that for almost all ω, all $i \geq i_1$ and all n*

$$\begin{aligned} |(F_\omega^n)_*(\lambda_\omega) \; - \; (F_\omega^n)_*(\lambda'_\omega)| &\leq 2(\lambda_\omega \times \lambda'_\omega)(\{T_{i,\omega} > n\}) \\ &\quad + 2\sum_{j=i}^{\infty} (1-\theta)^{j-i+1} (\lambda_\omega \times \lambda'_\omega)(\{T_{j,\omega} \leq n < T_{j+1,\omega}\}). \end{aligned} \tag{5.7}$$

Proof of Proposition 5.7. Just rewrite the proofs of Lemmas 3 (3') and 4 in [Yo2], remarking that the constants appearing there do not depend on ω in our context. $\square$

The following lemma is proved in the same way as Lemma 5.4 (see [Yo2, Sublemma 4]).

Lemma 5.8 (Relating joint stopping times and joint return times). *Let λ, λ' be absolutely continuous probability measures on $\{\Delta_\omega\}$ with densities φ, φ' in $\mathcal{F}_\beta^+ \cap \mathcal{L}_\infty^{\mathcal{K}_\omega}$. Then there is $C_{\varphi,\varphi'}$, depending only on the Lipschitz constants of φ, φ', so that for almost all ω, all i, each $\Gamma \in \hat{\Xi}_i$, and all n*

$$(\lambda \times \lambda')_\omega(\{T_{i+1,\omega} - T_{i,\omega} > n | \Gamma\}) \leq C_{\varphi,\varphi'}(m \times m)(\{T_{\sigma^{T_i}\omega} > n\}) .$$

Proposition 5.9 (Matching). *If (F, Δ) satisfies (A.I)–(A.VI) (in particular, all conditions in (A.V)), then there exist $\widetilde{C}_2(\epsilon) < \widetilde{C}_1(\epsilon)$, Ω_6 of full measure, and a random variable $n_6 : \Omega_6 \to \mathbb{Z}_+$ with $P(\{n_6(\omega) > n\}) \leq Ce^{-(n^{\frac{1}{16}}/\widetilde{C}_2(\epsilon))}$, such that, for each pair λ, λ' of absolutely continuous probability measures on $\{\Delta_\omega\}$ with densities φ and φ' in $\mathcal{F}_\beta^+ \cap \mathcal{L}_\infty^{\mathcal{K}_\omega}$ there is $C_{\lambda,\lambda'}(\epsilon)$, depending only on the Lipschitz constants of φ, φ', so that for each $\omega \in \Omega_6$ and $n \geq n_6(\omega)$,*

$$|(F_\omega^n)_*(\lambda_\omega) - (F_\omega^n)_*(\lambda'_\omega)| \leq C_{\lambda,\lambda'}(\epsilon)e^{-(n^{\frac{1}{16}}/\widetilde{C}_1(\epsilon))} .$$

Sketch of proof of Proposition 5.9. The proof follows that of Proposition 5.6, using Proposition 5.7 and Lemma 5.8. We just sketch how $n_6(\omega)$ is constructed.

Let $0 < s < 1/8$ and let $n_5(\omega)$, $\hat{\rho}$ be as in the proof of Proposition 5.6. The random variable $n_6(\omega)$ is characterized by the following property: For $n \geq n_6(\omega)$ and $i \leq n^s$

$$\sum_{j=0}^{i-1}(n_5(\sigma^{T_{j,\omega}}\omega))^{\frac{1}{8}} \leq \hat{\rho}n^{\frac{1}{8}} ,$$

for the "good" atoms of the partition $\hat{\Xi}_i^\omega$; additionally the mass of the "bad" atoms of the partition $\hat{\Xi}_i^\omega$ is less than $e^{-(n^{1/8-s}/C)}$. As in the proof of Proposition 5.6, the optimal choice is for $s = 1/8 - s$, i.e., $s = 1/16$. $\square$

5.D Future random correlations.

Our key lemma is now a corollary of Proposition 5.9:

Corollary 5.10 ("Future" correlations). *Assume (F, Δ) satisfies (A.I)–(A.VI) (in particular, all conditions in (A.V)) and let $\mathcal{K}_\omega$ be as in Theorem 3.2. There are $C(\epsilon)$, $v > 1$, and $\Omega_7 \subset \Omega_6$ of full measure, and for each $\omega \in \Omega_7$ there is $C(\omega)$ with*

$$P(\{C(\omega) > \ell\}) \leq \frac{C(\epsilon)}{\ell^v} ,$$

so that for each $\varphi \in \mathcal{L}_\infty^{\mathcal{K}_\omega}$, $\psi \in \mathcal{F}_\beta^{\mathcal{K}_\omega}$, and all n

$$\Big| \int \varphi_{\sigma^n\omega} \circ F_\omega^n \cdot \psi_\omega \, dm \;-\; \int \varphi_{\sigma^n\omega} d\mu_{\sigma^n\omega} \int \psi_\omega \, dm \Big| \\ \leq C(\omega) C(\epsilon) \, \|\varphi\|_{\mathcal{L}_\infty} \, \|\psi\|_{\mathcal{F}} \, e^{-(n^{\frac{1}{16}}/C(\epsilon))}.$$

Proof of Corollary 5.10. We start by showing that for all $\varphi \in \mathcal{L}_\infty^{\mathcal{K}_\omega}$, $\psi \in \mathcal{F}_\beta^{\mathcal{K}_\omega}$, and all n, there are $C(\omega)$ (as in the statement) and $C_{\varphi,\psi}(\epsilon) > 0$ such that

$$\left| \int \varphi_{\sigma^n\omega} \circ F_\omega^n \cdot \psi_\omega \, dm \;-\; \int \varphi_{\sigma^n\omega} \, d\mu_{\sigma^n\omega} \int \psi_\omega \, dm \right| \leq C(\omega) \, C_{\varphi,\psi}(\epsilon) \, e^{-(n^{\frac{1}{16}}/C(\epsilon))}. \quad (5.8)$$

Assume first that $\psi \in \mathcal{F}_\beta^+ \cap \mathcal{L}_\infty^{\mathcal{K}_\omega}$. Proposition 5.9 applied to $\lambda_\omega = \big(\int \psi_\omega \, dm\big)^{-1} \psi_\omega m$ and μ_ω gives that for $n \geq n_6(\omega)$,

$$\begin{aligned} \Big| \int \varphi_{\sigma^n\omega} \circ F_\omega^n \cdot \psi_\omega \, dm - \int \varphi_{\sigma^n\omega} \, d\mu_{\sigma^n\omega} \int \psi_\omega \, dm \Big| & \\ = \int \psi_\omega \, dm \; \Big| \int \varphi_{\sigma^n\omega} d\big[(F_\omega^n)_*(\lambda_\omega) - (F_\omega^n)_*(\mu_\omega)\big] \Big| & \\ \leq C_{\lambda,\mu}(\epsilon) \cdot \int \psi_\omega \, dm \cdot \sup |\varphi_{\sigma^n\omega}| e^{-(n^{\frac{1}{16}}/\widetilde{C}_1)} & \\ \leq C_{\lambda,\mu}(\epsilon) C_\psi m(\Delta_\omega) \mathcal{K}_\omega \, C'_\varphi \mathcal{K}_{\sigma^n\omega} e^{-(n^{\frac{1}{16}}/\widetilde{C}_1)} \, . & \end{aligned}$$

Now, define $n_7(\omega) = \inf\{k \geq n_6(\omega) \mid \mathcal{K}_{\sigma^k\omega} \leq k\}$. By (3.8) and the bounds on n_6 from Proposition 5.9, we get $P(\{n_7(\omega) > k\}) \leq e^{-(k^{\frac{1}{16}}/\widetilde{C}_2)}$. We find for $n > n_7(\omega)$,

$$\left| \int \varphi_{\sigma^n\omega} \circ F_\omega^n \cdot \psi_\omega \, dm \;-\; \int \varphi_{\sigma^n\omega} d\mu_{\sigma^n\omega} \int \psi_\omega \, dm \right| \leq \; C_{\varphi,\psi}(\epsilon) \mathcal{K}_\omega n \; e^{-(n^{\frac{1}{16}}/\widetilde{C}_1)}.$$

If $n \leq n_7(\omega)$,

$$\Big| \int \varphi_{\sigma^n\omega} \circ F_\omega^n \cdot \psi_\omega \, dm \;-\; \int \varphi_{\sigma^n\omega} d\mu_{\sigma^n\omega} \int \psi_\omega dm \Big| \\ \leq C_{\varphi,\psi}(\epsilon) \cdot C(\omega) e^{-(n^{\frac{1}{16}}/\widetilde{C}_1)},$$

setting

$$C(\omega) := e^{(n_6(\omega)^{\frac{1}{16}}/\widetilde{C}_1)} \cdot \mathcal{K}_\omega \cdot \max_{n \leq n_7(\omega)} \mathcal{K}_{\sigma^n\omega} \, .$$

This gives (5.8) if ψ belongs to $\mathcal{F}_\beta^+ \cap \mathcal{L}_\infty^{\mathcal{K}_\omega}$. For non negative real-valued $\psi \in \mathcal{F}_\beta^{\mathcal{K}_\omega}$, remark that $\tilde{\psi}_\omega = \psi_\omega + (C_\psi + 1)\mathcal{K}_\omega$ belongs to $\mathcal{F}_\beta^+ \cap \mathcal{L}_\infty^{\mathcal{K}_\omega}$ and apply the above estimate to $\tilde{\psi}$. General real-valued functions are decomposed into positive and negative parts. Complex-valued functions are decomposed into real and imaginary parts.

Next, we prove that $C(\omega)$ has the announced behaviour. Fix $0 < u < 1$ such that $\frac{\widetilde{C}_1(\epsilon)(1-u)}{\widetilde{C}_2(\epsilon)} > 1$, and use (3.8) and Proposition 5.9 again

$$
\begin{aligned}
&P(\{C(\omega) > \ell\}) \\
&\qquad \le P(\{\sup_{n \le n_7(\omega)} \mathcal{K}_{\sigma^n\omega} > \ell^{\frac{u}{2}}\}) + P(\{e^{n_6(\omega)^{\frac{1}{16}}/\widetilde{C}_1} > \ell^{1-u}\}) + P(\{\mathcal{K}_\omega > \ell^{\frac{u}{2}}\}) \\
&\qquad \le P(\{n_(\omega) > \ell\}) + \sum_{n=1}^{\ell} P(\{\mathcal{K}_{\sigma^n\omega} > \ell^{\frac{u}{2}}\}) \\
&\qquad\qquad + P(\{n_6(\omega) > [(1-u)\widetilde{C}_1 \log \ell]^{16}\}) + P(\{\mathcal{K}_\omega > \ell^{\frac{u}{2}}\}) \\
&\qquad \le e^{-(\ell^{\frac{1}{16}}/\widetilde{C}_2)} + \ell e^{-(\ell^{\frac{u}{8}}/C(\epsilon))} + e^{-[(\log \ell)\widetilde{C}_1(1-u)/\widetilde{C}_2]} + e^{-\ell^{\frac{u}{8}}/C(\epsilon)}.
\end{aligned}
$$

This proves the claim on the random variable $C(\omega)$, taking $v = \widetilde{C}_1(1-u)/\widetilde{C}_2 > 1$.

To conclude, it remains to show that $C_{\varphi\,\psi}(\epsilon) \le C(\epsilon)\|\varphi\|_{\mathcal{L}_\infty}\|\psi\|_{\mathcal{F}}$. We adapt to our random setting an argument of Collet [Co2] based on the uniform boundedness principle. Fix $\psi \in \mathcal{F}_\beta^{\mathcal{K}_\omega}$ and define

$$
p_{n,\omega}^{\psi}(\varphi) = \frac{e^{(n^{\frac{1}{16}}/\widetilde{C}_1)}}{C(\omega)} \left| \int \varphi_{\sigma^n\omega} \circ F_\omega^n \cdot \psi_\omega \, dm \;-\; \int \varphi_{\sigma^n\omega} \, d\mu_{\sigma^n\omega} \int \psi_\omega \, dm \right| .
$$

It follows from (5.8) that $\sup_{n,\omega\in\Omega_7} p_{n,\omega}^{\psi}(\varphi) < \infty$ for all $\varphi \in \mathcal{L}_\infty^{\mathcal{K}_\omega}$. The uniform boundedness principle gives a constant $D_\psi(\epsilon)$ such that

$$
\sup_{n,\omega\in\Omega_7, \|\varphi\|_{\mathcal{L}_\infty} \le 1} p_{n,\omega}^{\psi}(\varphi) \le D_\psi(\epsilon). \tag{5.9}
$$

For $n \in \mathbb{Z}_+$, $\omega \in \Omega_7$ and $\varphi \in \mathcal{L}_\infty^{\mathcal{K}_\omega}$ with $\|\varphi\|_{\mathcal{L}_\infty} \le 1$, set

$$
q_{n,\omega,\varphi}(\psi) = \frac{e^{(n^{\frac{1}{16}}/\widetilde{C}_1)}}{C(\omega)} \left| \int \varphi_{\sigma^n\omega} \circ F_\omega^n \cdot \psi_\omega \, dm \;-\; \int \varphi_{\sigma^n\omega} \, d\mu_{\sigma^n\omega} \int \psi_\omega \, dm \right| .
$$

It follows from (5.9) that for any $\psi \in \mathcal{F}_\beta^{\mathcal{K}_\omega}$,

$$
\sup_{n,\omega\in\Omega_7, \|\varphi\|_{\mathcal{L}_\infty} \le 1} q_{n,\omega,\varphi}(\psi) \le D_\psi(\epsilon)\,.
$$

Using once more the uniform boundedness principle, we conclude that there exists $C(\epsilon)$ so that

$$
\sup_{n,\omega\in\Omega_7, \|\varphi\|_{\mathcal{L}_\infty} \le 1, \|\psi\|_{\mathcal{F}} \le 1} q_{n,\omega,\varphi}(\psi) \le C(\epsilon).
$$

This ends the proof of Corollary 5.10. $\square$

6. Random coupling argument, "past" correlations

Assume (F, Δ) satisfies (A.I)–(A.VI) (for Lemmas 6.1 and 6.2 we shall not require (3.2) from (A.V)). The estimates for the "past" correlations are obtained by recycling the arguments of Section 5:

Lemma 6.1 (Lower bound for $P(\{T_\omega = \tau_i\})$). *Let λ, λ' be absolutely continuous probability measures on $\{\Delta_\omega\}$, with densities φ, φ' in $\mathcal{F}_\beta^+$. For each i, if $\Gamma \in \widetilde{\Xi}_i^{\sigma^{-n}\omega}$ is such that $(T_{\sigma^{-n}\omega})_{|\Gamma} > \tau_{i-1}$, then, associating $V^{\tau_i - \tau_{i-1}}_{\sigma^{\tau_{i-1}-n}\omega}$ to Γ as usual,*

$$(\lambda \times \lambda')(\{T_{\sigma^{-n}\omega} > \tau_i\}|\Gamma\}) \le 1 - V^{\tau_i-\tau_{i-1}}_{\sigma^{\tau_{i-1}-n}\omega}/C_{\lambda,\lambda'}(\epsilon)\,,$$

where $C_{\lambda,\lambda'}(\epsilon) > 1$ depends only on the Lipschitz constant of φ, φ'. This dependence may be removed if we consider $i \ge i_0(\lambda, \lambda')$.

Lemma 6.2 (Relating stopping times and return times). *Let λ, λ' be absolutely continuous probability measures on $\{\Delta_\omega\}$, with densities φ, φ' in $\mathcal{F}_\beta^+$. For each $\Gamma \in \widetilde{\Xi}_i^{\sigma^{-n}\omega}$, we have for all ℓ*

$$\begin{aligned}(\lambda \times \lambda')_{\sigma^{-n}\omega}(\{\tau_{i+1} - \tau_i > \ell_0 + \ell\} \mid \Gamma) \\ \le C_{\lambda,\lambda'}(\epsilon) m(\{R_{\sigma^{\tau_i+\ell_0-n}\omega} > \ell\}) \cdot m(\Delta_{\sigma^{\tau_i+\ell_0-n}\omega}).\end{aligned}$$

where $C_{\lambda,\lambda'}(\epsilon)$ depends on the Lipschitz constants of φ, φ'. This dependence may be removed if we consider $i \ge i_0(\lambda, \lambda')$.

Proposition 6.3 (Joint return time asymptotics). *For every pair λ, λ' of absolutely continuous probability measures on $\{\Delta_\omega\}$ having densities φ, φ' in $\mathcal{F}_\beta^+ \cap \mathcal{L}_\infty^{\mathcal{K}_\omega}$ there is $C_{\lambda,\lambda'}(\epsilon)$ so that for each $\omega \in \Omega_5$ and all $n > n_5(\omega)$*

$$(\lambda \times \lambda')_{\sigma^{-n}\omega}(\{T_{\sigma^{-n}\omega} > \ell\}) \le C_{\lambda,\lambda'}(\epsilon)\, e^{-(\ell^{\frac{1}{8}}/C(\epsilon))}\,.$$

Moreover, $C_{\lambda,\lambda'}(\epsilon)$ depends on λ and λ' only through the Lipschitz constants of φ, φ'.

Proof of Proposition 6.3. This is just Proposition 5.6 written for $\sigma^{-n}\omega$. $\square$

Proposition 6.4 (Matching). *There exist $\widetilde{C}_2(\epsilon) < \overline{C}_1(\epsilon)$, a subset $\Omega_8 \subset \Omega_5$ of full measure and a random variable $n_8 : \Omega_8 \to \mathbb{Z}_+$ with $P(\{n_8(\omega) > n\}) \le Ce^{-(n^{\frac{1}{16}}/\widetilde{C}_2(\epsilon))}$ such that for each pair λ, λ' of absolutely continuous probability measures on $\{\Delta_\omega\}$ with densities φ, φ' in $\mathcal{F}_\beta^+ \cap \mathcal{L}_\infty^{\mathcal{K}_\omega}$, there exists $C_{\lambda,\lambda'}(\epsilon)$, depending on λ and λ' only through the Lipschitz constants of φ, φ', such that, for each $\omega \in \Omega_8$ and all $n \ge n_8(\omega)$,*

$$\left|(F^n_{\sigma^{-n}\omega})_*(\lambda) - (F^n_{\sigma^{-n}\omega})_*(\lambda')\right| \le C_{\lambda,\lambda'}(\epsilon) e^{-(n^{\frac{1}{16}}/\overline{C}_1(\epsilon))}.$$

Proof of Proposition 6.4. The proof is along the lines of that of Proposition 5.9, we just discuss the random variable n_8. Let the sequence of successive joint entrance times

$T_{1,\omega}, \cdots, T_{k,\omega}, \cdots$, in $\Delta_{\omega,0} \times \Delta_{\omega,0}$ be as in Section 5. For fixed $i \leq n$, let $\hat{\Xi}_i^{\sigma^{-n}\omega}$ be the largest partition of $\Delta_{\omega,0} \times \Delta_{\omega,0}$ on which the $T_{1,\sigma^{-n}}, \cdots, T_{i,\sigma^{-n}\omega}$ are constant. Let $n_5(\omega)$ be as defined by Proposition 5.6. The random variable $n_8(\omega)$, defined on Ω_8, is such that, on the one hand, for $i \leq n^t$ (where $0 < t < 1/8$ will be fixed later on) and all $n \geq n_8(\omega)$

$$\sum_{j=0}^{i-1} (n_5(\sigma^{-n+T_{j,\sigma^{-n}\omega}}\omega))^{1/8} \leq \hat{\rho} n^{1/8}$$

for the "good" atoms of the partition $\hat{\Xi}_i^{\sigma^{-n}\omega}$, and, on the other hand, the mass of the "bad" atoms of the partition $\hat{\Xi}_i^{\sigma^{-n}\omega}$ is less than $e^{-n^{1/8-t}/C}$. Choose $t = 1/8 - t = 1/16$ to get the optimal rate. $\square$

Corollary 6.5 ("Past" correlations). *Let $\mathcal{K}_\omega$ be given by Theorem 3.2. There are $C(\epsilon)$, $v > 1$, $\Omega_9 \subset \Omega_8$ of full measure and a random variable $C(\omega)$ on Ω_9 satisfying $P(\{C(\omega) > \ell\}) \leq \frac{C}{\ell^v}$, and such that for each $\varphi \in \mathcal{L}_\infty^{\mathcal{K}_\omega}$, $\psi \in \mathcal{F}_\beta^{\mathcal{K}_\omega}$ and all n*

$$\Big| \int \varphi_\omega \circ F^n_{\sigma^{-n}\omega} \cdot \psi_{\sigma^{-n}\omega} \, dm - \int \varphi_\omega d\mu_\omega \int \psi_{\sigma^{-n}\omega} \, dm \Big| \leq C(\omega) C(\epsilon) \|\varphi\|_{\mathcal{L}_\infty} \|\psi\|_{\mathcal{F}} \, e^{-(n^{\frac{1}{16}}/C(\epsilon))} .$$

Proof of Corollary 6.5. As in the proof of Corollary 5.10, we show that

$$\left| \int \varphi_\omega \circ F^n_{\sigma^{-n}\omega} \cdot \psi_{\sigma^{-n}\omega} \, dm - \int \varphi_\omega d\mu_\omega \int \psi_{\sigma^{-n}\omega} \, dm \right| \leq C(\omega) C_{\varphi,\psi}(\epsilon) \, e^{-(n^{\frac{1}{16}}/C(\epsilon))} \tag{6.1}$$

and deduce the result from the uniform boundedness principle.

Let $\psi \in \mathcal{F}_\beta^+ \cap \mathcal{L}_\infty^{\mathcal{K}_\omega}$. Proposition 6.4 applied to μ_ω and $\lambda_\omega = (\int \psi_\omega \, dm)^{-1} \psi_\omega m$ implies that for $n \geq n_8(\omega)$,

$$\begin{aligned} \Big| \int \varphi_\omega \circ F^n_{\sigma^{-n}\omega} \cdot \psi_{\sigma^{-n}\omega} \, dm \;&-\; \int \varphi_\omega d\mu_\omega \int \psi_{\sigma^{-n}\omega} \, dm \Big| \\ &\leq C_{\lambda,\mu}(\epsilon) \cdot \int \psi_{\sigma^{-n}\omega} \, dm \cdot \sup |\varphi_\omega| \, e^{-(n^{\frac{1}{16}}/\widetilde{C}_1)} \\ &\leq C_{\lambda,\mu}(\epsilon) \, C_\psi m(\Delta_{\sigma^{-n}\omega}) \mathcal{K}_{\sigma^{-n}\omega} \, C'_\varphi \mathcal{K}_\omega \, e^{-(n^{\frac{1}{16}}/\overline{C}_1(\epsilon))} . \end{aligned}$$

Now, define $n_9(\omega) = \inf\{k \geq n_8(\omega) \mid \mathcal{K}_{\sigma^{-k}\omega} \leq k\}$. The properties of $\mathcal{K}_\omega$ (see (3.8)) and $n_8(\omega)$ give $P(\{n_9(\omega) > k\}) \leq e^{-(k^{\frac{1}{16}}/\widetilde{C}_2)}$.

Replacing $\widetilde{C}_1$ by a slightly larger positive number in $e^{-(n^{\frac{1}{16}}/\overline{C}_1)}$, we find for all $n \in \mathbb{Z}_+$

$$\Big| \int \varphi_\omega \circ F^n_{\sigma^{-n}\omega} \cdot \psi_{\sigma^{-n}\omega} \, dm - \int \varphi_\omega d\mu_\omega \int \psi_{\sigma^{-n}\omega} \, dm \Big| \leq \; C_{\varphi,\psi}(\epsilon) C(\omega) e^{-(n^{\frac{1}{16}}/C(\epsilon))} ,$$

where

$$C(\omega) := \max\Bigg(\mathcal{K}_\omega,\ \max_{n\le n_9(\omega)} e^{(n_8(\omega)^{\frac{1}{16}}/\overline{C}_1)}\times$$

$$\left|\int \varphi_\omega \circ F^n_{\sigma^{-n}\omega}\cdot \psi_{\sigma^{-n}\omega}\,dm - \int \varphi_\omega d\mu_\omega \int \psi_{\sigma^{-n}\omega}\,dm\right|\Bigg).$$

The claim on the distribution of $C(\omega)$ is proved as in Corollary 5.10. This gives (6.1) for $\psi \in \mathcal{F}^+_\beta \cap \mathcal{L}^{\mathcal{K}_\omega}_\infty$. For real-valued non negative $\psi \in \mathcal{F}^{\mathcal{K}_\omega}_\beta$, remark that $\tilde{\psi}_\omega = \psi_\omega + (C_\psi + 1)\mathcal{K}_\omega$ belongs to $\mathcal{F}^+_\beta \cap \mathcal{L}^{\mathcal{K}_\omega}_\infty$ and apply the above estimate to $\tilde{\psi}$. Complex-valued functions are decomposed as in Corollary 5.10. □

Part II: Random towers for random unimodal maps

7. Fubini and partitions via random hyperbolic times

From now on, we consider the concrete setting of (H1)–(H4). Our aim in Sections 7 and 8 is to construct a countable partition of a suitable subinterval $\Lambda \subset I$, with return times in order to build a tower satisfying Axioms (A.I)–(A.VI).

7.A Preliminary estimates.

In Lemmas 7.1 and 7.3, we extend to our situation (using techniques of Benedicks and Young [BeY]) basic estimates from Viana [V, Lemmas 2.4 and 2.5] and Alves [A, Lemma 2.1] proved there under a Misiurewicz assumption. Most of the ideas used go back to [BC1, BC2]. (We do not require the topological mixing assumption (H4) at this stage.)

By (H1), we may choose $K > \max(1, (\log\sup|f'|)/(\log\lambda))$ with $\lambda^K < 8$.

Lemma 7.1 (Starting in $(-\sqrt{\epsilon}, \sqrt{\epsilon})$). *Assume (H1), (H2), and (H3). For*

$$\frac{2\alpha}{\log\lambda} < \eta < \frac{1}{4},$$

there are a constant $C > 1$ and for each small enough $\epsilon > 0$ an integer $N(\epsilon)$ with

$$-C + \frac{\log(1/\epsilon)}{(K+1)\log\lambda} \le N(\epsilon) \le C + \frac{2\log(1/\epsilon)}{\log\lambda}$$

such that for all $\omega \in \Omega$ and each x with $|x| < 2\sqrt{\epsilon}$

$$\begin{cases} \left|\left(f^{N(\epsilon)}_\omega\right)'(x)\right| \ge |x|\epsilon^{-1+\eta}, \\ \left|f^j_\omega(x)\right| > \sqrt{\epsilon}, \forall j = 1, \dots, N(\epsilon). \end{cases}$$

In the proof of Lemma 7.4 below, it will be useful to take $\eta = \log\tau/(4\log 32)$ for $\tau > \lambda^{1/5}$ from Lemma 7.3. This is the reason for the condition on α in (H1). The lower bound $N(\epsilon) \ge \log(1/\epsilon)/\log 32$ (since $\lambda^{K+1} < 8\times 4$) is also convenient in the proof of Lemma 7.4.

To prove Lemma 7.1, we shall use the following result adapted from Lemma 4.4 in [BeY], which will also help to get the "large image" property in Lemma 7.10:

Sublemma 7.2 (Random bound period). *Assume (H1), (H2), (H3) and let* $\frac{2\alpha}{\log\lambda} < \eta < 1/4$*. For* k *such that* $e^{-k} < \delta$*, let* $J_{k,\epsilon}$ *be the interval*

$$J_{k,\epsilon} = \Big[-\epsilon + \min\Big(f\big(e^{-k}\big), f\big(-e^{-k}\big)\Big), f(0) + \epsilon \Big] ,$$

and let $p = p(k, \epsilon)$ *be the largest integer* p *such that*

$$\Big| \bigcup_{\omega \in \Omega} f^j_{\sigma\omega}(J_{k,\epsilon}) \Big| < \lambda^{-\eta j} \, , \forall\, j \in [0, p] \, . \tag{7.1}$$

Then there is $C > 1$*, independent of* δ*, such that for all small enough* ϵ*:*
(1) For all $\omega \in \Omega$*, all* $y \in J_{k,\epsilon}$ *and each* $0 \le j \le p(k, \epsilon)$

$$\frac{1}{C} \le \frac{|(f^j_{\sigma\omega})'(y)|}{|(f^j)'(f(0))|} \le C \, .$$

(2) $-C + \frac{\min(2k, \log(1/\epsilon))}{(K+1)\log\lambda} \le p(k,\epsilon) \le C + \frac{\min(2k,\log(1/\epsilon))}{\log\lambda}$.
(3) For all $\omega \in \Omega$ *and all* $y \in J_{k,\epsilon}$

$$\big| \big(f^{p(k,\epsilon)}_{\sigma\omega}\big)'(y) \big| \ge \frac{1}{C} \max(e^{(2-2\eta)k}, \epsilon^{-1+\eta}) \, .$$

Proof of Sublemma 7.2. This is an adaptation of the usual "bound period estimates" of [BC1, BC2]. The starting point is the claim that there is $C > 1$, independent of ϵ and δ, and such that for every $y, \tilde y \in J_{k,\epsilon}$, all $\omega, \tilde\omega \in \Omega$, and all $1 \le j \le p(k, \epsilon) + 1$

$$\big| f^j_{\sigma\omega}(y) - f^j_{\sigma\tilde\omega}(\tilde y) \big| \le C \big| (f^{j-1})'(f(0)) \big| \cdot \max(e^{-2k}, \epsilon) \, . \tag{7.2}$$

To check (7.2), we first verify inductively that

$$\big| f^j_{\sigma\omega}(y) - f^j_{\sigma\tilde\omega}(\tilde y) \big| \le \big[d_j(\cdots(d_2(d_1+1)\cdots)+1 \big] \, C \, \max(e^{-2k}, \epsilon) =: [m_j] \, C \, \max(e^{-2k}, \epsilon) \, ,$$

where $d_i = |f'_{\sigma^i\omega}(x_i)| = |f'(x_i)|$ for some $x_i \in [f^{i-1}_{\sigma\omega}(y), f^{i-1}_{\sigma\tilde\omega}(\tilde y)]$.

Then, to estimate m_j, we let $\hat d_i = |f'(f^i(0))|$, and we note that since $|f^i(0) - x_i| < \lambda^{-i\eta}$ for $1 \le i \le p+1$, by definition of p, and $|\hat d_i| \ge e^{-\alpha i}/C$ by (H1)(ii), standard arguments involving (H2) and using $e^{-j\eta\log\lambda} < e^{-2\alpha j}$ (see [BeY, Lemma 1.3]) give that there is $C > 1$ with

$$C^{-1} \le \frac{\prod_{i=1}^j d_i}{\prod_{i=1}^j \hat d_i} \le C \, , \forall\, 1 \le j \le p(k, \epsilon) + 1 \, . \tag{7.3}$$

In fact, the proof of (7.3) also gives assertion (1) of the sublemma. (Note that the proof of [BeY, Lemma 1.3] may require taking a smaller value of δ in (H2), in order to guarantee that $|f^j(0)| > \delta$ for $j \le M_0$, where M_0 is a large integer, making use of (H1)(ii).) Now, by definition and (H1)(i)

$$m_j = d_{j-1} m_{j-1} + 1 \le d_{j-1} m_{j-1} \big(1 + C\lambda^{-j}\big) ,$$

so that

$$m_j \le \Big(\prod_{i=1}^{j-1} d_j \Big) \prod_{i=1}^{j-1} (1 + C\lambda^{-i}) \, , \text{ showing our claim (7.2).}$$

We may now prove assertions (2) and (3) of the sublemma. Assumption (H1)(i), together with (1), that we already proved, and the fact that $|J_{k,\epsilon}| \ge \max(e^{-2k}, \epsilon)/C$, yield

$$\frac{\max(e^{-2k}, \epsilon)\, \lambda^{p-1}}{C} \le 1 \, ,$$

so that

$$p(k, \epsilon) \le 1 + \log(C \min(e^{2k}, \epsilon^{-1})) \frac{1}{\log \lambda} \, , \tag{7.4}$$

showing the upper bound in (2). For the lower bound, use $|J_{k,\epsilon}| \le C \max(e^{-2k}, \epsilon)$, the definition of $p(k, \epsilon)$ and $\lambda^K \lambda^\eta < \lambda^{K+1}$.

For (3), letting $1 \le j \le p(k, \epsilon) + 1$ it follows from (7.2) that for $y, \tilde y \in J_{k,\epsilon}$ and arbitrary $\omega, \tilde\omega \in \Omega$,

$$|f^j_{\sigma\omega}(y) - f^j_{\sigma\tilde\omega}(\tilde y)| \le C |(f^j)'(f(0))| \max(e^{-2k}, \epsilon) \le C^2 |(f^j_{\sigma\omega})'(y)| \max(e^{-2k}, \epsilon) \, .$$

Thus, the definition of $p(k, \epsilon)$ gives

$$C^2 |(f^{p(k,\epsilon)+1}_{\sigma\omega})'(y)| \cdot \max(e^{-2k}, \epsilon) \ge \lambda^{-\eta(p(k,\epsilon)+1)} \, .$$

Finally (7.4) implies

$$\lambda^{-\eta(p(k,\epsilon)+1)} \ge e^{-(\eta \log\lambda)\left[C + \log\left(\min\left(e^{2k}, \epsilon^{-1}\right)\right)\right]/\log\lambda} \ge \frac{\max(e^{-2\eta k}, \epsilon^\eta)}{C} \, ,$$

and we conclude that

$$|(f^{p(k,\epsilon)}_{\sigma\omega})'(y)| \ge \frac{1}{C} \max(e^{(1-\eta)2k}, \epsilon^{-1+\eta}) \, . \quad \square$$

Proof of Lemma 7.1. This will easily follow from Sublemma 7.2, taking $k = k(\epsilon) \in \mathbb{Z}_+$ maximal so that $\sqrt{\epsilon} < e^{-k}$. Indeed, for any $|x| < 2\sqrt{\epsilon}$, writing $y = f_\omega(x) \in J_{k,\epsilon}$, and setting $N(\epsilon) = p(k(\epsilon), \epsilon) + 1$ we get from (3), that for each ω

$$|(f^{N(\epsilon)}_\omega)'(x)| = |(f^{p(k(\epsilon))}_{\sigma\omega})'(y)| |f'_\omega(x)| \ge C |x| \epsilon^{-1+\eta} \, , \tag{7.5}$$

for some constant C, independent of ϵ, δ, ω, and which may be removed by working with a slightly smaller η in Sublemma 7.2 and taking small enough ϵ.

To check the second assertion, we decompose for each $1 \le j \le N(\epsilon)$

$$|f_\omega^j(x)| \ge |f^j(0)| - |f^j(0) - f_\omega^j(x)|\,.$$

Now, there are two cases. Either $j \le \log(1/\epsilon)/(4K\log\lambda)$, and then by using (H1)(ii) and Sublemma 7.2(2) (recall (7.2))

$$|f^j(0)| - |f^j(0) - f_\omega^j(x)| \ge e^{-\alpha N(\epsilon)} - C\epsilon|(f^j)'(f(0))| \ge \epsilon^\eta - C\epsilon^{3/4} > \sqrt{\epsilon}\,,$$

since $\eta < 1/4$, if $\epsilon > 0$ is small enough. The other possibility is $j > \log(1/\epsilon)/(4K\log\lambda)$, but then, using (H1)(ii) and the definition of $p(k(\epsilon))$, we get for small enough ϵ

$$|f^j(0)| - |f^j(0) - f_\omega^j(x)| \ge e^{-\alpha j} - e^{-2\alpha j} \ge \epsilon^\eta(1 - \epsilon^{\alpha/(4K\log\lambda)}) > \sqrt{\epsilon}\,. \quad \square$$

We now divert to verify the statement about varying δ in (H2)(i),(ii).

Remark. If there is $\delta = \delta_1$ so that (H1) holds with λ_1 satisfying $\lambda_1 > e^{20\alpha}$ and (H2) holds for a fixed $\delta = \delta_1$ and $\lambda = \lambda_1$ then for all $\delta = \delta_2 < \delta_1$ (H1) and (H2)(i),(ii) hold with $\lambda = \lambda_2 = \lambda_1^{\frac{1}{2}-4\alpha}$.

Sketch of proof. Take a point $x \notin (-\delta_2, \delta_2)$. If $x, fx, \ldots, f^{M-1}x \notin (-\delta_1, \delta_1)$ there is nothing to prove. Suppose that $k < M-1$ is the first index so that $f^k(x) \in (-\delta_2, \delta_2) \setminus (-\delta_1, \delta_1)$. Then by (H2)(ii) for $\delta = \delta_2$, $|(f^j)'(x)| \ge \lambda_1^k$. With $y = f^k(x)$ and the bound period $p = p(y)$ defined in the usual way it is easy to verify that $|(f^{p+1})'(y)| \ge \lambda_1^{(\frac{1}{2}-4\alpha)(p+1)}$. We conclude that with $\lambda_2 = \lambda_1^{\frac{1}{2}-4\alpha}$, $|(f^{k+p+1})'(x)| \ge \lambda_2^{k+p+1}$. Moreover with an argument similar to that in the proof of the second assertion in Lemma 7.1, $f^{k+j}(x)$ will never hit $(-\delta_2, \delta_2)$ for $j \le p$. We conclude that (H2)(i) holds with $M = M(\delta_2) = M(\delta_1) + p(\delta_2)$.

The proof of (H2)(ii) uses the same type of arguments. Each bound period of length p_i following a return y_i to $(-\delta_1, \delta_1) \setminus (-\delta_2, \delta_2)$ gives a derivative contribution $|(f^{p_i+1})'(y_i)| \ge \lambda_2^{p_i+1}$. The derivative during the "free" period following each bound period of this type lasting until the next return to $(-\delta_1, \delta_1)$ (and eventually to $(-\delta_2, \delta_2)$) is estimated using (H2)(ii) with $\delta = \delta_1$. $\square$

Lemma 7.3 ("Outside" lemma). *Let f satisfy (H1), (H2), and (H3) and assume $2\alpha/\log\lambda < \eta < 1/10$. There are $C > 1$ and $\tau > \lambda^{1/5} > 1$ such that for all $\epsilon > 0$, all $\omega \in \Omega$, $x \in I$, and $k \in \mathbb{Z}_+$*

$$|f_\omega^j(x)| \ge \sqrt{\epsilon}/2\,, \forall j = 0, \ldots, k-1 \Longrightarrow |(f_\omega^k)'(x)| \ge \frac{\sqrt{\epsilon}\tau^k}{C}\,. \tag{7.6}$$

There is $0<\delta_1<\delta$ *(independent of* ϵ, ω*) such that*

$$|f^j_\omega(x)|\ge \sqrt{\epsilon}/2\,,\forall j=0,\dots,k-1 \text{ and } |f^k_\omega(x)|<\delta_1 \Longrightarrow |(f^k_\omega)'(x)|\ge \frac{\tau^k}{C}\,. \tag{7.7}$$

Proof of Lemma 7.3. We claim that it suffices to see that there are $0<\delta_1<<\delta$ and $\tilde\tau>\lambda^{1/5}$ such that if $\sqrt{\epsilon}/2<|x|<\delta_1$ then there is $\tilde p(x)\le C\log(1/\epsilon)$ with

$$|f^j_\omega(x)|>\delta_1\,,\forall 0\le j\le \tilde p-1 \text{ and } \prod_{j=0}^{\tilde p-1}|f'_{\sigma^j\omega}(f^j_\omega(x))|\ge \tilde\tau^{\tilde p}\,,\forall\omega\in\Omega\,. \tag{7.8}$$

Indeed, (H2)(i) and (ii) imply by a continuity argument that for small enough ϵ (and up to slightly reducing λ) for each ω and y if y, $f_\omega(y),\dots,f^{n-1}_\omega(y)\notin(-\delta,\delta)$ then

$$|(f^n_\omega)'(y)|\ge \lambda^n/C\,.$$

If, additionally, $f^n_\omega(y)\in(-\delta,\delta)$ then $|(f^n_\omega)'(y)|\ge\lambda^n$. Using this fact and (7.8) (which plays the role of Lemma 2.4(b) in [V]), Lemma 7.3 may be proved as Lemma 2.5 in [V] using ideas going back to [BC1, BC2].

But now, (7.8) may be obtained for any $\tau<\lambda^\xi$ if $2\xi<1/2-\eta$, by the arguments used to show Sublemma 7.2(3), taking $k=k(\epsilon)$ maximal so that $\sqrt{\epsilon}<e^{-k}$ and considering $y\in J_{k(\epsilon),\epsilon}\setminus\hat J_\epsilon$ with $\hat J_\epsilon=[-\epsilon+f(\sqrt{\epsilon}/2),f(0)+\epsilon]$ (see [BeY, Lemma 4.4 (ii)]). □

7.B Estimating bad sets.

We now prepare the construction of the random dynamical partitions of the interval, in view of checking the tower axioms of Section 3. We start with the exponential partition $\mathcal{Q}$ of I (modulo zero measure sets) into intervals defined for $r\in\mathbb{Z}$ by $I_r=(\sqrt{\epsilon}e^{-r},\sqrt{\epsilon}e^{-(r-1)})$, $r\ge1$, $I_r=-I_{-r}$, $r\le-1$, $I_0^+=(\sqrt{\epsilon},\sqrt{\epsilon}e)$, $I_0^-=-I_0^+$, $I^+=(\sqrt{\epsilon}e,1)$, $I^-=-I^+$. For $|r|\ge1$ we write $I_r^+=I_r\cup I_{r+1}\cup I_{r-1}$. For $m\ge1$, we also introduce the functions $r_m:\Omega\times I\to\mathbb{Z}_+$, by setting $r_m(\omega,x)=|r|$ if $f^m_\omega(x)\in I_r$ and 0 otherwise, and sets

$$G_m(\omega,x)=G^\epsilon_m(\omega,x)=\left\{0\le j\le m\mid r_j(\omega,x)\ge\max\left(1,\left(\frac12-2\eta\right)\log\frac1\epsilon\right)\right\}. \tag{7.9}$$

Recall that $(2\alpha/\log\lambda)<\eta<1/10$ appeared in Lemmas 7.1 and 7.3. In view of the proof of Lemma 7.4, we take $\eta=\log\tau/(4\log 32)$ for $\tau>\lambda^{1/5}$ from Lemma 7.3 (since $5\cdot8\cdot\log(32)<200$, assumption (H1) guarantees that we may do this).

The reader is invited to check (see [V, § 2.4], and also [A, § 2]) that for suitably small $c>0$, large $C>1$, small $\epsilon>0$, Lemma 7.1 and the definition of $G_n(\omega,x)$ imply that for each large enough $n\gg C\log(1/\epsilon)$ and all (ω,x) for which

$$\sum_{j\in G^\epsilon_n(\omega,x)} r_j(\omega,x)\le cn\,, \tag{7.10}$$

we have $|(f^n_\omega)'(x)| > e^{n/C}$. Hint: The key step is the first of the following bounds, recorded here for future use,

$$\begin{cases} |(f^n_\omega)'(x)| \geq \exp\left(4cn - \sum_{j\in G_n(\omega,x)} r_j(\omega,x) - 2\log\frac{1}{\epsilon}\right), \\ |f^n_\omega(x)| < \sqrt{\epsilon} \Longrightarrow |(f^n_\omega)'(x)| \geq \exp\left(4cn - \sum_{j\in G_n(\omega,x)} r_j(\omega,x) - C\right). \end{cases} \tag{7.11}$$

Our next aim is to show that for all x the set of ω such that (7.10) is violated has small measure. The i.i.d. setting together with the assumption on ν_ϵ give:

Lemma 7.4 (Estimates on bad ω-sets). *Let c be as in* (7.10). *There are $C(\epsilon) > 1$, $\gamma(\epsilon) > \frac{1}{C\log(1/\epsilon)}$, and for each $x \in I$ and all $n \geq 1$ sets $E_n(x) \subset \Omega$ with $P(E_n(x)) \leq C(\epsilon)e^{-\gamma(\epsilon)n}$, such that if $\omega \notin E_n(x)$ then*

$$\sum_{\substack{j\in G^\epsilon_n(\omega,x)\\ j\neq 0}} r_j(\omega,x) \leq \frac{cn}{2}.$$

Proof of Lemma 7.4. The crucial point is the fact that there are $C > 0$ and $0 < \beta < 1$ so that for small enough ϵ, there is $\mathcal{M}(\epsilon) \sim C\log(1/\epsilon)$, so that for each interval I_r with $|r| \geq (1/2 - 2\eta)\log(1/\epsilon)$, and all x, ω

$$P(\{\omega \in \Omega \mid f^{\mathcal{M}(\epsilon)}_\omega(x) \in I_r\}) \leq Ce^{-4\beta|r|}. \tag{7.12}$$

(Note that an obvious upper bound is $(C/\epsilon)\sqrt{\epsilon}e^{-r}$ if $r > \log(1/\sqrt{\epsilon})$, with C the constant from (2.1). We need the better estimate (7.12) to deal with $(1/2 - 2\eta)\log(1/\epsilon) \leq r \leq (1/2)\log(1/\epsilon)$.) See Lemmas 2.3 and 2.6, and especially the bound on line 3 of p. 77 in [V] (note that this bound is in fact a conditional probability) for deterministic analogues of (7.12), obtained using a notion of admissible curves which we do not require.

Let us sketch how to adapt the proof of Lemma 2.6 in [V] to obtain (7.12). We start by observing that (2.1) implies that there are constants $C_1 > 1$ and $C_2 > 1$ so that for each $\epsilon > 0$ there are subsets $H_1 = H_1(\epsilon)$, $H_2 = H_2(\epsilon)$ of $[-\epsilon,\epsilon]$, with $\nu_\epsilon(H_i) > 1/C_1$ for $i = 1,2$, and the distance $d(H_1,H_2) > \epsilon/C_2$. This immediately implies that $|f_\omega(x) - f_{\tilde\omega}(x)| > \epsilon/C_2$ if $\omega_0 \in H_1$ and $\tilde\omega_0 \in H_2$. (This is Lemma 2.7 in [V] with $C_1 = 16$ and $C_2 = 100$.) Then, taking $\mathcal{M} = \mathcal{M}(\epsilon)$ to be the maximum integer so that $32^{\mathcal{M}(\epsilon)}\epsilon \leq 1$, we observe that $\mathcal{M}(\epsilon)$ is smaller than the constant $N(\epsilon)$ from Lemma 7.1. Since our choice of η and $\mathcal{M}$ implies

$$r + \mathcal{M}(\epsilon)\log\tau - \frac{1}{2}\log\frac{1}{\epsilon} \geq \eta r,$$

for all $r \geq (1/2 - 2\eta)\log(1/\epsilon)$, we may just follow the proof of Lemma 2.6 in [V], making use of (H1)(ii) in lieu of the finite postcritical assumption there (clearly, $\alpha < (\log 32)/4$), and of our Lemma 7.3 in place of his Lemma 2.5. (The fact that "our" C_2 depends on the constant in (2.1) and that the bound on the distortion is not necessarily ≤ 2 do not create difficulties if we replace $K = 400e^2$ in the analogue of Viana's K and $k(r)$ by an appropriate constant depending on f and C_2.)

Now, to deduce Lemma 7.4 from (7.12), we may simplify Viana's large deviation argument [V, Theorem A § 2.4]. In particular, our i.i.d. setting allows us to suppress the time-shift "$l = m - \mathcal{M}(\epsilon)$" (with $l \sim m \sim \sqrt{n}$) in [V]. As a consequence, we get exponential bounds (our rate depends on ϵ) instead of the stretched exponential bound in [V].

More precisely, we now sketch how (7.12) gives $\gamma(\epsilon) \ge C/\log(1/\epsilon)$ and $C > 1$ so that for each fixed small enough ϵ, all $x \in I$, and all $n \gg \log(1/\epsilon)$

$$P\Big(\Big\{\omega \mid \sum_{0\ne i \in G_n(\omega,x)} r_i(\omega,x) \ge \frac{cn}{2}\Big\}\Big) \le \frac{C\,\log(1/\epsilon)}{\sqrt{\epsilon}} e^{-\gamma(\epsilon)\,n}\,.$$

"Large deviations" here is just the remark that for any $\beta > 0$ (we shall take β as in (7.12)) and all $0 \le q \le \mathcal{M}(\epsilon) - 1$ (see Lemma 5.1 for a similar computation)

$$P\Big(\Big\{\omega \mid \sum_{\substack{i\in G_{n,q}(\omega,x)\\ i\ne 0}} r_i(\omega,x) \ge \frac{cn}{\mathcal{M}(\epsilon)}\Big\}\Big) \le e^{-\frac{\beta c n}{\mathcal{M}(\epsilon)}} \int_{\{G_{n,q}(\omega,x)\ne\emptyset\}} e^{\beta\sum_{i\in G_{n,q}(\omega,x)} r_i}\, dP(\omega)\,,$$

where $G_{n,q}(\omega,x)$ is the set of those $i \in G_n(\omega,x)$, $i \ne 0$, for which $i \equiv q$ modulo $\mathcal{M}(\epsilon)$. Thus, setting $\gamma(\epsilon) = c\beta/\mathcal{M}(\epsilon)$, it suffices to show

$$\int_{\Omega\cap\{G_{n,q}(\omega,x)\ne\emptyset\}} e^{\beta\sum_{i\in G_{n,q}(\omega,x)} r_i}\, dP(\omega) \le 1\,,$$

for some $\beta > 0$ and all $\epsilon > 0$, x, $0 \le q < \mathcal{M}(\epsilon) - 1$, and $n \gg \log(1/\epsilon)$. In order to obtain the above bound, we introduce some notation. For fixed ϵ, n, q, and x, ω, let $t(x,\omega) = t^{\epsilon,n,q}(x,\omega)$ be the cardinality of $G_{n,q}(\omega,x) = \{i_1 \le i_2 \ldots \le i_{t(x,\omega)}\}$, and set $\hat r_\ell = r_{\ell\mathcal{M}+q}$ if $r_{\ell\mathcal{M}+q} \ge (1/2 - 2\eta)\log(1/\epsilon)$ and $\hat r_\ell = 0$ otherwise.

Next, it is easy to deduce from (7.12) and independence that there is $C > 0$ so that for all ϵ, every $n \gg \log(1/\epsilon)$, each $0 \le q \le \mathcal{M}(\epsilon) - 1$, every $1 \le t \le n$, and any sequence ρ_i with either $\rho_i = 0$ or $\rho_i \ge (1/2 - 2\eta)\log(1/\epsilon)$,

$$\begin{aligned} P\Big(\Big\{\omega \mid t^{\epsilon,n,q}(x,\omega) = t \text{ and } \hat r_\ell = \rho_\ell\Big\}\Big) & \\ &\le \frac{Ce^{-\rho_t}}{\sqrt{\epsilon}} \prod_{\ell=1}^{t-1} P\Big(\Big\{\omega \mid f^{\mathcal{M}(\epsilon)}_{\sigma^{\ell\mathcal{M}+q}(\omega)}(x) \in I_{\rho_\ell}\Big\}\Big)\\ &\le \frac{C^t}{\sqrt{\epsilon}} e^{-4\beta\sum_\ell \rho_\ell}\,. \end{aligned}$$

(We used the trivial fact $(\ell+1)\mathcal{M}+q=\ell\mathcal{M}+q+\mathcal{M}$.) Thus

$$\int_{\Omega\cap\{G_{n,q}(\omega,x)\neq\emptyset\}} e^{\beta\sum_{i\in G_{n,q}(\omega,x)} r_i}\,dP(\omega)$$
$$\leq \sum_{\rho_\ell} C^t e^{-3\beta\sum_\ell \rho_\ell} \leq \sum_{t,R} \zeta(t,R)C^t e^{-3\beta R}\,,$$

where $\zeta(t,R)$ is the number of integer solutions of the equation $\sum_{\ell=1}^t \rho_\ell = R$ satisfying $\rho_\ell \geq (1/2-2\eta)\log(1/\epsilon)$ for all ℓ. Since $R/t \geq (1/2-2\eta)\log(1/\epsilon)$, taking $\epsilon>0$ small enough ensures that (recall $1\leq t\leq R$ and $R\geq(1/2-2\eta)\log(1/\epsilon)\gg 1$)

$$\sum_{t,R}\zeta(t,R)C^t e^{-3\beta R} \leq \sum_{t,R} e^{-\beta R} \leq \sum_R Re^{-\beta R} \leq 1\,. \quad \square$$

Corollary 7.5 (Bad (ω,x) sets). *Let c, $C=C(\epsilon)$ and $\gamma(\epsilon)$ be as in Lemma 7.4. For each $n\geq 1$ there is $E_n\subset\Omega\times I$ with $(P\times\mathrm{Leb})(E_n)\leq Ce^{-\gamma(\epsilon)n}$ such that for all $(\omega,x)\notin E_n$ condition* (7.10) *holds.*

Proof of Corollary 7.5. Just write $E_n=\{(\omega,x)\mid\omega\in E_n(x)\}\cup\{(\omega,x)\mid r_0(\omega,x)\geq cn/2\}$ and use Fubini to apply Lemma 7.4: $(P\times\mathrm{Leb})(E_n)\leq\int_I P(E_n(x))\,d\mathrm{Leb}+2\sqrt{\epsilon}e^{-cn/2}$. $\square$

Corollary 7.6 (Bad x sets). *Let $C=C(\epsilon)$ and $\gamma(\epsilon)$ be as in Lemma 7.4. For $\omega\in\Omega$, and $m\geq 1$, set $E_n(\omega)=\{x\in I\mid(x,\omega)\in E_n\}$. Then $P(\{\omega\in\Omega\mid\mathrm{Leb}(E_n(\omega))>\sqrt{Ce^{-\gamma(\epsilon)n}}\})\leq\sqrt{Ce^{-\gamma(\epsilon)n}}$.*

Proof of Corollary 7.6. This is Fubini again! Indeed, if $P(\{\omega\in\Omega\mid\mathrm{Leb}(E_n(\omega))>\sqrt{Ce^{-\gamma n}}\})>\sqrt{Ce^{-\gamma n}}$ then $(P\times\mathrm{Leb})(E_n)=\int_\Omega P(E_n(\omega))\,dP(\omega)$ would imply $(P\times\mathrm{Leb})(E_n)>\sqrt{Ce^{-\gamma n}}\sqrt{Ce^{-\gamma n}}$, contradicting Corollary 7.5. $\square$

Lemma 7.7 (Parameter exclusion – Waiting times). *Let $\gamma(\epsilon)$ be as in Lemma 7.4. There is $C=C(\epsilon)>1$ and a full measure subset $\Omega_0\subset\Omega$ such that for each $\omega\in\Omega_0$ there is $n_0(\omega)$ such that for all $m\geq n_0(\omega)$*

$$\mathrm{Leb}(E_m(\omega))<Ce^{-\frac{\gamma(\epsilon)}{2}m}\,.$$

Additionally, there are $C=C(\epsilon)>1$ and $\upsilon(\epsilon)>(C\log(\frac{1}{\epsilon}))^{-1}$ such that the random variable $n_0(\omega)$ satisfies for all $n\in\mathbb{Z}_+$

$$P\big(\{\omega\in\Omega\mid n_0(\omega)\geq n\}\big)\leq Ce^{-\upsilon(\epsilon)n}\,. \tag{7.13}$$

The lower bound $n_0(\omega)$ is called a *waiting time.* It will have to be modified before we reach the final waiting time function $n_3(\omega)$ which will play a role in the recurrence asymptotics of our random towers (see (A.V) in Subsection 3.B).

Proof of Lemma 7.7. Using $C = C(\epsilon)$ from Corollary 7.6, define for each n a "bad set"

$$B_n = \left\{ \omega \in \Omega \mid \exists m \geq n \, , \mathrm{Leb}(E_m(\omega)) > \sqrt{Ce^{-\gamma m}} \right\} .$$

Corollary 7.6 says that $P(B_n) \leq \sum_{k=n}^{\infty} C(\epsilon) e^{-\gamma(\epsilon)k/2}$. Therefore $\lim_{n\to\infty} P(B_n) = 0$. Setting $\Omega_0 = \bigcup_n (\Omega \setminus B_n)$, and for each $\omega \in \Omega_0$,

$$n_0(\omega) = \inf \left\{ n \in \mathbb{Z}_+ \mid \omega \notin B_n \right\} ,$$

we easily get (7.13). $\square$

Definition (Random hyperbolic (return) times). Fix $c' > c$. We say that m is a *hyperbolic time* for (ω, x) if for each $0 \leq k \leq m-1$ we have

$$\sum_{i \in G_m^\epsilon(\omega,x) \, , k \leq i \leq m-1} r_i(\omega, x) \leq c'(m-k) \, .$$

(This condition depends on ϵ through G_m^ϵ.) We say that m is a *hyperbolic return time* for (ω, x), or a *hyperbolic return* if m is a hyperbolic time and, additionally, $r_m(\omega, x) \geq 1$.

For $\omega \in \Omega$, a fixed $p_0(\epsilon)$ (the choice of p_0 occurs later in Lemmas 7.9 and 9.1), and all m we define

$$\begin{aligned} H_m(\omega) &= \{x \in I \mid m \text{ is the first hyperbolic time } \geq p_0 \text{ for } (\omega, x)\} \\ H_m^*(\omega) &= \{x \in I \mid m \text{ is the first hyperbolic return } \geq p_0 \text{ for } (\omega, x)\} \, . \end{aligned}$$

Finally, we set $E_m^*(\omega) = I \setminus \bigcup_{k=p_0}^m H_k^*(\omega)$.

Lemma 7.8 (Hyperbolic return estimates). *Let $0 < \upsilon(\epsilon) \leq \gamma(\epsilon)/2$ be as in Lemma 7.7. There are $0 < \zeta(\epsilon) < \upsilon(\epsilon)$ and $C(\epsilon) > 1$, such that for all $\omega \in \Omega_0$ and all $m \geq n_0(\omega) + C(\epsilon)$ we have* $\mathrm{Leb}(E_m^*(\omega)) \leq e^{-\zeta(\epsilon)m}$.

Proof of Lemma 7.8. Applying Pliss' Lemma as in [A, Proposition 2.6], we find

$$I \setminus E_m(\omega) \subset \bigcup_{k=p_0}^m H_k(\omega) \, , \qquad \forall m \geq p_0 \, .$$

Next, we shall show that there is $C > 1$ so that the proportion of points x in the complement of $E_m(\omega)$ such that $|f_\omega^{m+j}(x)| < \sqrt{\epsilon}$ for some $0 \leq j \leq m \cdot C \log(1/\epsilon)$ is larger than $1-(1-\sqrt{\epsilon})^m$. Of course, $m+j$ is then not only a return but also a hyperbolic return (use Lemma 7.3) for (x,ω), so that we get

$$\mathrm{Leb}\left(\bigcup_{k=p_0}^{m/(C\log(1/\epsilon))} H_k(\omega)\setminus\big(\bigcup_{k=p_0}^{m} H_k^*(\omega)\cap\bigcup_{k=p_0}^{m/(C\log(1/\epsilon))} H_k(\omega)\big)\right)\le(1-\sqrt{\epsilon})^m\,.$$

Therefore

$$\mathrm{Leb}(E_m^*(\omega)\setminus E_{m/(C\log(1/\epsilon))}(\omega))\le(1-\sqrt{\epsilon})^m\,,$$

showing Lemma 7.8 (by Lemma 7.7).

It remains to show the assertion on the proportion of good points. Let $x\notin E_m(\omega)$ and set $y=f_\omega^m(x)$. If $|y|>\sqrt{\epsilon}$ then $J_\epsilon=[y-\sqrt{\epsilon}/2,y+\sqrt{\epsilon}/2]\cap f_\omega^m(I\setminus E_m(\omega))$ does not intersect $(-\sqrt{\epsilon}/2,\sqrt{\epsilon}/2)$. Lemma 7.3 gives $\tau>1$ and $C>1$ so that if $|f^j_{\sigma^m\omega}(z)|>\sqrt{\epsilon}/2$ for $0\le j\le k-1$ then $|(f^k_{\sigma^m\omega})'(z)|>\sqrt{\epsilon}\tau^k/C$. If $(1-\epsilon)\tau>1$, there is C so that $C\sqrt{\epsilon}\sqrt{\epsilon}((1-\sqrt{\epsilon})\tau)^k>2=|I|$ for $k\ge C\log(1/\epsilon)$. The distortion $\sup|(f^j)'(z)/(f^j)'(w)|$, for $0\le j\le k$ is bounded by a constant independent of ϵ on J_ϵ (use Lemma 7.3 again). Hence, the proportion of points of J_ϵ which have never entered into $(-\sqrt{\epsilon},\sqrt{\epsilon})$ by the time $C\log(1/\epsilon)$ is smaller than $1-\sqrt{\epsilon}$. We may iterate this procedure m times. Points $y\in f_\omega^m(I\setminus E_m(\omega))$ not in J_ϵ are dealt with by a similar argument. □

7.C The random partitions.

The first step is to obtain for fixed $\omega\in\Omega$, and each $m\ge p_0$ a mod-0 partition of I into intervals

$$I=\bigcup_{k=p_0}^{m}\bigcup_{J\in\mathcal{R}_k(\omega)} J\ \cup\bigcup_{L\in\mathcal{S}_m(\omega)} L\,.$$

The families of intervals $\mathcal{R}_k=\mathcal{R}_k(\omega)$ and $\mathcal{S}_m(\omega)$ are constructed inductively, simplifying the strategy in [A, §3] (in particular the distinction between $\mathcal{R}_k$ and $\mathcal{R}_k^*$ does not exist here). We first list their key properties, valid for $p_0\le k\le m$ (recall the definitions given before Lemma 7.4):

- (P.I) $H_k^*(\omega)\subset\bigcup_{J\in\mathcal{R}_k(\omega)}J$ and $J\cap H_k^*(\omega)\ne\emptyset$ for each $J\in\mathcal{R}_k$. (In particular, if $\omega\in\Omega_0$ then Lemma 7.8 implies that $\mathrm{Leb}(\mathcal{S}_m(\omega))\le Ce^{-\zeta(\epsilon)m}$, if $n\ge n_0(\omega)$. As a consequence, $\bigcup_{k=p_0}^{\infty}\bigcup_{J\subset\mathcal{R}_k(\omega)}J$ is a partition of I modulo zero measure sets.)
- (P.II) For each $J\in\mathcal{R}_k(\omega)$ and $0\le j\le k-1$, there is $I_{r_j}\in\mathcal{Q}$ such that $f^j_\omega(J)\subset I^+_{r_j}$.
- (P.III) For each $J\in\mathcal{R}_k(\omega)$, there exist $0\le j\le k-1$ and I_{r_j} with $f^j_\omega(J)\supset I_{r_j}$.
- (P.IV) For each $J\in\mathcal{S}_m(\omega)$, either $J\in\mathcal{Q}$ or J is *subordinate* to some $J^*\in\mathcal{R}_\ell$ for some $\ell\le m$. (By definition, J is subordinate to $J^*\in\mathcal{R}_\ell$ if J and J^* have a common endpoint and there are $0\le j\le\ell-1$ and $r_j\ge1$ with $f^j_\omega(J)\supset I_{r_j+1}$ or $f^j(J)\supset I_{r_j-1}$ where $I_{r_j}\subset f^j_\omega(J^*)$.)

Construction of the initial partition:

First step: We first construct $\mathcal{R}_{p_0}$ and $\mathcal{S}_{p_0}$, by using an auxiliary sequence of families of intervals $\mathcal{J}_\ell$ for $1\le\ell\le p_0$. For this, start with the family of intervals $\mathcal{J}_1=$

$\{I_r \in \mathcal{Q} \mid I_r \cap H^*_{p_0}(\omega) \neq \emptyset\}$. For each $J_1 \in \mathcal{J}_1$, we consider $f_\omega(J_1)$. If it does not contain any interval of $\mathcal{Q}$ we put the interval J_1 in $\mathcal{J}_2$. Otherwise, we subdivide J_1 into subintervals having as image either exactly one element of $\mathcal{Q}$ or one element of $\mathcal{Q}$ and part of either of the elements of $\mathcal{Q}$ which intersect the boundary of $f_\omega(J_1)$, and we put into $\mathcal{J}_2$ those intervals in the decomposition which contain an element of $H^*_{p_0}(\omega)$. Then, for each $J_2 \in \mathcal{J}_2$ we consider $f^2_\omega(J_2)$, putting it into $\mathcal{J}_3$ if it contains no interval of $\mathcal{Q}$, and otherwise decomposing J_2 as in the first step and putting into $\mathcal{J}_3$ those subintervals which intersect $H^*_{p_0}(\omega)$. We continue in this way until reaching the iterate $f^{p_0-1}_\omega$, obtaining a family of intervals $\mathcal{J}_{p_0}$. We define $\mathcal{R}_{p_0} = \mathcal{J}_{p_0}$ and set

$$\mathcal{S}_{p_0} = (\mathcal{Q} \setminus \mathcal{J}_1) \cup \{\text{connected components of } J_1 \setminus \bigcup_{J \in \mathcal{J}_{p_0}} J \mid \forall J_1 \in \mathcal{J}_1\}\,.$$

Properties (P.I–IV) are satisfied by construction for $\mathcal{R}_{p_0}$ and $\mathcal{S}_{p_0}$ (we set $\mathcal{R}_\ell = \mathcal{J}_\ell$ for $1 \le \ell \le p_0 - 1$ in the formulation of (P.IV)).

Inductive step: Assume that $\mathcal{R}_k$, $p_0 \le k \le m$, and $\mathcal{S}_m$ have been defined and satisfy (P.I–IV). We shall construct $\mathcal{R}_{m+1}$ and $\mathcal{S}_{m+1}$. For this, let $J_m \in \mathcal{S}_m$. By construction, $J_m \subset I_r \in \mathcal{Q}$. If $J_m \cap H^*_{m+1}(\omega) = \emptyset$ we put this interval into $\mathcal{S}_{m+1}$ (no subdivision has been made, so that (P.IV) still holds). Otherwise, we observe that (P.IV) implies that there are $0 \le j \le m$ and $I_{r_j} \in \mathcal{Q}$ with $f^j_\omega(J_m) \supset I_{r_j}$ (indeed, if $J_m \in \mathcal{Q}$ we may just take $j = 0$ and otherwise we apply the definition of "subordinate"). We take the smallest such j and proceed as in the first step, decomposing J_m into subintervals having image either exactly one element of $\mathcal{Q}$ or one element of $\mathcal{Q}$ and part of one of the adjacent elements of $\mathcal{Q}$, putting into $\mathcal{S}_{m+1}$ the connected components of the complement of those intervals, $J'_{m,i}$ in the decomposition which contain a point in $H^*_{m+1}(\omega)$, and continuing the procedure until we exhaust all $j' \le m$ with $f^{j'}_\omega(J'_{m,i}) \supset I_{r_{j'}}$, defining thus $\mathcal{R}_{m+1}$ and $\mathcal{S}_{m+1}$. Properties (P.I–IV) hold by construction, and we are done.

Definition (Uniform contraction and bounded distortion). Let n, ω and an interval $J \subset I$ be such that f^n_ω is injective on J. We say that $f^n_\omega|_J$ enjoys uniform contraction along inverse branches for $0 < \beta < 1$ and $C > 1$ if for every $x \in J$ and all $0 \le j \le n-1$

$$\prod_{i=j}^{n-1} \left|f'(f^i_\omega(x))\right| \ge \frac{\beta^{j-n}}{C}\,. \tag{7.14}$$

We say that $\phi = f^n_\omega|_J$ enjoys bounded distortion for $\mathcal{K} > 1$ if for all $y \in f^n_\omega(J)$

$$\left|\frac{d}{dy}(\frac{1}{\phi'} \circ \phi^{-1})(y)\right| \cdot \left|\phi' \circ \phi^{-1}(y)\right| \le \mathcal{K}\,. \tag{7.15}$$

We list for further use the key property of the partition, adapted from [A].

Lemma 7.9 (Intermediate size – Bounded distortion – Uniform contraction). *There are $C>1$, $0<\beta<1$ and for each ϵ there are $p_0(\epsilon)\ge 1$ and $C(\epsilon)$ such that for all ω, each $m\ge p_0$, and every $J\in\mathcal{R}_m(\omega)$:*

(1) $f^m_\omega|_J$ is injective, $|f^m_\omega(J)|\ge \epsilon^{1-2\eta}/C$, and $f^m_\omega(J)|$ intersects $(-\sqrt{\epsilon},\sqrt{\epsilon})$.

(2) $f^m_\omega|_J$ enjoys uniform contraction along inverse branches for β and C.

(3) $f^m_\omega|_J$ enjoys bounded distortion for $C(\epsilon)$.

Proof of Lemma 7.9. Injectiveness is by construction. For the rest, we require in particular the following consequence of (P.I–P.II): For each $x\in J\in\mathcal{R}_m(\omega)$ there is $z\in J\cap H^*_m(\omega)$ with

$$r_i(\omega,x)\le r_i(\omega,z)+2\,,\ \forall 0\le i\le m-1\,,\ \text{and}\ r_i(\omega,x_j)\le r_i(\omega,z_j)+2\,,\forall 0\le i\le m-j-1\,,$$

where we set $x_j=f^j_\omega(x)$, $z_j=f^j_\omega(z)$. Assertion (2) on the contraction of inverse branches is then obtained from (7.11) (adapting the proof of Lemma 3.7 in [A]): It is not difficult to get (see [A, Lemma 2.3], observing that $m-j$ is a hyperbolic return for $(\sigma^j\omega,z_j)$ because m is a hyperbolic return for (ω,z))

$$\begin{aligned}\prod_{i=j}^{m-1}\left|f'(f^i_\omega(x))\right| &= \prod_{i=0}^{m-j-1}\left|f'(f^i_{\sigma^j\omega}(x_j))\right|\\ &\ge \exp\Bigg(3c(m-j)-\sum_{i\in G_{m-j}} r_i(\sigma^j\omega,z_j)-C\Bigg)\\ &\ge \exp\Big(3c(m-j)-c'(m-j)-C\Big)\ge \exp\Big(3c(m-j)/2-C\Big)\,.\end{aligned}\tag{7.16}$$

The claim on the length of the image follows from enhancing the bounds of [A, Proposition 4.8] by making use of the hyperbolic *returns.* Indeed, (P.III) implies that there is $0\le j\le m-1$ and I_{r_j} with $I_{r_j}\subset f^j_\omega(J)$. Then, by the mean value theorem, there is $x\in J$ with

$$|f^m_\omega(J)|=\left|(f^{m-j}_{\sigma^j\omega})'(f^j_\omega(x))\right|\cdot|f^j_\omega(J)|\,.$$

Next, applying (7.16),

$$\left|(f^{m-j}_{\sigma^j\omega})'(f^j_\omega(x))\right|\ge e^{2c(m-j)}/C\,.$$

It remains to obtain a lower bound for $|f^j_\omega(J)|$. For this, it suffices to control $|I_{r_j}|$. By construction, there is $x\in J$ with $r_j(\omega,x)=r_j$ and there is $y\in J\cap H^*_m(\omega)$ with $r_j=r_j(\omega,x)\le r_j(\omega,y)+2$. If $j\in G_m(\omega,y)$, since m is a hyperbolic time for (ω,y) we have $r_j(\omega,y)\le c'(m-j)$, so that, using $\eta<1/4$,

$$\begin{aligned}|I_{r_j}| &\ge \sqrt{\epsilon}\Big(e^{-r_j(\omega,y)-2}-e^{-r_j(\omega,y)-3}\Big)\\ &\ge \sqrt{\epsilon}(e^{-1}-e^{-2})e^{-c'(m-j)}\ge \epsilon^{1-2\eta}(e^{-1}-e^{-2})e^{-c'(m-j)}\,.\end{aligned}$$

If $j \notin G_m(\omega, y)$ then $r_j(\omega, y) \le (\frac{1}{2} - 2\eta)\log(1/\epsilon)$ and

$$|I_{r_j}| \ge \frac{e^{-2}}{2}\epsilon^{1-2\eta}\,.$$

Finally, the distortion control (3) with $C(\epsilon) \sim \epsilon^{-7/2}$ is obtained by a one-dimensional version of the proof of Proposition 4.2 in [A], adapting the estimates for the term A_2 there. (We leave the details to the reader.) $\Box$

Let us define the basic subintervals $\Lambda_\pm$ on which our random towers will be constructed. For this, we partition $(-\delta, \delta)$ (δ as in (H2) and small enough) into $\bigcup_{|k|\ge K_0} \hat I_k$ with $\hat I_k = (e^{-k-1}, e^{-k})$, $\hat I_{-k} = -\hat I_k$ and then we subdivide $\hat I_k = \bigcup_{\ell=1}^{k^2} \hat I_{k,\ell}$ so that the $\hat I_{k,\ell}$ are disjoint and $|\hat I_{k,\ell}| = k^{-2} \mid \hat I_k|$. (Note that ϵ does not intervene.) We set $\Lambda_\pm$ to be the rightmost and leftmost intervals of this partition of $(-\delta, \delta)$, i.e.,

$$\Lambda_+ = \hat I_{K_0, K_0^2}\,, \quad \Lambda_- = \hat I_{-K_0, 1}\,. \tag{7.17}$$

We also define $\tilde\Lambda_+$ to be the interval of length $3|\Lambda_+|$ centered at Λ_+, similarly for $\tilde\Lambda_-$.

We close this section with a lemma that will be instrumental to prove Lemma 8.1 (replacing ideas in the joint Appendix by Alves and Viana of a preprint version of [A] which circulated in 1997; we do *not* use the topological mixing assumption (H4)):

Lemma 7.10 (Large size of image). *Assume (H1)-(H3) and let $\beta < 1$ be as in Lemma 7.9. Then there is $C > 1$ and for every small enough ϵ and large enough $|k|$ there is a constant $C(k) > 1$ (independent of ϵ) so that for each $\omega \in \Omega$, and every interval $\hat I_{k,\ell}$ there are a time*

$$t(k) = t\big(\hat I_{k,\ell}, \omega\big) \le C|k|\,,$$

and a subinterval $\tilde U_\omega \subset \hat I_{k,\ell}$ such that

$$\begin{cases} |\tilde U_\omega| > 1/C(k)\,, \\ f_\omega^{t(k)}\big(\tilde U_\omega\big) = \tilde\Lambda_+ \ \text{or}\ \tilde\Lambda_-\,. \end{cases} \tag{7.18}$$

Furthermore, $\phi = f^t_\omega|_{\tilde U_\omega}$ is injective and enjoys both uniform contraction on backwards branches (7.14) *for C and β, and distortion bounds* (7.15) *for $\mathcal{K} = C(k)$.*

Proof of Lemma 7.10. We shall use again the random bound period ideas from [BeY]. We first state an easy consequence of Sublemma 7.2 (3). For every $1/4 > \eta' > \eta > 0$ (recall that η was fixed in the proof of Lemma 7.4) each small enough ϵ, all $\omega \in \Omega$, and every $\hat I_{k,\ell}$, taking $p(k, \epsilon)$ as in Sublemma 7.2:

$$\big|f_\omega^{p(k,\epsilon)+1}(\hat I_{k,\ell})\big| \ge e^{-2\eta'|k|}\,. \tag{7.19}$$

Indeed, just observe that

$$\left|f_\omega^{p(k,\epsilon)+1}(\hat I_{k,\ell})\right| \ge \inf\left|(f_{\sigma\omega}^{p(k,\epsilon)})'\right| \; \frac{e^{-|k|-1}}{C} \; \frac{e^{-|k|-1}}{k^2} \; \ge \; \frac{e^{(2-2\eta)|k|}}{C} \; \frac{e^{-2(|k|+1)}}{Ck^2} > e^{-2\eta'|k|} \, . \tag{7.20}$$

Next, we claim that there is an integer $i = i_0 \le C|k|$, so that for some k_1 and ℓ_1

$$f_\omega^{p(k,\epsilon)+1+i_0}(\hat I_{k,\ell}) \supset \hat I_{k_1,\ell_1-1} \cup \hat I_{k_1,\ell_1} \cup \hat I_{k_1,\ell_1+1} \, , \qquad \text{and } |k_1| \le 2\eta'|k| \tag{7.21}$$

(with the obvious interpretation if $\ell_1 = 1$ or $\ell_1 = k_1^2$).

To check (7.21) we first note that there is a first iterate $j_0 \le C|k|$ so that

$$f_\omega^{p(k,\epsilon)+1+j_0}(\hat I_{k,\ell}) \cap (-\delta,\delta) \ne \emptyset \, .$$

Indeed, while $f_\omega^{p(k,\epsilon)+1+i}(\hat I_{k,\ell})$ stays outside of $(-\delta,\delta)$ we have, setting $i = qM(\delta) + r$ with $0 \le r < M(\delta)$ and applying (H2)(i),

$$\left|f_\omega^{p(k,\epsilon)+1+i}(\hat I_{k,\ell})\right| \ge \frac{\lambda^{qM}}{(\min_{|x|\ge\delta}|f'(x)|)^M}\left|f_\omega^{p(k,\epsilon)+1}(\hat I_{k,\ell})\right| \ge \frac{\lambda^{qM}}{(\min_{|x|\ge\delta}|f'(x)|)^M}\, e^{-2\eta'|k|} \, .$$

Now, if $f_\omega^{p(k,\epsilon)+j_0}(\hat I_{k,\ell}) \subset (-\delta,\delta) \cup \Lambda_{++} \cup \Lambda_{--}$, where Λ_{++} is the interval to the right of Λ_+ in an augmented partition, and Λ_{--} is the corresponding interval to the left of Λ_-, we set $i' = j_0$, and by (H2)(ii)

$$\left|f_\omega^{p(k,\epsilon)+1+i'}(\hat I_{k,\ell})\right| \ge \lambda^{i'}\left|f_\omega^{p(k,\epsilon)+1}(\hat I_{k,\ell})\right| \ge \lambda^{i'} e^{-2\eta'|k|} \, . \tag{7.22}$$

In the other case, except if $f_\omega^{p(k,\epsilon)+1+j_0}(\hat I_{k,\ell})$ covers $\tilde\Lambda_+$ or $\tilde\Lambda_-$ (in which case we would stop, having proved Lemma 7.10), we replace $f_\omega^{p(k,\epsilon)+1+j_0}(\hat I_{k,\ell})$ by

$$f_\omega^{p(k,\epsilon)+1+j_0}(\hat I_{k,\ell}) \setminus (-\delta,\delta) \tag{7.23}$$

and continue iterating until we intersect $(-\delta,\delta)$ again. The loss in length caused by (7.23) is insignificant since there is a minimal time between successive returns to $(-\delta,\delta)$.

We may thus assume that we are in the situation (7.22) for some $i' \le C|k|$ and that there is (k',ℓ') with $|k'| \le \eta'|k|$ and

$$f_\omega^{p(k,\epsilon)+1+i'}(\hat I_{k,\ell}) \subsetneq \hat I_{k',\ell'-1} \cup \hat I_{k',\ell'} \cup \hat I_{k',\ell'+1} \tag{7.24}$$

(since otherwise (7.21) would be proved). Applying Lemma 7.2 (3) to $\hat I_{k',\ell'}$ we get (recall (7.20))

$$\left|f_\omega^{p(k,\epsilon)+1+i'+p(k',\epsilon)+1}(\hat I_{k,\ell})\right| \ge e^{-2\eta'|k|} \, .$$

Continuing the procedure, we eventually find subintervals $U_0 \subset \widetilde{U}_0 \subset \hat{I}_{k,\ell}$, an iterate $i = i_0$, and (k_1, ℓ_1) with $i_0 \le C|k|$ and $|k_1| \le 2\eta'|k|$,

$$f_\omega^{p(k,\epsilon)+1+i_0}(U_0) = \hat{I}_{k_1,\ell_1}\,,\ f_\omega^{p(k,\epsilon)+1+i_0}(\widetilde{U}_0) = \hat{I}_{k_1,\ell_1-1} \cup \hat{I}_{k_1,\ell_1} \cup \hat{I}_{k_1,\ell_1+1}\,, \tag{7.25}$$

ending the proof of (7.21). We take k_1 so that $|k_1|$ is minimal with the property (7.25).

We may now conclude the proof of Lemma 7.10: Repeating the procedure leading to (7.25), we obtain sequences

$$\begin{cases} U_1, U_2, \dots \widetilde{U}_1, \widetilde{U}_2, \dots\,, \\ k_0 = k\,, k_1\,, k_2\,, \dots\,,\ \text{with } |k_{m+1}| < 2\eta'|k_m|\,, \\ i_0, i_1, i_2, \dots\,,\ \text{with } i_m \le C|k_m|\,. \end{cases} \tag{7.26}$$

The only way this can stop is that the second line of (7.18) be satisfied. The total time spent before this happens is

$$\begin{aligned} t &= \sum_{m=0}^{s} (p(k_m, \epsilon) + 1 + i_m) \\ &\le C \sum_{m=0}^{s} |k_m| \le C \sum_{m=0}^{s} (2\eta')^m |k_0| \le C|k_0|\,. \end{aligned}$$

Since $s \le s(k) \le C|k_0|$, the lower bound on the length of U_ω follows from the remark and choice just after (7.23) and (7.25). The assertions on injectivity, distortion and contraction are immediate by construction, see in particular (7.24). $\square$

8. Escape and recurrence times asymptotics

Let $\Lambda_\pm$ and $\tilde{\Lambda}_\pm$ be defined by (7.17). We take as our reference interval $\Lambda = \Lambda_+ \subset I$, For small enough ϵ and for all $\omega \in \Omega$ we shall subdivide Λ into subintervals of points having the same return times to Λ, using the partitions $\mathcal{R}_m(\omega)$ and $\mathcal{S}_m(\omega)$ from the previous section. Our aim is to control asymptotically the measure of points having large return time. We first use Lemmas 7.9 and 7.10 to show:

Lemma 8.1. (Covering $\tilde{\Lambda}_\pm$ by iterating $J \in \mathcal{R}_m(\omega)$). *There are $C > 1$, and for each $\epsilon > 0$ a constant $C(\epsilon) > 1$ such that, for all ω, all $m \ge p_0$, and each interval J in $\mathcal{R}_m(\omega)$, the following holds:*

There are a subinterval $\widetilde{J} \subset f_\omega^m(J)$ and an iterate $t(J) \le C \log(1/\epsilon)$ such that $|\widetilde{J}| \ge C(\epsilon)^{-1}$ and for which $f_{\sigma^m\omega}^t$ maps $\widetilde{J}$ injectively onto either $\tilde{\Lambda}_+$ or $\tilde{\Lambda}_-$.

Furthermore, the restriction of $\phi = f_{\sigma^m\omega}^t$ on $\widetilde{J}$ enjoys both distortion bounds (7.15) for $\mathcal{K} = C(\epsilon)$ and uniform contraction on backwards branches (7.14) for the constant $\beta < 1$ from Lemma 7.9.

Proof of Lemma 8.1. By Lemma 7.9, the interval $f_\omega^m(J)$ has length $> \epsilon^{1-2\eta}/C$ and intersects $(-\sqrt{\epsilon}, \sqrt{\epsilon})$. It thus contains an interval $J' \subset (-2\sqrt{\epsilon}, 2\sqrt{\epsilon})$ of length $> \epsilon^{1-2\eta}/C$, disjoint from $(-\epsilon^{1-2\eta}/C, \epsilon^{1-2\eta}/C)$. Now an easy modification of the beginning of the proof of Lemma 7.10 may be applied to J', giving an iterate $t_0 \le C\log(1/\epsilon)$ and a subinterval $J'' \subset J'$ with $|J''| > 1/C(\epsilon)$ and such that $f^{t_0}_{\sigma^m\omega}(J'') = \hat{I}_{k,\ell}$ injectively, with $|k| \le C\log(1/\epsilon)$ minimal for this property, and good distorsion and expansion for the restriction to J'' of this t_0th iterate. (In particular, (7.20) is replaced by the observation that $|f^p_{\sigma^m\omega}(J')| > \epsilon^{1-3\eta}/C$.) We may then apply Lemma 7.10 to $\hat{I}_{k,\ell}$ and get a subinterval $\widetilde{U} \subset \hat{I}_{k,\ell}$ and a time $t_1 \le C\log(1/\epsilon)$ so that $|f^{t_1}_{\sigma^{m+t_0}\omega}(\widetilde{U})|$ is exactly one of the intervals $\tilde{\Lambda}_\pm$. Take $t(J) = t_0 + t_1$ and $\widetilde{J} = J'' \cap (f^{t_0}_{\sigma^m\omega})^{-1}(\widetilde{U})$. The assertions on the length of $\widetilde{J}$, distortion, and contraction follow from Lemmas 7.9 and 7.10. $\square$

Definition (Escape time). For $\omega \in \Omega$, $m \ge p_0$ and $J \in \mathcal{R}_m(\omega)$, let $t(J)$ be as given by Lemma 8.1. We say (J,ω) has (equivalently, (x,ω) for all points $x \in J$ have) *escaped* at time $m + t(J)$. (By Lemma 8.1, $f^{m+t}_\omega(J)$ contains $\tilde{\Lambda}_+$ or $\tilde{\Lambda}_-$, and we have good distortion and expansion control along the way.)

Lemmas 7.8 and 8.1 together with the remark in Property (P.I) immediately imply:

Corollary 8.2. (Basic escape time asymptotics). *For all* $\omega \in \Omega_0$ *and* $m \ge n_0(\omega) + 2C\log(1/\epsilon)$

$$\mathrm{Leb}\Big(\big\{(x,\omega) \in I \mid x \textit{ escapes at time } \ge m\big\}\Big) \le C\exp\left(-\zeta(\epsilon)\Big(m - 2C\log\Big(\frac{1}{\epsilon}\Big)\Big)\right).$$

Proof of Corollary 8.2. If $m \ge n_0(\omega) + 2C\log(1/\epsilon)$,

$$\begin{aligned}
&\mathrm{Leb}(\{(x,\omega) \in I \mid x \text{ escapes at time } \ge m\}) \\
&\qquad \le \mathrm{Leb}\Bigg(\Bigg\{I \setminus \bigcup_{k=p_0}^{m-2C\log(1/\epsilon)} \mathcal{R}_k(\omega)\Bigg\}\Bigg) \\
&\qquad \le \mathrm{Leb}(\{\mathcal{S}_{m-2C\log(1/\epsilon)}(\omega)\}) \le C\exp(-\zeta(m - 2C\log(1/\epsilon))\,. \quad \square
\end{aligned}$$

For each $m \ge p_0$, we shall define the return times of all $x \in J \in \mathcal{R}_m(\omega)$ abstractly (and independently of ϵ).

Definition (Return time–Partition $\Lambda_i(\omega)$– Abstract return time R_ω). Fix $\omega \in \Omega$, $m \ge p_0(\epsilon)$. For $x \in J \in \mathcal{R}_m(\omega)$, consider the set of all those $t \ge m$ such that f^t_ω maps J injectively onto an interval containing $\tilde{\Lambda}$ and for which there exists a nontrivial interval $\hat{J} = \hat{J}(t) \subset J$ containing x with $f^t_\omega(\hat{J}) = \Lambda$ and $f^t_\omega|_{\hat{J}}$ enjoys bounded distortion (7.15) and uniform contraction on inverse branches (7.14), with the constants from Lemma 8.1. The *return time* $R_\omega(x)$ is then the minimum of those t which appear. It is infinite if the set is empty.

For each ω, define a countable partition of Λ into subintervals $\{\Lambda_i = \Lambda_i(\omega) \mid i \in \mathbb{Z}_+\}$, by considering the connected components of the sets $\{\{x \in \Lambda \mid R_\omega(x) = r\} \mid r \ge p_0\}$.

Proposition 8.3 shows in particular that for $\omega \in \Omega_0$, the $\Lambda_i(\omega)$ form a partition of Λ modulo zero measure sets, and that the return times are almost everywhere defined:

Proposition 8.3 (Return time asymptotics). *There exists $\Omega_1 \subset \Omega_0$ of full measure, a random variable $n_1(\omega)$, and constants $C(\epsilon) \geq 1$, $C_1(\epsilon) > C_2(\epsilon) > 1$ such that for all $\omega \in \Omega_1$, and all $\ell \geq n_1(\omega)$,*

$$\mathrm{Leb}(\{x \in \Lambda \mid R_\omega(x) > \ell\}) < C(\epsilon) e^{-(\ell^{\frac{1}{4}}/C_1(\epsilon))},$$

and

$$P(\{\omega \mid n_1(\omega) > \ell\}) < C(\epsilon) e^{-(\ell^{\frac{1}{4}}/C_2(\epsilon))}.$$

We may replace the right-hand-sides in both inequalities by $C(\epsilon)e^{-\ell^u}$ for $0 < u < 1/4$.

The fact that $C_2(\epsilon) < C_1(\epsilon)$ are crucial to obtain the asymptotics (2.5) for C_ω (see Corollary 5.10).

Proof of Proposition 8.3. We first estimate auxiliary concrete (ϵ-dependent) return times $\widehat{R}_\omega(x)$, corresponding to the first moment when one of the $\Lambda_\pm$ is guaranteed by Lemma 8.1 to be "well" covered (with good expansion and distortion control). After that we shall define second auxiliary concrete return times $R^*_\omega(x)$ corresponding to the first moment when $\Lambda = \Lambda_+$ is well-covered and estimate them using the information on the $\widehat{R}_\omega(x)$. Since, by definition, the "abstract" return times satisfy $R_\omega \leq R^*_\omega$, this will prove Proposition 8.3.

Good returns to $\Lambda_+ \cup \Lambda_-$ (estimating $\widehat{R}_\omega$):

Fix $\omega \in \Omega$. For each $m \geq p_0$, and $J \in \mathcal{R}_m(\omega)$, we now define the auxiliary return time $\widehat{R}_\omega(x) \in \mathbb{Z}_+ \cup \{\infty\}$ of all $x \in J$ inductively. Let $t(J)$ and $\widetilde{J}$ be as in Lemma 8.1. Then, if $f^m_\omega(x) \in \widetilde{J}$, and $f^{m+t}_\omega(x) \in \Lambda_+$ or Λ_- we set

$$\widehat{R}_\omega(x) = m + t(J).$$

If $f^m_\omega(x) \in \widetilde{J}$, but $f^{m+t}_\omega(x) \notin \Lambda_\pm$ (for all r) then

$$\widehat{R}_\omega(x) = m + t(J) + \widehat{R}_{\sigma^{m+t}\omega}\big(f^{m+t}_\omega(x)\big).$$

Finally, if $f^m_\omega(x) \notin \widetilde{J}$, we set

$$\widehat{R}_\omega(x) = m + \widehat{R}_{\sigma^m\omega}\big(f^m_\omega(x)\big).$$

We introduce a sequence of *stopping times* $\widehat{T}_i = \widehat{T}_{\omega,i} : \Lambda_+ \cup \Lambda_- \to \{0, \ldots, n\} \cup \{\infty\}$ with

$$0 \equiv \widehat{T}_{\omega,0} \leq \widehat{T}_{\omega,1}(x) < \widehat{T}_{\omega,2}(x) < \cdots < \widehat{T}_{\omega,k_{\max}(x)}(x) = \widehat{R}_\omega(x), \tag{8.1}$$

such that for all ℓ, $k \in \mathbb{Z}_+$

$$\begin{aligned}\big\{x \in \Lambda_+ \cup \Lambda_- \mid \widehat{R}_\omega(x) > \ell\big\} \\ \subset \big\{x \in \Lambda_+ \cup \Lambda_- \mid k \geq k_{\max}(x), \exists i \leq k_{\max}(x) - 1, \widehat{T}_{\omega,i+1}(x) > \ell\big\} \\ \cup \big\{x \in \Lambda_+ \cup \Lambda_- \mid \widehat{R}_\omega(x) > \widehat{T}_{\omega,k}(x)\big\}.\end{aligned} \tag{8.2}$$

Using standard ideas, it will be easy to bound the mass of the second set in the above decomposition by showing that the probability that $\widehat{T}_{\omega,k} < \widehat{R}_\omega$ (that is, $k < k_{\max}(x)$) decays exponentially fast in k. That is, we shall find $\theta = \theta(\epsilon) < 1$ so that for all $k \in \mathbb{Z}_+$ and all $\omega \in \Omega_0$

$$\mathrm{Leb}\Big(\big\{x \in \Lambda_+ \cup \Lambda_- \mid \widehat{R}_\omega(x) > \widehat{T}_{\omega,k}(x)\big\}\Big) \le \theta^k \,. \tag{8.3}$$

Then, using the basic bound on escape times from Corollary 8.2, we shall control the mass of the first set. More precisely, we shall exhibit a random variable $\hat{n}_1(\omega)$ on a full measure set $\widehat{\Omega}_1$ (with controlled distribution, see (8.8)), and $C(\epsilon) \ge 1$, so that for $\ell > \hat{n}_1(\omega)$

$$\begin{aligned}\mathrm{Leb}\Big(\big\{x \in \Lambda_+ \cup \Lambda_- \mid \sqrt{\ell} \ge k_{\max}(x) \\ \text{and } \exists i \le k_{\max}(x) - 1 \text{ with } \widehat{T}_{\omega,i+1}(x) > \ell\big\}\Big) \le C(\epsilon) e^{-\sqrt{\ell}/C(\epsilon)} \,.\end{aligned} \tag{8.4}$$

Putting together (8.4) and (8.3) for $k = \sqrt{\ell}$ proves that there is $C_3(\epsilon) \ge 1$, so that for all $\ell > \hat{n}_1(\omega)$

$$\mathrm{Leb}\big(\big\{x \in \Lambda_+ \cup \Lambda_- \mid \widehat{R}_\omega(x) > \ell\big\}\big) \le C(\epsilon) e^{-\sqrt{\ell}/C_3(\epsilon)} \,. \tag{8.5}$$

Let us now define the stopping times, using again the notation from Lemma 8.1. We say that $\widehat{T}_{\omega,1}$ is defined at $x \in \Lambda_+ \cup \Lambda_-$ if there is $m_1 \ge p_0$ and $J_1 \in \mathcal{R}_{m_1}(\omega)$ with $x \in J_1$ (hence, $f^{m+t}(J_1)$ contains $f^{m+t}(x)$ and covers $\widetilde{\Lambda}_+$ or $\widetilde{\Lambda}_-$). We then set

$$\widehat{T}_{\omega,1}(x) = \begin{cases} m & \text{if } f^m_\omega x \notin \widetilde{J}, \\ m + t(J_1) & \text{otherwise.} \end{cases}$$

Clearly, $\widehat{R}_\omega(x) \ge \widehat{T}_{\omega,1}(x)$, and equality is only possible in the second case: There, at time $\widehat{T}_1(x)$, part of the component of $f_\omega^{\widehat{T}_1(x)}(J_1)$ containing $f_\omega^{\widehat{T}_1(x)}(x)$ returns to $\Lambda_+ \cup \Lambda_-$. We shall estimate the "overflowing parts" using the distortion control from Lemmas 7.9 and 8.1. For this, let $\Theta_{\omega,1} = \{x \in \Lambda_+ \cup \Lambda_- \mid \widehat{T}_{\omega,1}(x) \text{ is defined}\}$. For $x \in \Theta_{\omega,1} \setminus \{R(x) = \widehat{T}_1(x)\}$, we say that $\widehat{T}_2$ is defined at x if there are $m_2 > p_0$ and $J_2 \in \mathcal{R}_{m_2}(\sigma^{\widehat{T}_1(x)}\omega)$ with $f_\omega^{\widehat{T}_1(x)}(x) \in J_2$, setting $\widehat{T}_2(x)$ to be either $\widehat{T}_1(x) + m_2$, or $\widehat{T}_1(x) + m_2 + t(J_2)$. For general $k \ge 2$, we let $\Theta_{\omega,k} = \{x \mid \widehat{T}_{\omega,k}(x) \text{ is defined}\}$, and we define $\widehat{T}_{\omega,k+1}$ on $\Theta_{\omega,k+1} \subset \Theta_{\omega,k} \setminus \{\widehat{R}_\omega(x) = \widehat{T}_{\omega,k}(x)\}$ if there is $m_k \ge p_0$ and $J_k \in \mathcal{R}_{m_k}(\sigma^{\widehat{T}_k(x)}\omega)$ with $f_\omega^{\widehat{T}_k(x)}(x) \in J_k$. The relation (8.1) (and thus (8.2)) is an immediate consequence of the definition.

Estimate (8.3) *for* $\widehat{R}_\omega$*:*

The estimate (8.3) can be restated as $\mathrm{Leb}(\Theta_{\omega,k}) \le \theta^k$ for some $0 < \theta < 1$ and all $k \in \mathbb{Z}_+$, $n \in \mathbb{Z}_+$, $\omega \in \Omega_0$. This exponential bound will be an easy consequence of

Lemma 8.1. Indeed, for all $\omega \in \Omega_0$, n', and $p_0 \le m$, if J is an interval of $\mathcal{R}_m(\sigma^{n'}\omega)$, the uniform distortion bounds from Lemma 8.1 imply (using the notation there) that

$$\begin{aligned}
\mathrm{Leb}\big(L' := \widetilde{J} \cap (f^t)^{-1}_{\sigma^{n'+m}\omega}(\Lambda_+ \cup \Lambda_-)\big) &> \frac{1}{C(\epsilon)} \frac{|\Lambda_+ \cup \Lambda_-|}{2} \mathrm{Leb}(\widetilde{J}) \\
&\ge \frac{1}{C(\epsilon)^2} \frac{|\Lambda_+ \cup \Lambda_-|}{2}, \\
\mathrm{Leb}\big(J \cap (f^m)^{-1}_{\sigma^{n'}\omega}(L')\big) &> \frac{1}{C(\epsilon)} \frac{1}{C(\epsilon)^2} \frac{|\Lambda_+ \cup \Lambda_-|}{4} \mathrm{Leb}(J)\,.
\end{aligned}$$

(In the above bounds, J may be replaced by a subinterval $L \subset J$ with $|L| \ge |J|/C$, up to adapting the constants correspondingly.)

Therefore, setting $n'(x) = \widehat{T}_{\omega,k-1}(x)$ for $x \in \Theta_{\omega,k-1}$, we have

$$\frac{\mathrm{Leb}\big(f_\omega^{\widehat{T}_{k-1}}(\Theta_{\omega,k-1}) \cap \{y \in \Theta_{\sigma^{n'}\omega,1} \mid \widehat{R}_{\sigma^{n'}\omega}(y) = \widehat{T}_{\sigma^{n'}\omega,1}(y)\}\big)}{\mathrm{Leb}\big(f_\omega^{\widehat{T}_{k-1}}(\Theta_{\omega,k-1})\big)} > \frac{|\Lambda_+ \cup \Lambda_-|}{4C^3(\epsilon)} > 0\,.$$

Since $\Theta_{\omega,k} \subset \Theta_{\omega,k-1} \cap \{\widehat{R}_{\sigma^{n'}\omega} \circ f_\omega^{n'} > \widehat{T}_{\sigma^{n'}\omega,1} \circ f_\omega^{n'}\}$, setting

$$\theta = 1 - \frac{|\Lambda_+ \cup \Lambda_-|}{4C(\epsilon)^6} < 1\,,$$

one more (inductive) application of the distortion bounds yields $\mathrm{Leb}(\Theta_{\omega,k}) \le \theta^k$, as claimed. (Note that θ is uniform in ω but tends to 1 as $|\Lambda_+ \cup \Lambda_-| \to 0$ or $\epsilon \to 0$.)

Estimate (8.4) *for* $\widehat{R}_\omega$*:*

We now move to the estimate (8.4). For fixed $\ell, i \ge 1$, fixed $0 = \tau_0 \le p_0 \le \tau_1 < \tau_2 < \cdots < \tau_i \le \ell$, and $\tau \le \ell$, define $k(\tau) = \max\{0 \le k \le i \mid \tau_k \le \tau\}$ and

$$\begin{aligned}
A_\tau(\tau_1, \dots, \tau_i) = \{x \in \Lambda_+ \cup \Lambda_- \mid k(\tau) \le k_{\max}(x) - 1\,, \widehat{T}_{\omega,k(\tau)+1}(x) > \tau\,, \\
\text{and } \widehat{T}_{\omega,j}(x) = \tau_j\,, \forall \tau_j \le \tau\}\,.
\end{aligned}$$

Applying the absolute bound in Corollary 8.2 we find that, whenever $\tau_1 - 1 > n_0(\omega) + 2C\log(1/\epsilon)$,

$$\begin{aligned}
\mathrm{Leb}(A_{\tau_1 - 1}(\tau_1, \dots, \tau_i)) &= \mathrm{Leb}(\{x \in \Lambda_+ \cup \Lambda_- \mid \widehat{T}_{\omega,1}(x) > \tau_1 - 1\}) \\
&\le C e^{-\zeta(\tau_1 - 1) - 2C\log(1/\epsilon)}\,.
\end{aligned}$$

For $j \ge 2$, we let L be a component of $A_{\tau_{j-1}}(\tau_1, \dots, \tau_i)$ with $\widehat{T}_{\omega,j-1}|L = \tau_{j-1}$, and decompose $L - \{\widehat{R}_\omega = \tau_{j-1}\}$ into connected components $\bigcup_r L_r$ (with possible times $\widehat{T}_{j-1} = m$, and $m + t$). We apply again the absolute bounds from Corollary 8.2 to

$\ell = \tau_j$ and $f^{\tau_j - 1}(L_r)$ and get, using once more the distortion control in Lemma 8.1 when pulling back that whenever $\tau_j - \tau_{j-1} > n_0(\sigma^{\tau_{j-1}}\omega) + 2C\log(1/\epsilon)$

$$\mathrm{Leb}(L_r \cap A_{\tau_j - 1}(\tau_1, \dots, \tau_i)) \le C(\epsilon) \frac{\mathrm{Leb}(L_r)}{\mathrm{Leb}(f_\omega^{\tau_j - 1}(L_r))} e^{-\zeta(\tau_j - 1 - \tau_{j-1}) - 2C\log(1/\epsilon)} .$$

If $\tau_j - \tau_{j-1} \le n_0(\sigma^{\tau_{j-1}}\omega) + 2C\log(1/\epsilon)$, we only have, by the distortion control from Lemma 8.1, that

$$\mathrm{Leb}(L_r \cap A_{\tau_j - 1}(\tau_1, \dots, \tau_i)) \le C(\epsilon) \frac{\mathrm{Leb}(L_r)}{\mathrm{Leb}(f_\omega^{\tau_j - 1}(L_r))} .$$

Thus, by definition of the L_r and A_τ, and using the "large image" properties in Lemma 8.1, there is $C(\epsilon)$ such that for all $j \ge 2$,

$$\begin{cases} \frac{\mathrm{Leb}(A_{\tau_j - 1}(\tau_1, \dots, \tau_i))}{\mathrm{Leb}(A_{\tau_{j-1} - 1}(\tau_1, \dots, \tau_i))} \le C(\epsilon) e^{-\zeta(\tau_j - \tau_{j-1} - 1)} & \text{if } \tau_j - \tau_{j-1} - 1 \ge n_0(\sigma^{\tau_{j-1}}\omega) , \\ \frac{\mathrm{Leb}(A_{\tau_j - 1}(\tau_1, \dots, \tau_i))}{\mathrm{Leb}(A_{\tau_{j-1} - 1}(\tau_1, \dots, \tau_i))} \le C(\epsilon) & \text{if } \tau_j - \tau_{j-1} - 1 < n_0(\sigma^{\tau_{j-1}}\omega) . \end{cases}$$

Therefore for any $0 < \tau_1 < \cdots < \tau_i \le \ell$

$$\mathrm{Leb}(A_\ell(\tau_1, \dots, \tau_i)) \le C(\epsilon)^i e^{-\zeta\ell} \cdot \exp\left[\zeta(\epsilon) \sum_{\tau_j - \tau_{j-1} - 1 \le n_0(\sigma^{\tau_{j-1}}\omega)} (\tau_j - \tau_{j-1})\right],$$

and (we shall soon choose $k = k(\ell)$)

$$\begin{aligned} \mathrm{Leb}\big(\{k \ge k_{\max} , \exists i \le k_{\max} - 1 , \widehat{T}_{\omega,i+1} > \ell\}\big) &\le \sum_{i=0}^{k} \sum_{0 \le \tau_1 < \cdots < \tau_i \le \ell} \mathrm{Leb}(A_\ell(\tau_1, \dots, \tau_i)) \\ &\le \sum_{i=0}^{k} \sum_{0 \le \tau_1 < \cdots < \tau_i \le \ell} C(\epsilon)^i e^{-\zeta\ell} \cdot \exp\left[\zeta \sum_{j=1}^{i} n_0(\sigma^{\tau_{j-1}}\omega)\right] . \end{aligned} \tag{8.6}$$

We now estimate the last factor in (8.6), i.e., the effect of the random waiting times: This is where we shall lose the exponential decay. Fix $0 < \rho < 1$. Since $P(\{n_0(\omega) > n\}) \le Ce^{-\zeta n}$, for each fixed $1 \le i \le k$ and $\tau_1, \cdots, \tau_i$,

$$\begin{aligned} P\left(\left\{\sum_{j=1}^{i} n_0(\sigma^{\tau_{j-1}}\omega) > \rho\,\ell\right\}\right) &\le \sum_{j=1}^{i} P\left(\left\{n_0(\sigma^{\tau_{j-1}}\omega) > \frac{\rho\,\ell}{i}\right\}\right) \\ &\le C(\epsilon)\, k\, e^{-\zeta(\epsilon)\,\frac{\rho\,\ell}{k}} . \end{aligned} \tag{8.7}$$

Consider the partition of Λ into maximal atoms $\Gamma_\omega = \Gamma_\omega(k)$ on which the $\widehat{T}_{\omega,j}(x)$ are constant for $0 \le j \le k$. We will say that such an atom Γ is (ℓ, k)-*good* if for all $x \in \Gamma_\omega$ and $i \le k$,

$$\sum_{j=1}^{i} n_0(\sigma^{\widehat{T}_{\omega,j-1}}\omega) \le \rho\ell .$$

The other atoms are called (ℓ,k)-*bad.* Defining $M_{\ell,k} \subset \Omega \times I$ to be the set of (ω, x) such that x belongs to an (ℓ,k)-bad Γ_ω, (8.7) implies that $(P \times \mathrm{Leb})(M_{\ell,k}) \le Cke^{-\frac{\zeta\rho\ell}{k}}$. Using a Fubini argument as in Corollaries 7.5–7.6, we get that the set $M'_{\ell,k}$ of ω such that $\int \chi_{M_{\ell,k}}(\omega, x)\, d\mathrm{Leb}(x) > ke^{-\frac{\zeta\rho}{3}\frac{\ell}{k}}$ has P-measure smaller than $e^{-\frac{2\zeta\rho}{3}\frac{\ell}{k}}$. Therefore, there is a set of full measure $\widehat\Omega_1 \subset \Omega_0$ such that for each $\omega \in \widehat\Omega_1$, there exists $\hat n_1(\omega) \ge n_0(\omega)$ with the property that $\omega \notin M'_{\ell,k}$ for all $\ell \ge \hat n_1(\omega)$. Now, for $\omega \in \widehat\Omega_1$ and $\ell \ge \hat n_1(\omega)$

$$\begin{aligned}
&\mathrm{Leb}(\{k \ge k_{\max}, \exists i \le k_{\max} - 1, \widehat T_{\omega,i+1} > \ell\}) \\
&\qquad \le \sum_{i=0}^{k} \sum_{\substack{0 \le \tau_1 < \cdots < \tau_i \le \ell \\ (\ell,k)\text{-good } \Gamma_\omega}} \mathrm{Leb}(A_\ell(\tau_1, \ldots, \tau_i) \cap \Gamma_\omega) + \sum_{(\ell,k)\text{-bad } \Gamma_\omega} \mathrm{Leb}(\Gamma_\omega)\,.
\end{aligned}$$

Therefore, taking $k = \sqrt{\ell}$, applying (8.6), and using the Stirling formula we get for $1/2 < v < 1$ and $\ell \ge \hat n_1(\omega)$

$$\begin{aligned}
&\mathrm{Leb}\big(\{k = \sqrt{\ell} \ge k_{\max}, \exists i \le k_{\max}, \widehat T_{\omega,i} > \ell\}\big) \\
&\qquad \le \sqrt{\ell} e^{\ell^v} [C(\epsilon)]^{\sqrt{\ell}} e^{-\ell(1-\rho)\zeta(\epsilon)} + \sqrt{\ell} e^{-\frac{\zeta(\epsilon)\rho}{3}\sqrt{\ell}} \\
&\qquad \le C(\epsilon) e^{-(\sqrt{\ell}/C_3(\epsilon))}\,.
\end{aligned}$$

Combining this with (8.3) ends the proof of the bound (8.5) for the return times $\widehat R_\omega$. Moreover, we may estimate $P(\{\hat n_1(\omega) > \ell\})$:

$$\begin{aligned}
P\big(\{\omega \mid \hat n_1(\omega) > \ell\}\big) &\le P\big(\{\exists j > \ell \mid \omega \in M'_{j,\sqrt{\ell}}\}\big) + P\Big(\{n_0(\omega) > \ell\}\Big) \\
&\le \sum_{j > \ell} e^{-\frac{2\zeta(\epsilon)\rho}{3}\sqrt{j}} + Ce^{-\zeta(\epsilon)\ell} \le C(\epsilon)\, e^{-(\sqrt{\ell}/C_4(\epsilon))}\,.
\end{aligned} \tag{8.8}$$

Note that $C_4(\epsilon)^{-1} > C_3(\epsilon)^{-1}$.

Good returns to Λ_+ (estimating R^*_ω):

For $x \in \Lambda_+ \cup \Lambda_-$ we now consider the "concrete" return times $R^*_\omega(x)$ to $\Lambda = \Lambda_+$. As observed in the beginning of the proof, the abstract times satisfy $R_\omega(x) \le R^*_\omega(x)$. To prove the desired asymptotics for $R^*_\omega(x)$, following § 7.6 in [Yo1], we introduce second stopping times $S_{\omega,i}$ on $\Lambda_+ \cup \Lambda_-$ by setting $S_{\omega,0} \equiv 0$, and

$$S_{\omega,k}(x) = S_{\omega,k-1}(x) + \widehat R_{\sigma^{S_{\omega,k-1}}(\omega)}(f_\omega^{S_{k-1}}(x))\,.$$

If Ξ is the partition $\Lambda_+ \cup \Lambda_-$, and if we define $\Xi_k(\omega) = \Xi \bigvee f_\omega^{-S_1}(\Xi) \bigvee \cdots \bigvee f_\omega^{-S_{k-1}}(\Xi)$, then f^{S_k} maps each element ξ of $\Xi_k(\omega)$ onto Λ_+ or Λ_-, and $f_\omega^{S_k}$ restricted to each such ξ has bounded distortion and uniform contraction in the sense of Lemma 7.9. With the help of ideas already discussed, these two facts yield the following two claims:

(i) The map $f_{\sigma^j\omega}^{\widehat{R}_{\sigma^j\omega}}$ behaves like an irreducible two-state random Markov chain. Consider for a moment the unperturbed map f, writing R^* and S_k for its return and stopping time. Since the intervals $\Lambda_\pm$ are independent of ϵ there are ϵ-independent return times T_+ and T_- with

$$\min\left(\mathrm{Leb}\Big(\{x \in \Lambda_+ \mid R^*(x) = S_{T_+}(x)\}\Big), \mathrm{Leb}\Big(\{x \in \Lambda_- \mid R^*(x) = S_{T_-}(x)\}\Big)\right) > 0\,.$$

Thus, if ϵ is small enough,

$$\inf_{\pm,\omega\in\Omega_1} \mathrm{Leb}\big\{x \in \Lambda_\pm \mid R^*_\omega(x) = S_{\omega,T_\pm}(x)\big\} \geq \frac{1}{C} > 0\,.$$

Hence, there is $K_0 \geq 1$ so that for all ω and k,

$$\mathrm{Leb}\Big(\{x \in \Lambda \mid R^*_\omega > S_{\omega,kK_0}\}\Big) \leq \Big(1 - \frac{1}{C}\Big)^k\,. \tag{8.9}$$

Note also for further use that if (H4) holds, then there is $N_1(f,\Lambda)$ so that $(q - 1/C, q + 1/C) \subset f^n(\Lambda)$ for all $n = N_1(f,\Lambda)$, and thus for $n \geq N_1(f,\Lambda)$, where $q > 0$ is the repelling fixed point of f. (Indeed, take A to be the interior of Λ and, for B, take first $B_1 = (q - 2/C, q - 1/C)$, and then $B_2 = (q + 1/C, q + 2/C)$. For large enough $C \geq 1$, topological mixing gives $L(\Lambda, C)$ so that $f^\ell(\Lambda)$ intersects both B_1 and B_2 for all $\ell \geq L(\Lambda, C)$. Since $f^\ell(\Lambda)$ is connected, it must contain $(q - 1/C, q + 1/C)$ for all $\ell \geq L(\Lambda)$. Take $N_1 = L(\Lambda)$.) If ϵ is small enough this consequence of (H4) also holds for f^n_ω. Clearly, there is $N_2(f,\Lambda)$ so that f^{N_2} sends a subinterval of $(q - 1/C, q + 1/C)$ injectively onto $\widetilde{\Lambda}$ with bounded distorsion and uniform expansion. Thus, if $\epsilon > 0$ is small enough, for all $n \geq \widehat{N} = \max(p_0(\epsilon), N_1 + N_2)$

$$\inf_\omega \mathrm{Leb}\Big(\{x \in \Lambda \mid R^*_\omega(x) = S_{\omega,1}(x) = n\}\Big) \geq \mathrm{Leb}\Big(\{\Lambda \cap f_\omega^{-1}(\Lambda)\}\Big) \geq \frac{1}{C}\,. \tag{8.10}$$

(ii) The tail estimate already obtained for $\widehat{R}_\omega$ gives $C(\epsilon) > 1$ such that for all $\omega \in \widehat{\Omega}_1$, $x \in \Lambda$, $\ell \geq \hat{n}_1(\sigma^{S_{\omega,k}}(\omega))$, $k \in \mathbb{Z}_+$, writing $\xi_k(x)$ for the atom of $\Xi_k(\omega)$ containing x,

$$\frac{\mathrm{Leb}(\{y \in \xi_k(x) \mid S_{\omega,k+1} - S_{\omega,k} > \ell\})}{\mathrm{Leb}(\xi_k(x))} < C(\epsilon)\, e^{-(\sqrt{\ell}/C_1(\epsilon))}\,.$$

Therefore, similarly as in the proof of (8.4), we find a set Ω_1 of full measure and $n_1 : \Omega_1 \to \mathbb{Z}_+$ with $n_1(\omega) \geq \hat{n}_1(\omega)$ such that for all $\ell \geq n_1(\omega)$ and $0 < w < 1/2$,

$$\mathrm{Leb}\big(\{x \in \Lambda \mid S_{\omega,[\ell^w]} > \ell\}\big) < (C(\epsilon))^{\ell^w} \ell^w e^{-(\sqrt{\ell}/C_3'(\epsilon,w))} + C(\epsilon,w) e^{-(\ell^{1/2-w}\rho/(3C_4(\epsilon))}\,. \tag{8.11}$$

Combining (8.9) for $k = [\ell^w/K_0]$ with (8.11), the optimal choice being for $w = 1/2 - w = 1/4$, gives the first inequality of Proposition 8.3. The claim on $P(\{n_1(\omega) > \ell\})$ is proved just like the estimate on $P(\{\hat{n}_1(\omega) > \ell\})$. $\square$

Let us see how the results in Sections 7 and 8 allow us to construct random towers satisfying the axioms in Section 3. We already have a shift σ on the probability space $(P = \prod_{\mathbb{Z}} \nu_\epsilon, \Omega = [-\epsilon, \epsilon]^{\mathbb{Z}})$, for almost each ω a countable partition $\Lambda = \cup_j \Lambda_j(\omega)$ of the subinterval $\Lambda \subset I$ endowed with $m =$ Lebesgue measure, and a function $R_\omega : \Lambda \to \mathbb{Z}_+ \cup \{\infty\}$. The return map $f_\omega^R : \Lambda \to \Lambda$ is just $f_\omega^R(x) = f_\omega^{R_\omega(x)}(x)$. Clearly, the projection $\pi_\omega : \Delta_\omega \to I$ defined by $\pi_\omega(x, \ell) = f^\ell_{\sigma^{-\ell}\omega}(x)$ satisfies $f_\omega \circ \pi_\omega = \pi_{\sigma\omega} \circ F_\omega$ and $\pi_\omega(\Delta_\omega) = \bigcup_{\ell \ge 0} f^\ell_{\sigma^{-\ell}\omega}(\bigcup_j \Lambda_j(\sigma^{-\ell}\omega)) = \bigcup_{\ell \ge 0} f^\ell_{\sigma^{-\ell}\omega}(\Lambda)$.

We also denote by Leb the lift of Lebesgue measure on Δ_ω (suppressing the dependence on ω from the notation) and by d the lift to Δ_ω of the distance $d(x, y) = |x - y|$.

We may now check the axioms from Section 3:

(A.I–III) By construction.

(A.IV) [**Bounded distortion**] Lemmas 7.9–7.10 and 8.1 give $C(\epsilon) > 1$ and $0 < \beta < 1$ (β is independent of ϵ) such that (3.1) holds.

(A.V) [**Return times asymptotics**] For small enough ϵ, Proposition 8.3 gives Ω_1 of full measure and a random variable n_1 on Ω_3 so that (3.2) holds for each $\omega \in \Omega_1$. We apply Remark 3.1, choosing $\widehat{N} = \widehat{N}(\epsilon)$ from (8.10), see the proof of Proposition 4.3.

(A.VI) [**Gcd(Return times)=1**] By (8.10) the stronger property (that there is $\widehat{N}(\epsilon)$ so that $\mathrm{Leb}\{x \in \Lambda \mid R_\omega(x) = r\} > 0$ for each $r \ge \widehat{N}$) holds.

In combination with Corollaries 5.10 and 6.5, Lemma 9.1, Corollary 9.2, and Remark 9.3 give our main theorem:

Lemma 9.1 (Lifting bounded and Lipschitz functions). *There is $p_0(\epsilon)$ so that if $\inf_\omega \inf R_\omega \ge p_0(\epsilon)$ then for each Lipschitz $\phi : I \to \mathbb{C}$, the family of lifted functions $\tilde\phi_\omega = \phi \circ \pi_\omega : \Delta_\omega \to \mathbb{C}$ belongs to $\mathcal{F}_\beta^{\mathcal{K}_\omega}$, for $\mathcal{K}_\omega$ from Theorem 3.2. Furthermore, $C_{\tilde\phi}$ is bounded by an expression depending only on ϵ and (linearly) on the Lipschitz constant of ϕ. If ϕ is bounded on I then $\tilde\phi \in \mathcal{L}_\infty^{\mathcal{K}_\omega}$ and $\sup_\Delta |\tilde\phi| \le \sup |\phi|$.*

Proof of Lemma 9.1. The claim on bounded functions is trivial and we concentrate on Lipschitz functions. The statement is an immediate corollary of the following assertion: There is $C(\epsilon) > 0$ so that for all x, y in Λ, and ℓ for which

$$R_{\sigma^{-\ell}\omega}(x)\, , R_{\sigma^{-\ell}\omega}(y) \ge \ell\, , \text{ and } s_{\sigma^{-\ell}\omega}\big((x,0),(y,0)\big) \ge \ell\, ,$$

we have,

$$d\big(f^\ell_{\sigma^{-\ell}\omega}(x),\ f^\ell_{\sigma^{-\ell}\omega}(y)\big) \le C(\epsilon)\beta^{s_{\sigma^{-\ell}\omega}((x,0),(y,0))-\ell} = C(\epsilon)\beta^{s_\omega((x,\ell),(y,\ell))}\, .$$

To check the assertion, first assume that $s_{\sigma^{-\ell}\omega}((x,0),(y,0)) = p = R_{\sigma^{-\ell}\omega}(x) \ge \ell$. By Proposition 8.3, we have uniform backwards contraction: for all $0 \le j \le p$ and z such that $(z, 0)$ belongs to the same element of $\mathcal{Z}$ as $(x, 0)$ and $(y, 0)$,

$$\Big|\big(f^{p-j}_{\sigma^{-\ell+j}\omega}\big)'\big(f^j_{\sigma^{-\ell}\omega}(z)\big)\Big| \ge \frac{\beta^{j-p}}{C(\epsilon)}.$$

Let $x_\ell = f^\ell_{\sigma^{-\ell}\omega}(x)$, $y_\ell = f^\ell_{\sigma^{-\ell}\omega}(y)$, we have

$$d\big(f^{p-\ell}_\omega(x_\ell),\ f^{p-\ell}_\omega(y_\ell)\big) \ge \frac{\beta^{\ell-p}}{C(\epsilon)}\ d(x_\ell, y_\ell).$$

In general, decompose $s_{\sigma^{-\ell}\omega}((x,0),(y,0)) = p$ into the sum of successive return times of $(x,0)$ and $(y,0)$ to $\Delta_{\omega,0}$, invoking uniform backwards contraction successively and assuming that the minimal return time $p_0(\epsilon)$ has been chosen large enough to guarantee that $C(\epsilon)\beta^{p_0(\epsilon)} < 1$ (where $0 < \beta < 1$ and $C(\epsilon)$ are the contraction and distortion constants from Lemmas 7.9–7.10: there is no loophole here, as increasing p_0 when defining the partition for a fixed ϵ does not make $C(\epsilon)$ or β larger). $\square$

Corollary 9.2 (to Theorem 3.2) (Unique absolutely continuous measures, classical correlations). *The family $\{(\pi_\omega)_*\mu_\omega\}$ gives the unique absolutely continuous sample stationary measures $\{h_\omega \,\mathrm{Leb}\}$ for $\{f_\omega : I \to I\}$. For almost all ω, all Lipschitz ψ and bounded φ on I, we have for all $n \in \mathbb{Z}_+$*

$$\int_I (\varphi \circ f^n_\omega)\psi\, h_\omega\, d\mathrm{Leb} = \int_{\Delta_\omega} (\varphi \circ \pi_{\sigma^n\omega} \circ F^n_\omega)(\psi \circ \pi_\omega)\, d\mu_\omega$$
$$\int_I (\varphi \circ f^n_{\sigma^{-n}\omega})\psi\, h_{\sigma^{-n}\omega}\, d\mathrm{Leb} = \int_{\Delta_{\sigma^{-n}\omega}} (\varphi \circ \pi_\omega \circ F^n_{\sigma^{-n}\omega})(\psi \circ \pi_{\sigma^{-n}\omega})\, d\mu_{\sigma^{-n}\omega}$$

Proof of Corollary 9.2. The equality $(f_\omega)_*(\mu_\omega) = \mu_{\sigma\omega}$ is clear, absolute continuity follows from $((\pi_\omega)_*\mu_\omega)(E) = \sum_{\ell=0}^\infty \mu_\omega(f^{-\ell}_{\sigma^{-\ell}\omega}|_{\Delta_{\omega,\ell}}(E))$ and absolute continuity of μ_ω (using the asymptotics (A.V) as in the end of the proof of Theorem 3.2). Note that $h_\omega\,\mathrm{Leb}$ is strictly positive on $(f^2_{\sigma^{-2}\omega}(0), f_{\sigma^{-1}\omega}(0))$. The two equalities follow by definition (note that they imply the stretched exponential bounds on the past and future "classical" correlation functions by Corollaries 5.10 and 6.5).

To obtain uniqueness, observe that the measure $h_\omega\,\mathrm{Leb}\,P$ on $I \times \Omega$ is an invariant ergodic probability measure for the skew product $(x,\omega) \mapsto (f_\omega x, \sigma\omega)$ (it is in fact mixing). If $\{\tilde\mu_\omega\}$ is an arbitrary family of absolutely continuous sample stationary probability measures for $\{f_\omega\}$, it also gives rise to an invariant probability measure for the skew product. If $\tilde\mu_\omega\, P \ne h_\omega\,\mathrm{Leb}\,P$, then there exist Borel sets $E_\omega \subset [f^2_{\sigma^{-2}\omega}(0), f_{\sigma^{-1}\omega}(0)]$ with $\int \chi_{E_\omega} h_\omega d\mathrm{Leb} = 0$ (so that $\mathrm{Leb}(E_\omega) = 0$) but $\tilde\mu_\omega(E_\omega) > 0$ for almost all ω. This would contradict the assumed absolute continuity of the measures $\tilde\mu_\omega$. $\square$

Remark 9.3 (Convergence to equilibrium and operational correlations). Let $\tilde m_\omega$ be the Lebesgue measure restricted to $\Delta_{0,\omega} = \Lambda \times \{0\}$. By definition, $(\pi_\omega)_*\tilde m_\omega = \mathrm{Leb}|_\Lambda$, and it immediately follows from Corollary 5.10 and the results already shown that

$$|(f^n_\omega)_*(\varphi \mathrm{Leb}|_\Lambda) - h_{\sigma^n\omega}\mathrm{Leb} \int_\Lambda \varphi\, d\mathrm{Leb}|$$

satisfies stretched exponential bounds, for each Lipschitz φ. (The obvious analogous result holds for the "past" convergence to equilibrium, by Corollary 6.5.) To extend this beyond the subinterval Λ, and to estimate the operational correlations, we use the following consequence of Section 7: For each $\epsilon > 0$ there are $0 < \epsilon' < \epsilon$ and $\widetilde{N}(\epsilon) \in \mathbb{Z}_+$ so that $I' = [f^2(0)+\epsilon', f(0)-\epsilon'] \subset f_\omega^{\widetilde{N}(\epsilon)}(\Lambda)$, with only one fold, for all ω. This allows us to write Lebesgue measure restricted to $[f^2(0)+\epsilon', f(0)-\epsilon']$ as the push forward by π_ω of Lebesgue measure on Δ_ω restricted to a union $\mathcal{U}_\omega = F^{\widetilde{N}(\epsilon)}_{\sigma^{\widetilde{N}}\omega} \Delta_{\sigma^{\widetilde{N}}\omega, 0}$ of elements of $\mathcal{Z}_\omega$ such that $\pi_\omega : \mathcal{U}_\omega \to [f(0)+\epsilon', f^2(0)-\epsilon']$ is bijective. We may thus exploit the bounds in Corollaries 5.10 and 6.5 to estimate $|\int_{I'}(\psi \circ f_\omega^n)\varphi \, d\mathrm{Leb} - \int_I \psi h_{\sigma^n\omega} \, d\mathrm{Leb} \int_{I'} \varphi \, d\mathrm{Leb}|$, as well as the past correlation, for Lipschitz φ and bounded ψ. The constant ϵ' can be chosen so that, simultaneously, $I'' = I \setminus \mathrm{int}(I')$ has the property that $f_\omega(I'') \subset \mathrm{int}(I')$ for all ω, with $\inf |f'_\omega|_{I''}|$ bounded away from 1. Decomposing $\int_I \cdot = \int_{I'} \cdot + \int_{I''} \cdot$, we may conclude the proof of our main theorem. $\square$

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