LINEAR RESPONSE DESPITE CRITICAL POINTS

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Many Happy Returns to Nonlinearity!

ABSTRACT. Recent results and open problems about the differentiability (or lack thereof) of SRB measures as functions of the dynamics.

1. Introduction

Publicationwise, I am slightly younger than Nonlinearity: My first paper (joint with Eckmann and Ruelle) appeared in the second volume of the journal in 1989. Over the years, I came to appreciate the journal, not only as a reader/author but also as a referee and editorial board member. So I was delighted by the invitation to write down a "personal selection of problems" for the 20th birthday volume, especially since the Editors did not "expect ... systematic lists, or encyclopedic [articles] in coverage," which I took as an encouragement to let my hair down: In this note, I make very subjective choices, ask questions which seem currently beyond our reach, and provide a ludicrously incomplete bibliography.

2. SRB measures as functions of the dynamics

Let $f : M \to M$ be a smooth discrete-time dynamical system (on a compact Riemann manifold M) admitting a unique SRB measure [25] μ. Assume that $t \mapsto f_t$ is a smooth path through $f = f_0$ and that there exists a large enough [1] set Λ, containing 0 as an accumulation point, so that f_t admits an SRB measure μ_t for each $t \in \Lambda$. One asks how smooth the map $t \mapsto \mu_t$ is at 0, in particular whether it is differentiable.[2] David Ruelle has widely advertised this question, we refer to the introduction of [20] for motivation, in particular from nonequilibrium statistical mechanics. (See also [18], e.g., for related work on transport coefficients.)

We assume throughout, not only that f_0 has a unique SRB measure μ_0, but also that (f_0, μ_0) enjoys strong statistical properties, in particular it is mixing with fast decay of correlation, i.e., for smooth φ, ψ, we have (dx denotes Lebesgue measure)

(1) $\quad \int \varphi \circ f_0^n \psi \, d\mu_0$ converges fast (at least summably) to $\int \varphi \, d\mu_0 \int \psi \, d\mu_0$,

since the difficulties we want to highlight already appear in this case.

If Λ does not contain a neighbourhood 0, differentiability should be understood [21] in the sense of Whitney on Λ. In general, μ_t is not absolutely continuous with

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[1] Large enough can mean that 0 is a Lebesgue density point in Λ.

[2] We concentrate here on the first derivative to avoid complicated formulas.

respect to Lebesgue, so its norm should be taken in some distributional sense. In some cases, this can be done by using the anisotropic Banach spaces introduced recently [11, 16], but one can simply fix a smooth observable φ and consider the map $t \mapsto \int \varphi \, d\mu_t$. If the SRB measure μ_t admits a derivative at 0 (as just defined), one would like to express this derivative in terms of the *infinitesimal perturbation* $\partial_t f_t|_{t=0}$, that is, to write a *linear response formula.* We assume throughout that [3]

(2) there exists a smooth vector field X so that $\partial f_t|_{t=0} = X \circ f_0$.

Let $\mathcal{L}_t$ be the transfer operator associated to f_t (in particular, $\mathcal{L}_t^*$ preserves Lebesgue measure; in the invertible case, $\mathcal{L}_t \varphi = |\det D(f_t)^{-1}|\varphi \circ f_t^{-1}$). We write $\mu_t = \rho_t \, dx$, with $\mathcal{L}_t \rho_t = \rho_t$, letting $\mathcal{L}_t$ act on distributions if necessary [16, 11]. Applying naively standard perturbation theory, and computing $\partial_t \mathcal{L}_t|_{t=0}$, one obtains the linear response formula

$$\partial_t \rho_t|_{t=0} = -(\mathrm{id} - \mathcal{L}_0)^{-1}((X\rho_0)') = -\sum_{j=0}^{\infty} \mathcal{L}_0^j((X\rho_0)') \, , \tag{3}$$

where we use the notation (divergence and gradient are in the sense of distributions)

$$(X\rho_0)' = X'\rho_0 + X\rho_0' \text{ with } X'\rho_0 = \mathrm{div} X \, \rho_0 \text{ and } X\rho_0' = \langle X, \mathrm{grad}\, \rho_0 \rangle \, . \tag{4}$$

If f_t is a C^2 family of C^3 uniformly expanding maps (or C^3 Anosov diffeomorphisms), then (3) is rigourous. However, (3) must not be taken literally in general: Even if $\mathcal{L}_0$ has a spectral gap and the convergence in (1) is exponential, there may exist arbitrarily smooth X such that

(5) $\mathcal{L}_0^j(X\rho_0')$ does not converge summably (as a distribution) to $\rho_0 \int X\rho_0' \, dx$.

This occurs already for piecewise expanding interval maps, where ρ_0' involves a sum of dirac masses along the postcritical orbit. Using $\mathcal{L}_0^*(dx) = dx$, and integrating by parts, (3) gives another formal sum for the derivative of the SRB measure:

$$\partial_t \int \varphi \rho_t \, dx|_{t=0} = \sum_{j=0}^{\infty} \int \langle X, \mathrm{grad}(\varphi \circ f_0^j) \rangle \, \rho_0 \, dx \, . \tag{6}$$

Ruelle [21] proposed the right-hand-side of (6) as a candidate for the linear response formula, pointing out that the series may be divergent (this is connected to (5)). In view of resumming (6), Ruelle suggested to study the *susceptibility function*

$$\Psi(z) = \sum_{j=0}^{\infty} \int z^j \langle X, \mathrm{grad}(\varphi \circ f_0^j) \rangle \, d\mu_0 \, . \tag{7}$$

Another viewpoint is to consider $(\mathrm{id} - z\mathcal{L}_0)^{-1}((X\rho_0)') = \sum_{j=0}^{\infty} z^j \mathcal{L}_0^j((X\rho_0)')$.

We do not know any path f_t (satisfying our standing assumptions) so that the right-hand-side of (6) diverges for some smooth φ *but* $\int \varphi \, d\mu_t$ is Whitney-differentiable at 0, where $\partial f_t|_{t=0} = X \circ f_0$. (See also question (5) in § 3.2.)

Dolgopyat [15] obtained a linear response formula (6) (as a convergent sum) for some partially hyperbolic diffeomorphisms f_t, which may be structurally unstable. On the other hand, the SRB measure is not always differentiable in the simple setting of piecewise expanding interval maps ([5, 7] and references therein, see also § 3.1), where structural unstability originates from the presence of a *critical point.* Under our conditions, in particular (1), we expect that *failure of (Whitney)*

[3] If f_0 is invertible with f_0^{-1} smooth, just set $X = \partial f_t|_{t=0} \circ f_0^{-1}$.

differentiability of the SRB measure (that is, existence of smooth φ and of a path f_t so that $\int \varphi \, d\mu_t$ is not Whitney differentiable at 0 on any large Λ) is possible if and only if *critical points are present.* We do *not* have a general definition of critical points, and we shall restrict to settings where a set $\mathcal{C}_0$ of primitive critical points is available. Roughly speaking, c is a primitive critical point for f if (c belongs to the nonwandering set and) the hyperbolicity built up along the past orbit of c is destroyed when going from c to $f(c)$, e.g. because f is not smooth enough at c, or because Df not invertible at c, or more generally if there is a *folding* (e.g. a primitive homoclinic tangency) at c.

However, even when critical points exist, there may be paths f_t for which linear response holds. In view of the results in [7, 22, 9], we expect that, in many interesting settings with critical points, linear response holds if and only if [4]

(8) f_t is tangent (to some finite order) to the topological class of f_0 at $t = 0$.

A first order *horizontality* condition on f_t as in (8) should depend only on [5] the infinitesimal perturbation $\partial_t f_t|_{t=0} = X \circ f_0$. The reader is warned that the "smooth deformation" theory implicit in the present discussion exists only in the piecewise expanding interval case [7, 8], and for some analytic interval maps (see [4] and references therein to Lyubich's work). Before moving to specific examples in the next sections, we briefly explain how to formally derive necessary (first order) horizontality conditions.

Assume (see [22, 9]) that f_t stays within the topological class of $f = f_0$ for all small enough t, that is, there exist homeomorphisms h_t (with $h_0 = \mathrm{id}$) so that

$$h_t \circ f = f_t \circ h_t \, . \tag{9}$$

It is well-known that regularity of the map $t \mapsto h_t$ is of the utmost relevance. Differentiating formally both sides of (9), and letting α be the (formal) derivative $\partial_t h_t|_{t=0}$ (the *infinitesimal conjugacy*), we obtain the *twisted cohomological equation*

$$X(f(x)) = \alpha(f(x)) - Df(x)(\alpha(x)) \, , \quad \forall x \notin \mathcal{C}_0 \, . \tag{10}$$

If f is smooth at $c \in \mathcal{C}_0$, but Df is not invertible there, or is invertible but "exchanges stable and unstable directions" (see e.g. [12]), we can write (10) at $x = c$ but it is essentially useless there (see the examples in the next sections). This is why we exclude all points in $\mathcal{C}_0$, or in other words we become "amnesic" after crossing a critical point. In the one-dimensional cases treated so far [7, 22], $\mathcal{C}_0 = \{c\}$, α is continuous, and $\alpha(c) = 0$ is automatic [6]. Let us then assume, as a rule of thumb, that α is a continuous vector field which vanishes identically on $\mathcal{C}_0$ (see however §4.1–4.2), and see where formal manipulations lead us. First (10) implies

$$X(f(c)) = \alpha(f(c)) \, , \quad \forall c \in \mathcal{C}_0 \, . \tag{11}$$

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[4] As Dolgopyat explained to me, some of his examples [15, §2.3] enjoying linear response are *not* tangent to the topological class. There are no critical points in [15]. My intuition is that structural stability is violated there to first order, but "along neutral directions" so that linear response is still possible.

[5] Beware that, even in dimension one, we do not always know that first order tangency is enough, see questions (1–2) of § 3.2.

[6] In the piecewise smooth case, letting c_t be the critical point of f_t, we get $c_t = c$ from $f_t = f + tX + O(t^2)$ (in fact $\alpha(c) = 0$ is necessary to get a continuous solution to (10)), and in the smooth case $h_t(c) - c = O(t^2)$ if $h_t \circ f = f_t \circ h_t$.

For smooth one-dimensional maps (11) is just the classical *horizontality* condition (see (13) below). If f is invertible, (10) also gives

$$X(c) = -Df(f^{-1}(c))\alpha(f^{-1}(c))\,, \quad \forall c \in \mathcal{C}_0 \setminus f(\mathcal{C}_0)\,. \tag{12}$$

If f is invertible [7] we can rewrite (10) as $X = (\mathrm{id} - f^*)\alpha$, over $M \setminus f(\mathcal{C}_0)$. (In the Anosov case, where $\mathcal{C}_0 = \emptyset$, Moser [19] used this equation to give a proof of structural stability [8].) Our philosophy can be phrased as follows: We look for a space $\mathcal{H}$ of "horizontal" smooth vector fields X, so that $\mathrm{id} - f^*$ has a left inverse (see also (15) below) from $\mathcal{H}$ into $\mathcal{B}$, hoping that $\mathcal{B}$ will be nice. (Regularity of α is desirable [10] although not always necessary [7].) In view of (11–12), very roughly, the larger the set of (primitive) critical points, the higher the codimension of the set of horizontal vector fields X. (We expect the critical set to be thin in view of our fast mixing assumption (1). In all examples below, $\mathcal{C}_0$ has zero Lebesgue measure.)

As a final remark, the Viana and Hénon maps examples below indicate that a properly defined critical set may live in the tangent or cotangent bundle of the manifold, involving some notion of wave front set for the SRB measure.

3. Results and questions for one-dimensional dynamics

3.1. **Recent results.** We recall very recent developments in one-dimensional dynamics [7, 8, 9, 22]. Consider a piecewise expanding and piecewise C^3 interval map f with a single critical point [9] c, where the derivative is not defined. The identity (11) for a solution of (10) is equivalent (see [7] and references therein) with the following well-known horizontality condition in smooth one-dimensional dynamics

$$\sum_{j=0}^{\infty} \frac{X(f^j(f(c)))}{(f^j)'(f(c))} = 0\,. \tag{13}$$

The above (codimension-one) condition holds [8] for $X \circ f = \partial_t f_t|_{t=0}$ if and only if the C^2 path f_t is tangent to the topological class of $f = f_0$ at $t = 0$. It is necessary and sufficient [7] to ensure differentiability of the SRB measure $\mu_t = \rho_t\,dx$ of f_t at $t = 0$. (If the condition does not hold, the modulus of continuity $t \ln t$ found by Keller can sometimes [5, 7] be attained.) If X is horizontal, (10) admits a continuous solution α and, decomposing [5] $\rho_0 = \rho_{reg} + \rho_{sing}$, where $\rho'_{reg} \in BV$ and ρ_{sing} is an exponentially decaying sum of Heaviside jumps along $f^j(c)$, $j \geq 1$, we have [7] a linear response formula (in the sense of Radon measures)

$$\partial_t \rho_t\, dx|_{t=0} = -\alpha\rho'_{sing} - (\mathrm{id} - \mathcal{L}_0)^{-1}(X'\rho_{sing} + (X\rho_{reg})')\,dx\,, \tag{14}$$

compatible with a resummation of (7) at $z = 1$. Finally, for horizontal X, abusing slightly notation, [5, Lemma 4.1] and [7] give

$$(\mathrm{id} - \mathcal{L}_0)\partial_t\rho_t|_{t=0} = (X\rho_0)'\,. \tag{15}$$

Ruelle [22] considered a smooth path f_t, within the topological class of an analytic S-unimodal Misiurewicz map $f = f_0$, with $f'(c) = 0$, $f''(c) \neq 0$. Then, (13) holds. He proved that a resummation $\Psi(X, z)$ of $\Psi(z)$, or equivalently of $(\mathrm{id} - z\mathcal{L}_0)^{-1}((X\rho_0)')$, corresponding to a left-inverse [10] of $(\mathrm{id} - z\mathcal{L}_0)$ on a suitable space, is holomorphic [22, §17–19] in an annulus $1 - \eta < |z| < 1 + \eta$, for $\eta > 0$.

[7] If f is noninvertible, but Df is invertible write $(\mathrm{id} - (f^{-1})^*)\alpha = -(f^{-1})^*X$, on $M \setminus \mathcal{C}_0$.

[8] I thank Daniel Smania for mentioning this reference.

[9] We assume that f is continuous and c is not periodic.

[10] This is analogous to (15).

In addition he [22] obtained differentiability of the SRB measure and the linear response formula $\Psi(X,1) = \partial_t \int \varphi \rho_t \, dx|_{t=0}$ for smooth φ. He uses a decomposition $\rho_0 = \rho_{reg} + \rho_{sing}$, where $\mathcal{L}_0^j(X\rho'_{reg})$ converges exponentially to $\rho_0 \int X \rho'_{reg} \, dx$, and ρ_{sing} is an exponentially decaying sum of explicit functions ("spikes") along the postcritical orbit.

For holomorphic families f_t of quadratic-like holomorphic S-unimodal Collet-Eckmann maps which are real analytic on the interval, and real analytic φ, the map $t \mapsto \int \varphi \rho_t \, dx$ [9] is real analytic.

3.2. **Open problems for interval dynamics.** The case of smooth nonuniformly hyperbolic interval maps is very far from being understood. We restrict for simplicity to unimodal maps [11]. Let f_t be a C^s path of C^r S-unimodal maps, with $f = f_0$ Collet–Eckmann, and $f''(c) \neq 0$. Whitney continuity of $R(t) = \int \varphi \, d\mu_t$ at $t = 0 \in \Lambda$ was proved by various authors on various sets Λ, see [7, 22] for references. Recalling our notation (2), here are some open questions (taking r, s large enough, Λ "large", and φ smooth enough):

1) *Smooth deformations:* Does (13) imply that (10) admits a continuous solution α? In the affirmative: does α have a (Hölder?) modulus of continuity? is $\alpha(c) = 0$? do there exist homeomorphisms h_t so that $h_t \circ f = \tilde{f}_t \circ h_t$ with $\alpha = \partial_t h_t|_{t=0}$ and $\partial_t f_t|_{t=0} = \partial_t \tilde{f}_t|_{t=0}$? (If the answer to any of these questions is negative, we can consider higher-order horizontality conditions, or even assume that f_t stays in the topological class of f_0.)

2) *Linear response:* Does (13) imply Whitney differentiability [12] of $R(t)$ at $t = 0$ on Λ, as conjectured in [7, Conj. A']? Or do we need a tangency of higher order to the topological class of f_0? Is (13) (or some higher order condition) necessary to obtain Whitney differentiability?

3) *Transversal regularity:* How smooth is $R(t)$ at $t = 0$ in the transversal direction, i.e., when (13) does *not* hold? (We conjectured [5, Conj. B] that $R(t)$ is Whitney β-Holder for all $\beta < 1/2$. N. Dobbs and M. Todd pointed out to me that this cannot hold if Λ is the set of *all* Collet-Eckmann parameters: We now ask the question for a smaller large set Λ.)

4) *Beyond nondegenerate Collet-Eckmann:* What if $f''(c) = 0$ but $f^{(q)}(c) \neq 0$ for some $q > 2$? What if $(f^j)'(f(c))$ does not grow exponentially, but grows fast enough [13] so that f admits a unique SRB measure with summable decay of correlations?

5) In [7] we give a sufficient condition (stronger than (13)) for *abelianity* of $\Psi(z)$ (from (7)) in the piecewise expanding case. Is this condition necessary? Is there any Collet–Eckmann analogue (involving perhaps Borel monogenic extensions)?

4. Open problems in higher dimensions

4.1. **All exponents positive.** Assume that $f = f_0 : M \to M$ is a continuous and *piecewise smooth uniformly expanding map.* That is, there exists a finite partition of the manifold M into compact domains with "nice" (e.g. piecewise C^1) boundaries, so that $f = f_0$ restricted to each domain is $C^{2+\delta}$ and locally uniformly expanding. Additional complexity and transversality conditions (which are generic) have been shown by various authors (see e.g. the references in [6]) to ensure the existence of finitely many ergodic absolutely continuous invariant measures, which are the SRB

[11] For multimodal maps, the horizontality condition will have higher codimension

[12] We expect the linear response formula to be given by the candidate $\Psi(X,1)$ in [22, §17–18].

measures, and which enjoy exponential decay of correlations (up to taking a finite factor). Our standing assumption (1) on (f_0, μ_0) is thus reasonable.

As a working definition, let the set $\mathcal{C}_0$ of *primitive critical points* be the union of the boundaries of the domains of smoothness of f. In view of (10), we require for the moment that $\alpha(c) = 0$ if $c \in \mathcal{C}_0$, in the hope of getting α continuous. [13] The unique bounded function α satisfying (10) and $\alpha|_{\mathcal{C}_0} \equiv 0$ is

$$\alpha(x) = -\sum_{j=1}^{N(x)} (D(f^j)_x)^{-1}(X(f^j(x)))\,, \tag{16}$$

where $N(x) = \infty$ if $f^j(x) \notin \mathcal{C}_0$ for all $j \geq 0$, and otherwise $N(x)$ is the smallest $j \geq 0$ so that $f^j(x) \in \mathcal{C}_0$. (Note [14] that $N(x) = \infty$ on a large set.)

We thus obtain a conjectural *horizontality condition* from (10) and (11):

$$\mathcal{J}_f(X)(x) := \sum_{j=0}^{N(x)} (D(f^j)_x)^{-1}(X(f^j(x)))\,, \quad \mathcal{J}_f(X)(f(c)) = 0\,, \quad \forall c \in \mathcal{C}_0\,. \tag{17}$$

Clearly, $\mathcal{J}_f : X \mapsto \mathcal{J}_f(X)$ is a nontrivial linear operator from the space of smooth vector fields on M to the space of bounded vector fields on M. Here are open questions for generic piecewise smooth expanding maps, using our standard notations:

0) If α extends continuously from $\{x \mid N(x) = \infty\}$ to M, do we have $\alpha|_{\mathcal{C}_0} \equiv 0$? If not, is it possible to weaken the requirement that α vanishes on all of $\mathcal{C}_0$? (This would impact on some of the questions below.)

1.a) Assume (17). Does (10) admit a continuous solution α? Is it Hölder? Is there a path $\tilde f_t$ in the topological class of $f = f_0$, via homeomorphisms h_t, so that $\partial_t \tilde f_t|_{t=0} = \partial_t f_t|_{t=0}$ and $\alpha = \partial_t h_t|_{t=0}$? If the answer to any of these questions is negative, use higher order horizontality, or even assume f_t is topologically conjugated to $f = f_0$ via h_t for all small enough t.

1.b) Denote by $\mathcal{I}$ the restriction map from M to $f(\mathcal{C}_0)$. Can one describe the "size" of $\mathrm{Ker}(\mathcal{I} \circ \mathcal{J}_f)$ ($f(\mathcal{C}_0)$ is a an uncountable set, but also a finite union of codimension-one manifolds)? Does $\mathrm{Ker}(\mathcal{I} \circ \mathcal{J}_f)$ depend continuously on f?

2) Does (17) imply that μ_t depends differentiably on t, as a Radon measure or as a distribution (of order one?)?

2.a) If yes, do we get a linear response formula compatible with a resummation of $(\mathrm{id} - z\mathcal{L}_0)^{-1}((X\rho_0)')$ or of $\Psi(z)$? (Can we decompose $\rho_0 = \rho_{reg} + \rho_{sing}$ into a function ρ_{reg}, so that $\mathcal{L}_0^j(X\rho'_{reg})$ converges summably to $\rho_0 \int X\rho'_{reg}\,dx$, and a function $\rho_{sing} = \sum_{\ell \geq 1} \rho_{sing,\ell}$, with $\rho'_{sing,\ell}$ a distribution supported on $f^\ell(\mathcal{C}_0)$?) Can we get a more "compact" linear response formula, in the spirit of (14)? Is (17) (maybe ignoring subsets of $\mathcal{C}_0$ which are negligible with respect to ρ'_0) necessary to obtain α continuous? to get linear response?

2.b) If not, what about higher order horizontality conditions?

Some *nonuniformly expanding maps with singularities,* satisfying recurrence conditions, were proved by Alves–Bonatti–Viana to admit absolutely continuous SRB measures which enjoy [3] polynomial decay of correlations. The density $\rho_t \in L^1(dx)$ depends continuously on t [1]. The set of critical points (or singularities) $\mathcal{C}_0$ is well-defined, and one can ask the same questions as for piecewise expanding maps.

[13] Note also that (2) yields $Df_t = Df + tX' \circ fDf + O(t^2)$, so that the discontinuity set of Df_t and Df coincide if X is smooth, see footnote 16.

[14] If $N(x) = \infty$ (16) gives a solution of (10) at x without assuming $\alpha|_{\mathcal{C}_0} \equiv 0$

We consider next the smooth nonuniformly expanding maps known as *Viana maps* [23]. Here, $f = f_0$ is everywhere C^3 on a cylinder, and the critical set $\mathcal{C}_0$ is a well-identified curve, consisting of those x so that $Df(x)$ is not invertible. ($N(c) < \infty$ for $c \in \mathcal{C}_0$ is possible.) Existence and t-continuity of ρ_t as an element of $L^1(dx)$ is known (see references of [1]). The measure $\rho_0 dx$ is mixing and its correlations decay [3] rapidly (although probably not exponentially). Viana maps are robust, and differentiability of ρ_t under a horizontality condition possibly holds again in the usual sense. However, this setting is much more difficult than piecewise expanding maps: First, the series (16) is a priori convergent [23] only on a full measure subset of $\{N(x) = \infty\}$. If a continuous extension of α exists then it is not clear that it vanishes on $\mathcal{C}_0$. Assuming this is the case, (17) is a priori convergent only on a subset $\mathcal{V}$ of full measure of $f(\mathcal{C}_0)$ (taking a constant admissible curve in [23]). Defining horizontality by considering only $\mathcal{V}$, one can ask the same questions as before (although $\mathcal{J}$ is not a priori an operator mapping smooth vector fields to bounded vector fields, and $\rho_{sing,\ell}$ may not be supported on $f^\ell(\mathcal{C}_0)$). In addition, we ask one more question, assuming that $\mathcal{C}_0 = \{(\theta, x) \mid x = 0\}$ and the vertical bundle $E^{wu} = \{(v_\theta, v_x) \mid v_\theta = 0\}$ is invariant: Is it enough to require that the x-coordinate α^{wu} of α vanishes on $\mathcal{C}_0$? Then horizontality would read $\alpha^{wu}(f(c)) = X^{wu}(f(c))$, that is, $\sum_{j=0}^{N(f(c))}(D(f^j|_{E^{wu}})_{f(c))})^{-1}(X(f^{j+1}c)) = 0$, for almost all $c \in \mathcal{C}_0$.

Up to now, we considered differentiability in the usual sense in this section. It would be much more challenging to study *nonrobust nonuniformly expanding maps* (such as a skew products of two unimodal maps, assuming, e.g., that they are both Misiurewicz). To our knowledge, the existence of SRB measures is still an open problem here, so that it seems too early to ask about Whitney differentiability of the SRB measures. The very definition of (primitive) critical points is also a hurdle, and it is not clear that this critical set will be thin.

4.2. **In the presence of negative exponents.** Let us start with continuous and invertible *piecewise smooth uniformly hyperbolic maps:* Assume there are two smooth families of transversal closed cones C^u and C^s in the tangent bundle of a manifold M, and a finite partition of M into compact domains with "nice" boundaries, so that $f = f_0 : M \to M$ restricted to each domain is a local $C^{2+\delta}$ diffeomorphism, with $Df(C^u)$ mapped in the interior of C^u and $Df^{-1}(C^s)$ mapped in the interior of C^s. Complexity and transversality conditions (see [14, 6] and references therein) ensure the existence of finitely many SRB measures, which enjoy exponential decay of correlations (up to a finite factor). Our standing assumption (1) on (f_0, μ_0) is thus reasonable. In dimension two, Demers–Liverani [14] showed that the SRB measure of such f, viewed as a suitable distribution, depends continuously on f [15]. As for piecewise expanding maps, we define the set $\mathcal{C}_0$ of primitive critical points to be the union of the boundaries of the domains of smoothness of f.

Let us look for a vector field α, whose coefficients are functions vanishing at $\mathcal{C}_0$, solving (10). [16] The series (16) does not always converge. Inspired by [19], we choose measurable bundles F^u and F^s, defined over $M \setminus \mathcal{C}_0$, respectively $M \setminus f(\mathcal{C}_0)$, with $TM_{M\setminus(\mathcal{C}_0 \cup f(\mathcal{C}_0))} = F^u \oplus F^s$, and so that, for x, $f(x) \notin \mathcal{C}_0$ and y, $f^{-1}(y) \notin f(\mathcal{C}_0)$

$$Df(F^u(x)) = F^u(f(x)), \quad Df^{-1}(F^s(y)) = F^s(f^{-1}(y)),$$

[15]The distance between two maps depends on the distance between their critical sets and their C^2 distance outside of a neighbourhood of the critical sets.

[16]Again, requiring α to vanish on all of $\mathcal{C}_0$ may be too stringent.

and that, for all x, y, so that $f^j(x) \notin \mathcal{C}_0$ $f^{-j}(y) \notin f(\mathcal{C}_0)$ for all $j \ge 0$,

$$\sum_{j=1}^{\infty} \|(D(f^j|_{F^u})_x)^{-1}\| < \infty\,, \quad \sum_{j=0}^{\infty} \|(D(f^{-j}|_{F^s})_y)^{-1}\| < \infty\,.$$

If x is a periodic point whose orbit does not intersect $\mathcal{C}_0$, then we must take $F^u(x) = E^u(x)$ and $F^s(x) = E^s(x)$. More generally, if the forward orbit of y does not meet $\mathcal{C}_0$, then $F^s(y) = E^s(y)$, and if the past orbit of x does not meet $\mathcal{C}_0$, then $F^u(x) = E^u(x)$. However, F^u on the forward orbit of $\mathcal{C}_0$ and F^s on the backward orbit of $\mathcal{C}_0$ are not defined uniquely: We are asking for *forward* and *backwards expansion,* and invariance is not required across $\mathcal{C}_0$.

Then, a solution (α^u, α^s) of (10) is given over $M \setminus (\mathcal{C}_0 \cup f(\mathcal{C}_0))$ by

$$\begin{cases} \alpha^u(x) &= -\sum_{j=1}^{N(x)} (D(f^j|_{F^u})_x)^{-1}(X(f^j(x)))\,, \\ \alpha^s(x) &= \sum_{j=0}^{N^-(x)-1} (D(f^{-j}|_{F^s})_x)^{-1}(X(f^{-j}(x)))\,, \end{cases} \tag{18}$$

where [17] $N(x)$ is as in Subsection 4.1, $N^-(x) = \infty$ if $f^{-j}(x) \notin \mathcal{C}_0$ for all $j \ge 0$, and otherwise $N^-(x)$ is the smallest integer $j \ge 0$ so that $f^{-j}(x) \in \mathcal{C}_0$. In view of (11) for α^u, (12) for α^s, the conjectural *horizontality condition* is

(19)

$$\begin{cases} \mathcal{J}^u_f(X)(x) &:= \sum_{j=0}^{N(x)} (D(f^j|_{F^u})_x)^{-1}(X(f^j(x)), \qquad \mathcal{J}^u_f(X)(f(c)) = 0\,, \forall c \in \mathcal{C}_0, \\ \mathcal{J}^s_f(X)(x) &:= \sum_{j=0}^{N^-(f^{-1}(x))} (D(f^{-j}|_{F^s})_x)^{-1}(X(f^{-j}(x)), \; \mathcal{J}^s_f(X)(c) = 0\,, \forall c \in \mathcal{C}_0. \end{cases}$$

(Note that $c = f^{-1}(f(c))$, and the critical set of f^{-1} is $f(\mathcal{C}_0)$.) A priori, the horizontality condition (19) depends on the choice of F^u over $f(\mathcal{C}_0) \setminus \mathcal{C}_0$ and F^s over $\mathcal{C}_0 \setminus f(\mathcal{C}_0)$.

Replacing $\mathcal{J}$ by the pair $(\mathcal{J}^u, \mathcal{J}^s)$ we can ask the same questions as in Subsection 4.1 (taking again Λ a neighbourhood of 0, and allowing additional generic restrictions on the maps if necessary), with two essential changes: First, differentiability of μ_t should be in a distributional sense, by using either smooth observables, or anisotropic norms as in [6, 14]. (The "regular" term in the possible decomposition of ρ_0 should be meant in a distributional sense.) Second, and more importantly, we must determine whether "horizontality" means that (19) holds for a specific choice of F^u and F^s (if they exist, do such good F^u and F^s have continuity properties, at least on $f(\mathcal{C}_0)$, respectively $\mathcal{C}_0$? are they unique?), or for all F^u and F^s satisfying our conditions. (As before, maybe (19) on a large subset of $\mathcal{C}_0$ is enough.) Within the conjugacy class of f_t, the fact that α is well-defined at periodic orbits may let us bypass some of these difficulties, see Section 5.

Last, but not least, consider a (say, two-dimensional) *Hénon-like attractor* f_0 of Benedicks-Carleson type (see [24] and references therein), for which a unique exponentially mixing SRB measure exists. Recently, Alves–Carvalho–Freitas [2] proved that the SRB measure $\mu_{a,b}$ of the Hénon attractor $H_{a,b}$ depends continuously on (a, b) in the sense of Whitney, on the set Λ of Benedicks-Carleson parameters. [18] Here, the primitive critical set $\mathcal{C}_0$ of f_0 is a well-defined Cantor set, which appears to lie within a Hölder curve (by the – highly non trivial – construction, see e.g.

[17] If $N^-(x) = N(x) = \infty$ then (18) does not use $\alpha|_{\mathcal{C}_0} \equiv 0$

[18] Just as we cannot afford to stay within the logistic family $a - x^2$ if we require horizontality, Whitney differentiability of SRB measures within the Hénon family $H_{a,b}$, or even another admissible family [24], may be impossible. Also, the critical set may move too fast.

[24]). We have $N(c) = \infty$ and $N^-(f^{-1}(c)) = \infty$ for each $c \in \mathcal{C}_0$. Although the primitive critical set $\mathcal{C}_0(f^{-1})$ of f^{-1} is not formally defined, one may reasonably guess that it is $f(\mathcal{C}_0)$. Of course, it is not clear how to define $F^u(x)$ and $F^s(x)$, and thus our candidate (18) for $\alpha(x)$, for arbitrary x. (Periodic points x, e.g., are fine since they are all hyperbolic [24].) However, for any $c \in \mathcal{C}_0$, we can select $F^u(f(c))$ (and thus define $\alpha^u(f(c))$ and $\mathcal{J}^u_f(X)(c)$ via (18)), as we explain next. Beware that $f(c)$ does not necessarily possess a local unstable manifold (the local stable manifold does not seem to be a problem, by [12]). However, we are concerned with *forward* expansion. Setting v to be the unit vertical vector, there is (see e.g. [12]) $\lambda < 1$ so that $\|(Df^j)(c)(v)\| \ge \lambda^{-j}$, for all $c \in \mathcal{C}_0$ and $j \ge 0$. We may thus take $F^u(f(c)) = Df(c)(v)$, so that F^u is even smooth on $f(\mathcal{C}_0)$. Then, the series for $\alpha^u(f(c))$ is summable. Since f^{-1} expands volume, the existence of $F^s(c)$ is obvious for any $c \in \mathcal{C}_0$, defining $\alpha^s(c)$ and $\mathcal{J}^s_f(X)(c)$. In particular, the *horizontality condition* (19) corresponding to this choice of F^u and F^s is well-defined.

One can now ask the same questions as for piecewise hyperbolic maps, aiming only at Whitney differentiability on a large set Λ [19]. The requirement that α vanishes on $\mathcal{C}_0$ may be too strong for Hénon-like maps (Df is everywhere smooth and invertible), is it possible to focus on α^u (beware that Df sends $E^u(c)$ to $E^s(f(c))$ for $c \in \mathcal{C}_0$) in the spirit of our question about α^{wu} for the Viana map? Or need we consider also $\alpha^s(x)$, but only for x in the basin of the attractor? In which sense is a continuous solution α of (10) (if it exists) unique? To end by a litote, it seems in any case that linear response for Hénon maps is not an easy problem.

5. Is differentiability of the SRB measure determined by the periodic point structure?

In view of [17, 9], we ask the following question. We use our standard notations, considering families of invertible maps f_t as in §3–4. Let φ be a real observable. For small $t \in \Lambda$, let $\mathrm{Per}(f_t, p) = \{x \mid f_t^p(x) = x\,,\ f_t^j(x) \notin \mathcal{C}_0(f_t)\,,\ \forall j \in \mathbb{Z}\}$, and set for small s

$$\zeta(s,t,z) = \exp \sum_{p=1}^{\infty} \frac{z^p}{p} \sum_{y \in \mathrm{Per}(f_t,p)} \frac{e^{s \sum_{k=0}^{p-1} \varphi(f_t^k(y))}}{|\mathrm{Det} D(f_t^p)|_{E^u_{f_t}}(y)|}\,. \tag{20}$$

Let $\lambda_{s,t}^{-1}$ be the radius of convergence of $\zeta(s,t,z)$. Does fast enough decay of correlations for the SRB measure of f_t imply that $s \mapsto \log \lambda_{s,t}$ is differentiable at $s = 0$ (in the usual sense), with $\partial_s \log \lambda_{s,t}|_{s=0} = \int \varphi\, \mu_t$? (This would imply an equidistribution result.) In this case (Whitney) differentiability of the SRB measure could be read from mixed differentiability properties of $\log \lambda_{s,t}$. If we assume that $h_t \circ f = f_t \circ h_t$, we have

$$\zeta(s,t,z) := \exp \sum_{p=1}^{\infty} \frac{z^p}{p} \sum_{x \in \mathrm{Per}(f_0,p)} \frac{e^{s \sum_{k=0}^{p-1} \varphi(h_t(f^k(x)))}}{|\mathrm{Det} D(f_t^p)|_{E^u_{f_t}}(h_t(x))|}\,, \tag{21}$$

so that one expects

$$\log \lambda_{s,t} = \lim_{p \to \infty} \frac{1}{p} \log \sum_{x \in \mathrm{Per}(f_0,p)} \frac{e^{s \sum_{k=0}^{p-1} \varphi(h_t(f^k(x)))}}{|\mathrm{Det} D(f_t^p)|_{E^u_{f_t}}(h_t(x))|}\,.$$

[19] We could also fix the conjugacy class (in the attractor?) of a good Hénon-type map, hoping to get differentiability in the usual sense.

If $t \mapsto h_t(x)$ is differentiable on Λ for $x \in \cup_{p\geq 1}\mathrm{Per}(f_0, p)$, under which assumptions can we recover a linear response formula by taking derivatives $\partial_t \partial_s|_{(0,0)} \log \lambda_{s,t}$?

References

1. J.F. Alves, *Strong statistical stability of non-uniformly expanding maps,* Nonlinearity **17** (2004) 1193–1215.
2. J.F. Alves, M. Carvalho, and J.M. Freitas, *Statistical Stability for Hénon maps of the Benedicks-Carleson type,* `arXiv:math/0610602` (2006).
3. J.F. Alves, S. Luzzatto, and V. Pinheiro, *Markov structures and decay of correlations for non-uniformly expanding dynamical systems,* Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005) 817–839.
4. A. Avila, M. Lyubich, and W. de Melo, *Regular or stochastic dynamics in real analytic families of unimodal maps,* Invent. Math. **154** (2003) 451–550.
5. V. Baladi, *On the susceptibility function of piecewise expanding interval maps,* Comm. Math. Phys. **275** (2007) 839–859.
6. V. Baladi and S. Gouëzel, *Good Banach spaces for piecewise hyperbolic maps via interpolation,* `arXiv:math/0711.1960` (2007).
7. V. Baladi and D. Smania, *Linear response for piecewise expanding unimodal maps,* Nonlinearity **21** (2008) 677–711.
8. V. Baladi and D. Smania, *Smooth deformations of piecewise expanding unimodal maps,* `arXiv:math/0710.1845` (2007).
9. V. Baladi and D. Smania, *Analyticity of the SRB measure for holomorphic families of quadratic-like Collet-Eckmann maps,* `arXiv:math/0801.3356` (2008).
10. V. Baladi and D. Smania, *Alternative proofs of linear response for piecewise expanding unimodal maps,* `arXiv:math/0803.3344` (2008).
11. V. Baladi and M. Tsujii, *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms,* Annales de l'Institut Fourier **57** (2007) 127–154.
12. M. Benedicks and L.-S. Young, *Markov extensions and decay of correlations for certain Hénon maps,* Géométrie complexe et systèmes dynamiques, Astérisque **261** (2000) 13–56.
13. H. Bruin, S. Luzzatto, and S. Van Strien, *Decay of correlations in one-dimensional dynamics,* Ann. Sci. École Norm. Sup. **36** (2003) 621–646.
14. M. Demers and C. Liverani, *Stability of statistical properties in two-dimensional piecewise hyperbolic maps,* `arXiv:math/0602067` (2006) to appear Trans. Amer. Math. Soc.
15. D. Dolgopyat, *On differentiability of SRB states for partially hyperbolic systems,* Invent. Math. **155** (2004) 389–449.
16. S. Gouëzel and C. Liverani, *Banach spaces adapted to Anosov systems,* Ergodic Theory Dynam. Systems **26** (2006) 189–218.
17. A. Katok, G. Knieper, M. Pollicott, and H. Weiss, *Differentiability and analyticity of topological entropy for Anosov and geodesic flows,* Invent. Math. **98** (1989) 581–597.
18. G. Keller, P.J. Howard, and R. Klages, *Continuity properties of transport coefficients in simple maps,* `arxiv.org` (2008).
19. J. Moser, *On a theorem of Anosov,* J. Differential Equations **5** (1969) 411–440.
20. D. Ruelle, *Differentiation of SRB states,* Comm. Math. Phys. **187** (1997) 227–241.
21. D. Ruelle, *Application of hyperbolic dynamics to physics: some problems and conjectures,* Bull. Amer. Math. Soc. **41** (2004) 275–278.
22. D. Ruelle, *Structure and f-dependence of the a.c.i.m. for a unimodal map f of Misiurewicz type,* `arXiv:math/0710.2015` (2007).
23. M. Viana, *Multidimensional nonhyperbolic attractors,* Publ. Math. I.H.É.S. **85** (1997) 63–96.
24. Q. Wang and L.S. Young, *Strange attractors with one direction of instability,* Comm. Math. Phys. **218** (2001) 1–97.
25. L.-S. Young, *What are SRB measures, and which dynamical systems have them?* J. Statist. Phys. **108** (2002) 733–754.

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