Floquet spectrum of weakly coupled map lattices

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Revised - February 2001

ABSTRACT. We consider weakly coupled analytic expanding circle maps on the lattice $\mathbb{Z}^D$ (for $D \geq 1$), with small coupling strength ϵ and summable decay of the two-sites coupling. We study the spectrum of the associated (Perron-Frobenius) transfer operators $\mathcal{L}_\epsilon$. On a suitable Banach space, perturbation theory applied to the difference of a high iterate n_0 of the transfer operators $\mathcal{L}_0^{n_0}$ and $\mathcal{L}_\epsilon^{n_0}$ yields localisation of the full spectrum of $\mathcal{L}_\epsilon^{n_0}$. The time-transfer operator $\mathcal{L}_\epsilon^{n_0}$ commutes with the spatial translations, and we provide a description of part of their joint eigenvalues, more precisely of the "nonresonant" spectrum of $\mathcal{L}_\epsilon^{n_0}$ restricted to the eigenspaces of the spatial translations. In particular, we exhibit smooth curves of eigenvalues and eigenspaces of $\mathcal{L}_\epsilon^{n_0}$ as functions of the eigenvalues $e^{i\alpha}$ ("crystal momenta") of the spatial translations.

1. INTRODUCTION

Transfer operators are bounded linear operators $\mathcal{L}$, which are usually defined through composition by a discrete-time dynamical system $F : X \to X$, and multiplication by a weight function G on X:

$$\mathcal{L}\phi(x) = \sum_{Fy=x} G(y)\phi(y)\,,$$

where the observable $\phi : X \to \mathbb{C}$ belongs to a suitable Banach space $\mathcal{B}$. The weight, or potential, G is often a positive function related to some reference measure m by a "conformality" condition:

$$\int_X \psi \circ F\phi\, dm = \int_X \psi\mathcal{L}\phi\, dm\,.$$

In particular, if X is a (finite-dimensional) compact manifold, m is often Lebesgue measure and G the Jacobian. Then, if F is sufficiently smooth and hyperbolic, such a transfer operator enjoys Perron-Frobenius type properties, the maximal eigenfunction being related to an F-invariant probability measure with good thermodynamic and/or ergodic properties, and the spectral gap corresponding to the exponential rate of decay

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1991 *Mathematics Subject Classification.* 37C30 37D20 37L60.

Both authors thank David Ruelle and the I.H.É.S. where this project was started during a visit in 1998. Support by the PRODYN programme of the European Science Foundation is also gratefully acknowledged. V.B. was partially supported by the Fonds National Suisse de la Recherche Scientifique and is grateful to IMPA for its hospitality in the final phase of this work.

of correlations for test functions in $\mathcal{B}$. Even in this case, transfer operators are usually, however, very far from being finite rank: compactness essentially only holds if F is analytic, while a weaker quasicompactness property is the best one can do in finite differentiability. That part of the spectrum which consists of isolated eigenvalues of finite multiplicity sometimes has a dynamical interpretation, and can often be described by a dynamical Fredholm determinant or zeta function. For the rich array of results obtained since the early work of Ruelle in the 1960's, we refer, e.g., to the monograph [B] and references therein.

One aim of this article is to see how much of the above picture remains true in infinite-dimensional dynamics, focusing on one of the most tractable situations: weakly coupled map lattices.

Coupled map lattices (CMLs) have been an object of rigorous studies since the seminal article of Bunimovich and Sinai [BS]. In this work, we shall consider the situation of weakly coupled analytic circle maps $\mathcal{F}_\epsilon$, the lattice being just $\mathbb{Z}^D$, in the footsteps of Bricmont and Kupiainen [BK1]. We refer to the book of Kaneko [K], the survey of Bunimovich [Bu] or the introductions, e.g., of the more recent articles [BIJ] or [FR] for overview and motivation, or discussion of other settings (in particular the work of Pesin-Sinai [PS], Keller and Künzle [KK], Volevich [V], Jiang [J], Jiang-Pesin [JP], or the article of Bricmont-Kupiainen [BK2] in a nonanalytic framework).

Bricmont and Kupiainen [BK1], in a remarkable paper, constructed a Sinai-Ruelle-Bowen measure (limit of $(\mathcal{F}_\epsilon^*)^n(\otimes_{\mathbb{Z}^D}\text{Lebesgue})$) and proved the exponential decay of its correlation functions for analytic observables depending on finitely many sites. (They were supposing exponential decay of the couplings, an assumption which has been weakened only very recently by Rugh [R], see below.) Their main technical tools were the transfer operators $L_{\Lambda,\epsilon}$ associated to the truncations of the system on finite subsets $\Lambda \subset \mathbb{Z}^D$ of the lattice, acting on holomorphic and bounded functions on finite-dimensional polyannuli. They showed bounds on their spectral gaps, uniform in the size $|\Lambda|$. This approach forced them to consider only local observables, and the prefactor in the correlation decay bounds grows exponentially with the size of Λ. It is therefore not only for esthetical reasons that one would like to define a Banach space of observables depending on *all lattice* sites, and a transfer operator associated to the *full* (non truncated) coupled dynamics.

There are two obvious caveats in this enterprise. Firstly, in the translation-invariant case the SRB measure (which is ergodic) is absolutely continuous with respect to the product $(\otimes_{\mathbb{Z}^D}\text{Lebesgue})$ of Lebesgue measure on the individual sites only if it coincides with this product. So a Banach space which is big enough to contain the SRB fixed point of the transfer operator cannot be a subset of $L^1(\otimes_{\mathbb{Z}^D}\text{Lebesgue})$. Secondly, an element $\lambda \neq 1$ of the spectrum should not be expected to be an eigenvalue of finite multiplicity, even for the uncoupled transfer operator. Indeed, restricting the uncoupled operator to a finite subset $\Lambda \subset \mathbb{Z}^D$, we obtain a finite tensor product, with eigenvalues $\prod_{k=1}^K \lambda_{i_k}$, $K \le |\Lambda|$, where the λ_i are eigenvalues of the single-site operator L^{n_0}, and the multiplicity, e.g., of a simple eigenvalue λ_i of L^{n_0} viewed as an eigenvalue (for $K = 1$) of $\otimes_\Lambda L^{n_0}$ coincides with the cardinality of Λ. (Writing h for the fixed point of L^{n_0} and φ_i for the eigenfunction of L^{n_0} and λ_i, just note that for each $p \in \Lambda$ the tensor product $\varphi_i(x_p) \otimes_{q\in\Lambda\setminus\{p\}} h(x_q)$ gives a different eigenfunction.) An obvious solution to the second difficulty is to use translation invariance to restrict the transfer operator to the (invariant) eigenspaces of the spatial translations σ^*. The spectrum of each such restriction should be more tractable, in particular the "single-site" or "nonresonant" eigenvalues of the uncoupled operator ($K = 1$ in the analysis above for the uncoupled operator) should have finite multiplicity, giving some hope that a perturbative analysis might lead to the construction of nontrivial eigenvalues and eigenfunctions for the coupled operator $\mathcal{L}_\epsilon$. It is this "Floquet spectrum" strategy (suggested to the first author by D. Ruelle in 1996) which we are able, finally, to carry out successfully in the present article. This is Theorem 1, stated in Section 2, which describes the Floquet eigenvalues generated by "nonresonant" eigenvalues of the full uncoupled operator $\mathcal{L}_0$.

A few words about our use of the Floquet spectrum terminology ([JS, Chapter 9], see [FKT] for a recent example) are perhaps appropriate here. In fact, we hesitated between "Bloch decomposition" (see [AC] for a recent implementation of this idea to Schrödinger operators) and "Floquet spectrum." We opted for the second expression, which seems to be more frequent in the rigorous literature. We would like to point out that in our situation *all* complex frequencies or multifrequencies (sometimes called crystal momenta elsewhere, they are denoted α below) are present because of the infinite-dimensional nature of the system.

A brief history of the "full" transfer operator for CMLs

Keller and Künzle [KK] were the first to let the transfer operator associated to the full (infinite-dimensional) coupled dynamics act on a Banach space, in the setting of coupled non Markov piecewise monotone interval maps, with bounded variation norms. Unfortunately, they were not able to prove the existence of a spectral gap for this full transfer operator (this remains to our knowledge an open question in this framework). In the easier situation of coupled analytic expanding circle maps from [BK1], assuming furthermore that $D = 1$, Baladi et al. [BIJ] were able to construct a Banach space on which the full transfer operator has a simple eigenvalue at 1 and enjoys a spectral gap. (This yields exponential decay of correlations for a class of nonlocal observables.) Sadly, this Banach space is not spatially homogeneous, so that the spatial translation σ^* does not define a bounded operator on it. For this reason, it was not possible to go "beyond the first gap" for the Banach space of [BIJ].

In the same paper [BIJ], a translationally invariant Fréchet space $\mathcal{F}$ was introduced on which the coupled transfer operator enjoyed a spectral gap property (for all $D \geq 1$), except, however, for the possible presence of continuous spectrum, which also obstructed further spectral analysis.

This frustrating situation improved greatly when Fischer and Rugh [FR], using the same basic assumptions from [BK1], but exploiting a much simpler cluster expansion formalism (inspired by a paper of Maes and van Moffaert [MM] in a random noise framework), showed that the Banach space subset of $\mathcal{F}$ defined by a natural norm was preserved by a suitable iterate of $\mathcal{L}_\epsilon^{n_0}$, and that this iterate enjoyed a spectral gap. (As Fischer and Rugh were mostly concerned with ergodic properties of the dynamics, these facts are not explicitly stated in [FR], but can easily be deduced from their results.) The Banach space analysed in [FR] appears to be the "correct" invariant space: neither too big, nor too small, and translationally invariant. In the present work, we carry out the Floquet spectrum program sketched above for this Banach space, not only under the hypotheses of [FR], but in fact in a more general setting which was introduced very recently by Rugh [R]. Rugh also considered weakly coupled expanding analytic circle maps, and he studied the transfer operator acting on the Banach space from [FR]. The main novelty, for our present purposes, is that exponential decay of the coupling is not necessary: only a summability condition (see (2.1–2.2)) is required.

Sketch of the paper

The article is organised as follows. In Section 2, after recalling the restriction of the setting of [R] (translationally invariant coupled maps on a lattice) which is relevant here, we first state Theorem 0 (proved in the appendix), which exploits the results of [R] to describe the spectrum of the uncoupled transfer operator and show that the coupled and uncoupled operator are close in operator norm when the coupling strength ϵ is small. Next, we describe the eigenspaces X^α of σ^* as well as the eigenspaces of the restriction of the uncoupled transfer operator to the various X^α (see in particular Lemma 1). We then state our main result, Theorem 1, which describes the nonresonant part of the spectrum of $\mathcal{L}_\epsilon^{n_0}$ restricted to each eigenspace X^α of σ^* (claim (1), which can be deduced from Theorem 0 and perturbation theory), as well as the α-dependence of the corresponding eigenvalues (claim (2)).

Section 3 is devoted to an elementary abstract functional analytical argument inspired from [BY] which gives, in particular, a very explicit proof of Theorem 1(1).

The results from Section 3 are used in Section 4 to yield a matrix description of the restriction of $\mathcal{L}_\epsilon^{n_0}$ to an invariant space, expressed as a graph over the corresponding invariant space for $\mathcal{L}_0^{n_0}$. We exploit this description to show Theorem 1 (2). The key for this is Proposition 3, which embodies that information from the coupling decay which is relevant.

Acknowledgements We are grateful to the referee for many thoughtful comments which improved the presentation.

2. Setting and formal statement of results

We start by recalling the settings of [FR] and [R], keeping as close as possible to the notation of [R] (the main exceptions are our use of n, n_0 for the time parameter instead of Rugh's T, T_0, ϵ instead of κ for the coupling strength, and Θ instead of ϑ for the Banach space parameter). In fact, we are essentially restricting the framework [R] to the case when the infinite set $\Omega = \mathbb{Z}^D$, with a translationally invariant coupling which enjoys some spatial decay. We concentrate on pair couplings with exponential or polynomial decay for simplicity: m-point couplings (allowing m to be unbounded) with other kinds of decay may be treated by exactly the same methods.

Tori and polyannuli

We view the circle $S^1 = \mathbb{R}/\mathbb{Z}$ as a subset of the complex cylinder $\mathcal{C} = \mathbb{C}/\mathbb{Z}$. For $\rho \geq 0$, we define the closed annulus in the cylinder,

$$A[\rho] = \{z \in \mathbb{C}/\mathbb{Z} : |\mathrm{Im} z| \leq \rho\} \subset \mathcal{C} .$$

For each integer $D \geq 1$, the infinite torus $S_{\mathbb{Z}^D} = \prod_{p\in\mathbb{Z}^D} S^1$ and the infinite polyannulus $A_{\mathbb{Z}^D} = \prod_{p\in\mathbb{Z}^D} A[\rho]$ are both compact for the product topology.

Let $\mathcal{S}$ denote the family of all finite subsets, including the empty set, of the lattice $\mathbb{Z}^D$. For $\Lambda \in \mathcal{S}$ we write $S_\Lambda = \prod_{p\in\Lambda} S^1$ and $A_\Lambda = \prod_{p\in\Lambda} A[\rho]$ for the Λ-torus and the Λ-annulus, respectively. $H_\Lambda = H(A_\Lambda)$ denotes the space of complex valued functions, holomorphic on the interior of A_Λ and continuous on A_Λ. In the case $\Lambda = \emptyset$, we let $H_\emptyset = \mathbb{C}$.

Each H_Λ is a Banach space for the uniform norm denoted $|\cdot|$. Through coordinate projections we obtain natural inclusions: $j_{\Lambda,K} : H_K \hookrightarrow H_\Lambda$ whenever $K \subset \Lambda (\in \mathcal{S})$ and also $j_\Lambda : H_\Lambda \hookrightarrow C(A_{\mathbb{Z}^D})$, where $C(A_{\mathbb{Z}^D})$ is the set of continuous functions on $A_{\mathbb{Z}^D}$. We denote by $H(A_{\mathbb{Z}^D})$ the closure of $\cup_\Lambda j_\Lambda H_\Lambda$ in $C(A_{\mathbb{Z}^D})$, for the supremum norm.

Analytic expanding circle map f, uncoupled map $\mathcal{F}$

Definition. For $\rho > 0$ and $\lambda > 1$ we say that $f : A[\rho] \to \mathcal{C}$ is a *real analytic, (ρ, λ)-expanding map of the circle* if

(1) f is holomorphic in Int $A[\rho]$ and continuous on $A[\rho]$.
(2) $f(S^1) = S^1$. (Real-analyticity).
(3) The intersection $f\partial A[\rho] \cap A[\lambda\rho]$ is empty. (Expansion).

A real analytic, (ρ, λ)-expanding map of the circle is real analytic expanding on the circle in the usual metric sense, and λ gives a lower bound for the expansion constant (see Appendix A in [R]).

We shall choose a single real analytic, (ρ, λ)-expanding map f. The uncoupled system is the direct product $\mathcal{F} = (f)_{p\in\mathbb{Z}^D}$, which leaves the infinite (real) torus invariant.

Coupling g, coupling-strength ϵ, and coupled map $\mathcal{F}_\epsilon$

It is useful to introduce now the notation $\sigma_p : \mathbb{Z}^D \to \mathbb{Z}^D$ ($p \in \mathbb{Z}^D$) for the translation $\sigma_p(k) = k + p$. The theory in [R] is valid for a general class of summable couplings. For the sake of simplicity we restrict here to pair-couplings. Let us first describe an *exponentially decaying* setting in which d will denote a translationally invariant metric on the lattice, to be specified later.

For fixed $\xi \in (0, 1)$, let $H_{\xi,0} \subset H(A_{\mathbb{Z}^D})$ be those ϕ which may be written as

$$\phi = \sum_{q\in\mathbb{Z}^D} j_{\{0,q\}}\phi_{\{0,q\}} \, ,$$

the sum being uniformly converging with each $\phi_{\{0,q\}} \in H_{\{0,q\}}$, and where

$$\sum_{q\in\mathbb{Z}^D} \xi^{-d(0,q)} |\phi_{\{0,q\}}| < \infty \, .$$

Let $|\phi|_{\xi,0}$ denote the infimum of the above sum over all possible decompositions of ϕ. The parameter ξ quantifies an exponential spatial decay of the terms $\phi_{\{p,q\}}$ with respect to the distinguished point 0 and the metric d.

In our context there are two particular choices for the metric d which will be of interest. First, we may take the Euclidean metric, i.e., $d(p, q) = \|p - q\|$, for which the decay is truly exponential. In this case the above sum becomes

$$\sum_{q\in\mathbb{Z}^D} \xi^{-\|q\|} |\phi_{\{0,q\}}| \, . \tag{2.1}$$

Our second choice, cf. also [R, Section 6], is to take a "renormalized" metric, $d(p, q) = \log(1 + \|p - q\|)$. Setting $P = -\log(\xi) > 0$, the identity $\xi^{d(p,q)} = (1 + \|p - q\|)^{-P}$ shows that we are in fact describing a *polynomial* decay with respect to the Euclidean metric. The above sum then reads

$$\sum_{q\in\mathbb{Z}^D} (1+\|q\|)^P |\phi_{\{0,q\}}| \,. \tag{2.2}$$

We shall consider a translationally invariant system of couplings $g_{\mathbb{Z}^D} = (g_p)_{p\in\mathbb{Z}^D}$, where $g_p = g_0 \circ \sigma_p$ with g_0 in $H_{\xi,0}$, either for the Euclidean metric (exponential case (2.1)) or for the renormalized Euclidean metric (polynomial case (2.2)), mapping the real torus to the real line, and so that

$$|g_0|_{\xi,0} \le \epsilon \,.$$

We say that the coupling decays exponentially with rate ξ (respectively polynomially with exponent $P = \log(1/\xi)$) and *(spatial) coupling strength* ϵ.

The *coupled system* $\mathcal{F}_\epsilon = (F_{\epsilon,p})_{p\in\mathbb{Z}^D} : A_{\mathbb{Z}^D} \to \mathcal{C}^{\mathbb{Z}^D}$ is obtained by setting

$$F_{\epsilon,p} : A_{\mathbb{Z}^D} \to \mathcal{C}\,, \quad F_{\epsilon,p} : z \mapsto f(z_p) + g_p(z) \ , p \in \mathbb{Z}^D \,.$$

The infinite real torus is invariant under $\mathcal{F}_\epsilon$, by definition.

Boundary condition $\bar{x}$, truncated map $F_{\Lambda,\epsilon}$

Let $\bar{x} \in S_{\mathbb{Z}^D}$ denote an arbitrary but fixed reference point. For $\Lambda \in \mathcal{S}$ we define a holomorphic injection, $i_\Lambda : A_\Lambda \to A_{\mathbb{Z}^D}$, $i_\Lambda(z_\Lambda) = (z_\Lambda, \bar{x}_{\Lambda^c})$ and a natural projection, $q_\Lambda : A_{\mathbb{Z}^D} \to A_\Lambda$, $q_\Lambda(z_{\mathbb{Z}^D}) = z_\Lambda$, all expressed in natural coordinates on the annuli. The map q_Λ is the left-inverse of i_Λ while $r_\Lambda = i_\Lambda \circ q_\Lambda : A_{\mathbb{Z}^D} \to A_{\mathbb{Z}^D}$ is a projection which gives boundary conditions outside Λ. We will use the same notation when considering the restriction of the above maps to the tori $S_{\mathbb{Z}^D}$ and S_Λ.

Using the above notation, we define for $\Lambda \in \mathcal{S}$ the Λ-truncated coupled map, $F_{\Lambda,\epsilon} : S_\Lambda \to S_\Lambda$ (and $F_{\Lambda,\epsilon} : A_\Lambda \to \mathcal{C}^\Lambda$) by setting $F_{\Lambda,\epsilon} = q_\Lambda \circ \mathcal{F}_\epsilon \circ i_\Lambda$. When $w_\Lambda \in A_\Lambda$ we denote by $F^{-1}_{\Lambda,\epsilon}(w_\Lambda)$ the (finite) set of inverse images in A_Λ obtained by solving $w_\Lambda = F_{\Lambda,\epsilon}(z_\Lambda)$ for $z_\Lambda \in A_\Lambda$. The condition $\epsilon < (\lambda-1)\rho$ on the coupling strength implies [R, Appendix E] that local inverses indeed exist and are real analytic in w_Λ. In particular, $F_{\Lambda,\epsilon}$ is non-singular and therefore has a well-defined orientation on S_Λ.

Truncated transfer operator $L_{\Lambda,\epsilon}$, single-site gap η

The transfer operator $L_{\Lambda,\epsilon}$ associated with the truncated dynamical system $(S_\Lambda, F_{\Lambda,\epsilon})$ and the Lebesgue measure m_Λ on the Λ-torus is defined by the identification,

$$\int_{S_\Lambda} \psi\, L_{\Lambda,\epsilon}\phi \, dm_\Lambda \equiv \int_{S_\Lambda} \psi \circ F_{\Lambda,\epsilon}\, \phi \, dm_\Lambda \,,$$

for $\psi \in L^\infty(S_\Lambda)$ and $\phi \in L^1(S_\Lambda)$. By standard arguments, this defines a bounded linear operator, $L_{\Lambda,\epsilon}$, on $L^1(S_\Lambda)$. This operator may be extended to a compact (in fact nuclear) operator on H_Λ.

We shall write L for the single-site operator associated to f acting on $H = H(A[\rho])$, and we denote by $\eta' < 1$ the maximal modulus of its eigenvalues different from 1, which is a simple eigenvalue with a positive eigenfunction $h \in H$. It may happen (nongenerically) that $\eta' = 0$, e.g. for the dynamics $z \mapsto z^2$. The present work is only interesting if $\eta' > 0$, we shall make this assumption and fix $\eta' < \eta < 1$. Note that

$c_h' := \sup_n \|L^n(1)\|$ and $c_r := \sup_n \|(L - h\int \cdot)^n\|/(\eta)^n$ are finite. Since $L^n(1)$ converges to h in the supremum norm on S^1, for any $c_h > c_h'$ we have $\|h\|_H \le c_h$, up to taking smaller $\rho > 0$ if necessary.

Banach space $\mathcal{M}_\theta$

Given a non-empty $\Lambda \in \mathcal{S}$, let $z = (z_p)_{p\in\Lambda}$ be natural coordinates on the Λ-torus S_Λ. For every pair $K \subset \Lambda \in \mathcal{S}$ we define a linear operator $\pi_{K,\Lambda} : H_\Lambda \to H_K$ by "integrating away" the coordinates outside K :

$$\pi_{K,\Lambda}\phi_\Lambda(z_K) = \int_{S_{\Lambda\setminus K}} \phi_\Lambda(z_\Lambda)\, dm_{\Lambda\setminus K}(z_{\Lambda\setminus K})\,, \quad \phi_\Lambda \in H_\Lambda\,.$$

Such operators are norm-contracting and Fubini's theorem shows that $\pi_{L,K}\pi_{K,\Lambda} = \pi_{L,\Lambda}$ if $L \subset K \subset \Lambda$. Hence the family $(H_\Lambda, \pi_{K,\Lambda})$ is projective. We denote by $\mathcal{M}$ its projective limit. An element $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{S}} \in \mathcal{M}$ satisfies $\pi_{K,\Lambda}\phi_\Lambda = \phi_K$ whenever $K \subset \Lambda \in \mathcal{S}$. We write $\pi_\Lambda\phi$ for the natural projection $\phi \in \mathcal{M} \mapsto \phi_\Lambda \in H_\Lambda$. From the defining equation it is clear that if $K \subset \Lambda$, $a_K \in H_K$, and $\phi_\Lambda \in H_\Lambda$ then

$$\pi_{K,\Lambda}((j_{\Lambda,K}a_K)\phi_\Lambda) = a_K \pi_{K,\Lambda}\phi_\Lambda\,,$$

simply because a_K does not depend on the variables which are "integrated away."

We introduce for $\theta \in (0,1)$

$$\|\phi\|_\theta = \sup_{\Lambda\in\mathcal{S}} \theta^{|\Lambda|}|\phi_\Lambda| \in [0,+\infty]\,, \quad \phi \in \mathcal{M}\,, \tag{2.3}$$

and define $\mathcal{M}_\theta$ to be the set of those ϕ with finite norm. The norm of the natural projection $\pi_\Lambda : \mathcal{M}_\theta \to H_\Lambda$ is given by

$$\|\pi_\Lambda\|_\theta = \theta^{-|\Lambda|}\,.$$

Full transfer operator $\mathcal{L}_\epsilon$, choice of $0 < \theta < \Theta$

In the following we use the constants $\rho > 0$, $\lambda > 1$, $\eta < 1$, c_h, and c_r introduced above and fix $\theta \in (0,1)$ and $\epsilon_0 > 0$ (denoted κ in [R]) so that the condition TR in [R, Definition 4.18] is verified for some $1 < \gamma < \eta^{-1}$ (the constant denoted C_β there is our $C(\epsilon)$ defined in (2.4) below). For a suitable value of $0 < \theta < \Theta$ [R, Theorem 2.1, Lemmas 4.20 and 4.25], we may define for each $n \ge 0$ a bounded linear operator $\mathcal{L}_\epsilon^{(n)} : \mathcal{M}_\Theta \to \mathcal{M}_\theta$ by the requirement that for all $0 < \epsilon < \epsilon_0$ and each finite $K \subset \mathbb{Z}^D$:

$$\pi_K \mathcal{L}_\epsilon^{(n)} = \lim_{\Lambda\to\mathbb{Z}^D} \pi_{K,\Lambda} \circ L^n_{\Lambda,\epsilon} \circ \pi_\Lambda\,.$$

Furthermore, there is n_0 finite, such that for $n \ge n_0$, $\mathcal{L}_\epsilon^{(n)}$ maps $\mathcal{M}_\Theta$ into $\mathcal{M}_\Theta$ itself so that we may consider the spectrum of this operator.

Rugh [R] gives another characterisation of $\mathcal{L}_\epsilon^{(n)}$ as a sum over configurations (see his Lemma 4.25). This allows to bypass the boundary condition $\bar{x}$ and is convenient to obtain bounds (some of which are quoted and used below), but since our current aim is to reduce technicalities as much as possible we content ourselves here with the naive definition and refer to [R] for more.

If $n, N \ge n_0$ we have that $\mathcal{L}_\epsilon^{(N+n)} = \mathcal{L}_\epsilon^{(N)}\mathcal{L}_\epsilon^{(n)}$ is a bounded operator on $\mathcal{M}_\Theta$. It is thus not a (serious) abuse of notation to write $\mathcal{L}_\epsilon^n$, without parentheses, for $n \ge n_0$.

Uncoupled spectrum and perturbative bounds

Let $\mathrm{sp}(\cdot)$ denote the spectrum of an operator. We state here extensions of the results from [R] required for our present purposes (the proofs are given in the appendix):

Theorem 0 (Uncoupled spectrum and perturbation). *Let θ, Θ, ϵ_0, and n_0 be as above.*

(1) *If the coupling strength $0 \leq \epsilon \leq \epsilon_0$ the operators $\mathcal{L}_\epsilon^n : \mathcal{M}_\Theta \to \mathcal{M}_\theta$ and $\mathcal{L}_\epsilon^n : \mathcal{M}_\Theta \to \mathcal{M}_\Theta$ are bounded for all $n \geq 0$ and $n \geq n_0$, respectively.*

(2) *Let $\mathrm{sp}(L^{n_0}) = \{1, \lambda_j\,, j = 1, 2, \ldots, |\lambda_j| < 1\}$ denote the nonzero spectrum of the compact single-site operator L^{n_0} on H. Then the nonzero spectrum of $\mathcal{L}_0^{n_0}$ on $\mathcal{M}_\Theta$ consists in the discrete set*

$$\Sigma_0 = \{1, \prod_{k=1}^{K} \lambda_{j_k}\,, K = 1, 2, \ldots, 1 \leq j_1 \leq j_2 \leq \ldots \leq j_K\}\,,$$

where 1 is a simple eigenvalue, and all other points are eigenvalues of infinite multiplicity. Furthermore, the image of the spectral projector associated [Ka, III.6.4] to any domain $\mathcal{D} \subset \mathbb{C}$, with $0 \notin \mathcal{D}$ and $\partial\mathcal{D}$ a union of rectifiable curves disjoint from Σ_0, consists of the generalised eigenvectors $\{\varphi \in \mathcal{M}_\Theta \mid \exists \ell \geq 1\,, \exists z \in \Sigma_0 \cap \mathcal{D}\,, s.t.\ (z - \mathcal{L}_0^{n_0})^\ell \varphi = 0\}$.

(3) *Define for $0 \leq \epsilon \leq \epsilon_0$ the function,*

$$C(\epsilon) = \frac{e^{2\pi\epsilon}}{e^{2\pi(\lambda-1)\rho} - e^{2\pi\epsilon}} - \frac{1}{e^{2\pi(\lambda-1)\rho} - 1}\,. \tag{2.4}$$

Then we have the following bound on the perturbation for all $n \geq n_0$:

$$\|\mathcal{L}_0^n - \mathcal{L}_\epsilon^n\|_{\mathcal{M}_\Theta} \leq C(\epsilon)/C(\epsilon_0)\,. \tag{2.5}$$

The function $C(\epsilon)$ is denoted C_β in [R], see Lemma 3.4 there.

Corollary of Theorem 0 (Perturbation theory). *Since 1 is an eigenvalue of $\mathcal{L}_\epsilon^{n_0}$, it follows from (2.5) that for small enough $0 \leq \epsilon \leq \epsilon_0$, the spectrum of the bounded operator $\mathcal{L}_\epsilon^{n_0} : \mathcal{M}_\Theta \to \mathcal{M}_\Theta$ consists in $\{1\} \cup \Sigma_\epsilon$, where Σ_ϵ is a subset of a disc of radius η. The maximal eigenvalue 1 is simple, with an eigenfunction invariant under spatial translations, and there are no other spectral points of $\mathcal{L}_\epsilon^{n_0}$ on the unit circle. For each $\delta > 0$ there is a function $C_\delta(\epsilon) = \mathcal{O}(\epsilon)$ as $\epsilon \to 0$ and such that $d(z, \Sigma_0) < C_\delta(\epsilon)$ for each $z \in \Sigma_\epsilon$ with $\|z\| > \delta$.*

In fact, the fixed point of $\mathcal{L}_\epsilon^{n_0}$ in $\mathcal{M}_\Theta$ gives rise to an $\mathcal{F}_\epsilon$-invariant finite positive Borel measure on the infinite torus which is the SRB measure. We refer to [BK1], [BIJ], [FR], and [R] for this claim and for ergodic-theoretical properties of $\mathcal{F}_\epsilon$ related to the spectral properties stated in Theorem 0 (quantified time-mixing for observables in a space related to the coupling strength).

Floquet spectrum of $\mathcal{L}_\epsilon$, resonant spectrum $\mathcal{R}$

For simplicity, consider $D = 1$ (the extension to $D \geq 2$ is straightforward and left to the reader). Write σ for the spatial translation (to the left, say) on $\mathbb{Z}$, and also $\sigma : A_\mathbb{Z} \to A_\mathbb{Z}$ for the associated shift on the product annulus. This shift defines in a natural way an operator σ^* on $\mathcal{M}_\Theta$. To be more precise, let K be a finite subset of our lattice and denote by $\sigma_K : A_K \to A_{\sigma(K)}$ the action of the shift on the truncated annulus. For a projective family, $\phi \in \mathcal{M}_\Theta$, we may for each finite subset, $K \subset \mathbb{Z}$, associate a pull-back, $(\sigma^*\phi)_K \equiv (\pi_{\sigma(K)}\phi) \circ \sigma_K$. Applying Fubini, it is readily seen that if $K \subset \Lambda \in \mathcal{S}$

then $\pi_{K,\Lambda}(\sigma^*\phi)_\Lambda = \pi_{K,\Lambda}(\phi_{\sigma(\Lambda)} \circ \sigma_\Lambda) = (\pi_{\sigma(K),\sigma(\Lambda)}\phi_{\sigma(\Lambda)}) \circ \sigma_K = \phi_{\sigma(K)} \circ \sigma_K = (\sigma^*\phi)_K$ (i.e., the shift and the projection commute). The same is true for the inverse operator. Thus, the family $(\sigma^*\phi)_K$, $K \in \mathcal{S}$ is projective and clearly has the same norm in $\mathcal{M}_\Theta$ as ϕ. Hence, $\sigma^* : \mathcal{M}_\Theta \to \mathcal{M}_\Theta$ is an isometry.

It is not difficult to see that every frequency $e^{i\alpha}$, $\alpha \in [0, 2\pi)$ is an eigenvalue of σ^* of infinite multiplicity. Letting $\pi_\emptyset : \mathcal{M}_\Theta \to \mathbb{C}$ be the projection to the empty set, we denote by

$$X^\alpha = \{\varphi \in \mathcal{M}_\Theta \mid \sigma^*\varphi = e^{i\alpha}\varphi \, , \pi_\emptyset(\varphi) = 0\}$$

the corresponding eigenspace in the kernel of Lebesgue. The set X^α is obviously a complete subspace of $\mathcal{M}_\Theta$.

Note (although we shall not use this fact here) that $\|(\sigma^*)^n\| = 1$ for all n implies that X^α coincides with the generalised eigenspace $\{\varphi \in \mathcal{M}_\Theta \mid \exists \ell \geq 1 \, , (\sigma^* - e^{i\alpha}\mathrm{Id})^\ell \varphi = 0 \, , \pi_\emptyset(\varphi) = 0\}$). For example, if φ is not an eigenvector but is a generalised eigenvector for $\ell = 2$ and $e^{i\alpha}$, then one easily shows by induction that $(\sigma^*)^r\varphi = re^{i(r-1)\alpha}(\sigma^*\varphi - e^{i\alpha}\varphi) + e^{ir\alpha}\varphi$, for all $r \geq 2$, contradicting $\|(\sigma^*)^n\| = 1$ for all n.

Our aim in the present work is to describe in more detail the structure of the spectrum of $\mathcal{L}_\epsilon^{n_0}$, using our spatially invariant setting. Choosing a translationally invariant reference point $\bar{x}$ for the boundary condition, the truncated dynamical system becomes spatially invariant and the very definition of the truncated operator ensures that $(L_{\sigma(\Lambda),\epsilon}\phi_{\sigma(\Lambda)}) \circ \sigma_\Lambda = L_{\Lambda,\epsilon}(\phi_{\sigma(\Lambda)} \circ \sigma_\Lambda)$ for $\Lambda \in \mathcal{S}$. Therefore $\pi_{K,\Lambda}L^n_{\Lambda,\epsilon}(\phi_{\sigma(\Lambda)} \circ \sigma_\Lambda) = (\pi_{\sigma(K),\sigma(\Lambda)}L^n_{\sigma(\Lambda),\epsilon}\phi_{\sigma(\Lambda)}) \circ \sigma_K$ for $K \subset \Lambda \in \mathcal{S}$ and any $n \geq 0$,. Taking the limit $\Lambda \to \mathbb{Z}$ (which also implies $\sigma(\Lambda) \to \mathbb{Z}$) we obtain $\pi_K(\mathcal{L}_\epsilon^n(\sigma^*\phi)) = (\pi_{\sigma(K)}(\mathcal{L}_\epsilon^n\phi)) \circ \sigma_K = \pi_K(\sigma^*(\mathcal{L}_\epsilon^n\phi))$, i.e., that σ^* and $\mathcal{L}_\epsilon^n$ commute as operators on $\mathcal{M}_\Theta$:

$$\mathcal{L}_\epsilon^n \circ \sigma^* = \sigma^* \circ \mathcal{L}_\epsilon^n \, , \; \forall \; 0 \leq \epsilon < \epsilon_0 \, , n \geq n_0 \, .$$

In particular, $\mathcal{L}_\epsilon^{n_0}$ sends X^α into itself.

Recalling the notation $\{\lambda_j\}$ and Σ_0 from Theorem 0, we define the *"resonant eigenvalues"* of $\mathcal{L}_0^{n_0}$ to be:

$$\mathcal{R} = \{z \in \Sigma_0 \mid z = \prod_{k=1}^{K} \lambda_{j_k} \, , K \geq 2 \, , j_1 \leq \ldots \leq j_K\} \cup \{0\} \, .$$

We also call resonant those eigenvalues of L^{n_0} which belong to $\mathcal{R}$ (this happens if an eigenvalue of L^{n_0} coincides with a product of other eigenvalues). We have:

Lemma 1 (Nonresonant spectrum of $\mathcal{L}_0^{n_0}|X^\alpha$). *Except for the resonant eigenvalues in $\mathcal{R}$ and, possibly $\{1\}$, for each $\alpha \in [0, 2\pi)$ the nonzero spectrum of $\mathcal{L}_0^{n_0}|X^\alpha$ is the same (in particular, consists of eigenvalues with the same finite algebraic multiplicities) as the non resonant spectrum of L^{n_0} on H. For each $\lambda \in \mathrm{sp}(L^{n_0}) \setminus \{1, 0\}$, a bijection between any basis of the generalised eigenspace of L^{n_0} and a basis for that of $\mathcal{L}_0^{n_0}|_{X^\alpha}$ is given by:*

$$\begin{cases} \varphi \mapsto \varphi^\alpha = \sum_{k\in\mathbb{Z}^D} e^{i\alpha k}\varphi_0 \circ \sigma^k \in \mathcal{M}_\Theta \\ \text{where } \varphi_0(x) = \varphi(x_0) \otimes_{\ell\in\mathbb{Z}^D\setminus\{0\}} h(x_\ell) \, . \end{cases} \tag{2.6}$$

More precisely, the projective families defined by (2.6) should be understood as follows. For each finite $\Lambda \subset \mathbb{Z}^D$ we set $(\varphi^\alpha)|_\Lambda = \sum_{k\in\Lambda} e^{i\alpha k}\varphi_{k,\Lambda}$ where $\varphi_{k,\Lambda}(x) = \varphi(x_k) \otimes_{\ell\in\Lambda\setminus\{k\}} h(x_\ell)$. (Note that $\int_{S^1} \varphi \, dm = 0$ and $\sup_\Lambda \sup |(\varphi^\alpha)|_\Lambda|/|\Lambda| < \infty$.)

Proof of Lemma 1. First note that, since a spectral projector $\Pi^\alpha_{\mathcal{D}}$ for $\mathcal{L}_0^{n_0}|_{X^\alpha}$ is just the restriction to X^α of the corresponding spectral projector for $\mathcal{L}_0^{n_0}$, the last statement of Theorem 0 (2) implies that the image of $\Pi^\alpha_{\mathcal{D}}$ is a vector space of generalised eigenvectors in X^α. Fix a nonresonant eigenvalue $\lambda \neq 1$ for L^{n_0} and let φ be an element of a basis of a generalised eigenspace for the compact operator L^{n_0} and λ, in particular $(L^{n_0} - \lambda)^\ell \varphi = 0$ for some $\ell \geq 1$ and λ. Obviously, $\varphi^\alpha \in X^\alpha$ defined by (2.6) is a generalised eigenvector of $\mathcal{L}_0^{n_0}$ for λ, and $\lambda \notin \mathcal{R}$ by the choice of λ. To check that the image of a linearly independent set by the map (2.6) is a linearly independent set, use that $\sum_i \upsilon_i \varphi_i^\alpha = 0$ implies $\sum_i \upsilon_i \pi_{\{0\}} \varphi_i^\alpha = \sum_i \upsilon_i \varphi_i = 0$. To check that the image of (2.6) spans the generalised eigenspace for $\mathcal{L}_0^{n_0}|_{X^\alpha}$ and λ, first note that if ψ^α is a generalised eigenvector for $\mathcal{L}_0^{n_0}|_{X^\alpha}$ and $\lambda \notin \mathcal{R}$, then there is some nonempty finite Λ so that $\psi^\alpha_\Lambda \neq 0$ is a generalised eigenfunction for $L^{n_0}_\Lambda$ and λ. The tensor product expression for $L^{n_0}_\Lambda$ and elementary algebra imply that there are $\upsilon_{t,p} \in \mathbb{C}$, for $t = 1, \ldots, m$ and $p \in \Lambda$, with $\psi^\alpha_\Lambda = \sum_{p\in\Lambda} \sum_{t=1}^m \upsilon_{t,p} \hat{\psi}_t(x_p) \otimes_{k\in\Lambda\setminus\{p\}} h(x_k)$, where $\{\hat{\psi}_s\,, s = 1, \ldots, m\}$ is a basis of the m-dimensional space of generalised eigenfunctions for (L^{n_0}, λ) (observe that $m = m_\lambda$ does not depend on Λ). Now, since $\psi^\alpha \in X^\alpha$, linear independence yields $\upsilon_{t,p} = \upsilon_{t,q} e^{i\alpha(p-q)}$ so that, writing $\upsilon_t = \upsilon_{t,p_0}$ for some arbitrarily chosen $p_0 \in \Lambda$, we find $\psi^\alpha = \sum_{t=1}^m \upsilon_t \hat{\psi}^\alpha_t \circ \sigma_{p_0}$. $\square$

Recall that $\mathrm{sp}(\mathcal{L}_\epsilon^{n_0}) \subset \{1\} \cup \{|z| < \eta\}$ with 1 a simple translationally invariant eigenvalue. The key result of this work is the following description of part of the spectrum of $\mathcal{L}_\epsilon^{n_0}|_{X^\alpha}$ (see also the remark after the statement, and note that similar properties of the spectrum of $\mathcal{L}_\epsilon^{n}$ on X^α for $n > n_0$ may easily be formulated and proved):

Theorem 1. *Let $\Delta \subset \{z \mid |z| < 1\}$ be a complex neighbourhood of a finite subset of the nonresonant spectrum $\mathrm{sp}(\mathcal{L}_0^{n_0}) \setminus \mathcal{R}$ of $\mathcal{L}_0^{n_0}$ such that $\Delta \cap \mathcal{R} = \emptyset$. There is $C > 0$ so that for each small enough coupling strength $0 < \epsilon < \epsilon_0$:*

(1) *For every $\alpha \in [0, 2\pi)$, the spectrum $\mathrm{sp}(\mathcal{L}_\epsilon^{n_0}|_{X^\alpha}) \cap \Delta$ consists in finitely many eigenvalues $\{\lambda_\ell^\epsilon(\alpha)\,, \ell \in \mathcal{E}(\alpha)\}$ of finite multiplicity. To each $\lambda_j \in \Delta \cap \mathrm{sp}(\mathcal{L}_0^{n_0})$ is associated a finite set $\{\lambda_\ell^\epsilon(\alpha)\,, \ell \in \mathcal{E}_j(\alpha)\}$ so that the $\mathcal{E}_j(\alpha)$ form a partition of $\mathcal{E}(\alpha)$, and the sum of the algebraic multiplicities of the $\lambda_\ell^\epsilon(\alpha)$ for $\ell \in \mathcal{E}_j(\alpha)$ coincide with the algebraic multiplicity M_j of λ_j for L^{n_0}.*
Furthermore, $|\lambda_\ell^\epsilon(\alpha) - \lambda_j| \to 0$, uniformly, as $\epsilon \to 0$ for all $\ell \in \mathcal{E}_j(\alpha)$. Finally, if φ^α is associated to φ (with $(\lambda_j - L^{n_0})^k \varphi = 0$) as in (2.6) then there are $\varphi^\alpha_{\epsilon,\ell} \in X^\alpha$ with $(\lambda_\ell^\epsilon - \mathcal{L}_\epsilon^{n_0})^{k_\ell} \varphi^\alpha_{\epsilon,\ell} = 0$ so that $\| \sum_{\ell\in\mathcal{E}_j(\alpha)} \varphi^\alpha_{\epsilon,\ell} - \varphi^\alpha \|_\Theta \to 0$, uniformly, as $\epsilon \to 0$. More precisely, we have

$$\begin{cases} |\lambda_\ell^\epsilon(\alpha) - \lambda_j)| \leq C \cdot \left(\frac{C(\epsilon)}{C(\epsilon_0)}\right)^{1/M_j}, \forall \ell \\ \| \sum_{\ell\in\mathcal{E}_j(\alpha)} \varphi^\alpha_{\epsilon,\ell} - \varphi^\alpha \|_\Theta \leq C \cdot \left(\frac{C(\epsilon)}{C(\epsilon_0)}\right)^{1/M_j} . \end{cases} \tag{2.7}$$

(2) *If the coupling decays exponentially, then for each j the map*

$$\alpha \mapsto \sum_{\ell\in\mathcal{E}_j(\alpha)} \lambda_\ell^\epsilon(\alpha) \tag{2.8}$$

is analytic. If the coupling decays polynomially with exponent $P > D$, then each map (2.8) is $(P - D)$-times differentiable. Analogous statements, in the appropriate topology, hold for the (sums of) eigenfunctions and eigenfunctionals.

Theorem 1(1) is obtained through perturbation theory [Ka], and it will be proved in Section 3. Theorem 1 (2) is proved in Section 4, using information from Section 3.

Remark (Joint eigenvalues vs. joint spectrum). Since there is no obvious spectral decomposition of σ^* acting on the Banach space $\mathcal{M}_\Theta$ (indeed, spectral projectors associated to the eigenvalues of σ^* are not available), Theorem 1 does not give the full joint spectrum of the commuting pair $(\mathcal{L}_\epsilon^{n_0}, \sigma^*)$ on $\mathcal{M}_\Theta$, even restricting to Δ. We refer, e.g., to [MPR] for a list of definitions of joint spectrum of commuting operators in infinite dimension. If there are just two operators, say $\mathcal{L}_\epsilon^{n_0}$ and σ^*, the easiest to state is that of the Harte spectrum, given as follows: $(\nu, \lambda) \in \mathbb{C}^2$ belongs to the *joint Harte spectrum* of $(\mathcal{L}_\epsilon^{n_0}, \sigma^*)$ on $\mathcal{M}_\Theta$ if at least one of the two equations $(\sigma^* - \nu)A_1 + (\mathcal{L}_\epsilon^{n_0} - \lambda)A_2 = \mathrm{Id}$, $A_1(\sigma^* - \nu) + A_2(\mathcal{L}_\epsilon^{n_0} - \lambda) = \mathrm{Id}$ has no bounded operator solutions A_1, A_2. For commuting finite matrices B_1, B_2, one defines the set of *joint eigenvalues* to be those $(\nu, \lambda) \in \mathbb{C}^2$ so that there is a vector x with $B_1 x = \nu x$ and $B_2 x = \lambda x$ (see, e.g., [BB]). Extending this definition to bounded linear operators, Theorem 1 indeed describes the intersection of $\Delta \times \mathbb{C} \subset \mathbb{C}^2$ with the set of joint eigenvalues of the commuting pair $(\mathcal{L}_\epsilon^{n_0}, \sigma^*)$ on $\mathcal{M}_\Theta$.

3. Existence of eigenvalues: Elementary perturbation theory

We formulate some abstract results in an axiomatic setting: Let $(X, \|\cdot\|)$ be a complex Banach space, and let $\{T_\epsilon, 0 \le \epsilon \le \epsilon_0\}$ be a family of bounded linear operators on X. We make the following two assumptions about T_0 and T_ϵ for $\epsilon \ge 0$:

There is $\kappa_0 > 0$ so that the spectrum of T_0 decomposes as $\Sigma_0 \cup \Sigma_1$, with

$$\inf_{z_0 \in \Sigma_0, z_1 \in \Sigma_1} |z_0 - z_1| > \kappa_0 \,. \tag{3.1}$$

$$\lim_{\epsilon \to 0} \|T_\epsilon - T_0\| = 0 \,. \tag{3.2}$$

The main result of this section follows:

Proposition 1 (Perturbation theory). *Let $\{T_\epsilon, \epsilon \ge 0\}$ satisfy (3.1–3.2). Then for all sufficiently small $\epsilon > 0$, there are $C > 1$ and a decomposition of $\mathrm{sp}(T_\epsilon)$ into $\Sigma_0^\epsilon \cup \Sigma_1^\epsilon$ such that*

1. $\inf_{z_0 \in \Sigma_0^\epsilon, z_1 \in \Sigma_1^\epsilon} |z_0 - z_1| > \kappa_0/3$.
2. *Let $\pi_0^\epsilon : X_0^\epsilon \oplus X_1^\epsilon \to X_0^\epsilon$ be the projection associated with the spectral decomposition of T_ϵ. Then $\|\pi_0 - \pi_0^\epsilon\| \le C \cdot \|T_\epsilon - T_0\|$.*
3. *Let X_i be the image of X by the spectral projector of T_0 corresponding to Σ_i. The invariant space X_0^ϵ may be written as the graph of a linear map $S_\epsilon : X_0 \to X_1$, with $\|S_\epsilon\| \le C \cdot \|T_\epsilon - T_0\|$.*

It follows from Proposition 1 (2) that, if we assume additionally that

$$\dim(X_0) < \infty \text{ (in particular, } X_0 \text{ is a generalised eigenspace)}, \tag{3.3}$$

then $\dim(X_0^\epsilon) = \dim(X_0)$ is finite for all small enough ϵ.

Clearly, statements (1) and (2) of Proposition 1 can be obtained from standard perturbation theory, see, e.g., [Ka, Section IV.3.5]. The third claim allows us to define $\widehat{T}_\epsilon : X_0 \to X_0$ by

$$\widehat{T}_\epsilon(x) = \pi_0 \circ T_\epsilon(x + S_\epsilon(x)) \,. \tag{3.4}$$

The map $\widehat{T}_\epsilon$ will be convenient to show (2.7) in Theorem 1 (1) (see the following Proposition 2), and also to prove Theorem 1 (2) (see Section 4).

Proposition 2 (Finite-dimensional perturbation). *Assume that $\{T_\epsilon, \epsilon \geq 0\}$ satisfies (3.1–3.3). Let M denote the maximum algebraic multiplicity of $\lambda \in \mathrm{sp}(T_0|_{X_0})$. Then there is $C > 0$ so that for all small enough ϵ*

(1) $\inf_{z_0 \in (\mathrm{sp}(T_0|_{X_0}), z_\epsilon \in \mathrm{sp}(T_\epsilon|_{X_0^\epsilon})} |z_0 - z_\epsilon| \leq C \cdot \left\| T_\epsilon - T_0 \right\|^{1/M}$.

(2) *If $x_j \in X_0$ is an eigenvector for T_0 with eigenvalue λ_j and algebraic multiplicity M_j, then T_ϵ has at most M_j eigenvectors $x^\epsilon_{j,\ell} \in X_0^\epsilon$ with eigenvalues $\lambda^\epsilon_{j,\ell}$ such that*

$$|\lambda^\epsilon_{j,\ell} - \lambda_j| \leq C \cdot \left\| T_\epsilon - T_0 \right\|^{1/M_j}, \forall \ell,$$
$$\| \sum_\ell x^\epsilon_{j,\ell} - x_j \| \leq C \cdot \left\| T_\epsilon - T_0 \right\|^{1/M_j}.$$

Proof of Theorem 1 (1). Taking $X = X^\alpha$ for some $\alpha \in [0, 2\pi)$ and applying Theorem 0 and Lemma 1, assumptions (3.1–3.3) are satisfied for $T_0 = \mathcal{L}^{n_0}|_{X^\alpha}$ and $T_\epsilon = \mathcal{L}_\epsilon^{n_0}|_{X^\alpha}$. Proposition 2 gives (2.7) while Proposition 1 implies all other claims. $\square$

We now prove Propositions 1 and 2. We let $\pi_i : X \to X_i$ $(i = 0, 1)$ denote the projections on X_i, i.e., $x = \pi_0(x) + \pi_1(x) \in X_0 \oplus X_1$.

Proof of Proposition 1.

(1) Let $\lambda \in \mathbb{C}$ satisfy $d(\lambda, \mathrm{sp}(T_0)) > \kappa_0/4$. To show that $\lambda \notin \mathrm{sp}(T_\epsilon)$ (if ϵ is small enough) it suffices to prove that the resolvent $R(\lambda, T_\epsilon)$ is a bounded operator. If the following sum converges it coincides with the resolvent (see, e.g., [BY, (2.1)]):

$$R(\lambda, T_\epsilon) = \sum_{j=0}^{\infty} \big(R(\lambda, T_0) \circ (T_\epsilon - T_0)\big)^j \cdot R(\lambda, T_0). \tag{3.5}$$

It is easily seen that $\sigma_0 := \sup_{\{\lambda | d(\lambda, sp(T_0)) > \kappa_0/4\}} \|R(\lambda, T_0)\| < \infty$. Therefore, it suffices to take ϵ small enough so that $\|T_\epsilon - T_0\| < 1/\sigma_0$. Setting

$$\Sigma_0^\epsilon = \{z \in \sigma(T_\epsilon) \mid d(z, \Sigma_0) < \kappa_0/3\}, \quad \Sigma_1^\epsilon = \{z \in \sigma(T_\epsilon) \mid d(z, \Sigma_1) < \kappa_0/3\},$$

our assumption (3.1) allows us to conclude.

(2) Let $\mathcal{D} \subset \mathbb{C}$ contain $\Sigma_0 \cup \Sigma_0^\epsilon$, be disjoint from a $\kappa_0/3$ neighbourhood of $\{0\} \cup \Sigma_1 \cup \Sigma_1^\epsilon$, and be such that $\partial\mathcal{D}$ is a union of rectifiable curves with finite total length (such a domain $\mathcal{D}$ exists by (1)). Then we have

$$\pi_0 = \frac{1}{2i\pi} \int_{\partial\mathcal{D}} R(\lambda, T_0)\, d\lambda \qquad \pi_0^\epsilon = \frac{1}{2i\pi} \int_{\partial\mathcal{D}} R(\lambda, T_\epsilon)\, d\lambda.$$

We may thus estimate $\|\pi_0 - \pi_0^\epsilon\|$ by

$$\begin{aligned} \|\pi_0 - \pi_0^\epsilon\| &\leq \frac{1}{2\pi} \int_{\partial\mathcal{D}} \|R(\lambda, T_0) - R(\lambda, T_\epsilon)\|\, d\lambda \\ &\leq \frac{1}{2\pi} \text{length } (\partial\mathcal{D}) \max_{\lambda \in \partial\mathcal{D}} \|R(\lambda, T_0) - R(\lambda, T_\epsilon)\| \end{aligned} \tag{3.6}$$

Using (3.5), we find

$$\|R(\lambda, T_0) - R(\lambda, T_\epsilon)\| \le \sum_{j=1}^{\infty} \|R(\lambda, T_0)\|^{j+1} \cdot \|T_\epsilon - T_0\|^j \,.$$

Statement (2) immediately follows from (3.2).

(3) To prove the last claim, take ϵ small and x in X_0^ϵ. Since $\|x - \pi_0(x)\| \le \|\pi_0^\epsilon - \pi_0\|\|x\|$, it follows that if $x = (x_0, x_1) \in X_0 \oplus X_1$, then $\|x_1\| \ll \|x_0\|$. This inequality implies in particular that π_0 is injective on X_0^ϵ so that S_ϵ is well defined. The estimate on $\|S_\epsilon\|$ follows from

$$\begin{aligned} \|x_1\| = \|x - \pi_0(x)\| &\le C\|T_\epsilon - T_0\|\|x\| \\ &\le C\frac{\|T_\epsilon - T_0\|}{1 - C\|T_\epsilon - T_0\|}\|x_0\| \,. \quad \square \end{aligned}$$

Proof of Proposition 2. Using the map $\widehat{T}_\epsilon : X_0 \to X_0$ from (3.4), we have for $x \in X_0$ with $\|x\| = 1$,

$$\begin{aligned} \|\widehat{T}_\epsilon(x) - T_0(x)\| &\le \|\pi_0\| \cdot (\|T_\epsilon(x) - T_0(x)\| + \|T_\epsilon(S_\epsilon(x))\|) \\ &\le C \cdot \|\pi_0\| \cdot \Big(\|T_\epsilon - T_0\| + \|T_\epsilon\| \cdot \|T_\epsilon - T_0\|\Big) \,. \end{aligned} \tag{3.7}$$

The assertions in Proposition 2 follow immediately. (See, e.g., [W, Chapter 2] and [Ka, II.5].) $\square$

4. Smoothness of eigenvalue curves

Returning now to the setting of Section 2, we show here how the information from Section 3 can be combined with further crucial quantitative bounds on $\mathcal{L}_\epsilon^{n_0}$ (Proposition 3) to show Theorem 1(2). Recall from Section 2 that the distance $d(p, q)$ in the definition of the spatial coupling strength is either the Euclidean metric (exponential decay with rate $\xi < 1$) or the renormalized Euclidean metric (polynomial decay with exponent $P = \log(1/\xi)$) on the lattice.

Proposition 3. *Let n_0 and ϵ_0 be given by Theorem 0, and take $\epsilon < \epsilon_0$. Let Π_0 be a spectral projector for a subset of a disc of radius strictly less that 1 in the spectrum of $\mathcal{L}_0^{n_0}$ and let Π_0^ϵ be the corresponding spectral projector in the spectrum of $\mathcal{L}_\epsilon^{n_0}$. For $k \in \mathbb{Z}^D$ let $Z_k = \{\varphi \in \mathcal{M}_\theta \mid \pi_\Lambda \varphi = 0\,, \forall \Lambda \text{ s.t. } k \notin \Lambda\}$. Then there is $C > 0$ so that for all $k \in \mathbb{Z}^D$,*

$$\max(\|\pi_{\{0\}} \circ (\Pi_0^\epsilon)_{|Z_k}\|, \|\pi_{\{0\}} \circ (\mathcal{L}_\epsilon^{n_0} \circ \Pi_0^\epsilon)_{|Z_k}\|) \le C\xi^{d(0,k)} \,. \tag{4.1}$$

Proof of Proposition 3. To simplify notation, we write $\mathcal{L}_0$ and $\mathcal{L}_\epsilon$ instead of $\mathcal{L}_0^{n_0}$ and $\mathcal{L}_\epsilon^{n_0}$ in this proof. We shall concentrate on bounding the second expression in the max, a simpler version of our arguments gives the other estimate.

We shall represent Π_0^ϵ as a suitable contour integral $\int_{\partial\mathcal{D}} \cdot \, d\lambda$ of the resolvent $R(\lambda,\mathcal{L}_\epsilon)$ and use (3.5) again:

$$R(\lambda,\mathcal{L}_\epsilon) = \sum_{j=0}^{\infty} \bigl(R(\lambda,\mathcal{L}_0)\circ(\mathcal{L}_\epsilon-\mathcal{L}_0)\bigr)^j \circ R(\lambda,\mathcal{L}_0)\,.$$

The resolvent $R(\lambda,\mathcal{L}_0)$ preserves Z_k, and we shall in fact show that for $j=0,1,\ldots$:

$$\|\pi_{\{0\}}\circ\mathcal{L}_\epsilon\circ(R(\lambda,\mathcal{L}_0)\circ(\mathcal{L}_\epsilon-\mathcal{L}_0))^j_{|Z_k}\| \le \Theta^{-1}\gamma^{-n_0}\xi^{d(0,k)}(\sigma_0\gamma^{-n_0}(C(\epsilon)/C(\epsilon_0)))^j\,, \quad (4.2)$$

where $\sigma_0=\sup_{\lambda\in\partial\mathcal{D}}\|R(\lambda,\mathcal{L}_0)\|$ and $\gamma>1$ is again the constant from TR in [R, Definition 4.18].

Let us first consider $j=1$. For Λ a finite subset of our lattice, we consider the operator $\pi_\Lambda\circ R(\lambda,\mathcal{L}_0)\circ(\mathcal{L}_\epsilon-\mathcal{L}_0)$. By Lemma A.1 from the appendix, we may write $\pi_\Lambda\circ R(\lambda,\mathcal{L}_0)=R(\lambda,L_\Lambda)\circ\pi_\Lambda$ where $R(\lambda,L_\Lambda)$ is the (bounded) resolvent of L_Λ acting on H_Λ. The operator $\mathcal{L}_\epsilon$ is defined in [R, Section 4.7] through a sum of configurational operators $\pi_\Lambda\circ\mathcal{L}_\epsilon=\sum_{\mathcal{C}\in\mathcal{C}[\Lambda,n_0]}L_\Lambda[\mathcal{C}]:\mathcal{M}_\Theta\to H_\Lambda$. We may therefore write:

$$\pi_\Lambda\circ R(\lambda,\mathcal{L}_0)\circ(\mathcal{L}_\epsilon-\mathcal{L}_0)=R(\lambda,L_\Lambda)\circ\sum_{\mathcal{C}\in\mathcal{C}^*[\Lambda,n_0]}L_\Lambda[\mathcal{C}]\,,$$

where $\mathcal{C}^*[\Lambda,n_0]$ represents the configurations which do not consist only of initial-leaves and end-leaves (see the proof of Theorem 0 (2) in the appendix). The bounds on the configurational operators (see also the appendix) imply furthermore that

$$\sum_{\mathcal{C}\in\mathcal{C}^*[\Lambda,n_0]}\|\,R(\lambda,L_\Lambda)\circ L_\Lambda[\mathcal{C}]\,\|\le\frac{C(\epsilon)}{C(\epsilon_0)}\|R(\lambda,\mathcal{L}_0)\|\Theta^{-|\Lambda|}\,.$$

The proof of this bound is perhaps slightly trickier than it looks. The reason is that the a priori bounds for $R(\lambda,L_\Lambda)$ are useless here. Instead one should use the same expansion as in the proof of Lemma A.1, integrate each term in the resolvent separately (i.e., each term in $U_\Lambda^{<m}$ and $U_\Lambda^{\ge m}$), and combine with the configurational expansion of $\mathcal{L}_\epsilon-\mathcal{L}_0$. The terms in this expansion of the projector involve in particular projections to subsets J of Λ. Therefore, when composing with a configurational operator $L_\Lambda[C]$ such a term vanishes unless C is a configuration over J (we refer to [R] for the terminology). Summing up what remains yields the above inequality.

When acting on the kernel Ker Leb $=\{\varphi\mid\pi_\emptyset=0\}$ of the Lebesgue measure and taking the spatial decay of the couplings into account, more can be said. First, the "size" [R, Definition 4.11, Lemma 4.25] of each configuration must be at least n_0 in order to get a non-vanishing contribution. Second, we may introduce an "interaction-radius", $\mathrm{rad}(\mathcal{C})$, of a configuration, $\mathcal{C}$: First, we map $\mathcal{C}$ into a collection of trees y_p, $p\in\Lambda$ [R, Section 4.3, Definition 4.9]. A branching (in a tree) of a point q into a set K is associated with the interaction-radius, $\mathrm{rad}(q,K)=\max\{d(q,r)\mid r\in\ K\cup\{q\}\}$ and we define $\mathrm{rad}(\mathcal{C})$ to be the sum of interaction-radii over all branchings and all trees.

As in the proofs of temporal [R, Lemma 4.25] and spatial [R, Section 5.1] decay, we then obtain the bound,

$$\sum_{\mathcal{C}\in\mathcal{C}^*[\Lambda,n_0]}\|\,R(\lambda,L_\Lambda)\circ L_\Lambda[\mathcal{C}]_{|\mathrm{Ker\ Leb}}\,\|\,\xi^{-\mathrm{rad}(\mathcal{C})}\cdot\gamma^{n_0}\le\frac{C(\epsilon)}{C(\epsilon_0)}\|R(\lambda,\mathcal{L}_0)\|\Theta^{-|\Lambda|}\,.$$

We wish to iterate this type of argument when taking powers $j \geq 2$ of the operator difference composed with the resolvent. (The case $j = 0$ is easier and left to the reader.) In order to do so, we note that both $\mathcal{L}_\epsilon - \mathcal{L}_0$ and $R(\lambda, \mathcal{L}_0)$ map the kernel of Lebesgue measure into itself ([R, Lemma 4.25] and the fact that no projector involved includes the eigenvalue 1). In the product, $\pi_\Lambda \circ \mathcal{L}_\epsilon \circ (R(\lambda, \mathcal{L}_0) \circ (\mathcal{L}_\epsilon - \mathcal{L}_0))^j$, we introduce the configurational expansion in each factor, $\mathcal{L}_\epsilon - \mathcal{L}_0$. We then obtain a sequence of finite subsets, $\Lambda_0 \equiv \Lambda \subset \Lambda_1 \subset \cdots \subset \Lambda_j$ and configurations $\mathcal{C}_i \in \mathcal{C}^*[\Lambda_i, n_0]$, $i = 0, ..., j-1$ where each configuration expands Λ_i into Λ_{i+1}. It is important here to notice that each occurrence of $R(\lambda, \mathcal{L}_0)$ does not expand a given finite subset (since $\pi_K \circ R(\lambda, \mathcal{L}_0) = R(\lambda, L_K) \circ \pi_K$). When acting on Z_k we note that non-vanishing contributions can only occur provided $k \in \Lambda_j$, i.e., the very last expansion of the original set, Λ, has to contain the distinguished point, k. This, in turn, implies that the sequence of configurations must include a sequence of "trees" which "connects" by a path (of branchings) the point k with a point in Λ. In the product we therefore obtain factors of ξ raised to the power $\mathrm{rad}(\mathcal{C}_0) + \cdots \mathrm{rad}(\mathcal{C}_{j-1}) \geq d(k, \Lambda)$. This finally implies

$$\|\pi_\Lambda \circ \mathcal{L}_\epsilon \circ (R(\lambda, \mathcal{L}_0) \circ (\mathcal{L}_\epsilon - \mathcal{L}_0))^j_{|Z_k}\| \leq \Theta^{-|\Lambda|} \gamma^{-n_0} (\sigma_0 \gamma^{-n_0} (C(\epsilon)/C(\epsilon_0)))^j \xi^{d(k,\Lambda)} \,.$$

The claim (4.2) now follows if for the set Λ we take the origin of our lattice.

To end the proof of Proposition 3, just note that $\sum_j (\sigma_0 \gamma^{-n_0} C(\epsilon)/C(\epsilon_0))^j$ remains uniformly bounded for small ϵ. $\square$

Proof of Theorem 1 (2). To simplify notation, we write $\mathcal{L}_0$ and $\mathcal{L}_\epsilon$ instead of $\mathcal{L}_0^{n_0}$ and $\mathcal{L}_\epsilon^{n_0}$ in this proof. We shall use the notation from Section 3, taking again $T_0 = \mathcal{L}|_{X^\alpha}$ $T_\epsilon = \mathcal{L}_\epsilon|_{X^\alpha}$, and $X = X^\alpha$ with the norm induced by $\mathcal{M}_\Theta$, writing also $X_0^\alpha = X_0$, $X_1^\alpha = X_1$. Note that $\pi_0 = \pi_0^\alpha$ and $\pi_1 = \pi_1^\alpha$ are just the restrictions to X^α of the spectral projections Π_0, Π_1 of $\mathcal{L}_0 : \mathcal{M}_\Theta \to \mathcal{M}_\Theta$ associated to the partition $\Delta \cup (\mathbb{C} \setminus \Delta)$. Theorem 0 (2) guarantees that the image of π_0^α is a generalised eigenspace, and since Δ is disjoint from a neighbourhood of $\mathcal{R}$, Lemma 1 says that this eigenspace is finite-dimensional and in bijection with the eigenspace associated to L^{n_0} and Δ. Using also Theorem 0 (3), we see that conditions (3.1–3.3) are satisfied. Note also that $\|\pi_i^\alpha\| \leq \|\Pi_i\|$ for $i = 0, 1$ and all α.

We work with the finite-dimensional operator $\widehat{T}_\epsilon = \widehat{T}_\epsilon(\alpha)$ from (3.4). Since $\widehat{T}_\epsilon(x_0) = \pi_0 T_\epsilon(x_0 + S_\epsilon(x_0)) = \lambda x_0$ if $x_0 + S_\epsilon(x_0)$ is an eigenvector of T_ϵ for the eigenvalue λ, the relevant eigenvalues λ_ϵ^α of $\mathcal{L}_\epsilon|_{X^\alpha}$ are just the eigenvalues of any finite matrix representing $\widehat{T}_\epsilon(\alpha)$ on the finite-dimensional space X_0^α. Therefore, to prove our claim it suffices (by classical results, see, e.g., [W, Chapter 2], [DS, Theorem VII.6.9]) to check that the coefficients of such a matrix depend analytically (respectively differentiably) on $\alpha \in [0, 2\pi)$. By Lemma 1, these coefficients may be indexed by the following finite basis of generalised eigenvectors of $\mathcal{L}_0$ in X^α (for eigenvalues in Δ):

$$\begin{cases} \varphi_q^\alpha = \sum_{k \in \mathbb{Z}^D} e^{i\alpha k} \varphi_q \circ \sigma^k \,, \quad \varphi_q(z) = \hat{\varphi}_q(z_0) \otimes_{\ell \in \mathbb{Z}^D \setminus \{0\}} h(z_\ell) \,, \\ \{\hat{\varphi}_q\} \text{ a basis of generalised eigenvectors for } L \text{ for eigenvalues in } \Delta \,. \end{cases} \tag{4.3}$$

Just like for the more precise definition for (2.6) given after Lemma 1, the expression in (4.3) means that for each $\Lambda \subset \mathbb{Z}^D$ we define $(\varphi_q^\alpha)|_\Lambda = \sum_{k \in \Lambda} e^{i\alpha k} \varphi_{q,k,\Lambda}$ with $\varphi_{q,k,\Lambda}(z) = \hat{\varphi}_q(z_k) \otimes_{\ell \in \Lambda \setminus \{k\}} h(z_\ell)$ (note that $\int_{S^1} \hat{\varphi}_q \, dm = 0$). (Recall also that we in fact restrict to $D = 1$ for simplicity, the extension to the general case is straightforward, with α in a D-dimensional torus and using that $1/(\|x\|^t(1+\|x\|)^P)$ is integrable on $\mathbb{R}^D$ if and only if $t > D - P$.) To compute these coefficients, we may also use a finite basis of generalised eigenfunctionals of $\mathcal{L}_0^*$ in $(X^\alpha)^*$

$$\nu_n = \hat{\nu}_n \otimes_{k \in \mathbb{Z}^D \setminus \{0\}} \int d\mathrm{Leb}\,, \quad \{\hat{\nu}_n\} \text{ basis of eigenvectors for } L^* \text{ for eigenvalues in } \Delta.$$

Note that Proposition 1 may be applied to the spectral decomposition of $\mathcal{L}_0^{n_0}$ associated to $\Delta \cup (\mathbb{C} \setminus \Delta)$ on $\mathcal{M}_\Theta$, giving projectors Π_i^ϵ for $\mathcal{L}_\epsilon$. Recall from Section 3 that $S_\epsilon = S_\epsilon(\alpha) : X_0^\alpha \to X_1^\alpha$ is such that $X_0^{\alpha,\epsilon} = \Pi_0^\epsilon(X_0^\alpha) = X_0^\alpha + S_\epsilon(X_0^\alpha)$. Now, the operator π_0 is invertible when acting on the finite-dimensional space $\pi_0^\epsilon(X_0^\alpha)$. (In fact, $\pi_0(\pi_0^\epsilon x) = \pi_0^\epsilon(x) + (\pi_0 - \pi_0^\epsilon)(\pi_0^\epsilon x)$ so that $\pi_0|_{\pi_0^\epsilon(X_0^\alpha)}$ is a small perturbation of the identity.) Thus, we may define $\mathcal{Q}_\epsilon = \mathcal{Q}_\epsilon(\alpha) : X_0^\alpha \to \pi_0^\epsilon(X_0^\alpha)$ by

$$\mathcal{Q}_\epsilon(\alpha) = \mathrm{Id} + S_\epsilon(\alpha) = (\pi_0|_{\pi_0^\epsilon(X_0^\alpha)})^{-1}\,. \tag{4.4}$$

For each fixed α, the coefficient $\widehat{T}_{\epsilon,nq}(\alpha)$ of the matrix of $\widehat{T}_\epsilon$ in the chosen bases can then be expressed as:

$$\widehat{T}_{\epsilon,nq}(\alpha) = \nu_n \mathcal{L}_\epsilon \mathcal{Q}_\epsilon\big(\sum_{k \in \mathbb{Z}^D} e^{i\alpha k} \varphi_q \circ \sigma^k\big)\,. \tag{4.5}$$

Denoting by $\mathcal{Q}_{rq}(\alpha)$ the coefficients of the matrix of $\mathcal{Q}$ in the bases given by (4.3) and its image under π_0^ϵ, we finally get (the sum over r is finite)

$$\begin{aligned}\widehat{T}_{\epsilon,nq}(\alpha) &= \nu_n \mathcal{L}_\epsilon \sum_r \mathcal{Q}_{rq}(\alpha) \pi_0^\epsilon \sum_{\ell \in \mathbb{Z}^D} e^{i\alpha\ell}(\varphi_r \circ \sigma^\ell) \\ &= \sum_r \mathcal{Q}_{rq}(\alpha) \sum_{\ell \in \mathbb{Z}^D} e^{i\alpha\ell} \nu_n(\mathcal{L}_\epsilon \Pi_0^\epsilon)(\varphi_r \circ \sigma^\ell)\,.\end{aligned}$$

Formally, we may write the derivatives of the interior sum over ℓ as:

$$\frac{d^s}{d\alpha^s} \sum_{\ell \in \mathbb{Z}^D} e^{i\alpha\ell} \nu_n(\mathcal{L}_\epsilon \Pi_0^\epsilon)(\varphi_r \circ \sigma^\ell) = \sum_{\ell \in \mathbb{Z}^D} \ell^s e^{i\alpha\ell} \nu_n(\mathcal{L}_\epsilon \Pi_0^\epsilon)(\varphi_r \circ \sigma^\ell)\,.$$

By definition (see also the remarks after (4.3)), $\varphi_r \in Z_0$ so that $\varphi_r \circ \sigma^\ell \in Z_\ell$. The upper bounds given by Proposition 3,

$$|\nu_n(\mathcal{L}_\epsilon \Pi_0^\epsilon)(\varphi_r \circ \sigma^\ell)| \le C \cdot \begin{cases} \xi^{|\ell|} \|\varphi_r\|\,, & \text{for exponential coupling} \\ (1+|\ell|)^{-P} \|\varphi_r\|\,, & \text{for polynomial coupling,} \end{cases}$$

are good enough for our purposes.

By the Leibniz formula, it only remains to prove that each coefficient map $\alpha \mapsto \mathcal{Q}_{rq}(\alpha)$ is analytic (respectively P times differentiable). It is equivalent to prove the same claim for the coefficients $\widetilde{\mathcal{Q}}_{qr}(\alpha)$ of the inverse map $\mathcal{Q}_\epsilon(\alpha)^{-1} = \pi_0|_{\pi_0^\epsilon(X_0^\alpha)}$. We have

$$\frac{d^s}{d\alpha^s} \widetilde{Q}_{qr}(\alpha) = \sum_{k \in \mathbb{Z}^D} k^s e^{i\alpha k} \nu_q \Pi_0^\epsilon(\varphi_r \circ \sigma^k)\,,$$

so that our task reduces to finding good bounds for $|\nu_q \Pi_0^\epsilon(\varphi_r \circ \sigma^k)|$ for all $k \in \mathbb{Z}$. But this is again Proposition 3. $\square$

Appendix

Proof of Theorem 0. The proof of (1) may be found in [R], so that we first concentrate on (2). We recall some results from [R, Appendix B]: For each finite $\Lambda \subset \mathbb{Z}^D$, the tensor product $\otimes_{p\in\Lambda} H_p$ is dense in H_Λ. Also, if $J \subset \Lambda$ then a bounded operator T_J on H_J has a natural norm-preserving extension to a bounded operator T_J^Λ acting on H_Λ (see Theorem B.4 in [R, Appendix B]). The extended operator may be defined as $T_J^\Lambda(\phi_J \otimes \phi_{\Lambda\setminus J}) \equiv (T_J\phi_J)\otimes\phi_{\Lambda\setminus J}$ when acting on a direct tensor product and extended by continuity. The truncated unperturbed Perron-Frobenius operator L_Λ acting on H_Λ is itself a tensor product, $L_\Lambda = \otimes_{p\in\Lambda} L_p$, of the operators acting at the individual sites (here, for simplicity, we do not write L_p^Λ for the extension of $L : H \to H$).

In the rest of this proof we write L_Λ instead of $L_\Lambda^{n_0}$ (and $\mathcal{L}_0$, $\mathcal{L}_\epsilon$ for $\mathcal{L}_0^{n_0}$, $\mathcal{L}_\epsilon^{n_0}$) in order to simplify notation, and we let $P_p = h_p \otimes m_p$ denote the principal eigenprojection (which yields the single-site SRB measure) of L_p. Thus, $L_p \circ P_p = P_p \circ L_p = P_p$. Denoting $P_J = \otimes_{p\in J} P_p$ and $(1-P)_J = \otimes_{p\in J}(1-P_p)$, let us introduce another extension of $T_J : H_J \to H_J$ through:

$$\mathrm{Ext}_\Lambda(T_J) = T_J \otimes P_{\Lambda\setminus J} = T_J^\Lambda P_{\Lambda\setminus J}^\Lambda\,.$$

We end these preliminaries by a bound for $\sum_{J\subset\Lambda, |J|=k} \|\mathrm{Ext}_\Lambda((1-P)_J)\circ\pi_\Lambda\|$. For this, we start by noting that for $J \subset \Lambda$

$$\mathrm{Ext}_\Lambda((1-P)_J) = \bigotimes_{p\in J}(1-P_p)\otimes P_{\Lambda\setminus J} = \sum_{I\subset J}(-1)^{|I|} P_I \otimes P_{\Lambda\setminus J}.$$

Next, define for $k \in \mathbb{N}$ the constants $\alpha_k = \alpha_k(\Theta, c_h) = \max_{\ell\geq 0}\binom{\ell}{k}(c_h\Theta+1)^k(c_h\Theta)^{\ell-k}$, where $c_h > 1$ is associated as above to the single-site operator L (equivalently, to L^{n_0}) and $0 < \Theta < 1$ is chosen as in Section 2 (see also [R, Theorem 2.1, Lemmas 4.20 and 4.25]). Since $c_h\Theta < 1$ these constants are finite, though not uniformly bounded in k. Using again [R, Theorem B.4], we get, recalling also the definition of c_h,the estimate (with $K = J \setminus I$),

$$\begin{aligned}
\Theta^{|\Lambda|}\sum_{J\subset\Lambda,|J|=k}\|\mathrm{Ext}_\Lambda((1-P)_J)\circ\pi_\Lambda\phi\| &\leq \Theta^{|\Lambda|}\sum_{J\subset\Lambda,|J|=k}\sum_{I\subset J}\|P_I\otimes P_{\Lambda\setminus J}\circ\pi_\Lambda\phi\| \\
&\leq \Theta^{|\Lambda|}\sum_{J\subset\Lambda,|J|=k} c_h^{|\Lambda\setminus J|}\sum_{K\subset J} c_h^{|J\setminus K|}\|\phi_K\| \\
&\leq \Theta^{|\Lambda|}\sum_{J\subset\Lambda,|J|=k} c_h^{|\Lambda\setminus J|}\sum_{K\subset J}\Theta^{-|K|}c_h^{|J\setminus K|}\|\phi\|_\Theta \\
&\leq \sum_{J\subset\Lambda,|J|=k}(c_h\Theta)^{|\Lambda\setminus J|}\sum_{K\subset J}(c_h\Theta)^{|J\setminus K|}\|\phi\|_\Theta \\
&\leq \sum_{J\subset\Lambda,|J|=k}(c_h\Theta+1)^{|J|}(c_h\Theta)^{|\Lambda\setminus J|}\|\phi\|_\Theta \\
&\leq \alpha_k\|\phi\|_\Theta\,.
\end{aligned} \tag{A.1}$$

We will use the following lemma:

Lemma A.1. *For $\lambda \notin \Sigma_0$, $\lambda - \mathcal{L}_0$ is invertible in $\mathcal{M}_\Theta$. We have the following expansion for the marginals of the resolvent:*

$$\pi_\Lambda \circ R(\lambda, \mathcal{L}_0) = \sum_{J \subset \Lambda} \mathrm{Ext}_\Lambda(R(\lambda, L_J)(1-P)_J)\,.$$

Proof of Lemma A.1. The identity $m_p L_p = m_p$ and the fact that L_Λ is a tensor product implies for $K \subset \Lambda$ that $\pi_{K,\Lambda} L_\Lambda = L_K \pi_{K,\Lambda}$. When $\lambda \notin \Sigma_0$ then the definition of the resolvent $R(\lambda, L_\Lambda) = (\lambda - L_\Lambda)^{-1}$ yields $\pi_{K,\Lambda} R(\lambda, L_\Lambda) = R(\lambda, L_K)\pi_{K,\Lambda}$. The properties of P_p also ensure that $L_\Lambda \mathrm{Ext}_\Lambda((1-P)_K) = \mathrm{Ext}_\Lambda(L_K(1-P)_K)$ and therefore also

$$R(\lambda, L_\Lambda)\mathrm{Ext}_\Lambda((1-P)_K) = \mathrm{Ext}_\Lambda(R(\lambda, L_K)(1-P)_K)\,. \tag{A.2}$$

Note, that this is consistent with $R(\lambda, L_\Lambda)P_\Lambda = (\lambda-1)^{-1}P_\Lambda$ since for $K = \emptyset$ we define $L_\emptyset$ and $(1-P)_\emptyset : \mathbb{C} \to \mathbb{C}$ to be the identity and $\mathrm{Ext}_\Lambda(\mathrm{Id}) = P_\Lambda$.

Our goal is to extend the family of resolvents $R(\lambda, L_\Lambda)$ to a bounded operator acting on the space $\mathcal{M}_\Theta$. We may clearly define $R(\lambda, L_\Lambda) \circ \pi_\Lambda : \mathcal{M}_\Theta \to H_\Lambda$. This family is projective but a priori it is not clear that it is bounded in the $\mathcal{M}_\Theta$ norm. Let us first note that $1_\Lambda = \otimes_{p\in\Lambda}(P_p + (1-P)_p) = \sum_{J\subset\Lambda} \mathrm{Ext}_\Lambda((1-P)_J)$. Acting on each individual term with the resolvent of L_Λ we obtain the right hand side in Lemma A.1. In general, however, this expansion is not going to be $\mathcal{M}_\Theta$ convergent so we need to be more careful. Instead we choose $m = m(|\lambda|) \in \mathbb{N}$ such that

$$|\lambda| > \eta^m \tag{A.3}$$

and we split the decomposition of 1_Λ just introduced, into two sums,

$$1_\Lambda = \sum_{J\subset\Lambda, |J|<m} \mathrm{Ext}_\Lambda((1-P)_J) + \sum_{J\subset\Lambda, |J|\geq m} \mathrm{Ext}_\Lambda((1-P)_J)\,. \tag{A.4}$$

We shall deal with these two terms separately. For the first term we note that by (A.2) we get, using again [R, Theorem B.4], the bound $\|R(\lambda, L_\Lambda)\mathrm{Ext}_\Lambda((1-P)_J)\circ\pi_\Lambda\| \leq \|R(\lambda, L_J)\|\,\|\mathrm{Ext}_\Lambda((1-P)_J)\circ\pi_\Lambda\|$. Here $\|R(\lambda, L_J)\| = \|R(\lambda, L^{\otimes|J|})\|$ depends on J only through its cardinality. The operator,

$$U_\Lambda^{<m} = R(\lambda, L_\Lambda) \sum_{J\subset\Lambda,|J|<m} \mathrm{Ext}_\Lambda((1-P)_J) = \sum_{J\subset\Lambda,|J|<m} \mathrm{Ext}_\Lambda(R(\lambda, L_J)(1-P)_J)\,, \tag{A.5}$$

is clearly bounded on H_Λ. If we combine with (A.1) we see that we also have the norm bound,

$$\|\Theta^{|\Lambda|} U_\Lambda^{<m} \circ \pi_\Lambda\| \leq \sum_{k=0}^{m-1} \|R(\lambda, L^{\otimes k})\| \cdot \alpha_k(\Theta, c_h)\,, \tag{A.6}$$

uniformly in Λ. By commutativity of the involved operators we also have

$$(\lambda - L_\Lambda)U_\Lambda^{<m} = U_\Lambda^{<m}(\lambda - L_\Lambda) = \sum_{J\subset\Lambda,|J|<m} \mathrm{Ext}_\Lambda((1-P)_J)\,. \tag{A.7}$$

For the second term in (A.4) we shall use the following "large deviations" bound (see Lemma 4.15 in [R] for a similar computation): if a, $b > 0$ and $\gamma > 1$ are so that $\gamma a + b \le 1$, then

$$\sum_{J\subset\Lambda\,,|J|\ge m} a^{|J|}b^{|\Lambda\setminus J|} \le \sum_{J\subset\Lambda} a^{|J|}b^{|\Lambda\setminus J|}\gamma^{|J|-m} \le (\gamma a+b)^{|\Lambda|}\gamma^{-m} \le \gamma^{-m}\,. \tag{A.8}$$

Taking $\gamma > 1$ such that condition TR in [R, Definition 4.18], holds, as before, our choices above for Θ (see [R, Lemma 4.20]) and n_0 (see [R, Lemma 4.21], recalling that the notation η represents in fact η^{n_0}) ensure that $c_r\,\gamma\eta(1+c_h\Theta)+c_h\Theta \le 1$. Then the "large deviations" bound gives (note the extra factor of $\eta^{\ell-1}$),

$$\sum_{J\subset\Lambda,|J|\ge m} (c_r\eta^\ell(1+c_h\Theta))^{|J|}(c_h\Theta)^{|\Lambda\setminus J|} \le \eta^{\ell m}\,(\gamma\eta)^{-m}\,. \tag{A.9}$$

Hence, using

$$\begin{aligned}\|L_\Lambda^\ell \mathrm{Ext}_\Lambda((1-P)_J)\circ\pi_\Lambda\| &= \|\mathrm{Ext}_\Lambda(L_J^\ell(1-P)_J)\circ\pi_\Lambda\|\\ &\le (c_r\eta^\ell)^{|J|}\|\mathrm{Ext}_\lambda((1-P)_J)\circ\pi_\Lambda\|\,,\end{aligned}$$

we get by a calculation similar to (A.1)

$$\Theta^{|\Lambda|}\sum_{J\subset\Lambda,|J|\ge m}\|L_\Lambda^\ell\mathrm{Ext}_\Lambda((1-P)_J)\circ\pi_\Lambda\| \le \eta^{\ell m}\,(\gamma\eta)^{-m}\,. \tag{A.10}$$

Therefore, since $|\lambda| > \eta^m$, the following sum is absolutely convergent on H_Λ (but the bound depends on the size of Λ):

$$A = \sum_{\ell=1}^\infty \frac{1}{\lambda^{\ell+1}} \sum_{J\subset\Lambda,|J|\ge m} L_\Lambda^\ell \mathrm{Ext}_\Lambda((1-P)_J)\,.$$

More precisely, the above calculations show that $\Theta^{|\Lambda|}\,A\circ\pi_\Lambda : \mathcal{M}_\Theta \to H_\Lambda$ is bounded in norm by

$$\sum_{\ell=1}^\infty \frac{\eta^{\ell m}}{|\lambda|^{\ell+1}}(\gamma\eta)^{-m} = \frac{1}{\gamma^m|\lambda|}\frac{1}{|\lambda|-\eta^m}\,,$$

uniformly in Λ. Finally, using (A.4), we find (corresponding to the "missing term" $\ell = 0$ in A) that

$$\Theta^{|\Lambda|}\frac{1}{\lambda}\sum_{J\subset\Lambda,|J|\ge m}\mathrm{Ext}_\Lambda((1-P)_J)\circ\pi_\Lambda = \Theta^{|\Lambda|}\frac{1}{\lambda}(1_\Lambda - \sum_{J\subset\Lambda,|J|<m}\mathrm{Ext}_\Lambda((1-P)_J))\circ\pi_\Lambda\,,$$

which by (A.1) is bounded by

$$\frac{1}{|\lambda|}(1+\sum_{k=0}^{m-1}\alpha_k)\,,$$

again uniformly in Λ. We may therefore define a bounded linear operator $U_\Lambda^{\geq m} : H_\Lambda \to H_\Lambda$ through

$$U_\Lambda^{\geq m} = \sum_{\ell=0}^{\infty} \frac{1}{\lambda^{\ell+1}} \sum_{J\subset\Lambda, |J|\geq m} L_\Lambda^\ell \mathrm{Ext}_\Lambda((1-P)_J)\,.$$

Now, $\Theta^{|\Lambda|} U_\Lambda^{\geq m} \circ \pi_\Lambda$ is absolutely bounded by

$$\frac{1}{|\lambda|}\Big(\frac{1}{\gamma^m}\frac{1}{|\lambda|-\eta^m} + 1 + \sum_{k=0}^{m-1}\alpha_k\Big)\,. \tag{A.11}$$

Uniform convergence and continuity allow us to interchange the sum and the action of the operator to obtain

$$(\lambda - L_\Lambda)U_\Lambda^{\geq m} = U_\Lambda^{\geq m}(\lambda - L_\Lambda) = \sum_{J\subset\Lambda, |J|\geq m} \mathrm{Ext}_\Lambda((1-P)_J)\,. \tag{A.12}$$

Combining the above, in particular (A.7) and (A.12) we see that by setting

$$U_\Lambda = U_\Lambda^{<m} + U_\Lambda^{\geq m}, \tag{A.13}$$

we obtain a bounded linear operator on H_Λ for which

$$U_\Lambda(\lambda - L_\Lambda) = (\lambda - L_\Lambda)U_\Lambda = 1_\Lambda\,,$$

thus showing that $U_\Lambda \equiv R(\lambda, L_\Lambda)$ is indeed the resolvent of L_Λ. In particular,

$$\Theta^{|\Lambda|}\|R(\lambda, L_\Lambda)\circ\pi_\Lambda\|$$

is uniformly bounded in Λ. As already mentioned, the family $R(\lambda, L_\Lambda)\circ\pi_\Lambda : \mathcal{M}_\Theta \to H_\Lambda$ is projective. Hence, if we set

$$(U\phi)_\Lambda \equiv R(\lambda, L_\Lambda)\phi_\Lambda\,,$$

then U is a linear operator on $\mathcal{M}_\Theta$ bounded in norm by the sum of (A.6) and (A.11). Finally $\pi_\Lambda\circ U\circ(\lambda-\mathcal{L}_0) = U_\Lambda\circ(\lambda-L_\Lambda)\circ\pi_\Lambda = \pi_\Lambda$ and $\pi_\Lambda\circ(\lambda-\mathcal{L}_0)\circ U = (\lambda-L_\Lambda)\circ U_\Lambda\circ\pi_\Lambda = \pi_\Lambda$, demonstrating that U is precisely the resolvent $R(\lambda, \mathcal{L}_0)$ which is therefore a bounded linear operator on $\mathcal{M}_\Theta$. The finite marginals, $\pi_\Lambda\circ R(\lambda, \mathcal{L}_0)$, may be computed from the expansion of the identity which completes the proof of Lemma A.1. □

End of the proof of Theorem 0. To end the proof of the unperturbed case (2), we must prove the claim on the image of the spectral projection. (The fact that every $\lambda\in\Sigma_0$ is an eigenvalue of $\mathcal{L}_0$ on $\mathcal{M}_\Theta$ is obvious, but not sufficient to prove the claim.) For this, we apply Theorem VII.3.24 in [DS] and need to check that every nonzero eigenvalue λ_0 of $\mathcal{L}_0$ on $\mathcal{M}_\Theta$ is a pole of finite order of the operator-valued resolvent function $\lambda\mapsto R(\lambda, \mathcal{L}_0)$. (See also [DS, Theorem VII.3.18] and its proof.) Clearly, it suffices to show that there is $\ell(\lambda_0)$ such that for each finite subset $\Lambda\subset\mathbb{Z}^D$, the point λ_0 is a pole of finite order at most $\ell(\lambda_0)$ of the operator-valued resolvent function $\lambda\mapsto R(\lambda, L_\Lambda)$, and that the projective family of operator-valued residues $W_\Lambda^{\lambda_0}$ of this pole define a bounded linear operator W^{λ_0} on $\mathcal{M}_\Theta$ by setting $(W^{\lambda_0}(\phi))_\Lambda = W_\Lambda^{\lambda_0}\phi_\Lambda$. The first claim follows from the fact that the index ℓ of the eigenvalue $\lambda_0 = \lambda_1 \cdots \lambda_m$ of L_Λ is the same as the index of λ_0 for $L_{\Lambda'}$ if λ_0 is also an eigenvalue of $L_{\Lambda'}$ (e.g. if $\Lambda \subset \Lambda'$). The second can be obtained by large deviations arguments similar to those used above (the combinatorics is slightly more involved), using $m \geq 1$ so that $\eta^{m+1} < |z| < \eta^m$.

The proof of (3) is short but we need to go a bit deeper into the construction in [R]. Let $0 < \epsilon \leq \epsilon_0$. The generating functions $u_p^n(\gamma)$ [R, Definition 4.16] defined for each fixed $0 \leq \epsilon \leq \epsilon_0$ verify the following bound [R, Lemma 4.20] for all $\gamma > 1$, close enough to 1:

$$u_p^n(1) \leq u_p^n(\gamma) \leq A + B\ C(\epsilon) \leq \Theta^{-1}.$$

Here A and B are positive constants, related to the single-site unperturbed operators (initial- and end-leaves) and the couplings (branchings), respectively and $C(\epsilon)$ is the function defined in our Theorem 0 above. (The bound in [R, Lemma 4.20] is stated for $u_p^n(\gamma)$ with some $\gamma > 1$, but the functions $u_p^n(\gamma)$ are monotone increasing). The Perron-Frobenius operator when projected to H_Λ is bounded by [R, Lemma 4.22]:

$$\|\pi_\Lambda \circ \mathcal{L}_\epsilon\| \leq \prod_{p\in\Lambda} u_p^n(1) \leq \prod_{p\in\Lambda}(A + B\ C(\epsilon)) \leq \Theta^{-|\Lambda|}\,.$$

If we now consider the perturbation alone, then in the proof of [R, Lemma 4.22] we should omit all contributions associated with a product of trees related to the *unperturbed* operator (trees consisting of initial-leaves and end-leaves only [R, Definition 4.9, Lemma 4.14]). This amounts to subtracting from the above expression precisely the contributions coming from such product trees, in other words

$$\|\pi_\Lambda \circ (\mathcal{L}_\epsilon - \mathcal{L}_0)\| \leq \prod_{p\in\Lambda}(A + B\ C(\epsilon)) - \prod_{p\in\Lambda} A\,. \tag{A.14}$$

Expanding the product we see that at least one factor of $C(\epsilon)$ occurs in each term on the right hand side. On the other hand, we know that each factor, $(A + B\ C(\epsilon_0))$, (note here the occurrence of ϵ_0 rather than ϵ) is bounded by Θ^{-1}. Hence, applying a large deviation argument similar to (A.8) (with $m = 1$ and $\gamma = C(\epsilon_0)/C(\epsilon)$), we see that

$$\Theta^{|\Lambda|}\|\pi_\Lambda \circ (\mathcal{L}_\epsilon - \mathcal{L}_0)\| \leq C(\epsilon)/C(\epsilon_0)\,, \tag{A.15}$$

from which we obtain the desired bound. □

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