Finite-dimensional functional analysis applied to transfer operators for infinite-dimensional maps

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January 2001

Abstract. We describe a simple approach to perturbative analysis of Perron-Frobenius operators, and study of the Floquet spectrum of the transfer operators of weakly coupled analytic maps on an infinite lattice: we are able to go beyond the first spectral gap and to exhibit smooth curves of eigenvalues and eigenvectors as functions of the crystal momenta. This talk given on January 23, 2001, at the session on Rapidity of convergence to equilibrium or stationary states, Journées Systèmes Aléatoires Inhomogènes, (Université de Cergy-Pontoise, France) describes joint work with H.-H. Rugh, Cergy-Pontoise. The detailed proofs of the results announced here, as well as further statements and references, may be found in [BR].

Extended abstract

The infinite-dimensional systems in the title are weakly coupled map lattices (CMLs), where the lattice is $\mathbb{Z}^D$ for some $D \geq 1$. Let us only describe here the simplest case where our theorems apply: we consider a complex annulus $\mathcal{A} \subset \mathbb{C}$ containing the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and the infinite torus $\mathcal{X} = \times_{\mathbb{Z}^D} S^1$. Letting $f : S^1 \to S^1$ be a real-analytic expanding (i.e. $\inf |f'| > 1$, the reader may think of a small analytic perturbation of $z \mapsto z^2$) transformation, we define the uncoupled map $\mathcal{F}_0 : \mathcal{X} \to \mathcal{X}$ by setting $(\mathcal{F}_0(z))_p = f(z_p)$ for every $p \in \mathbb{Z}^D$ and $z \in \mathcal{X}$. For small $\epsilon > 0$ and a coupling-decay parameter $0 < \xi < 1$, we introduce a coupling map $\mathcal{T}_\epsilon : \mathcal{X} \to \mathcal{X}$:

$$(\mathcal{T}_\epsilon(z))_p = z_p e^{i\epsilon \sum_{q \in \mathbb{Z}^D} g_{p-q}(z_p, z_q)} \, , |z_p| = 1 \, , p \in \mathbb{Z}^D \, ,$$

where the pair-couplings g_ℓ are real analytic on $S^1 \times S^1$, extend holomorphically to $\mathcal{A} \times \mathcal{A}$, and satisfy the bounds, for some $0 < C_g < \infty$,

$$\sup_{\ell \in \mathbb{Z}^D} \sup_{(z,w) \in \mathcal{A} \times \mathcal{A}} |g_\ell(z,w)| \leq C_g \, \xi^{d(0,\ell)} \, , \tag{1}$$

for a suitable distance function d on $\mathbb{Z}^D \times \mathbb{Z}^D$. We recover the *exponential* decay assumed by Bricmont and Kupiainen in their pioneering work [BK] if $d(p,q) = \|p - q\|$ is the usual euclidean metric, but we may also consider *polynomial* decay of degree $P > D$ by taking $d(p,q) = \log(1 + \|p - q\|)$ and setting $P = -\log(\xi)$. (We emphasize that the nonexponential decay can be treated due to very recent breakthroughs by Rugh [Ru].) Finally, the coupled map is defined on $\mathcal{X}$ by $\mathcal{F}_\epsilon = \mathcal{T}_\epsilon \circ \mathcal{F}_0$.

Assuming exponential decay of couplings, Bricmont and Kupiainen constructed a Sinai-Ruelle-Bowen measure:

Theorem (Bricmont-Kupiainen [BK]). *For small enough $\epsilon > 0$, there is a unique $\mathcal{F}_\epsilon$-invariant Borel probability measure μ_ϵ on $\mathcal{X}$ which satisfies all three conditions:*

(1) *For continuous $\varphi : \mathcal{X} \to \mathbb{C}$: $\lim_{n\to\infty} \int \varphi \circ \mathcal{F}_\epsilon^n \times_{\mathbb{Z}^D} dLebesgue = \int \varphi \, d\mu_\epsilon$.*

(2) *For each finite $\Lambda \subset \mathbb{Z}^D$, the marginal $\mu_{\epsilon,\Lambda}$ of μ_ϵ on $\times_\Lambda S^1$ is absolutely continuous with respect to $\times_\Lambda$Lebesgue and the logarithm of its Radon-Nikodyn derivative is real-analytic.*

(3) *There is $\rho < 1$ so that for each finite $\Lambda \subset \mathbb{Z}^D$ and all real analytic $\varphi, \psi : \times_\Lambda S^1 \to \mathbb{C}$ there are $C(\Lambda)$ and $C(\varphi, \psi)$ so that for all $n \ge 1$*

$$|\int \varphi \circ \mathcal{F}_\epsilon^n \, \psi \, d\mu_\epsilon - \int \varphi \, d\mu_\epsilon \int \psi \, d\mu_\epsilon| \le C(\Lambda) C(\varphi, \psi) \, \rho^n \,. \tag{2}$$

The main technical tools used in [BK] were the transfer operators $L_{\Lambda,\epsilon}$ associated to the truncations of the system on finite subsets $\Lambda \subset \mathbb{Z}^D$, acting on holomorphic and bounded functions on finite-dimensional polyannuli $\times_\Lambda \mathcal{A}$. Bricmont and Kupiainen showed bounds on their spectral gaps, uniform in the size $|\Lambda|$, but the prefactor $C(\Lambda)$ in the correlation decay bounds (2) grows exponentially with the size of Λ. It is therefore not only for esthetical reasons that one asks:

Question I. *Can one define a Banach space $\mathcal{B}$ of observables depending on all lattice sites, and a transfer operator $\mathcal{L}_\epsilon$ associated to the full (non truncated) coupled dynamics by*

$$\int_{\mathcal{X}} \varphi \, \mathcal{L}_\epsilon^n(\psi) \, \times_{\mathbb{Z}^D} dLebesgue \; = \int_{\mathcal{X}} \varphi \circ \mathcal{F}_\epsilon^n \, \psi \, \times_{\mathbb{Z}^D} dLebesgue \;,$$

and which posesses a spectral gap on $\mathcal{B}$?

To answer to the above question we must face the following difficulty: in the translation-invariant case the SRB measure (which is ergodic) is absolutely continuous with respect to the product ($\times_{\mathbb{Z}^D}$Lebesgue) only if it coincides with this product. So a Banach space which is big enough to contain the SRB fixed point of the transfer operator cannot be a subset of $L^1(\times_{\mathbb{Z}^D}\text{Lebesgue})$.

Such a Banach space $\mathcal{M}_\Theta$ was introduced by Fischer and Rugh [FR] in a paper where they gave a positive answer to Question I in the exponentially decaying coupling case. (For technical reasons, one must consider iterates $\mathcal{L}_\epsilon^n$ for $n \ge n_0$.) For a well-chosen $\Theta < 1$ (depending on $\mathcal{F}_\epsilon$) this space consists in all projective families

$$\{\varphi = (\varphi_\Lambda : \times_\Lambda \mathcal{A} \to \mathbb{C}, \Lambda \subset \mathbb{Z}^D \text{finite}) \mid \varphi_\Lambda \text{ holomorphic and bounded,}$$
$$\forall \text{ finite } \Lambda \subset \Lambda' \subset \mathbb{Z}^D \,,\, \varphi_\Lambda = \int \varphi_{\Lambda'} \, \times_{\Lambda' \setminus \Lambda} d\text{Lebesgue} \,\}$$

with finite norm

$$||\varphi||_\Theta = \sup_\Lambda \Theta^{|\Lambda|} \sup_{\times_\Lambda \mathcal{A}} |\varphi_\Lambda| \,.$$

In finite-dimensional dynamics, the transfer operator usually has some isolated eigenvalues of finite multiplicity beyond the first gap. They are sometimes called Ruelle resonances, and may be proved to be in bijection with the poles of the Fourier transform of the correlation (2) for suitable observables φ, ψ (see, e.g. [B]). In particular, they describe faster rates of decay in test function subspaces of finite codimension. Encouraged by the results of Fischer and Rugh [FR], we ask:

Question II. *Can one describe the spectrum of $\mathcal{L}_\epsilon$ on $\mathcal{M}_\Theta$ beyond the first gap?*

To address this second question, we must be aware of the fact that an element $\lambda \neq 1$ of the spectrum should not be expected to be an eigenvalue of finite multiplicity, even for the uncoupled transfer operator. Indeed, restricting the uncoupled operator to the dynamics on $\times_\Lambda S^1$ for a finite $\Lambda \subset \mathbb{Z}^D$, we obtain a finite tensor product, with eigenvalues $\prod_{k=1}^K \lambda_{i_k}$, $K \leq |\Lambda|$, where $\{1\} \cup \{\lambda_i \, , i \geq 1\}$ are the eigenvalues of the single-site operator L^{n_0}, and the multiplicity, e.g., of a simple eigenvalue λ_i of L viewed as an eigenvalue of $\otimes_\Lambda L$ coincides with $|\Lambda|$. (Writing h for the fixed point of L^{n_0} and φ_i for the eigenfunction of L^{n_0} and λ_i, just note that for each $p \in \Lambda$ the tensor product $\varphi_i(z_p) \otimes_{q \in \Lambda \setminus \{p\}} h(z_q)$ gives an independent eigenfunction.) One may however restrict both coupled and uncoupled transfer operators to the (invariant) eigenspaces of the spatial translations, and the spectrum of each such restriction should be more tractable, giving hope that a perturbative analysis might lead to the construction of nontrivial eigenvalues and eigenfunctions. It is this "Floquet spectrum" strategy which is carried out successfully in [BR] (see Theorem 1 below). Let us first state a preliminary result, which is deduced from quantitative estimates in [Ru] (where the analysis of [BK] and [FR] was extended to nonnecessarily exponentially decaying couplings):

Theorem 0 (Uncoupled spectrum and perturbation [BR]).

(1) *The nonzero spectrum of $\mathcal{L}_0^{n_0}$ on $\mathcal{M}_\Theta$ consists in*

$$\Sigma_0 = \{1, \prod_{k=1}^K \lambda_{j_k} \, , K = 1, 2, \dots, 1 \leq j_1 \leq j_2 \leq \dots \leq j_K\} \cup \{0\} \, .$$

1 is a simple eigenvalue, and all other eigenvalues have infinite multiplicity.

(2)
$$\|\mathcal{L}_0^{n_0} - \mathcal{L}_\epsilon^{n_0}\|_{\mathcal{M}_\Theta} \leq \mathcal{O}(\epsilon) \, . \tag{3}$$

To state the main theorem, let us enter in more details, considering for simplicity the case $D = 1$ and writing σ for the spatial translation on $\mathbb{Z}$ which defines in a natural way an isometric operator σ^* on $\mathcal{M}_\Theta$. It is not difficult to see that every frequency $e^{i\alpha}$, $\alpha \in [0, 2\pi)$ is an eigenvalue (crystal momentum) of σ^* of infinite multiplicity. Let

$$X^\alpha = \{\varphi \in \mathcal{M}_\Theta \mid \sigma^* \varphi = e^{i\alpha} \varphi \, , \int_{\mathcal{X}} \varphi \times_{\mathbb{Z}} d\text{Lebesgue} = 0\}$$

be the corresponding eigenspace. The definitions ensure that σ^* and $\mathcal{L}_\epsilon^n$ commute as operators on $\mathcal{M}_\Theta$, in particular, $\mathcal{L}_\epsilon^{n_0}$ sends X^α into itself.

We define the *resonant spectrum* of $\mathcal{L}_0^{n_0}$ to be:

$$\mathcal{R} = \{z \in \Sigma_0 \mid z = \prod_{k=1}^K \lambda_{j_k} \, , K \geq 2 \, , j_1 \leq \dots \leq j_K\} \cup \{0\} \, .$$

It is not very difficult to show that, except for the eigenvalues in $\mathcal{R}$ and, possibly $\{1\}$, the nonzero spectrum of $\mathcal{L}_0^{n_0} | X^\alpha$ is the same as the nonzero spectrum of L^{n_0}. In fact for each $\lambda \in \mathrm{sp}(L^{n_0}) \setminus \{1, 0\}$, a bijection between any basis of the generalised eigenspace of L^{n_0} and a basis for that of $\mathcal{L}_0^{n_0}|_{X^\alpha}$ is given by :

$$\varphi \mapsto \varphi^\alpha = \sum_{k \in \mathbb{Z}} e^{i\alpha k} \varphi_0 \circ \sigma^k \in \mathcal{M}_\Theta \text{ where } \varphi_0(z_k) = \varphi(z_0) \otimes_{k \in \mathbb{Z} \setminus \{0\}} h(z_k) \, . \tag{4}$$

Our key result is the following description of part of the spectrum of $\mathcal{L}_\epsilon^{n_0}|_{X^\alpha}$:

Theorem 1 (Floquet spectrum of uncoupled operator [BR]). *Let $\Delta \subset \{|z| < 1\}$ be a complex neighbourhood of a finite subset of the nonresonant spectrum* $\mathrm{sp}(\mathcal{L}_0^{n_0}) \setminus \mathcal{R}$ *of $\mathcal{L}_0^{n_0}$ such that $\Delta \cap \mathcal{R} = \emptyset$. Then for each small enough ϵ:*

(1) *For every $\alpha \in [0, 2\pi)$, the spectrum* $\mathrm{sp}(\mathcal{L}_\epsilon^{n_0}|_{X^\alpha}) \cap \Delta$ *consists in finitely many eigenvalues $\{\lambda_\ell^\epsilon(\alpha)\,, \ell \in \mathcal{E}(\alpha)\}$ of finite multiplicity. To each $\lambda_j \in \Delta \cap \mathrm{sp}(\mathcal{L}_0^{n_0})$ is associated $\{\lambda_\ell^\epsilon(\alpha)\,, \ell \in \mathcal{E}_j(\alpha)\}$ so that the $\mathcal{E}_j(\alpha)$ form a partition of $\mathcal{E}(\alpha)$, and the sum of the algebraic multiplicities of the $\lambda_\ell^\epsilon(\alpha)$ for $\ell \in \mathcal{E}_j(\alpha)$ coincide with the multiplicity of λ_j for L^{n_0}. Furthermore, $|\lambda_\ell^\epsilon(\alpha) - \lambda_j| \to 0$, uniformly, as $\epsilon \to 0$ for all $\ell \in \mathcal{E}_j(\alpha)$.*

(2) *If the couplings* (1) *decay exponentially, then for each j the map*

$$\alpha \mapsto \sum_{\ell \in \mathcal{E}_j(\alpha)} \lambda_\ell^\epsilon(\alpha) \tag{5}$$

is analytic. If the couplings decay polynomially with exponent $P > D$, then each map (5) *is $(P - D)$-times differentiable.*

Sketch of proof (see [BR]). As indicated in the title, the trick is to reduce to a finite matrix describing the part of the spectrum of $\mathcal{L}_\epsilon^{n_0}|_{X^\alpha}$ we are interested in. For this, letting $X^\alpha = X_0^\alpha \oplus X_1^\alpha$ be the splitting associated to the decomposition of the spectrum of $\mathcal{L}_0^{n_0}|_{X^\alpha}$ corresponding to $(\Delta, \mathrm{sp}(\mathcal{L}_0^{n_0}|_{X^\alpha}) \setminus \Delta)$, we obtain by (3) and perturbation theory a spectral decomposition $X^\alpha = X_{0,\epsilon}^\alpha \oplus X_{1,\epsilon}^\alpha$ for $\mathcal{L}_\epsilon^{n_0}|_{X^\alpha}$. We define an operator $\hat{\mathcal{L}}_\epsilon^{n_0}$ on the finite-dimensional vector space X_0^α by

$$\hat{\mathcal{L}}_\epsilon^{n_0} = \Pi_0^\alpha \circ \mathcal{L}_\epsilon^{n_0} \circ (\Pi_0^\alpha|_{\Pi_{0,\epsilon}^\alpha(X_0^\alpha)})^{-1} \,,$$

where the projectors Π_0^α and $\Pi_{0,\epsilon}^\alpha$ are those associated to the spectral decompositions above. The coefficients of $\hat{\mathcal{L}}_\epsilon^{n_0}$ in appropriate natural bases involving (4) are shown to depend smoothly on α by adapting some estimates from [Ru], ending our sketch of the proof.

References

[B] V. Baladi, *Positive transfer operators and decay of correlations*, World Scientific, Singapore, 2000.

[BR] V. Baladi and H.-H. Rugh, *Floquet spectrum of weakly coupled map lattices*, IHES preprint (2001), `www.ma.utexas.edu/mp_arc`, submitted for publication.

[BK] J. Bricmont and A. Kupiainen, *Coupled analytic maps*, Nonlinearity **8** (1995), 379–396.

[FR] T. Fischer and H.H. Rugh, *Transfer operators for coupled analytic maps*, Ergodic Theory Dynam. Systems **20** (2000), 109–144.

[Ru] H.H. Rugh, *Coupled maps and analytic function spaces*, Preprint Université de Cergy-Pontoise (2000), `www.ma.utexas.edu/mp_arc`, submitted for publication.

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