LINEAR RESPONSE FORMULA FOR PIECEWISE EXPANDING UNIMODAL MAPS

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ABSTRACT. The average $\mathcal{R}(t) = \int \varphi \, d\mu_t$ of a smooth function φ with respect to the SRB measure μ_t of a smooth one-parameter family f_t of piecewise expanding interval maps is not always Lipschitz [4], [19]. We prove that if f_t is tangent to the topological class of f, and if $\partial_t f_t|_{t=0} = X \circ f$, then $\mathcal{R}(t)$ is differentiable at zero, and $\mathcal{R}'(0)$ coincides with the resummation proposed in [4] of the (a priori divergent) series $\sum_{n=0}^{\infty} \int X(y) \partial_y (\varphi \circ f^n)(y) \, d\mu_0(y)$ given by Ruelle's conjecture. In fact, we show that $t \mapsto \mu_t$ is differentiable within Radon measures. Linear response is violated if and only if f_t is transversal to the topological class of f.

1. INTRODUCTION

Let us call SRB measure for a dynamical system $f : \mathcal{M} \to \mathcal{M}$, on a manifold $\mathcal{M}$ endowed with Lebesgue measure, an f-invariant ergodic probability measure μ so that the set $\{x \in \mathcal{M} \mid \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi \, d\mu\}$ has positive Lebesgue measure, for continuous observables φ. (In fact this defines a *physical measure,* see e.g. [32].) If f_t is a smooth one-parameter family with $f_0 = f$, and each f_t admits a unique SRB measure μ_t, it is natural to ask how μ_t depends on t. More precisely, one studies, for fixed smooth enough φ, the function $\mathcal{R}(t) = \int \varphi \, d\mu_t$.

If f is a sufficiently smooth uniformly hyperbolic diffeomorphism restricted to a transitive attractor, Ruelle [22]–[25] proved that $\mathcal{R}(t)$ is differentiable at $t = 0$. In addition, Ruelle gave an explicit formula for $\mathcal{R}'(0)$, depending on f_t only through its linear part (the "infinitesimal deformation") $v = \partial_t f_t|_{t=0}$. For obvious reasons, this formula is called the *linear response formula.* See [14, Cor. 1 p. 595] – noting that f and ρ in the statement there need in fact only be Hölder – for a previous results in continuous-time the Anosov setting, without an explicit formula for $\mathcal{R}'(0)$. We refer to the introductions of [9], [8], [4], for a discussion of more references regarding linear response for hyperbolic dynamical systems, including [8], [7], [12], and applications to statistical mechanics [11].

A much more difficult situation consists in studying nonuniformly hyperbolic interval maps f, e.g. smooth unimodal maps. For some of these maps, in particular those which satisfy the Collet-Eckmann condition, there exists a unique SRB measure μ. Two new difficulties are that structural stability does not hold (in a

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rather drastic way[1]), and that f_t will not always have an SRB measure even if f has one. In this setting, Ruelle ([26], [27]) has outlined a program, for infinitesimal deformations of the form $v = X \circ f$. He proposed $\Psi(1)$, where

$$\Psi(z) = \sum_{n=0}^{\infty} \int z^n X(y) \partial_y (\varphi \circ f^n)(y) \, d\mu_0(y) \,, \tag{1}$$

is the "susceptibility function," [2] as a candidate for the derivative, in the sense of Whitney's extension, of $\mathcal{R}(t)$ at $t = 0$. (We refer e.g. to the introduction of [4] for more details.) Beware that the series (1) may diverge at $z = 1$ so that $\Psi(1)$ needs to be suitably interpreted.

In this paper, just like in [4], we consider a simpler situation which exhibits however a similar bifurcation structure (in particular structural stability does not hold and infinitely many symbols may be required to code the dynamics): piecewise expanding interval maps. For such maps, it has been known for some time that μ_t exists for all t, and, under mild assumptions, that $\mathcal{R}(t)$ has modulus of continuity $O(t \ln |t|)$ (see (7) below and the references given there). We view the setting of piecewise expanding interval maps as a laboratory in which to test our ideas about smooth deformations. The arguments are free from technicalities, but exhibit most of the features that will appear in the Collet-Eckmann case.

Let us recall now recent results in this piecewise expanding setting. Assuming that $\partial_t f_t|_{t=0} = X \circ f$, a function $(f, X) \mapsto \mathcal{J}(f, X)$ was introduced in [4] (see (41)). There exist ([4], [19]) examples of piecewise expanding unimodal interval maps f_t so that $\mathcal{R}(t)$ is not Lipschitz. For these counterexamples, it turns out that $\mathcal{J}(f, X) \neq 0$. The function $\Psi(z)$ is holomorphic [4] in the open unit disc. In addition, if $\mathcal{J}(f, X) = 0$ and f is Markov (i.e., the postcritical orbit is finite) then $\Psi(z)$ is holomorphic at $z = 1$ ([4]). If $\mathcal{J}(f, X) = 0$ but f is not Markov a resummation Ψ_1 was devised [4] for the possibly divergent series $\Psi(1)$ (see Proposition 4.3 below). In view of the above facts (see also [4, Remark 4.5]), a modification of Ruelle's conjecture, was proposed in [4, Conjecture A] for perturbations of piecewise expanding or Collet-Eckmann f, assuming in addition that each f_t *is topologically conjugated to* f.

The main result of this paper is the proof of Conjecture A from [4] in the piecewise expanding setting. In fact, we prove a slightly stronger result (Theorem 5.1): It is enough to assume that f_t is *tangent* to the topological class of f (see §2.1). Also, the observable φ need only be continuous, so that in fact we prove that $t \mapsto \mu_t$ is differentiable into Radon measures. The interpretation of $\Psi(1)$ in Theorem 5.1 is in the sense of Ψ_1 from [4], and we find a more compact expression for Ψ_1, as well as a condition ensuring that Ψ_1 is the abelian limit of $\Psi(z)$ (Proposition 4.6).

Our approach to prove Theorem 5.1 is a perturbative spectral analysis (via resolvents) of transfer operators, on suitable spaces, adapted from those in [4]. [3] (In spirit, this is somewhat similar to the work of Butterley-Liverani [8].) To perform

[1]As was explained to us by D. Dolgopyat, the examples in [9, Section 2.3(B)] may fail to be structurally stable. However, shadowing holds for a sufficiently large measure of points so that Theorem 1–Proposition 2.6 of [9] provide a linear response formula in the sense of Whitney.

[2]Since $\Psi(e^{i\omega})$ is the Fourier transform of the "linear response" [23], it is natural to consider the variable ω, but we prefer to work with the variable $z = e^{i\omega}$.

[3]The spaces in [4] were inspired by what Ruelle told us about his then ongoing work on the nonuniformly expanding case [28].

this analysis, we use the Keller-Liverani [16] results together with smooth motions (Proposition 2.4) and the twisted cohomological equation for f and $X \circ f$. The novelty of this work resides in the combination of these two ingredients. A key new ingredient in the implementation of our ideas is the use of the isometry G_t in the proof of Theorem 5.1: this isometry is the device which allows us to use the same Banach space for the transfer operators of all perturbations, by forcing the singularities (here, jumps) to lie on a prescribed set.

We next summarise informally the picture for piecewise expanding, piecewise smooth unimodal maps (see § 2.1 for assumptions). If the critical point is not periodic, noting $f^0 = \mathrm{id}$, we say that v is horizontal for f if $\sum_{j=0}^{\infty} \frac{v(f^j(c))}{(f^j)'(f(c))} = 0$ (see (9) for the periodic case). Then:

- (i) $\mathcal{J}(f, X) = 0$ if and only if X is horizontal for f (Corollary 2.6).
- (ii) $X \circ f$ is horizontal for f if and only if the candidate Ψ_1 from [4] for the derivative is well-defined (Proposition 4.3 from [4], Proposition 4.5).
- (iii) If f_t is tangent to the topological class of f then $\partial_t f_t|_{t=0}$ is horizontal for f (Corollary 2.6).
- (iv) If v is horizontal for f, then any f_t with $\partial_t f_t|_{t=0} = v$ is tangent to the topological class of f. (Theorem 2.8 below, to appear in [5].)
- (v) If f_t is stably mixing [4] and tangent to the topological class of f with $\partial_t f_t|_{t=0} = X \circ f$, then $\mathcal{R}(t)$ is differentiable at $t = 0$, and the linear response formula $\mathcal{R}'(0) = \Psi_1$ holds (Theorem 5.1).
- (vi) If $\partial_t f_t|_{t=0}$ is not horizontal and c is not periodic for f then there exists C^∞ observable φ so that $\mathcal{R}(t)$ is not Lipschitz (Theorem 7.1, see [4], [19] for isolated examples).

In view of the results of the present paper, we expect that the following strengthening of Conjecture A [4] in the Collet-Eckmann case holds:

Conjecture A$'$. Let f be a mixing smooth Collet-Eckmann unimodal map with a nonflat critical point. Let f_t be a smooth perturbation, with $f_0 = f$ and $\partial_t f_t|_{t=0} = X \circ f$, which is *tangent to the topological class of* f*,* (i.e., so that there exists $\tilde{f}_t$ such that $|\tilde{f}_t - f_t| = O(t^2)$ and each $\tilde{f}_t$ is topologically conjugated to f). Then $\mathcal{R}(t)$ is differentiable at 0 *in the sense of Whitney* for all smooth observables φ, and $\mathcal{R}'(0) = \Psi(1)$ (the infinite sum being suitably interpreted).

In particular, if f_t remains in the topological class of a Collet-Eckmann map f, Conjecture A' is just [4, Conjecture A], where differentiability of $\mathcal{R}(t)$ is foreseen in the usual sense. We expect (see Conjecture B in [4]) that paths f_t which are *not* tangent to conjugacy classes give rise to $\mathcal{R}(t)$ which are in general Hölder but not Lipschitz in the sense of Whitney. Note that topological classes are called hybrid classes in this context, and they form a well understood lamination for smooth maps with a quadratic critical point (see [17], [2] and references therein).

This work is about the linear response. One can also wonder about formulas for the derivatives of higher order of $\mathcal{R}(t)$ (see [24]). Indeed, we expect that a suitable modification of the proof of Theorem 5.1 will give, if f_t is a C^{r_0, r_0+1} perturbation, tangent up to order $r_0 - 1$ to the topological class of a stably mixing piecewise expanding unimodal map f (i.e., we replace $|f_t - \tilde{f}_t| = 0(t^2)$ by $O(t^{r_0})$ for $r_0 \geq 3$ in

[4]Beware that if f is not stably mixing, then there exist f_t with $\partial_t f_t|_{t=0} = X \circ f$ horizontal and $\Psi(z)$ holomorphic at 0, but $\mathcal{R}(t)$ not Lipschitz.

§ 2.1), that $\mathcal{R}(t)$ has a Taylor series of degree $r_0 - 1$ at 0, with explicit coefficients (in the spirit of [24]). The coefficients will be related to twisted cohomological equations for derivatives of higher order of h_t (see the proof of Proposition 2.4). In the Collet-Eckmann setting, if f_t is tangent to the hybrid class of f up to order $r_0 - 1$, then we expect that higher order derivatives and Taylor series of degree $r_0 - 1$ should be attainable, of course in the sense of Whitney. (If f_t lies in the hybrid class, we expect a Taylor series in the usual calculus sense.)

The paper is organised as follows: Section 2 contains definitions, and the essential result on the "smooth motions" $h_t(x)$ (Proposition 2.4). The infinitesimal conjugacy $\alpha = \partial_t h_t|_{t=0}$ is introduced there. In Section 3, we recall the decomposition of the invariant density from [4], we adapt results from [16] on the transfer operators to reduce from families tangent to the topological class to families within the topological class (Proposition 3.3), and we introduce appropriate spaces $\mathcal{B}_t$ for transfer operators (Subsection 3.3) of sums of a "smooth" function with a sum of jumps along the postcritical orbit. In Section 4, we recall information from [4] on the susceptibility function $\Psi(z)$ and the candidate Ψ_1 for the derivative of $\mathcal{R}(t)$. We prove Theorem 5.1 in Section 5, combining the main ingredients (Proposition 2.4, Proposition 3.3, and the spectral analysis on the function spaces $\mathcal{B}_t$ from Subsection 3.3). The proof uses strongly the perturbation theory from Keller and Liverani [16] (we need to extend their result slightly, see Appendix B). Section 6 contains (Theorem 6.2) a simpler formula for $\mathcal{R}'(0)$, which is true if and only if α is absolutely continuous (a rare event). Theorem 7.1 in the last section shows that the condition to be tangent to the topological class is necessary.

After the first version of the present paper was made public, David Ruelle sent us a copy of [29], which contains in particular a proof of [4, Conjecture A] under the additional assumptions that f_0 is analytic and has a nonrecurrent postcritical orbit. We hope that injecting in our argument tools analogous to those developed there should eventually give a proof of Conjecture A' for Collet–Eckmann maps.

2. The setting, the twisted cohomological equation and the infinitesimal conjugacy α

2.1. **Piecewise expanding C^r unimodal maps and their perturbations.** If $K \subset \mathbb{R}$ is a compact interval and $\ell \geq 0$, we let $C^\ell(K)$ denote the set of functions on K which extend to C^ℓ functions in an open neighbourhood of K. In this work, we consider the following objects:

Definition. For an integer $r \geq 1$, a *piecewise expanding C^r unimodal map* is a continuous map $f : I \to I$, where $I = [a, b]$, so that f is strictly increasing on $I_+ = [a, c]$, strictly decreasing on $I_- = [c, b]$ ($a < c < b$), with $f(a) = f(b) = a$; and for $\sigma = \pm$, the map $f|_{I_\sigma}$ extends to a C^r map on a neighbourhood of I_σ, with [5] $\inf |f'|_{I_\sigma}| > 1$.
A piecewise expanding C^r unimodal map f is *good* if either c is not periodic under f or $\inf |(f^{n_1})'| > 2$, where $n_1 \geq 2$ is the minimal period of c; it is *mixing* if f is topologically mixing on $[f^2(c), f(c)]$.

[5]A prime denotes derivation with respect to $x \in I$, a priori in the sense of distributions.

Beware that a piecewise expanding C^r unimodal map f is only continuous, and never C^1 (it is piecewise C^r). We restrict to unimodal (as opposed to multimodal) to avoid unessential combinatorical difficulties.

Given a piecewise expanding C^r unimodal map f, we shall use the following notation: The point c will be called the *critical point* of f. We write $c_k = f^k(c)$ for $k \geq 0$. We say that c is *preperiodic* if it is not periodic but there exist $n_0 \geq 1$ and $n_1 \geq 1$ so that c_{n_0} is periodic of minimal period n_1 (we take n_0 minimal for this property and our assumptions imply $n_0 \geq 2$). If c is periodic for f of minimal period $n_1 \geq 2$ we set (by convention) $n_0 = 1$. If c is preperiodic or periodic for f, we set

$$N_f := n_0 + n_1 - 1 \geq 2 \,. \tag{2}$$

(If c is periodic we have $N_f = n_1$.) If c is neither preperiodic nor periodic for f, we set $N_f = \infty$.

Define $J := (-\infty, f(c)]$ and $\chi : \mathbb{R} \to \{0, 1, 1/2\}$ by

$$\chi(x) = 0 \text{ if } x \notin J\,, \quad \chi(x) = 1 \text{ if } x \in \operatorname{int} J\,, \quad \chi(f(c)) = \frac{1}{2}\,. \tag{3}$$

The two inverse branches of f, a priori defined on $[f(a), f(c)]$ and $[f(b), f(c)]$, may be extended to maps $\psi_+ : J \to \mathbb{R}_-$ and $\psi_- : \to \mathbb{R}_+$ in $C^r(J)$, with $\sup |\psi'_\sigma| < 1$ for $\sigma = \pm$. We set

$$\lambda_0 = \lim_{n\to\infty} (\sup(|(f^{-n})'|))^{1/n}\,, \quad \Lambda_0 = \lim_{n\to\infty} (\sup |(f^n)'|)^{1/n}\,. \tag{4}$$

and choose

$$\lambda \in (\lambda_0, 1)\,, \quad \Lambda > \Lambda_0\,.$$

Definition. Let $r \geq r_0 \geq 2$ be integers. For a piecewise expanding C^r unimodal map f, a $C^{r_0,r}$ *perturbation* of f is a family of piecewise expanding C^r unimodal maps $f_t : I \to I$, $|t| < \epsilon$, with $f_0 = f$, and satisfying the following properties: There exists a neighbourhood $\mathcal{I}_\sigma$ of I_σ, $\sigma = \pm$, so that the C^r norm of the extension of $f_t|_{I_\sigma}$ to $\mathcal{I}_\sigma$ is uniformly bounded for small $|t|$, and so that

$$\|(f - f_t)|_{\mathcal{I}_\sigma}\|_{C^{r-1}} = O(t)\,. \tag{5}$$

The map $(x, t) \mapsto f_t(x)$, extends to a C^{r_0} function on a neighbourhood of $(I_+ \cup I_-) \times \{0\}$. The *infinitesimal deformation* of the perturbation f_t is defined by

$$v = \partial_t f_t|_{t=0}\,. \tag{6}$$

Our assumptions imply that the infinitesimal deformation satisfies $v(a) = v(b) = 0$ and, if $f(c) = b$, also $v(c) = 0$.

If f_t is a $C^{2,2}$ perturbation of a piecewise expanding C^2 unimodal map, then each f_t (for small enough t) admits an absolutely continuous invariant probability measure (see e.g. [4] for references), with a density ρ_t which is of bounded variation. In fact, there is only one absolutely continuous invariant probability measure. Each ρ_t is continuous on the complement of the at most countable set $\{f_t^k(c) \mid k \geq 1\}$, and it is supported in $[f_t^2(c), f_t(c)] \subset [a, b]$ (we extend it by zero on $\mathbb{R}$). If f is good and mixing, then f_t is mixing and the absolutely continuous invariant measure is mixing. (If f is mixing, but not good, f_t need not be mixing.) In other words, assuming that f is good and mixing implies that f is stably mixing (we do not claim the converse), in addition, denoting by $|\varphi|_{L^1(Leb)}$ the $L^1(\mathbb{R}, \text{Lebesgue})$ norm of φ, by [16, Prop. 7] (by uniform Lasota-Yorke estimates, see [16, Remarks 1, 5]), we have

$$|\rho_t - \rho_0|_{L^1(Leb)} = 0(t \ln |t|) \,. \tag{7}$$

If f is not good, the function $t \mapsto \rho_t$ need not be continuous. (This is germane to the fact that mixing is not necessarily preserved if f is not good. See [15] for an illuminating multimodal example.) See also Remark 3.4.

Remark 2.1. Note that Ruelle's conjecture offers a candidate for the derivative of

$$\mathcal{R}(t) = \int \varphi \, \rho_t \, dx \tag{8}$$

only if $\partial_t f_t|_{t=0} = X \circ f$. (See also Remark 4.1.)

Definition. For integers $r \geq r_0 \geq 2$, and a piecewise expanding C^r unimodal map f, a $C^{r_0,r}$ *perturbation of* f *tangent to the topological class of* f is a $C^{r_0,r}$ perturbation f_t of f so that there exist a $C^{2,2}$ perturbation $\tilde f_t$ of f with

$$\sup_x |\tilde f_t(x) - f_t(x)| = O(t^2)$$

and homeomorphisms h_t with $h_t(c) = c$ and $\tilde f_t = h_t \circ f \circ h_t^{-1}$.

Clearly, if f_t is a $C^{2,2}$ perturbation of f tangent to the topological class of f, then $v = \partial_t f_t|_{t=0} = \partial_t \tilde f_t|_{t=0}$. We shall see (Corollary 2.6) that the infinitesimal deformations v of tangent perturbations are *horizontal* for f:

Definition. A continuous $v : I \to \mathbb{R}$ is *horizontal* [6] for a piecewise expanding C^1 unimodal map f if, setting $M_f = n_1$ if c is periodic of minimal period $n_1 \geq 2$, and $M_f = +\infty$ otherwise,

$$\sum_{j=0}^{M_f-1} \frac{v(c_j)}{(f^j)'(c_1)} = 0 \,. \tag{9}$$

See also Subsection 2.3 for a discussion of perturbations f_t tangent to the topological class of f.

When considering $C^{2,2}$ perturbations f_t, we have in particular $\sup_x |f_t'(x) - f'(x)| = o(1)$ (considering the extensions to neighbourhoods of I_σ) and we shall implicitly restrict to ϵ small enough so that

$$\sup_{|t|<\epsilon} \lim_{n\to\infty} (\sup(|(f_t^{-n})'|))^{1/n} < \lambda \,, \quad \sup_{|t|<\epsilon} \lim_{n\to\infty} (\sup(|(\tilde f_t^{-n})'|))^{1/n} < \lambda \,, \tag{10}$$
$$\sup_{|t|<\epsilon} \lim_{n\to\infty} (\sup |(f_t^{n})'|)^{1/n} < \Lambda \,, \quad \sup_{|t|<\epsilon} \lim_{n\to\infty} (\sup |(\tilde f_t^{n})'|)^{1/n} < \Lambda \,.$$

2.2. **The twisted cohomological equation, the smooth motions $h_t(x)$, and the infinitesimal conjugacy α.**

In this section, we discuss the following *twisted cohomological equation* (TCE, see e.g. [30]) for piecewise expanding unimodal f and bounded v:

$$v(x) = \alpha(f(x)) - f'(x)\alpha(x) \,, \quad \forall x \in I \,, x \neq c \,. \tag{11}$$

Let us start with an easy lemma:

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[6] See [17], [2] and references therein for a motivation of this terminology.

Lemma 2.2. *Assume that f is a piecewise expanding C^1 unimodal map and that v is a bounded function on I. Then for every $\omega \in \mathbb{R}$ the unique bounded solution $\alpha_{(\omega)}$ to (11) which satisfies $\alpha_{(\omega)}(c) = \omega$ is given by:*

$$\alpha_{(\omega)}(x) = \begin{cases} -\sum_{j=0}^{\infty} \frac{v(f^j(x))}{(f^{j+1})'(x)}\,, & \text{if } f^j(x) \neq c\,, \forall j \geq 0\,, \\ \frac{\omega}{(f^\ell)'(x)} - \sum_{j=0}^{\ell-1} \frac{v(f^j(x))}{(f^{j+1})'(x)} & \text{if } \exists \ell \geq 1 \text{ s.t. } f^\ell(x) = c\,. \end{cases} \tag{12}$$

Remark 2.3. If (11) admits a continuous solution α, it is easy to see by taking limits as $x \to c$ from the left and from the right that $\alpha(c) = 0$ and $v(c) = \alpha(c_1)$. (In particular, there is at most one continuous solution to (11).) We shall not use this.

Proof. For x so that $f^\ell(x) \neq c$ for all $\ell \geq 0$ (12) defines a bounded solution uniquely on this set: Indeed any bounded solution satisfies $\beta = -v/f' - \ldots - v \circ f^{k-1}/(f^k)' + \beta \circ f^{k+1}/(f^k)'$; if $\beta(x) \neq \alpha_{(\omega)}(x)$, then we take k so that $K/(f^k)' < (\beta(x) - \alpha_{(\omega)}(x))/3$ with $K = \max(\sup|\beta|, \sup|\alpha_{(\omega)}|)$, and we get a contradiction. If $\beta(c) = \omega$, then for each x so that $f^\ell(x) = c$ we must have $\beta(x) = \alpha_{(\omega)}(x)$ as defined in (12). □

When v is the infinitesimal deformation of a perturbation f_t tangent to the topological class of f we shall relate solutions to (11) to the conjugacies h_t. The key ingredient for this is the following information about the smoothness of $t \mapsto h_t$:

Proposition 2.4. *Let $r_0 \geq 2$ be an integer. Assume that $\tilde{f}_t$ is a C^{r_0,r_0} perturbation of a piecewise expanding C^{r_0} unimodal map f, so that for each small t there exists a homeomorphism h_t with $h_t(c) = c$ and $\tilde{f}_t = h_t \circ f \circ h_t^{-1}$. Then for small enough ϵ, the map $(t,x) \mapsto h_t(x)$ is continuous from $(-\epsilon,\epsilon) \times I \to \mathbb{R}$ and the maps $t \mapsto h_t(x)$ are $C^{r_0-1+Lip}$ on $[-\epsilon,\epsilon]$, uniformly in $x \in I$. (I.e. $\sup_x \|h_\cdot(x)\|_{C^{r_0-1+Lip}([-\epsilon,\epsilon])} < \infty$.)*

Remark 2.5. Although the $h_t(x)$ cannot be called "holomorphic motions" (see e.g. [2]) they certainly be called "smooth motions"! Beware that the maps $t \mapsto h_t^{-1}(x)$ are in general not C^{1+Lip}, although it is easy to see that the map $t \mapsto h_t^{-1}(x)$ is differentiable at $t = 0$ with derivative $-\alpha(x)$ for all $x \in I$. Also, the maps $x \mapsto h_t(x)$, $x \mapsto h_t^{-1}(x)$ are in general not absolutely continuous (see Section 6).

It will then be easy to show:

Corollary 2.6. *Under the assumptions of Proposition 2.4 the bounded function $\alpha : I \to \mathbb{R}$ defined by $\alpha(x) = \partial_t h_t(x)|_{t=0}$ satisfies the TCE (11) for $v = \partial_t f_t|_{t=0}$. In addition, α is continuous, $\alpha(c) = 0$ and $v(c) - \alpha(c_1) = 0$, so that v is horizontal for f.*

Definition. Under the assumptions of Proposition 2.4, the function $\alpha = \partial_t h_t|_{t=0}$ is the *infinitesimal conjugacy* associated to the infinitesimal deformation v of f_t.

Remark 2.7. It follows from Corollary 2.6 that if f_t is a perturbation of f and $v = \partial_t f_t|_{t=0}$ is not horizontal for f, then there exist arbitrarily small t so that f and f_t are *not* topologically conjugated, in particular f is not structurally stable. See [1] for an analogous statement about rational maps.

Proof of Proposition 2.4. To simplify notation, we assume that $c = 0$ in this proof. Let $\mathcal{P}_t$ be the set of points which are either periodic or eventually periodic for $\tilde{f}_t$, and whose forward orbit under $\tilde{f}_t$ does not contain the turning point c. It is easy to see that $\mathcal{P}_t$ is dense in I. Let $\theta = \sup_{x,t} |\tilde f_t'(x)|^{-1}$. We first prove that $(t,x) \to h_t(x)$ is continuous. Fix (x_0, t_0) and let $\kappa > 0$. Pick $n \in \mathbb{N}$ and $\delta > 0$ such that $\theta^n + \frac{\delta}{1-\theta} < \kappa$. Choose $\eta_0 < \epsilon/2$ small enough such that if $|t - t_0| < \eta_0$ then

$$\sup_x |\tilde f_t(x) - \tilde f_{t_0}(x)| < \delta ,$$

and let η_1 be such that $|x - x_0| < \eta_1$ implies $f^k(x) \cdot f^k(x_0) \geq 0$, for every $k \leq n$. So $\tilde f_t^k(h_t(x)) \cdot \tilde f_t^k(h_t(x_0)) \geq 0$, for every $k \leq n$ and t. Of course $\tilde f_t^k(h_t(x_0)) \cdot \tilde f_{t_0}^k(h_{t_0}(x_0)) \geq 0$. By Lemma A.1, for every $(t,x) \in \{|t-t_0| < \eta_0\} \times \{|x - x_0| < \eta_1\}$ we have

$$|h(t,x) - h(t_0, x_0)| \leq \kappa .$$

In the remainder of this proof, $\partial_t^i h_t$ denotes $\partial_s^i h_s|_{s=t}$. The implicit function theorem tells us that if $p \in \mathcal{P}_0$ then $t \to h_t(p)$ is a C^{r_0} function. Differentiating the equation $h_t \circ f(p) = \tilde f_t \circ h_t(p)$ with respect to t we obtain

$$\partial_t h_t \circ f(p) = \partial_t \tilde f_t \circ h_t(p) + \tilde f_t'(h_t(p)) \partial h_t(p) . \tag{13}$$

In other words

$$\partial_t h_t \circ f(p) - \tilde f_t'(h_t(p)) \partial h_t(p) = \partial_t \tilde f_t \circ h_t(p) = F_1(p) .$$

Next, differentiating (13) r_0 times, we can easily prove that for each $i \leq r_0$

$$\partial_t^i h_t \circ f(p) - \tilde f_t'(h_t(p)) \partial_t^i h_t(p) = F_i(p) , \tag{14}$$

where the function F_i is a polynomial combination of compositions of (all) partial derivatives of $\tilde f_t(x)$ up to order i, including mixed ones, with the function h_t, and partial derivatives $\partial_t^j h_t$, for $j = 1, \dots, i-1$.

For every $q \in \mathcal{P}_t$, we have $q = h_t(p)$, with $p \in \mathcal{P}_0$. Define

$$\alpha_t^i(q) := \partial_t^i h_t(h_t^{-1}(q)).$$

Define $Q_i(q) = F_i(h_t^{-1}(q))$. From (14) we obtain the twisted cohomological equation

$$Q_i(q) = \alpha_t^i(\tilde f_t(q)) - \tilde f_t'(q) \cdot \alpha_t^i(q) . \tag{15}$$

Let call this equation TCE_i.

Note that F_1 is bounded on $\mathcal{P}_0$. We claim that

$$|F_i|_\infty < \infty$$

for every $i \leq r_0$. Indeed, suppose by induction that F_ℓ and $\partial_t^{\ell-1} h_t$ are bounded functions on $\mathcal{P}_0$, for every $\ell \leq i < r_0$. Then Q_i is bounded on $\mathcal{P}_t$, and the unique solution for TCE_i on $\mathcal{P}_t$ is given by the expression

$$\alpha_t^i(q) = -\sum_{j=0}^{\infty} \frac{Q_i(\tilde f_t^j(q))}{(\tilde f_t^{j+1})'(q)} .$$

The uniqueness of the solution follows from the fact that every point in $\mathcal{P}_t$ is eventually periodic.

In particular

$$\sup_{q \in \mathcal{P}_t} |\alpha_t^i(q)| \leq \frac{|Q_i|_\infty}{1 - \sup_x |\tilde f_t'(x)|^{-1}} . \tag{16}$$

It follows that $\partial_t^i h_t$ is bounded on $\mathcal{P}_0$, and hence F_i is bounded in the same domain. This concludes the inductive argument.

Then from (16) we have an upper bound for $|\partial_t^i h_t|$, for $i \le r_0$, which is uniform on $t \in [-\epsilon, \epsilon]$ (up to taking a smaller ϵ). So the family of functions $t \to h_t(p)$, with $p \in \mathcal{P}_0$ and $t \in [-\epsilon, \epsilon]$, is a bounded subset of $C^{r_0}([-\epsilon, \epsilon])$.

We claim that $t \mapsto h_t(x)$ is $C^{r_0-1+Lip}$ for every $x \in I$. Indeed, let $p_n \in \mathcal{P}_0$ be a sequence which converges to x. Of course the sequence of functions $t \mapsto h_t(p_n)$ converges to the function $t \mapsto h_t(x)$. Since every sequence in a bounded subset of $C^{r_0}([-\epsilon, \epsilon])$ has a subsequence which converges to a function in $C^{r_0-1+Lip}$, we conclude that $t \mapsto h_t(x)$ is $C^{r_0-1+Lip}$. □

Proof of Corollary 2.6. By differentiating $\tilde f_t \circ h_t = h_t \circ f$ with respect to t at $t = 0$, we see that $\alpha(x)$ satisfies (11) at all $x \neq c$. Since $h_t(c) = c$ for all c we have $\alpha(c) = 0$. To prove $v(c) = \alpha(c_1)$, we use $\tilde f_t \circ h_t(c) = h_t \circ f(c)$: The derivative with respect to t of the right-hand-side at $t = 0$ is just $\alpha(c_1)$. This implies that the left-hand-side is differentiable at $t = 0$, and, using $h_t(c) = c$, the derivative is

$$\lim_{t\to 0} \frac{\tilde f_t(h_t(c)) - \tilde f_t(c)}{t} + \lim_{t\to 0} \frac{\tilde f_t(c) - f(c)}{t} = 0 + v(c)\,.$$

□

2.3. **Perturbations f_t tangent to the topological class of f.** For $r \ge 2$ and a fixed piecewise expanding C^r unimodal map f, we may pick $h_t(x)$ with $h_t(c) = c$, so that $(x, t) \mapsto h_t(x)$ is C^r, and define $\tilde f_t := h_t \circ f \circ h_t^{-1}$. Then $\tilde f_t$ is a $C^{r,r}$ perturbation of f in its topological class. If we assume in addition that $h_t(c + x) = \mathcal{S} h_t(c - x)$, where the ($C^r$) symmetry $\mathcal{S}$ is such that $f(c + x) = f(\mathcal{S}(c - x))$, we can ensure that the infinitesimal deformation is of the form $v = X \circ f$. Since $x \mapsto h_t(x)$ is a diffeomorphism in this construction, it gives a conjugacy between the invariant densities $\tilde\rho_t$ of $\tilde f_t$ and ρ_0 of f. Thus differentiability of $\widetilde{\mathcal{R}}(t) = \int \varphi \tilde\rho_t \, dx$ can be obtained by relatively easy perturbation theory arguments on the transfer operator. Theorem 5.1 applies to *all* smooth perturbations f_t which are tangent to $\tilde f_t$, and we may choose f_t in such a way as to ensure that f_t and f are not topologically conjugated (by modifying the kneading invariant), or are not smoothly conjugated (by acting on the multipliers [18]).

In view of a more general and systematic description of perturbations tangent to the topological class, recall that Corollary 2.6 implies that if a $C^{2,2}$ perturbation f_t of a C^2 map f is tangent to the topological class of f, then its infinitesimal deformation v is horizontal. In the smooth nonuniformly hyperbolic case (see [17], [2] and references therein) a converse to this statement holds. The proof of the converse in our setting is given elsewhere:

Theorem 2.8. *(See [5]) For $r_0 \ge 2$, let f be a good piecewise expanding C^{r_0} unimodal map and let $v \in C^{r_0}(I)$ be horizontal for f and satisfy $v(a) = 0$, $v(b) = 0$, and, if $f(c) = b$, also $v(c) = 0$. Then there exists a family of piecewise expanding C^{r_0} unimodal maps $\tilde f_t : I \to I$, $|t| < \epsilon$, with $\tilde f_0 = f$, so that the map $(x, t) \mapsto \tilde f_t(x)$, extends to a $C^{r_0-1+Lip}$ function on a neighbourhood of $(I_+ \cup I_-) \times \{0\}$, and, in addition, $\partial_t \tilde f_t|_{t=0} = v$, and for each t there is a homeomorphism h_t with $h_t(c) = c$ and $\tilde f_t = h_t \circ f \circ h_t^{-1}$. The conjugacies h_t are in general not absolutely continuous.*

In particular, the above implies that any $C^{2,r}$ perturbation f_t of a piecewise expanding C^r unimodal map f ($r \ge 2$) so that $v = \partial_t f_t|_{t=0}$ is horizontal and $v \in C^2(I)$ is tangent to the topological class of f.

Note that there exist (many) $C^{2,r}$ perturbations f_t of mixing piecewise expanding C^r unimodal maps, and such that $v = \partial_t f_t|_{t=0}$ is C^r and horizontal (also if we require $v = X \circ f$). Indeed, the functional $L_f : v \mapsto v(c) - \alpha_{(0)}(c_1)$ is bounded and linear from $\{v \in C^r(I)\}$ to $\mathbb{R}$. So it has a codimension-one kernel.

3. Transfer operators and their spectra

3.1. **Definitions and previous results.** Recall that a point x is called regular for a function ϕ if $2\phi(x) = \lim_{y\uparrow x} \phi(y) + \lim_{y\downarrow x} \phi(y)$. If ϕ_1 and ϕ_2 are functions of bounded variation on $\mathbb{R}$ having at most regular discontinuities, the Leibniz formula says that $(\phi_1\phi_2)' = \phi_1'\phi_2 + \phi_1\phi_2'$, where both sides are a priori finite measures. (Viewing a function ϕ in BV as a measure means considering $\phi\, dx$.)

For a piecewise expanding C^2 unimodal map f, recalling (3), we introduce two linear operators:

$$\mathcal{L}_0\varphi(x) := \chi(x)\varphi(\psi_+(x)) - \chi(x)\varphi(\psi_-(x))\,, \tag{17}$$

and

$$\mathcal{L}_1\varphi(x) := \chi(x)\psi_+'(x)\varphi(\psi_+(x)) + \chi(x)|\psi_-'(x)|\varphi(\psi_-(x))\,. \tag{18}$$

Note that $\mathcal{L}_1$ is the usual (Perron-Frobenius) transfer operator for f, in particular, $\mathcal{L}_1\rho_0 = \rho_0$ and $\mathcal{L}_1^*(\text{Lebesgue}_{\mathbb{R}}) = \text{Lebesgue}_{\mathbb{R}}$. The operators $\mathcal{L}_0$ and $\mathcal{L}_1$ both act boundedly on the Banach space

$$BV = BV^{(0)} := \{\varphi : \mathbb{R} \to \mathbb{C} \mid \text{var}(\varphi) < \infty\,, \text{supp}(\varphi) \subset [a,b]\}/\sim\,,$$

endowed with the norm $\|\varphi\|_{BV} = \inf_{\phi\sim\varphi} \text{var}(\phi)$, where var denotes total variation and $\varphi_1 \sim \varphi_2$ if the bounded functions φ_1, φ_2 differ on an at most countable set. To get finer information on $\mathcal{L}_0$, we consider the smaller Banach space (see e.g. [21])

$$BV^{(1)} = \{\varphi : \mathbb{R} \to \mathbb{C} \mid \text{supp}(\varphi) \subset (-\infty, b]\,, \varphi' \in BV\}\,,$$

for the norm $\|\varphi\|_{BV^{(1)}} = \|\varphi'\|_{BV}$. If $\mathcal{L}$ is a bounded linear operator on a Banach space $\mathcal{B}$, we denote the spectrum of $\mathcal{L}$ by $\text{sp}(\mathcal{L})$, and we define $R_{ess}(\mathcal{L})$, the essential spectral radius of $\mathcal{L}$, to be

$$R_{ess}(\mathcal{L}) = \inf\{R \geq 0 \mid \text{sp}(\mathcal{L}) \cap \{|z| > R\} \text{ consists of isolated eigenvalues of finite multiplicity}\}\,.$$

Recalling the definition (4) of λ_0, we have the following key lemma (see [4], the claims on $\mathcal{L}_1$ on BV are classical):

Lemma 3.1. *Assume that f is a mixing piecewise expanding C^2 unimodal map. The essential spectral radius of $\mathcal{L}_1$ on BV is $\leq \lambda_0$. In addition, 1 is a maximal eigenvalue of $\mathcal{L}_1$, which is simple, for the eigenvector ρ_0, and there are no other eigenvalues of $\mathcal{L}_1$ of modulus 1 on BV. The spectral radius of $\mathcal{L}_0$ on BV is equal to 1. For any $\varphi \in BV^{(1)}$*

$$(\mathcal{L}_0\varphi)' = \mathcal{L}_1(\varphi')\,. \tag{19}$$

Finally, the spectrum of $\mathcal{L}_0$ on $BV^{(1)}$ and that of $\mathcal{L}_1$ on BV coincide.

For further use, associate to a mixing piecewise expanding C^2 unimodal map f

$$\tau_0 = \max\Big(\lambda_0\,,\ \sup\{|z| \mid z \in \text{sp}\,(\mathcal{L}_1|_{BV})\,,\ z \neq 1\}\Big)\,, \tag{20}$$

(note that $\tau_0 < 1$), and choose

$$\tau \in (\tau_0, 1) .$$

Set $H_u(x) = -1$ if $x < u$, $H_u(x) = 0$ if $x > u$ and $H_u(u) = -1/2$. If f is a piecewise expanding C^2 unimodal map, the invariant density of f is of bounded variation and thus decomposes uniquely [20] as $\rho_0 = \rho_{sal} + \rho_{reg}$ with ρ_{reg} continuous and ρ_{sal} the *saltus term* (recalling N_f from § 2.1):

$$\rho_{sal} = \sum_{n=1}^{N_f} s_n H_{c_n} , \tag{21}$$

with $s_n = \lim_{y \downarrow c_n} \rho_0(y) - \lim_{x \uparrow c_n} \rho_0(x)$. By [4, Prop. 3.3] we have[7]:

Proposition 3.2. *Let f be a mixing piecewise expanding C^3 unimodal map. Then ρ_{reg} from the decomposition (21) of the invariant density is an element of $BV^{(1)}$.*

(M. Misiurewicz pointed out to us the related work of [31].)

3.2. **Comparing the invariant densities of two tangent perturbations.** Our main result is about perturbations f_t which are tangent to the topological class of f_0. In this subsection, we prove Proposition 3.3 (using classical Banach spaces, and tools from Keller-Liverani [16]) which will allow us to reduce from this assumption to the hypothesis that f_t lies in the topological class of f_0.

We need more notation. Let f_t be a $C^{2,r}$ perturbation of a piecewise expanding C^r unimodal map ($r \geq 2$) Define $J_t := (-\infty, f_t(c)]$ and $\chi_t : \mathbb{R} \to \{0, 1, 1/2\}$ by

$$\chi_t(x) = 0 \text{ if } x \notin J_t , \quad \chi_t(x) = 1 \text{ if } x \in \operatorname{int} J_t , \quad \chi_t(f_t(c)) = \frac{1}{2} .$$

The two inverse branches of f_t, a priori defined on $[f_t(a), f_t(c)]$ and $[f_t(b), f_t(c)]$, may be extended to maps $\psi_{t,+} : J_t \to (-\infty, c]$ and $\psi_{t,-} : J_t \to [c, \infty)$ in $C^r(J_t)$, with $\sup |\psi'_{t,\sigma}| < 1$ for $\sigma = \pm$. Put

$$\mathcal{L}_{1,t}\varphi(x) := \chi_t(x)\psi'_{t,+}(x)\varphi(\psi_{t,+}(x)) + \chi_t(x)|\psi'_{t,-}(x)|\varphi(\psi_{t,-}(x)) . \tag{22}$$

Recall our choices $\lambda < 1$ from (4) and $\tau < 1$ from (20). Lemma 3.1 applies to $\mathcal{L}_{1,t}$. By [16] we may assume that t is small enough so that

$$\max\big(\lambda, \sup_t \sup\{|z| \mid z \in \operatorname{sp}(\mathcal{L}_{1,t}|_{BV}) , \ z \neq 1\}\big) < \tau .$$

We may now state the new result of this subsection:

Proposition 3.3. *Let f be a good mixing piecewise expanding C^2 unimodal map. Then for any $C \geq 1$ and every pair (f_t, g_t) of $C^{2,2}$ perturbations of f, and so that*

$$\sup_x |f_t(x) - g_t(x)| \leq Ct^2 , \ \forall |t| \leq \epsilon , \tag{23}$$

there exist $C_1 \geq 1$, $\epsilon_0 > 0$ and $\xi > 1$ so that, letting ρ_t and $\tilde\rho_t$ denote the respective invariant densities of f_t and g_t, we have

$$\|\rho_t - \tilde\rho_t\|_{L^1(Leb)} \leq C_1|t|^\xi , \quad \forall |t| \leq \epsilon_0 .$$

Remark 3.4. The assumption that f is good is crucial in the above proposition since otherwise we do not have uniform Lasota-Yorke bounds (26) in general.

[7] The proof there does not require that c is not periodic.

Proof. Recall $\lambda < 1$ from (4) (we require that (10) hold for g_t too). Denote by $\mathcal{L}_{1,t}$ the transfer operator of f_t, by $\widetilde{\mathcal{L}}_{1,t}$ the transfer operator of g_t, acting on BV. Each $\mathcal{L}_{1,t}$ and each $\widetilde{\mathcal{L}}_{1,t}$ has a simple maximal eigenvalue at $z = 1$ and essential spectral radius $\le \lambda$ for small enough t. Our assumptions ensure that

$$\|f_t(x)\|_{C^{1+Lip}(V)} \le C\,, \quad \|g_t(x)\|_{C^{1+Lip}(V)} \le C\,, \tag{24}$$

on a neighbourhood V of $(I_+ \cup I_-) \times \{0\}$. Also, there exist $\widetilde{C}$ and $\epsilon_1 > 0$ depending only on f and C so that (our assumptions imply that g_t and f_t satisfy (5))

$$\begin{aligned} &\sup_j \|\mathcal{L}^j_{1,t}\|_{L^1(Leb)} < \widetilde{C}\,, \quad \sup_j \|\tilde{\mathcal{L}}^j_{1,t}\|_{L^1(Leb)} < \widetilde{C}\,,\ \forall |t| \le \epsilon_1\,, \\ &\|\mathcal{L}_{1,t}(\varphi) - \mathcal{L}_{1,0}(\varphi)\|_{L^1(Leb)} \le \widetilde{C}|t|\|\varphi\|_{BV}\,,\ \forall \varphi \in BV\,,\ \forall |t| \le \epsilon_1\,, \\ &\|\tilde{\mathcal{L}}_{1,t}(\varphi) - \mathcal{L}_{1,0}(\varphi)\|_{L^1(Leb)} \le \widetilde{C}|t|\|\varphi\|_{BV}\,,\ \forall \varphi \in BV\,,\ \forall |t| \le \epsilon_1\,. \end{aligned} \tag{25}$$

also, since f is good [16, Remark 5],

$$\max(\|\mathcal{L}^j_{1,t}\varphi\|_{BV}, \|\tilde{\mathcal{L}}^j_{1,t}\|_{BV}) \le \widetilde{C}\lambda^j\|\varphi\|_{BV} + \widetilde{C}\|\varphi\|_{L^1}\,,\ \forall \varphi \in BV\,,\ \forall |t| \le \epsilon_1\,, \tag{26}$$

finally, (24) and (23) imply $\|(f_t - g_t)|_{I_\sigma}\|_{C^1} = O(t^2)$, with a constant depending only on f and C, and thus

$$\|\mathcal{L}_{1,t}(\varphi) - \tilde{\mathcal{L}}_{1,t}(\varphi)\|_{L^1(Leb)} \le \widetilde{C}t^2\|\varphi\|_{BV}\,,\ \forall \varphi \in BV\,,\ \forall |t| \le \epsilon_1\,. \tag{27}$$

It follows from (25–26) for $\widetilde{\mathcal{L}}_{1,t}$, $\mathcal{L}_1$, and [16, Theorem 1] that for each small enough $\delta > 0$ there are $\epsilon_2 > 0$ and $\widehat{C} \ge 1$, depending only on f and C so that

$$\|(z - \widetilde{\mathcal{L}}_{1,t})^{-1}\|_{BV} \le \widehat{C}\,, \forall |t| \le \epsilon_2\,, \forall z \text{ with } |z| \ge \tau + \delta \text{ and } |z-1| \ge \delta\,. \tag{28}$$

We claim that the above estimate together with (27) implies $\|\rho_t - \tilde{\rho}_t\|_{L^1(Leb)} = O(|t|^{2\eta})$ for any $\eta < 1$. Taking η so that $2\eta > 1$, the claim ends the proof.

To obtain the claim, we revisit the proof of [16, Theorem 1]. Following Keller–Liverani, we put $\mathcal{Q}_t = (z - \mathcal{L}_{1,t})$ and $\widetilde{\mathcal{Q}}_t = (z - \widetilde{\mathcal{L}}_{1,t})$. In the sense of formal power series in z, we have for all $|t| \le \epsilon$

$$\mathcal{Q}_t^{-1} - \widetilde{\mathcal{Q}}_t^{-1} = \mathcal{Q}_t^{-1}(\mathcal{L}_{1,t} - \widetilde{\mathcal{L}}_{1,t})\widetilde{\mathcal{Q}}_t^{-1}\,. \tag{29}$$

By (28) and (27), the second part of the proof of [16, Theorem 1] gives that for any $\eta < 1$ and $\gamma > 0$, there are constants $\epsilon_0 > 0$, $\widetilde{A} \ge 1$, $\widetilde{B} \ge 1$, depending only on η, $\widetilde{C}$ and γ, so that for any z satisfying $|z| \ge \tau + \gamma$ and $|z - 1| \ge \gamma$, all $\varphi \in BV$, and all $|t| \le \epsilon_0$,

$$\begin{aligned} \|\mathcal{Q}_t^{-1}(\varphi)\|_{L^1(Leb)} \le{}& 2(t^2)^\eta\big(\widetilde{A}\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV} + \widetilde{B}\big)\|\varphi\|_{BV} \\ &+ 2(t^2)^{\eta-1}\left(\widetilde{C}\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV} + \frac{\widetilde{C}}{1-\tau}\right)\|\varphi\|_{L^1(Leb)}\,. \end{aligned} \tag{30}$$

Applying the above estimate to $(\mathcal{L}_{1,t} - \widetilde{\mathcal{L}}_{1,t})\widetilde{\mathcal{Q}}_t^{-1}(\varphi)$ and using (29), we get

$$\begin{aligned} &\|(\mathcal{Q}_t^{-1} - \widetilde{\mathcal{Q}}_t^{-1})(\varphi)\|_{L^1} \\ &\quad \le 2|t|^{2\eta}(\|\mathcal{L}_{1,t}\|_{BV} + \|\widetilde{\mathcal{L}}_{1,t}\|_{BV})\big(\widetilde{A}\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV} + \widetilde{B}\big)\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV}\|\varphi\|_{BV} \\ &\qquad + 2C|t|^{2\eta}\left(\widetilde{C}\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV} + \frac{\widetilde{C}}{1-\tau}\right)\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV}\|\varphi\|_{BV}\,, \end{aligned} \tag{31}$$

for any $\varphi \in BV$. Writing the difference between the spectral projectors for the eigenvalue 1 of $\mathcal{L}_{1,t}$ and $\widetilde{\mathcal{L}}_{1,t}$ as a contour integral of the difference of the resolvents, this shows the claim. □

3.3. **Spaces of sums of smooth functions and postcritical jumps.** In this subsection we shall introduce Banach spaces $\mathcal{B}_t \subset BV$ and $\mathcal{B}_t^{Lip} \subset BV$ of functions with controlled jumps along the postcritical orbit, on which the transfer operators $\mathcal{L}_{1,t}$ have essential spectral radius $\le \lambda$, in view of the proof of our main theorem in Section 5.

Let f be a mixing piecewise expanding C^3 unimodal map. Recall that $N_f = n_0 + n_1 - 1$ if c is preperiodic, $N_f = n_1$ if c is periodic, and $N_f = \infty$ otherwise. Let $\widetilde{BV}$ be the Banach space of continuous functions of bounded variation supported in $[a, b]$, for the BV norm. Fix $\eta > 0$ small. Consider the Banach space $(\widehat{\mathcal{B}}, \|\cdot\|)$ of pairs $\phi = (\phi_{reg}, \phi_{sal})$ with $\phi_{reg} \in \widetilde{BV}$, and $\phi_{sal} = (u_k)_{k=1,\dots,N_f}$, normed by

$$\|\phi\| = \|\phi_{reg}\|_{BV} + |\phi_{sal}|_\eta \text{ with } |\phi_{sal}|_\eta = \sup_{1\le k\le N_f} (1+\eta)^k |u_k| \,, \tag{32}$$

and so that, in addition,

$$\phi_{reg}(x) = \sum_{k=1}^{N_f} u_k \,, \ \forall x < a \,. \tag{33}$$

We define $\Gamma = \Gamma_0 : \widehat{\mathcal{B}} \to BV$ by

$$\Gamma(\phi_{reg}, (u_k)_{k\ge 1}) = \phi_{reg} + \sum_{k=1}^{N_f} u_k H_{c_k} \,. \tag{34}$$

(In particular, $\mathrm{supp}(\Gamma(\phi)) \subset [a, b]$.) The map Γ is injective, and we define $\mathcal{B}_0 \subset BV$ to be the isometric image of $\widehat{\mathcal{B}}$ under Γ.

It is easy to see that $\rho_0 \in \mathcal{B}_0$. For $\phi = (\phi_{reg}, (u_k)_{k\ge 1}) \in \widehat{\mathcal{B}}$, we may decompose $\tilde\varphi = \mathcal{L}_1(\Gamma(\phi)) \in BV$ into $\tilde\varphi = \tilde\varphi_{reg} + \tilde\varphi_{sal}$. Then, we have

$$\tilde\varphi_{sal} = \sum_{k\ge 1} w_k H_{c_k} \,,$$

with (writing $f'(c_-) = \lim_{y\uparrow c} f'(y)$ and $f'(c_+) = \lim_{y\downarrow c} f'(y)$)

$$\begin{cases} w_k = \frac{u_{k-1}}{f'(c_{k-1})} \,, & k \ge 2 \,, \\ w_1 = -\big(\frac{1}{|f'(c_-)|} + \frac{1}{|f'(c_+)|}\big)\big(\phi_{reg}(c) + \sum_{k\ge 1, c_k > c} u_k\big) \,, & \end{cases} \tag{35}$$

if the postcritical orbit is infinite (i.e., $N_f = \infty$), while

$$\begin{cases} w_k = \frac{u_{k-1}}{f'(c_{k-1})} \,, & 2 \le k \le N_f \,, k \ne n_0 \\ w_{n_0} = \frac{u_{n_0-1}}{f'(c_{n_0-1})} + \frac{u_{n_0+n_1-1}}{f'(c_{n_0+n_1-1})} \,, & \text{if } n_0 \ne 1 \,, \\ w_1 = -\big(\frac{1}{|f'(c_-)|} + \frac{1}{|f'(c_+)|}\big)\big(\phi_{reg}(c) + \sum_{k\ge 1, c_k > c} u_k\big) \,, & \end{cases} \tag{36}$$

if $N_f < \infty$. Also, we find

$$
\begin{aligned}
\tilde{\varphi}_{reg} &= \mathcal{L}_1(\phi_{reg}) \\
&\quad + H_{c_1}\left(\frac{1}{|f'(c_-)|} + \frac{1}{|f'(c_+)|}\right) \cdot \left(\phi_{reg}(c) + \sum_{1\le k\le N_f, c_k > c} u_k\right) \\
&\quad + \sum_{k=2}^{N_f} u_{k-1}\left(\mathcal{L}_1(H_{c_{k-1}}) - \frac{H_{c_k}}{f'(c_{k-1})}\right) .
\end{aligned} \tag{37}
$$

It is thus not difficult to check that $\tilde{\varphi} \in \mathcal{B}_0$. We next prove that in fact $\mathcal{L}_1$ is bounded on $\mathcal{B}_0$ with essential spectral radius $\le \lambda$.

We shall use that if $\mathcal{L}$ is a bounded operator on a Banach space $\mathcal{B}$, and $\mathcal{K}$ is a compact operator on $\mathcal{B}$, then the essential spectral radii of $\mathcal{L}$ and $\mathcal{L} - \mathcal{K}$ coincide (see e.g. [10] or [13, Theorem IV.5.35]). This fact is behind most techniques to estimate the essential spectral radius: Lasota-Yorke or Doeblin-Fortet bounds, Hennion's theorem, the Nussbaum formula, see e.g. [3]. In view of this, recall that the BV-closed unit ball is compact for the $L^1(Leb)$ norm. (See e.g. [3, §3.2, Prop. 3.3] for a proof of this Arzelà-Ascoli type result). In view of obtaining compact perturbations if $N_f = \infty$, note that for any $\delta > 0$ there is $k_\delta = O(\ln(\delta^{-1}))$ so that for any $\phi = (\phi_{reg}, (u_k)_{k\ge 1}) \in \widehat{\mathcal{B}}$,

$$
\sum_{k \ge k_\delta} |u_k| \le \delta \sup_{k\ge 1}\big((1+\eta)^k |u_k|\big) . \tag{38}
$$

For $\varphi \in BV$, we write $\Pi_{reg}(\varphi) = \varphi_{reg} \in C^0$ and $\Pi_{sal}(\varphi) = \varphi_{sal}$. If $N_f \ne \infty$, the operator $\mathcal{K}_0(\varphi) = \Pi_{sal}(\mathcal{L}_1(\varphi))$ is finite rank on $\mathcal{B}_0$, and thus compact. If $N_f = \infty$, the operator

$$
\mathcal{K}_0(\varphi) = -H_{c_1}(\varphi_{reg}(c) + \sum_{k\ge 1, c_k > c} u_k)(|f'(c_-)|^{-1} + |f'(c_+)|^{-1})
$$

is rank one, and thus compact, while the operator $\Pi_{sal} \circ (\mathcal{L}_1 - \mathcal{K}_0)$ has norm bounded by $(1+\eta)\sup|f'|^{-1}$ by definition.

We next consider $\Pi_{reg} \circ \mathcal{L}_1$. If $N_f < \infty$, the second and third lines of (37) are finite rank contributions, which will be denoted by $\mathcal{K}_1(\phi)$. If $N_f = \infty$, since

$$
\sup_{k\ge 2}\left\| \mathcal{L}_1(H_{c_{k-1}}) - \frac{H_{c_k}}{f'(c_{k-1})} \right\|_{BV} < \infty ,
$$

then (38) implies that the second and third line of (37) give a compact contribution, also denoted by $\mathcal{K}_1(\phi)$.

Then, consider the Radon measure $(\Pi_{reg} \circ \mathcal{L}_1(\varphi) - \mathcal{K}_1(\varphi))'$. By the Leibniz formula we have, as Radon measures,

$$
\begin{aligned}
(\Pi_{reg} \circ \mathcal{L}_1(\varphi) - \mathcal{K}_1(\varphi))'(y) = \chi_J \Bigg(&\frac{f''(\psi_+(y))}{(f'(\psi_+(y)))^2}\varphi(\psi_+(y)) - \frac{f''(\psi_-(y))}{(f'(\psi_-(y)))^2}\varphi(\psi_-(y)) \\
&+ \frac{\varphi'(\psi_+(y))}{(f'(\psi_+(y)))^2} - \frac{\varphi'(\psi_-(y))}{(f'(\psi_-(y)))^2} \Bigg) .
\end{aligned} \tag{39}
$$

By the compact inclusion property mentioned above, the contribution φ_1 in the first line is compact, let us call $(\mathcal{K}_2(\varphi))' = \varphi_1$ the corresponding operator. Now, the operator $\varphi' \mapsto \mathcal{M}(\varphi') = (\Pi_{reg} \circ \mathcal{L}_1(\varphi) - \mathcal{K}_1(\varphi) - \mathcal{K}_2(\varphi))'$ is bounded on measures, with norm at most $\sup(|f'|^{-1})\|\mathcal{L}_1\|_\infty$ where $\|\mathcal{L}_1\|_\infty$ is the operator norm of

$\mathcal{L}_1$ acting on bounded functions. Applying the above argument to $\mathcal{L}_1^j$, and using $\sup_j \|\mathcal{L}_1^j\|_\infty < \infty$, we obtain for each $j \geq 1$ a decomposition $\mathcal{L}_1^j = \mathcal{K}^{(j)} + \mathcal{M}^{(j)}$ where $\mathcal{K}^{(j)}$ is compact on $\mathcal{B}_0$, and $\|\mathcal{M}^{(j)}\|_{\mathcal{B}_0} \leq C_0(1+\eta)^j \sup(|(f^j)'|^{-1})$. Therefore, the essential spectral radius of $\mathcal{L}_1$ on $\mathcal{B}_0$ is $\leq \lambda$.

Consider now the Banach space $(\widehat{\mathcal{B}}^{Lip}, \|\cdot\|)$ of pairs $\phi = (\phi_{reg}, \phi_{sal})$ with $\phi_{reg} \in Lip((-\infty, b])$, and $\phi_{sal} = (u_k)_{k=1,\ldots,N_f}$, normed by $\|\phi\| = \|\phi_{reg}\|_{Lip} + |\phi_{sal}|_\eta$ and so that $\phi_{reg}(x) = \sum_{k=1}^{N_f} u_k$ for all $x < a$ (in particular, ϕ_{reg} is constant on $(-\infty, a)$). Using Γ as above, we define a Banach space $\mathcal{B}_0^{Lip} \subset \mathcal{B}_0 \subset BV$. Since $\|\phi\|_{Lip} = \|\phi'\|_{L^\infty}$ and since the $Lip([a,b])$- closed unit ball is compact in the $L^\infty([a,b])$ topology, the same argument as above shows that $\mathcal{L}_1$ is bounded on $\mathcal{B}_0^{Lip}$, with essential spectral radius $\leq \lambda$. Since $BV^{(1)} \subset Lip$, we have that $\rho_0 \in \mathcal{B}_0^{Lip}$.

If f_t is a $C^{2,3}$ perturbation of f we may define $\mathcal{B}_t$ and $\mathcal{B}_t^{Lip}$ for each t by taking the isometric image in BV of $\widehat{\mathcal{B}}$, respectively $\widehat{\mathcal{B}}^{Lip}$ under Γ_t defined by

$$\Gamma_t\Big(\phi_{reg}, (u_k)_{k\geq 1}\Big) = \phi_{reg} + \sum_{k=1}^{\infty} u_k H_{c_{k,t}} .$$

The argument above shows that $\mathcal{L}_{1,t}$ has essential spectral radius bounded by λ on $\mathcal{B}_t$ and $\mathcal{B}_t^{Lip}$. Since each $\mathcal{B}_t$ and each $\mathcal{B}_t^{Lip}$ is a subset of BV and since $\rho_t \in \mathcal{B}_t^{Lip} \subset \mathcal{B}_t$, we have proved that outside of the disc of radius τ the spectrum of $\mathcal{L}_{1,t}$ on $\mathcal{B}_t$ or on $\mathcal{B}_t^{Lip}$ consists in a simple eigenvalue at 1, with corresponding spectral projector $\varphi \mapsto \rho_t \int \varphi \, dx$.

4. The susceptibility function and the candidate Ψ_1 for the derivative

The susceptibility function [27] associated to a piecewise expanding C^2 unimodal map f, a test function $\varphi \in C^1([a,b])$, and a deformation $v = X \circ f$ for $X \in C^1([a,b])$ is the formal power series

$$\Psi(z) = \sum_{n=0}^{\infty} \int z^n X(y)\rho_0(y)(\varphi \circ f^n)'(y)\, dy = \sum_{n=0}^{\infty} \int z^n \mathcal{L}_0^n(X\rho_0)(x)\varphi'(x)\, dx . \tag{40}$$

In this section, we recall in Proposition 4.3 the resummation Ψ_1 proposed in [4] for the a priori divergent series $\Psi(1)$ when $X \circ f$ is horizontal. In addition, we give in Lemma 4.4 an expression for Ψ_1 in terms of the infinitesimal conjugacy α from Section 2, and we show that Ψ_1 is not well-defined if $X \circ f$ is not horizontal (Proposition 4.5).

Remark 4.1. If the infinitesimal deformation v is not of the form $X \circ f$, the heuristic argument of Ruelle [23] suggests to define the susceptibility function as:

$$\Psi(z) = \sum_{n=0}^{\infty} \int z^n \mathcal{L}_1(v\rho_0)(y)(\varphi \circ f^n)'(y)\, dy = \sum_{n=0}^{\infty} \int z^n \mathcal{L}_0^n\big(\mathcal{L}_1(v\rho_0)\big)(x)\varphi'(x)\, dx .$$

The analysis of the above expressions produces additional difficulties, and will not be pursued here.

Since $X\rho_0 \in BV$, Lemma 3.1 implies that the power series $\Psi(z)$ extends to a holomorphic function in the open unit disc, and in this disc we have

$$\Psi(z) = \int (\mathrm{id} - z\mathcal{L}_0)^{-1}(X\rho_0)(x)\, \varphi'(x)\, dx .$$

Recalling the jumps s_n in the saltus term ρ_{sal} for ρ (see (21)), the weighted total jump of f defined in [4] is:

$$\mathcal{J}(f, X) = \sum_{n=1}^{N_f} s_n X(c_n) . \tag{41}$$

In [4], we resummed the possibly divergent series $\Psi(1)$ under the condition $\mathcal{J}(f, X) = 0$ (see Proposition 4.3 below). We have the following simple but enlightening lemma:

Lemma 4.2. *Assume that f is a piecewise expanding C^2 unimodal map f, and that $X : I \to \mathbb{R}$ is bounded. Define $\alpha_{(0)}(c_1)$ by (12) for $v = X \circ f$. Then*

$$\mathcal{J}(f, X) = s_1(X(c_1) - \alpha_{(0)}(c_1)) .$$

Since $s_1 < 0$, the lemma implies $\mathcal{J}(f, X) = 0$ if and only if $\alpha_{(0)}(c_1) = X(c_1)$, i.e., if and only if $X \circ f$ is horizontal for f.

Proof. If c is neither periodic nor preperiodic, then $s_k = f'(c_k)s_{k+1}$ for $k \geq 1$, and thus

$$\mathcal{J}(f, X) = s_1(X(c_1) - \alpha_{(0)}(c_1)) = s_1 \sum_{j \geq 0} \frac{X(f^j(c_1))}{(f^j)'(c_1)} \tag{42}$$

(see [4, Rem. 4.5]). The case of periodic c is similar using $s_k = f'(c_k)s_{k+1}$ for $1 \leq k \leq n_1 - 1$ and $M_f = n_1$.

If c is preperiodic, using $s_k = f'(c_k)s_{k+1}$ for $1 \leq k \leq n_0 + n_1 - 2$, $k \neq n_0 - 1$, and

$$s_{n_0} = \frac{s_{n_0-1}}{f'(c_{n_0-1})} + \frac{s_{n_0+n_1-1}}{f'(c_{n_0+n_1-1})} = \frac{s_{n_0-1}}{f'(c_{n_0-1})} + \frac{s_{n_0}}{(f^{n_1})'(c_{n_0})} ,$$

which implies $(1 - (f^{n_1})'(c_{n_0}))s_{n_0} = s_{n_0-1}/(f'(c_{n_0-1}))$ and thus

$$s_{n_0+j} = \frac{s_1}{(f^{n_0+j-1})'(c_1)} \frac{1}{1 - 1/(f^{n_1})'(c_{n_0})} , \quad 0 \leq j \leq n_1 - 1 ,$$

we get

$$\begin{aligned} \mathcal{J}(f, X) &= s_1 \Big(\sum_{n=0}^{n_0-2} \frac{X(f^n c_1)}{(f^n)'(c_1)} + \sum_{j=0}^{n_1-1} \frac{X(f^{n_0+j-1}(c_1))}{(f^{n_0+j-1})'(c_1)} \frac{1}{1 - 1/(f^{n_1})'(c_{n_0})} \Big) \\ &= s_1(X(c_1) - \alpha_{(0)}(c_1)) . \end{aligned}$$

□

We next recall the candidate Ψ_1 for the derivative of $t \mapsto \mathcal{R}(t)$ from Ruelle's conjecture as interpreted in [4]. Note that if $X \in C^2(f(I))$ satisfies $X(a) = 0$ then the function $\widetilde{X}$ defined by $\widetilde{X}(x) := X(x)$ for $x \geq a$ and $\widetilde{X}(x) := 0$ for $x \leq a$ is such that $\widetilde{X}'$ is of bounded variation, and $\widetilde{X}'\tilde{\rho}$ is supported in $[a, b]$ for any $\tilde{\rho}$ supported in $(-\infty, b]$. Recall M_f from (9). Then, by Proposition 3.2 and the properties of s_k from the proof of Lemma 4.2, putting together [4, Lemma 4.1, Proposition 4.4, Theorem 5.2] gives [8]:

[8]Theorem 5.2 in [4] also holds if c is periodic, with a similar proof.

Proposition 4.3. *Let f be a mixing piecewise expanding C^3 unimodal map. Let $X \in C^2(f(I))$ satisfy $X(a) = 0$ and $\mathcal{J}(f, X) = 0$. For $\varphi \in C^1([a, b])$ and $|z| < 1$:*

$$\Psi(z) = -\sum_{j=1}^{\infty} \varphi(c_j) \sum_{k=1}^{\min(j,M_f)} z^{j-k} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} - \int (\mathrm{id} - z\mathcal{L}_1)^{-1}(X'\rho_{sal} + (X\rho_{reg})')\varphi \, dx \,. \tag{43}$$

The second term in (43) extends to a holomorphic function in the open disc of radius λ_0^{-1}. If c is periodic or preperiodic then the first term of (43) is a rational function which is holomorphic at $z = 1$.

In addition, the following is a well-defined complex number

$$\Psi_1 = -\sum_{j=1}^{M_f} \varphi(c_j) \sum_{k=1}^{j} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} - \int (\mathrm{id} - \mathcal{L}_1)^{-1}(X'\rho_{sal} + (X\rho_{reg})')\varphi \, dx \,. \tag{44}$$

Note that if $\mathcal{J}(f, X) = 0$ (a codimension one condition on X) then $\Psi_1 = \Psi_1(\varphi)$ is well-defined even if φ is only continuous.

We have the following simpler expression for the first term of Ψ_1:

Lemma 4.4. *Let f be a mixing piecewise expanding C^3 unimodal map. Let $X \in C^2(f(I))$ satisfy $X(a) = 0$ and $\mathcal{J}(f, X) = 0$, and let $\varphi \in C^1([a, b])$. Then, setting $\alpha = \alpha_{(0)}$ from (12) for f and $v = X \circ f$,*

$$\Psi_1 = -\int \alpha\varphi \, \rho'_{sal} - \int (\mathrm{id} - \mathcal{L}_1)^{-1}(X'\rho_{sal} + (X\rho_{reg})')\varphi \, dx \,. \tag{45}$$

Proof. By Lemma 4.2 $X(c_1) = \alpha(c_1)$. Thus, by (49) the first term of Ψ_1 from (44) may be rewritten as a Stieltjes integral (α is continuous by Corollary 2.6)

$$\begin{aligned} -s_1 \sum_{j=1}^{M_f} \varphi(c_j) \left(X(c_1) - \alpha(c_1) + \frac{\alpha(c_j)}{(f^{j-1})'(c_1)} \right) &= -s_1 \sum_{j=1}^{M_f} \varphi(c_j) \frac{\alpha(c_j)}{(f^{j-1})'(c_1)} \\ &= -\int \alpha\varphi \, \rho'_{sal} \,. \end{aligned} \tag{46}$$

□

In fact, Ψ_1 is well-defined only if $\mathcal{J}(f, X) = 0$:

Proposition 4.5. *Let f be a mixing piecewise expanding C^3 unimodal map f, let $X \in C^2(f(I))$ satisfy $X(a) = 0$. For every $\varphi \in C^0([a, b])$ the following series converges*

$$-\sum_{j=1}^{\infty} \int \mathcal{L}_1^j((X\rho_{reg})')(x)\varphi(x) \, dx \,.$$

If $\mathcal{J}(f, X) \neq 0$ then Ψ_1 is not well-defined, in the following sense: There exists $\varphi \in C^\infty([a, b])$ so that, on the one hand, both series below diverge

$$-\sum_{j=1}^{\infty} \varphi(c_j) \sum_{k=1}^{\min(j,M_f)} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} - \sum_{j=1}^{\infty} \int \mathcal{L}_1^j(X'\rho_{sal})(x)\varphi(x) \, dx \,, \tag{47}$$

and on the other hand, the following series diverges

$$-\sum_{j=1}^{\infty}\left(\varphi(c_j)\sum_{k=1}^{\min(j,M_f)}\frac{s_1 X(c_k)}{(f^{k-1})'(c_1)}+\int \mathcal{L}_1^j(X'\rho_{sal})(x)\varphi(x)\,dx\right). \tag{48}$$

Proof. Since $\int((X\rho_{reg})')(x)\,dx=0$, the proof of [4, Proposition 4.4], implies

$$|-\int \mathcal{L}_1^j((X\rho_{reg})')(x)\varphi(x)\,dx|\le C\tau^j\,,$$

which gives the first claim.

To fix ideas assume that $\mathcal{J}(f,X)>0$. Recalling Lemma 4.2, note that if c is not periodic, then for each j

$$\sum_{k=1}^{j} s_k X(c_k)=X(c_1)-\alpha_{(0)}(c_1)+\frac{\alpha_{(0)}(c_j)}{(f^{j-1})'(c_1)}=\mathcal{J}(X,f)+\frac{\alpha_{(0)}(c_j)}{(f^{j-1})'(c_1)}\,. \tag{49}$$

By the proof of [4, Proposition 4.4],

$$|-\int \mathcal{L}_1^j(X'\rho_{sal})(x)\varphi(x)\,dx-\mathcal{J}(f,X)\int\varphi\rho_0\,dx|\le C\tau^j\,,$$

thus if $\int\varphi\rho_0\,dx>0$ then the second term in (47) diverges to $+\infty$. If c is not periodic and, in addition, $\inf_j\varphi(c_j)>\int\varphi\rho_0\,dx>0$ then the first term diverges to $-\infty$ (use (49)). Finally, for the same φ, if c is not periodic then (48) is $\mathcal{J}(f,X)\sum_j(-\varphi(c_j)+\int\varphi\rho_0\,dx)$, which clearly diverges to $-\infty$. The case of periodic c is similar. □

We end this section by discussing the relation between $\Psi(z)$ and Ψ_1 when $\mathcal{J}(f,X)=0$: If c is preperiodic or periodic, Ψ_1 is just the value at 1 of the holomorphic extension of $\Psi(z)$, and we have $\Psi_1=\lim_{z\to 1}\Psi(z)$. If c is neither periodic nor preperiodic we do not know if the resummation Ψ_1 for the possibly divergent series $\Psi(1)$ is always Abelian, i.e., if $\Psi_1=\lim_{z\in(0,1),z\to 1}\Psi(z)$, but we have the following sufficient codimension-two condition on X ensuring abelianity:

Proposition 4.6. *Let f be a mixing piecewise expanding C^3 unimodal map. Let $X\in C^2(f(I))$ satisfy $X(a)=0$, $\mathcal{J}(f,X)=0$, and, in addition,*

$$\sum_{j=1}^{\infty}\frac{j\ X(f^j(c_1))}{(f^j)'(c_1))}=0 \tag{50}$$

then $\Psi_1=\lim_{z\in(0,1),z\to 1}\Psi(z)$.

Proof. We may assume that the critical point c is not periodic, so that the following formal Laurent series is well-defined for $\ell\ge 1$:

$$\alpha(c_\ell,z)=-\sum_{j=1}^{\infty}\frac{X(f^j(c_\ell))}{z^j(f^j)'(c_\ell)}\,.$$

Clearly, $z\mapsto\alpha(c_1,z)$ is analytic in $\{z\in\mathbb{C}\mid |z|\min|f'|>1\}$. We have

$$\partial_z\alpha(c_1,z)|_{z=1}=\sum_{j=1}^{\infty}\frac{j\ X(f^j(c_1))}{(f^j)'(c_1)}\,. \tag{51}$$

Note for further use that if $X(c_1)=\alpha(c_1,1)$ (which is equivalent to $\mathcal{J}(f,X)=0$ by Lemma 4.2) and if (50) holds, then (51) implies

$$\alpha(c_1,z)=\alpha(c_1,1)+O(|1-z|^2)\,. \tag{52}$$

Now, using $\alpha(c_1, z)$, we may rewrite the coefficient of $\varphi(c_j)$ in the first term of $\Psi(z)$ from Proposition 4.3 as

$$
\begin{aligned}
s_1 z^{j-1} \sum_{k=1}^{j} & \frac{X(c_k)}{z^{k-1}(f^{k-1})'(c_1)} \\
&= s_1 z^{j-1} \Big(X(c_1) - \alpha(c_1, z) - \frac{1}{z^{j-1}(f^{j-1})'(c_1)} \sum_{m=1}^{\infty} \frac{X(c_{j+m})}{z^m (f^m)'(c_j)} \Big) \\
&= s_1 z^{j-1} \big(X(c_1) - \alpha(c_1, z) \big) - s_1 \frac{\alpha(c_j, z)}{(f^{j-1})'(c_1)} .
\end{aligned}
$$

Consequently, if $(\min |f'|)^{-1} < |z| < 1$, the first term of $\Psi(z)$ can be written as

$$
\begin{aligned}
\sum_{j=1}^{\infty} & \varphi(c_j) \sum_{k=1}^{j} z^{j-k} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} \\
&= \sum_{j=1}^{\infty} \varphi(c_j) \Big[s_1 z^{j-1} \big(X(c_1) - \alpha(c_1, z) \big) - s_1 \frac{\alpha(c_j, z)}{(f^{j-1})'(c_1)} \Big] \\
&= s_1 \Big[\big(X(c_1) - \alpha(c_1, z) \big) \sum_{j=1}^{\infty} \varphi(c_j) z^{j-1} - \sum_{j=1}^{\infty} \frac{\varphi(c_j)\ \alpha(c_j, z)}{(f^{j-1})'(c_1)} \Big] .
\end{aligned}
$$

It is easy to see that

$$
\lim_{|z|<1, z \to 1} \sum_{j=1}^{\infty} \frac{\varphi(c_j)\ \alpha(c_j, z)}{(f^{j-1})'(c_1)} = \sum_{j=1}^{\infty} \frac{\varphi(c_j)\ \alpha(c_j, 1)}{(f^{j-1})'(c_1)} .
$$

Note also that if $|z| < 1$ then $| \sum_{j=1}^{\infty} \varphi(c_j) z^{j-1} | \le \frac{\sup |\varphi|}{1-|z|}$.

Finally, (52) implies

$$
|\big(X(c_1) - \alpha(c_1, z) \big) \sum_{j=1}^{\infty} \varphi(c_j) z^{j-1}| = |\big(\alpha(c_1, 1) - \alpha(c_1, z) \big) \sum_{j=1}^{\infty} \varphi(c_j) z^{j-1}| \le C|z-1| .
$$

Putting together the above estimates, we find using (49)

$$
\begin{aligned}
\lim_{z \to 1^-} \sum_{j=1}^{\infty} \varphi(c_j) \sum_{k=1}^{j} z^{j-k} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} &= -s_1 \sum_{j=1}^{\infty} \frac{\varphi(c_j) \alpha(c_j, 1)}{(f^{j-1})'(c_1)} \\
&= \sum_{j=1}^{\infty} \varphi(c_j) \sum_{k=1}^{j} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} ,
\end{aligned}
$$

which immediately gives the claim. □

5. Proof of the main theorem

If f_t is a $C^{2,2}$ perturbation of a mixing piecewise expanding C^2 unimodal map f tangent to its topological class, then Corollary 2.6 gives that the infinitesimal deformation v is horizontal. If $v = X \circ f$, Lemma 4.2 thus implies that $\mathcal{J}(f, X) = 0$. Therefore, if $X \in C^2(f(I))$, a candidate Ψ_1 for the derivative is defined by Proposition 4.3 and Lemma 4.4. Our main result can now be stated:

Theorem 5.1. *Let f_t be a $C^{2,3}$ perturbation of a mixing piecewise expanding C^3 unimodal map f with infinitesimal deformation $v = X \circ f$ such that $X \in C^2(f(I))$. If f_0 is good and f_t is tangent to its topological class, or if $f_t = \tilde f_t$ lies in the topological class of f_0, then $t \mapsto \rho_t\, dx$ from $(-\epsilon, \epsilon)$ to Radon measures is differentiable at* 0*, and*

$$\partial_t(\rho_t\, dx)|_{t=0} = -\alpha \rho'_{sal} - (\mathrm{id} - \mathcal{L}_1)^{-1}(X'\rho_{sal} + (X\rho_{reg})')\, dx\, .$$

In particular, for any $\hat\varphi \in C^0([a,b])$, the map $\mathcal{R}(t) = \int \hat\varphi\, \rho_t\, dx$ is differentiable at $t = 0$, and $\mathcal{R}'(0) = \Psi_1(\hat\varphi)$.

Remark 5.2. See Theorem 7.1 for necessity of the condition $\mathcal{J}(f, X) = 0$ (which is equivalent to tangency to the topological class by Corollary 2.6).

Proof of Theorem 5.1. Since $\tilde f_t = f_t$ if f is not good, we may assume without loss of generality by Proposition 3.3 that $\tilde f_t = f_t = h_t \circ f \circ h_t^{-1}$ for all t. Also, since each ρ_t is a probability measure, we may restrict to continuous functions $\hat\varphi$ so that $\int \hat\varphi\, d\rho_0 = 0$. The proof will then be divided in three steps.

Step 1: Perturbation theory via resolvents.

Recall the spaces $\mathcal{B}_t = \Gamma_t(\widehat{\mathcal{B}})$ from Subsection 3.3, for a fixed $\eta > 0$, and define linear isometries $G_t = \Gamma_0 \circ \Gamma_t^{-1} : \mathcal{B}_t \to \mathcal{B}_0$. We decompose

$$\rho_t - \rho_0 = (G_t(\rho_t) - \rho_0) + (\rho_t - G_t(\rho_t))\, . \tag{53}$$

The second term may be analysed directly, noting that (as Radon measures)

$$\lim_{t\to 0} \frac{\rho_t - G_t(\rho_t)}{t} = \lim_{t\to 0} \frac{\rho_{sal,t} - \rho_{sal,t} \circ h_t}{t} = -\sum_{k=1}^{N_f} \alpha(c_k) s_k \delta_{c_k} = -\alpha \rho'_{sal}\, .$$

(We used that $c_{k,t} = h_t(c_k)$ implies $H_{c_k} = H_{c_{k,t}} \circ h_t$ and that $\lim_{t\to 0} s_{1,t} = s_1$[9] .) To study the first term in (53), set

$$\mathcal{P}_t = G_t \circ \mathcal{L}_{1,t} \circ G_t^{-1}\, , \quad \widehat{\mathcal{Q}}_t = \widehat{\mathcal{Q}}_t(z) = z - \mathcal{P}_t\, .$$

(Of course $\mathcal{P}_0 = \mathcal{L}_1$ and $\widehat{\mathcal{Q}}_0 = z - \mathcal{L}_1$.) The operator $\mathcal{P}_t$ on $\mathcal{B}_0$ is conjugated to $\mathcal{L}_{1,t}$ on $\mathcal{B}_t$ and therefore has the same spectrum. The fixed point of $\mathcal{P}_t$ is $G_t(\rho_t)$ and the fixed point of $\mathcal{P}_t^*$ is $\nu_t(\varphi) = \int G_t^{-1}(\varphi)\, dx$. We denote by $\widehat\Pi_t(\varphi) = G_t(\rho_t)\nu_t(\varphi)$ the corresponding spectral projector. Our strategy will be to use, as in Proposition 3.3,

$$\widehat{\mathcal{Q}}_t^{-1} - \widehat{\mathcal{Q}}_0^{-1} = \widehat{\mathcal{Q}}_t^{-1}(\mathcal{P}_t - \mathcal{P}_0)\widehat{\mathcal{Q}}_0^{-1}\, ,$$

in order to write $G_t(\rho_t)\nu_t(\varphi_0) - \rho_0 \int \varphi_0\, dx$ as a difference of spectral projectors applied to $\varphi_0 \in \widetilde{\mathcal{B}}_0$, where

$$\widetilde{\mathcal{B}}_0 = \{\varphi \in \mathcal{B}_0 \mid \varphi'_{reg} \in \mathcal{B}_0^{Lip}\} \text{ with the norm } \|\varphi'_{reg}\|_{\mathcal{B}_0^{Lip}} + \|\varphi\|_{\mathcal{B}_0}\, .$$

In fact, we do not need to perform the spectral analysis of $\mathcal{L}_1$ on $\widetilde{\mathcal{B}}_0$, since we shall work exclusively with $\rho_0 \in \widetilde{\mathcal{B}}_0$ (the fact that $\rho'_{reg} \in \mathcal{B}_0^{Lip}$, i.e., that all discontinuities of ρ'_{reg} lie on the postcritical orbit, that the jump at c_k is $O(\lambda^k)$, and that $(\rho_{reg})'_{reg} \in$ *Lip* is an easy consequence of the proof of [4, Proposition 3.3], noting in particular the uniform bound for $\Delta'_n(x)$ there – see also (70) and (71)).

[9]For this claim (which implies $\lim_{t\to 0} s_{k,t} = s_k$ for each fixed k), use that $\lim_{t\to 0} \int \varphi \rho_t\, dx = \int \varphi\rho\, dx$ for all bounded φ: Since $\lim_{t\to 0} c_{1,t} = c_1$, and $\sup_t \|\rho_{reg,t}\|_{Lip} < \infty$, while $|s_{k,t}| \le C\lambda^k$ uniformly in t, choosing for φ the characteristic function of a sufficiently small neighbourhood of c_1, we get a contradiction if $s_{1,t} \not\to s_1$.

Since $\int \rho_0 \, dx = 1$, noting that $\widehat{\mathcal{Q}}_0^{-1}(\rho_0) = \rho_0/(z-1)$, we find

$$G_t(\rho_t)\nu_t(\rho_0) - \rho_0 = -\frac{1}{2i\pi} \oint \frac{\widehat{\mathcal{Q}}_t^{-1}(z)}{z-1} (\mathcal{P}_t - \mathcal{P}_0)(\rho_0) \, dz = (\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t)(\mathcal{P}_t - \mathcal{P}_0)(\rho_0) \, , \tag{54}$$

where the contour is a circle centered at 1, outside of the disc of radius τ.

We shall also use the following norms on $\mathcal{B}_0$, for $j \ge 0$

$$|\varphi|_{weak,j} = \frac{\|\varphi_{reg}\|_{L^1(Leb)}}{2} + \frac{\max\{|\varphi_{reg}(y)| \mid y \in \cup_{0 \le \ell \le j} f^{-\ell}(c)\}}{2} + |\Gamma^{-1}(\varphi_{sal})|_\eta \, .$$

We have $|\varphi|_{weak,j} \le \|\varphi\|_{\mathcal{B}_0}$ for all $j \ge 0$. It is not difficult to see by adapting the estimates in Subsection 3.3 that there exist $\epsilon > 0$ and $C \ge 1$ so that, for all $|t| \le \epsilon$ all j, ℓ, all $\varphi \in \mathcal{B}_0$,

$$|\mathcal{P}_t^j(\varphi)|_{weak,\ell} \le C |\varphi|_{weak,\ell+j} \, , \quad \|\mathcal{P}_t^j(\varphi)\| \le C\lambda^j \|\varphi\| + C|\varphi|_{weak,j} \, . \tag{55}$$

(Uniformity in t of the constant C in the Lasota-Yorke estimate follows from the fact that f is good. The reason why $\sup_{\ell \le j} |\varphi_{reg}(f^{-\ell}(c))|$ appears in the weak norm is to take into account the compact operators $\mathcal{K}_0(\mathcal{L}_1^j)$ from the decomposition in § 3.3.) We shall see in Step 3 that for any fixed $j \ge 0$ there is a modulus of continuity $\delta_j(t) \ge 0$ (i.e., $\limsup_{t \to 0} \delta_j(t) = 0$) so that for each $\varphi \in \mathcal{B}_0$

$$|\mathcal{P}_t(\varphi) - \mathcal{P}_0(\varphi)|_{weak,j} \le \delta_j(t) \|\varphi\|_{\mathcal{B}_0} \, . \tag{56}$$

Therefore, the proof of [16, Theorem 1] (see Appendix B) gives $\epsilon_0 > 0$ so that

$$A_{\epsilon_0} := \sup_{|t| < \epsilon_0} \|(\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t)\|_{\mathcal{B}_0} < \infty \, . \tag{57}$$

Beware that it is not clear whether $|(\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t)(\varphi) - (\mathrm{id} - \mathcal{P}_0)^{-1}(\varphi)|_{weak,0}$ tends to zero uniformly in $\|\varphi\|_{\mathcal{B}_0} \le 1$ as $t \to 0$. This is why we next consider $\mathcal{P}_t$ acting on $\mathcal{B}_0^{Lip}$: By § 3.3, the essential spectral radius is $\le \lambda$, and the spectrum outside of the disc of radius τ consists in the eigenvalue 1, with projector $\widehat{\Pi}_t$. We introduce a weak norm on $\mathcal{B}_0^{Lip}$:

$$|\varphi|_{weak,\infty} = \|\varphi_{reg}\|_{L^\infty(Leb)} + |\Gamma^{-1}(\varphi_{sal})|_\eta \, .$$

Applying again the argument in § 3.3, we see that (55) holds for $\ell = \infty$. Clearly, $|\varphi|_{weak,j} \le |b-a| |\varphi|_{weak,\infty}$. In Step 3, we shall find $\widetilde{C} \ge 1$ so that for each $\varphi \in \mathcal{B}_0^{Lip}$

$$|\mathcal{P}_t(\varphi) - \mathcal{P}_0(\varphi)|_{weak,\infty} \le \widetilde{C} |t| \|\varphi\|_{\mathcal{B}_0^{Lip}} \, . \tag{58}$$

Then, setting

$$\mathcal{N}_t = (\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t) - (\mathrm{id} - \mathcal{P}_0)^{-1}(\mathrm{id} - \widehat{\Pi}_0) \, ,$$

(55) and (58) imply by [16, Theorem 1, Corollary 1] that there are $\widehat{C} \ge 1$ and $\xi > 0$ so that for each $\varphi \in \mathcal{B}_0^{Lip}$

$$|\mathcal{N}_t(\varphi)|_{weak,\infty} \le \widehat{C} |t|^\xi \|\varphi\|_{\mathcal{B}_0^{Lip}} \, . \tag{59}$$

If we knew that there existed $\mathcal{D} \in \mathcal{B}_0^{Lip}$ so that [10]

$$\|\mathcal{P}_t(\rho_0) - \mathcal{P}_0(\rho_0) - t\mathcal{D}\|_{\mathcal{B}_0} = O(t^2) \, , \tag{60}$$

[10]We emphasize that the norm in (60) is in $\mathcal{B}_0$, and a priori not in $\mathcal{B}_0^{Lip}$.

uniformly in small t (this will be shown in Step 2), then (54) and (59) would give

$$\partial_t(G_t(\rho_t)\nu_t(\rho_0))|_{t=0} = (\mathrm{id} - \mathcal{L}_1)^{-1}(\mathrm{id} - \widehat{\Pi}_0)(\mathcal{D})\,, \tag{61}$$

in $L^\infty(Leb)$: Indeed, write $(\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t) = \mathcal{N}_t + (\mathrm{id} - \mathcal{P}_0)^{-1}(\mathrm{id} - \widehat{\Pi}_0)$ and note that (57) implies

$$\begin{aligned} G_t(\rho_t)\nu_t(\rho_0) - \rho_0 &= (\mathcal{N}_t + (\mathrm{id} - \mathcal{P}_0)^{-1}(\mathrm{id} - \widehat{\Pi}_0))(t\mathcal{D} + O_{\mathcal{B}_0}(t^2)) \\ &= t\mathcal{N}_t(\mathcal{D}) + t(\mathrm{id} - \mathcal{P}_0)^{-1}(\mathrm{id} - \widehat{\Pi}_0)(\mathcal{D}) + A_{\epsilon_0}O(t^2)\,. \end{aligned} \tag{62}$$

Dividing by t and letting $t \to 0$, (59) gives the claim (61).

Note that $t \mapsto \nu_t(\rho_0)$ is differentiable at 0: As $\nu_t(\rho_0) = \int \rho_{sal} \circ h_t^{-1}\, dx + \int \rho_{reg}\, dx$, one easily sees that $\partial_t \nu_t(\rho_0)|_{t=0} = -\sum_{k=1}^{N_f} \alpha(c_k)s_k$. Then, by the Leibniz formula,

$$\partial_t(G_t\rho_t)|_{t=0} = \partial_t(G_t(\rho_t)\nu_t(\rho_0))|_{t=0} - \rho_0\, \partial_t(\nu_t(\rho_0))|_{t=0}\,. \tag{63}$$

Since our test functions satisfy $\int \hat{\varphi}\, d\rho_0 = 0$, we can ignore scalar multiples of ρ_0, and it only remains to show (56), (58), and (60) with

$$(\mathrm{id} - \widehat{\Pi}_0)(\mathcal{D}) = -X'\rho_0 - X\rho_{reg}'\,. \tag{64}$$

Step 2: Analysing the derivative of $t \mapsto \mathcal{P}_t(\rho_0)$.

In this step, we prove (60) and (64). By definition, for any $\varphi \in \mathcal{B}_0$

$$\mathcal{P}_t(\varphi) = (\mathcal{L}_{1,t}(\varphi_{sal} \circ h_t^{-1} + \varphi_{reg}))_{sal} \circ h_t + (\mathcal{L}_{1,t}(\varphi_{sal} \circ h_t^{-1} + \varphi_{reg}))_{reg}\,. \tag{65}$$

From now on, we assume that the postcritical orbit is infinite, to fix ideas. (The case of finite postcritical orbit is similar.) Recall (35). Noting that $c_k > c$ if and only if $c_{k,t} = f_t^k(c) > c$, and writing $\varphi_{sal} = \sum_k u_k H_{c_k}$, the contribution to $\mathcal{P}_t(\varphi) - \mathcal{P}_0(\varphi)$ from the first term in the right-hand-side of (65), i.e., $(\mathcal{P}_t(\varphi))_{sal} - \mathcal{P}_0(\varphi)_{sal}$, is just

$$\begin{aligned} &\sum_{k=2}^{N_f} u_{k-1}\left(\frac{1}{f_t'(c_{k-1,t})} - \frac{1}{f'(c_{k-1})}\right)H_{c_k} \\ &\qquad + (\varphi_{reg}(c) + \sum_{c_k > c} u_k)\left(\frac{1}{f_t'(c_-)} - \frac{1}{f'(c_-)} - \frac{1}{f_t'(c_+)} + \frac{1}{f'(c_+)}\right)H_{c_1}\,. \end{aligned} \tag{66}$$

Next, we find by (37) that the derivative of the second term $((\mathcal{P}_t(\varphi))_{reg} - \mathcal{P}_0(\varphi)_{reg})$ of (65), which is an atomless measure, coincides with

$$\begin{aligned} &(\mathcal{L}_{1,t}(\varphi_{reg}))'|_{(a,c_{1,t})} - (\mathcal{L}_1(\varphi_{reg}))'|_{(a,c_1)} \\ &\quad + \sum_{k=2, c_{k-1} > c}^{N_f} u_{k-1}\big((\mathcal{L}_{1,t}(H_{c_{k-1,t}}))'|_{(c_{k,t},c_{1,t})} - (\mathcal{L}_1(H_{c_{k-1}}))'|_{(c_k,c_1)}\big) \\ &\quad + \sum_{k=2}^{N_f} u_{k-1}\big((\mathcal{L}_{1,t}(H_{c_{k-1,t}}))'|_{(a,c_{k,t})} - (\mathcal{L}_1(H_{c_{k-1}}))'|_{(a,c_k)}\big)\,. \end{aligned} \tag{67}$$

Put $\varphi = \rho_0$, and consider first (66). Note that $c_{k,t} = h_t(c_k)$. Write

$$\frac{1}{f_t'(h_t(w))} - \frac{1}{f'(w)} = \frac{f'(w) - f_t'(h_t(w))}{f_t'(h_t(w))f'(w)}\,,$$

and decompose $f'(w) - f_t'(h_t(w)) = f'(w) - f_t'(w) + f_t'(w) - f_t'(h_t(w))$, with $f'(w) - f_t'(w) = -tX'(f(w))f'(w) + O(t^2)$, and $f_t'(w) - f_t'(h_t(w)) = -tf_t''(w)\alpha(w) + O(t^2)$.

Thus, we find, by using $(\mathcal{L}_1(\rho))_{sal} = \rho_{sal}$ and (11), that

$$\lim_{t\to 0}\frac{(\mathcal{P}_t(\rho_0))_{sal} - (\rho_0)_{sal}}{t} = -\sum_{k=1}^{N_f} X'(c_k)s_k H_{c_k} - \sum_{k=2}^{N_f}\frac{\alpha(c_{k-1})s_{k-1}f''(c_{k-1})}{(f'(c_{k-1}))^2}H_{c_k}$$

$$= -\sum_{k=1}^{N_f} X'(c_k)s_k H_{c_k} + \sum_{k=2}^{N_f}\frac{(X(c_k)-\alpha(c_k))s_{k-1}f''(c_{k-1})}{(f'(c_{k-1}))^3}H_{c_k}$$

$$= -(X'\rho)_{sal} + \sum_{k=1}^{N_f}(X(c_k)-\alpha(c_k))E_k H_{c_k}\,, \tag{68}$$

where we used $X(c_1)=\alpha(c_1)$ with (the choice of E_1 will become clear later on)

$$E_k = \frac{s_{k-1}f''(c_{k-1})}{(f'(c_{k-1}))^3}\,,\ k\ge 2\,, \tag{69}$$

$$E_1 = \left(-\frac{\rho_{reg}(c)f''(c_-)}{(f'(c_-))^3} + \frac{\rho_{reg}(c)f''(c_+)}{(f'(c_+))^3}\right) + \sum_{k\ge 2, c_{k-1}>c} s_{k-1}\left(\frac{f''(c_-)}{(f'(c_-))^3} - \frac{f''(c_+)}{(f'(c_+))^3}\right).$$

It will turn out essential to study $((\rho_{reg})')_{sal} = \sum_{k=1}^{N_k} s'_k H_{c_k}$. If $x\in[a,c_1)$ is not along the critical orbit we have

$$(\rho_{reg})'(x) = (\rho_0)'(x) = (\mathcal{L}_1(\rho_0))'(x) = \sum_{f(y)=x}\frac{(\rho_{reg})'(y)}{|f'(y)|f'(y)} - \frac{\rho_0(y)f''(y)}{|f'(y)|(f'(y))^2}\,. \tag{70}$$

(We used $(\rho_{reg})'(y) = (\rho_0)'(y)$ if y is not along the postcritical orbit.) Taking the difference between $(\rho_{reg})'(x)$ for $x\uparrow c_k$ and $x\downarrow c_k$, and recalling E_k from (69), we easily get from the previous identity that [11]

$$s'_k = E'_k - E_k\,,\ \text{with } E'_k = \frac{s'_{k-1}}{(f'(c_{k-1})^2)}\,,\ k\ge 2\,,\ E'_1 = -\frac{(\rho_{reg})'(c)}{(f'(c_-))^2} + \frac{(\rho_{reg})'(c)}{(f'(c_+))^2}\,. \tag{71}$$

We now consider $\lim_{t\to 0}\frac{1}{t}((\mathcal{P}_t(\rho_0))_{reg} - (\rho_0)_{reg})'$. We get two sorts of contributions to (67): For

$$x\in[\min(c_k,c_{k,t}),\max(c_k,c_{k,t})] \text{ or } x\in[\min(c_k,f_t(c_{k-1})),\max(c_k,f_t(c_{k-1}))]\,, \tag{72}$$

an atom may appear at c_k in the limit, we call such x singular points. For the other values of x, which we call the regular points, the limit will be a function. Recalling (69) and (71), we claim that the contribution of the singular points to $\lim_{t\to 0}((\mathcal{P}_t(\rho_0))_{reg} - (\rho_0)_{reg})|_{(a,b)})'/t$ is

$$\sum_{k=1}^{N_f}(\alpha(c_k)E_k - X(c_k)E'_k)\delta_{c_k}\,. \tag{73}$$

Indeed, if $k\ge 2$ and $c_{k,t}<c_k$ and $c_{k-1}<c$, we must consider the Radon measure

$$\varphi\mapsto -\frac{s_{k-1}}{t}\int_{c_{k,t}}^{c_k}\frac{f''(\psi_-(x))}{(f'(\psi_-(x)))^3}\varphi(x)\,dx = \alpha(c_k)s_{k-1}\frac{f''(c_{k-1})}{(f'(c_{k-1}))^3}\varphi(c_k) + O(t)\,,$$

[11] If c is periodic then $(\rho_{reg})'(c)$ may be undefined, but $(\rho_{reg})'(c_\pm)$ are both defined.

coming from $-(\mathcal{L}_1(H_{c_{k-1}}))'$ (we used $h_t(c_k) = c_{k,t}$). If $k \geq 2$, $c_{k,t} < c_k$, and $c_{k-1} > c$, we must consider the Radon measure

$$\varphi \mapsto -\frac{s_{k-1}}{t}\int_{c_{k,t}}^{c_k} \frac{f_t''(\psi_{t,+}(x))}{(f_t'(\psi_{t,+}(x)))^3}\varphi(x)\,dx = \alpha(c_k)s_{k-1}\frac{f''(c_{k-1})}{(f'(c_{k-1}))^3}\varphi(c_k) + O(t),$$

from $(\mathcal{L}_{1,t}(H_{c_{k-1,t}}))' - (\mathcal{L}_1(H_{c_{k-1}}))'$ (the corresponding term for the branches ψ_- and $\psi_{t,-}$ vanishes in the limit). For $k = 1$ and $c_{1,t} < c_1$ we must consider the three contributions given by, firstly,

$$\varphi \mapsto -\frac{1}{t}\int_{c_{1,t}}^{c_1} \frac{(\rho_{reg})'(\psi_-(x))}{(f'(\psi_-(x)))^2}\varphi(x)\,dx = \alpha(c_1)\frac{(\rho_{reg})'(c)}{(f'(c_-))^2}\varphi(c_1) + O(t)\,,$$

(recall also that $c_{1,t} = h_t(c_1)$ and $\alpha(c_1) = X(c_1)$), secondly,

$$\varphi \mapsto \frac{1}{t}\int_{c_{1,t}}^{c_1} \frac{\rho_{reg}(\psi_-(x))f''(\psi_-(x))}{(f'(\psi_-(x)))^3}\varphi(x)\,dx = \alpha(c_1)\frac{-\rho_{reg}(c)f''(c_-)}{(f'(c_-))^3}\varphi(c_1) + O(t)\,,$$

and thirdly, by the sum over those $j \geq 2$ so that $c_{j-1} > c$ of

$$\varphi \mapsto -\frac{s_{j-1}}{t}\int_{c_{1,t}}^{c_1} \frac{f''(\psi_-(x))}{(f'(\psi_-(x)))^3}\varphi(x)\,dx = \alpha(c_1)s_{j-1}\frac{f''(c_-)}{(f'(c_-))^3}\varphi(c_1) + O(t)\,,$$

as well as the corresponding three contributions for ψ_+. The cases $c_{k,t} > c_k$ are similar. For $k \geq 2$, we must also deal with the jump terms from $(\mathcal{L}_{1,t}(\rho_{reg}))' - (\mathcal{L}_1(\rho_{reg}))'$ (one at $f_t(c_{k-1})$ the other at c_k), which give, using $f_t(c_{k-1}) - f(c_{k-1}) = tX(c_k) + O(t^2)$:

$$\varphi \mapsto \frac{1}{t}\int_{f_t(c_{k-1})}^{c_k} \frac{s'_{k-1}}{(f'(c_{k-1}))^2}\varphi(x)\,dx = -X(c_k)\frac{s'_{k-1}}{(f'(c_{k-1}))^2}\varphi(c_k) + O(t)\,.$$

We move to the regular points: For small t, let $k_t \geq 2$ be so that $\sum_{k\geq k_t}|s_{k-1}| \leq t^2$ (clearly, $k_t = O(\ln|t|)$), and take I_t to be the union of the $O(k_t)$ intervals of singular points associated to $k \leq k_t$ via (72) (in particular, the Lebesgue measure of I_t is an $O(t\ln|t|)$). We have by definition

$$\|(\mathcal{P}_t(\rho_0))_{reg} - (\rho_0)_{reg} - (\mathcal{L}_{1,t}(\rho_0) - \mathcal{L}_1(\rho_0))_{reg}\|_{\mathcal{B}_0(I\setminus I_t)} = O(t^2)\,, \tag{74}$$

where $\|\phi_{reg}\|_{\mathcal{B}_0(I\setminus I_t)}$ is the norm of Radon measure $(\phi_{reg})'$ on the metric set $I \setminus I_t$. (For this, we use that $\sum_{k\geq k_t}|s_{k-1}|\|\mathcal{L}_{1,t}(H_{c_{k-1,t}}) - \mathcal{L}_{1,t}(H_{c_{k-1}})\|_{\mathcal{B}_0} = O(t^2)$, and that $\mathcal{L}_{1,t}(H_{c_{k-1,t}})(x) - \mathcal{L}_{1,t}(H_{c_{k-1}})(x) = 0$ for $k \leq k_t$ and $x \notin I_t$.) The contribution (73) takes care of $\|(\mathcal{P}_t(\rho_0))_{reg} - (\rho_0)_{reg}\|_{\mathcal{B}_0(I_t)}$ (note that $\sum_{k\geq k_t}|\alpha(c_k)E_k| + |X(c_k)E'_k| = O(t^2)$) so that we may concentrate on $(\mathcal{L}_{1,t}(\rho_0) - \mathcal{L}_1(\rho_0))_{reg}$ on $I \setminus I_t$.

Note that

$$f^{-1}(x) - f_t^{-1}(x) = t\frac{X(x)}{f'(f^{-1}(x))} + O(t^2)\,, \tag{75}$$

where we choose the same inverse branch for f_t and f. It follows that

$$\begin{aligned}\frac{\varphi(f_t^{-1}(x))}{|f_t'(f_t^{-1}(x))|} - \frac{\varphi(f^{-1}(x))}{|f'(f^{-1}(x))|} &= -tX'(x)\frac{\varphi(f^{-1}(x))}{|f'(f^{-1}(x))|}\\ &- tX(x)\left(\frac{\varphi'(f^{-1}(x))}{f'(f^{-1}(x))|f'(f^{-1}(x))|} + \frac{\varphi(f^{-1}(x))f''(f^{-1}(x))}{(f'(f^{-1}(x)))^2|f'(f^{-1}(x))|}\right) + O(t^2)\,,\end{aligned}$$

if φ is C^{1+Lip} at $f^{-1}(x)$, which gives, after summing over the two inverse branches,

$$-tX'(x)\mathcal{L}_1(\varphi)(x) - tX(x)(\mathcal{L}_1(\varphi))'(x) + O(t^2)\,. \tag{76}$$

Therefore, if $x \notin I_t$, and $x \neq c_k$ and $x \neq c_{k,t}$ for all $k \geq 1$, we have, decomposing $\rho_0 = \rho_{reg} + \sum_k s_k H_{c_k}$,

$$(\mathcal{L}_{1,t}(\rho_0) - \mathcal{L}_1(\rho_0))_{reg}(x) = -t(X'\rho_0 - X(\rho_0)')_{reg}(x) + O(t^2)$$

(77) $$= -t(X'\rho_0)_{reg}(x) - t(X(\rho_{reg})')_{reg}(x) + O(t^2)\,.$$

(The $O(t^2)$ term is in $\mathcal{B}_0$, not $\mathcal{B}_0^{Lip}$.) By continuity, (77) holds for all $x \notin I_t$.

The regular contribution to $\lim_{t\to 0} \big((\mathcal{P}_t(\rho_0))_{reg} - (\rho_0)_{reg}\big)/t$ is thus

(78) $$-\big(X'\rho_0 - (X'\rho_0)_{sal}\big) - \big(X(\rho_{reg})' - (X(\rho_{reg})')_{sal}\big)\,.$$

All together, we find from (68–73–78) and (71) (differentiating in $\mathcal{B}_0$)

$$\partial_t(\mathcal{P}_t(\rho_0))|_{t=0} = -X'\rho_{sal} - X'\rho_{reg} - X(\rho_{reg})' \in \mathcal{B}_0^{Lip}\,.$$

This establishes (60) and (64) (note that $\int X'\rho_{sal} + (X\rho_{reg})'\,dx = 0$).

Step 3: Proving the weak norm bounds necessary for [16]**.**

It remains to prove the bounds (56) and (58) for $\mathcal{P}_t(\varphi) - \mathcal{P}_0(\varphi)$. We start with (56). For the term corresponding to (66), since φ is not necessarily a fixed point of $\mathcal{L}_1$, we get in addition to (68) a term

$$(|\varphi_{reg}(c)| + \sum_{c_k > c} |u_k|)O(t) = O(t)|\varphi|_{weak,0}\,.$$

Next, consider (67). For the $L^1(Leb)$ norm of $(\mathcal{P}_t - \mathcal{P})_{reg}$, the singular contributions produce an $O(t \ln |t|)$ term: Indeed, by (38), up to an error $O(t)$ we may restrict to a finite set of c_ks, where the cardinality of this finite set is an $O(\ln |t|)$; for this finite set, the total Lebesgue measure of the intervals of singular points is an $O(t \ln |t|)$. For the regular contributions, although $\mathcal{L}_1(\varphi)$ is not equal to φ in general, and φ_{reg} is only continuous and of bounded variation, we get an $O(t)\|\varphi\|_{\mathcal{B}_0}$ contribution to the $L^1(Leb)$ norm of $(\mathcal{P}_t - \mathcal{P})_{reg}$: Indeed, the only delicate terms are of the form

$$\int h(y)(\varphi_{reg}(\psi_{+,t}(y)) - \varphi_{reg}(\psi_+(y)))\,dy\,,$$

with $|h| \leq \|f\|_{C^{1+Lip}}$, and similarly with ψ_-. Now we exploit that if $\phi \in BV$ and Ψ_t is C^2 with $|\Psi_t(x) - x| \leq C|t|$ and $|\Psi_t'(x) - 1| \leq C|t|$ then (use [15, Lemma 11] as in [15, Lemma 13])

$$\int |\phi(y) - \phi(\Psi_t(y))|\,dy = O(t)\|\phi\|_{BV}\,.$$

We must still bound $|\mathcal{P}_t(\varphi)_{reg}(y) - \mathcal{P}_0(\varphi)_{reg}(y)|$ for $y \in \mathcal{S}_j = \cup_{0\leq \ell \leq j} f^{-\ell}(c)$. We make no distinction between regular and singular points here. The contribution corresponding to differences between derivatives of f of f_t gives $O(t)$. Next, φ_{reg} is continuous by definition of $\mathcal{B}_0$. Writing $\tilde{\delta}_j(\cdot)$ for its worse modulus of continuity on the finite set $\mathcal{S}_j$, we get since $|c_k - c_{k,t}| = O(t)$ that

$$\sup_{y \in \mathcal{S}_j} |\mathcal{P}_t(\varphi)_{reg}(y) - \mathcal{P}_0(\varphi)_{reg}(y)| = O(\tilde{\delta}_j(t) + |t|)\,.$$

Finally, (58) can be proved by using the Lipschitz assumption on φ_{reg}, to simplify the argument for (56): The uniform modulus of continuity $\delta(t) = O(t)$ of φ_{reg} allows us to deal with the L^∞ norm in $|\cdot|_{weak,\infty}$. □

6. The derivative in terms of the infinitesimal conjugacy α

Let f_t be a $C^{2,2}$ perturbation tangent to the topological class of a mixing piecewise expanding C^2 unimodal map. We do not know whether $x \mapsto h_t(x)$ is quasisymmetric, as in the smooth expanding case. Note however that in general it is *not* absolutely continuous (see [18] for the nonuniformly expanding case). For similar reasons, $\alpha = \partial_t h_t|_{t=0}$ is in general not absolutely continuous. In this section, we shall see that absolute continuity of α is equivalent to a remarkable formula for $\Psi_1 = \mathcal{R}'(0)$ which can be "guessed" from the following easy lemma:

Lemma 6.1. *Assume that f_t is a $C^{2,2}$ perturbation tangent to the topological class of a piecewise expanding C^2 unimodal map f, with infinitesimal perturbation $v = X \circ f$. Then recalling $\alpha = \partial_t h_t|_{t=0}$ from Corollary 2.6, we have*

$$(\mathrm{id} - \mathcal{L}_0)(\alpha\rho_0) = X\rho_0 \,, \tag{79}$$

and $\sum_{k=0}^{n} \mathcal{L}_0^k(X\rho_0) = \alpha\rho_0 - \mathcal{L}_0^{n+1}(\alpha\rho_0)$.

The lemma gives that the partial sum of order n for the series $\Psi(z)$ at $z = 1$ is

$$\sum_{k=0}^{n} \int \mathcal{L}_0^k(X\rho_0)\varphi' \, dx = \int \varphi'\alpha\rho_0 - \int \varphi'\mathcal{L}_0^{n+1}(\alpha\rho_0)\, dx \,.$$

We do not claim that $\int \varphi'\mathcal{L}_0^{n+1}(\alpha\rho_0)\, dx$ converges as $n \to \infty$.

Proof. We know that $X(y) = \alpha(y) - f'(\psi(y))\alpha(\psi(y))$ where ψ is an arbitrary inverse branch of f. Multiply this by the positive number $\rho_0(\psi(y))/|f'(\psi(y))|$ and sum over inverse branches. Since ρ_0 is the invariant density, the sum of these positive numbers is $\rho_0(y)$, which gives the first claim. A telescopic sum gives the second claim. □

Theorem 6.2. *Assume that f_t is a $C^{2,3}$ perturbation tangent to the topological class of a mixing piecewise expanding C^3 unimodal map f with infinitesimal perturbation $v = X \circ f$ (in particular $\mathcal{J}(f, X) = 0$) so that $X \in C^2(f(I))$. If $\alpha = \partial_t h_t|_{t=0}$ is absolutely continuous then*

$$\Psi_1 = \int \varphi'\alpha\rho_0 \, dx \,, \quad \forall \varphi \in C^1([a, b]) \,. \tag{80}$$

Conversely, if (80) holds then $\alpha \in BV^{(1)}$ (in particular, α is absolutely continuous).

Theorem 6.2 will easily imply:

Corollary 6.3 (Derivative of the TCE)**.** *Under the assumptions of Theorem 6.2, if α is absolutely continuous, then*

$$(-\mathrm{id} + \mathcal{L}_1)(\alpha'\rho_0 + \alpha(\rho_{reg})') = X'\rho_0 + X(\rho_{reg})' \,. \tag{81}$$

Note that the proofs of Theorem 6.2 and Corollary 6.3 use the results from [4] (in particular Lemma 4.1, Prop. 4.4 there), Proposition 2.4, and the easy Lemma 6.1 but do not require any information from Sections 3, 4 or 5 of the present paper.

Proof of Corollary 6.3. Putting together (80) and (46) we get

$$\begin{aligned}\Psi_1 + \int \alpha\varphi(\rho_{sal})' &= \int \alpha\varphi'\rho_0 \, dx + \int \alpha\varphi(\rho_{sal})' \\ &= \int (\mathrm{id} - \mathcal{L}_1)^{-1}(X'\rho_{sal} + (X\rho_{reg})')\varphi \, dx \,.\end{aligned}$$

And, since the boundary term in the integration by parts vanishes,

$$\int \alpha\varphi'\rho_0\,dx + \int \alpha\varphi(\rho_{sal})' = \int \alpha\varphi(-\rho_0' + (\rho_{sal})') - \int \alpha'\varphi\rho_0\,dx$$
$$= -\int \alpha\varphi(\rho_{reg})'\,dx - \int \alpha'\varphi\rho_0\,dx\,.$$

□

Proof of Theorem 6.2. We suppose that c is neither periodic nor preperiodic (the other cases are easier). Recall that α is continuous by Corollary 2.6. Lemma 4.4 allows us to write Ψ_1 as

$$\Psi_1 = -\int \varphi\beta'\,, \tag{82}$$

where β' is a Stieltjes measure. In fact,

$$\beta' = \alpha(\rho_{sal})' + (\mathrm{id} - \mathcal{L}_1)^{-1}(X'\rho_{sal} + (X\rho_{reg})')\,dx\,.$$

The above implies that β' is the sum of an absolutely continuous measure with density of bounded variation, and a weighted sum of diracs along the postcritical orbit. Now by [4, Lemma 4.1], we know that $(\mathrm{id} - f_*)(\alpha\rho'_{sal}) = X\rho'_{sal}$. Thus

$$(\mathrm{id} - f_*)(\beta') = X(\rho_{sal})' + X'\rho_{sal} + (X\rho_{reg})' = (X\rho_0)'\,. \tag{83}$$

Integrating (82) by parts, we get (there are no boundary terms, see e.g. [4, Proof of Prop. 4.4, Theorem 5.1]),

$$\Psi_1 = \int \varphi'(x)B(x)\,dx\,,$$

where B is a function of bounded variation, supported in $[a, b]$, satisfying $B' = \beta'$. In particular, B is the sum of an element B_1 of $BV^{(1)}$ with a function with prescribed jumps along the postcritical orbit. It is easy to check that this function is in fact just the saltus of $\alpha\rho_{sal}$ (or, equivalently, the saltus of $\alpha\rho_0$). By (83) (and the fact that both $B(x)$ and $\rho_0(x)$ vanish for $x \ge b$) we get that

$$(\mathrm{id} - \mathcal{L}_0)B = X\rho_0\,. \tag{84}$$

Now, Lemma 6.1 implies that

$$(\mathrm{id} - \mathcal{L}_0)(\alpha\rho_0) = X\rho_0\,. \tag{85}$$

Putting together (84–85) and $B = B_1 + (\alpha\rho_0)_{sal}$, we get that

$$(\mathrm{id} - \mathcal{L}_0)(B_1 - (\alpha\rho_0)_{reg}) = 0\,. \tag{86}$$

After these preliminaries, we move on to the proof.

If α is absolutely continuous then $(\alpha\rho_0)_{reg}$ is absolutely continuous (because $\alpha \in BV \cap C^0$ and $((\alpha\rho_0)_{reg})' = \alpha'\rho_0 + \alpha(\rho_{reg})'$ is in $L^1(Leb)$). B_1 is absolutely continuous because it is in $BV^{(1)}$. The operator $\mathcal{L}_1$ acting on $L^1(Leb)$ has ρ_0 as unique fixed point, and thus $\mathcal{L}_0$ on the Banach space of absolutely continuous functions supported in $(-\infty, b]$ has $R_0(x) = -1 + \int_{-\infty}^x \rho_0(y)\,dy$ as unique fixed point. Thus (86) implies that $B_1 = (\alpha\rho_0)_{reg} + \kappa R_0$, so that $B = \alpha\rho_0 + \kappa R_0$. Since $B(x) = \alpha(x)\rho_0(x) = 0$ for $x \le a$ (use that $\int(X'\rho_{sal} + (X\rho_{reg})')dx = 0$ by $\mathcal{J}(f, X) = 0$), we have that $\kappa = 0$, proving (80).

We next prove the converse. If (80) holds then $B = \alpha\rho_0 = \alpha\rho_{sal} + \alpha\rho_{reg}$ is in BV by the preliminary remarks. Since ρ_0 is bounded from below on $[c_2, c_1]$, this implies that $\alpha|_{[c_2,c_1]}$ is in BV. The preliminaries also give $B-(\alpha\rho_0)_{sal} = (\alpha\rho_0)_{reg} \in BV^{(1)}$, i.e., $\alpha'\rho_0 + \alpha(\rho_{reg})' \in BV$, which implies that $\alpha'\rho_0 \in BV$ (since $\alpha \in BV$). Using again $\inf_{[c_2,c_1]} \rho_0 > 0$ we get that $\alpha' \in BV$, i.e., $\alpha \in BV^{(1)}$. □

7. Necessity of the horizontality condition

There exist examples of perturbations f_t of good mixing piecewise expanding C^∞ unimodal maps f with c preperiodic, $v = X \circ f$ and $\mathcal{J}(f, X) \neq 0$ so that $\mathcal{R}(t)$ is not Lipschitz for some $\varphi \in C^\infty([a, b])$ ([4, §6] and [19], see also [4, Remark 6.3]). Theorem 7.1 below shows the lack of Lipschitz regularity of $\mathcal{R}(t)$ for all perturbations f_t so that the infinitesimal deformation is not horizontal (we require that c be nonperiodic and, if c recurrs to itself, $f'(c_-) = -f'(c_+)$). The proof of Theorem 7.1 hinges on a careful rereading of the proof of Theorem 5.1.

Theorem 7.1. *Let f_t be a $C^{2,3}$ perturbation of a mixing piecewise expanding C^3 unimodal map f with infinitesimal deformation $v = X \circ f$ such that $X \in C^2(f(I))$ but v is not horizontal for $f_0 = f$, and assume that c is not periodic for f. If*

$$\gamma = \inf d(f^j(c), c) = 0,$$

we assume in addition that $\lim_{x\to c, x<c} f'(x) = -\lim_{x\to c, x>c} f'(x)$.

If the postcritical orbit of f_0 is not dense in $[c_2, c_1]$ then there exist $\varphi \in C^\infty(I)$ and $K > 0$, so that, for any sequence $t_n \to 0$ so that the postcritcal orbit of each f_{t_n} is infinite, there is $n_0 \geq 1$ so that

$$|\int \varphi\, \rho_{t_n}\, dx - \int \varphi\, \rho_0\, dx| \geq K|t_n||\ln|t_n||\,, \quad \forall n \geq n_0\,.$$

If the postcritical orbit of f_0 is infinite but not dense, the above holds for any sequence $t_n \to 0$ with c not periodic under f_{t_n}.

If the postcritical orbit of f_0 is dense in $[c_2, c_1]$ then there exists $\varphi \in C^\infty(I)$ so that for any sequence $t_n \to 0$ so that c not periodic under f_{t_n}, we have

$$\lim_{n\to\infty} |t_n^{-1}(\int \varphi\, \rho_{t_n}\, dx - \int \varphi\, \rho_0\, dx)| \to \infty\,.$$

We expect that if c is periodic, but $f = f_0$ is good and $\lim_{x\to c, x<c} f'(x) = -\lim_{x\to c, x>c} f'(x)$, v is not horizontal, then there exists $\varphi \in C^\infty(I)$ so that the function $\int \varphi\rho_t\, dx$ is not Lipschitz at $t = 0$.

Existence of sequences t_n as in Theorem 7.1 is guaranteed by the following easy lemma:

Lemma 7.2. *Let f_t be a $C^{2,2}$ perturbation of a mixing piecewise expanding C^2 unimodal map f with infinitesimal deformation v. If v is not horizontal for f_0 then there is a sequence $t_n \to 0$ so that c has an infinite forward orbit for each f_{t_n}.*

Proof of Lemma 7.2. First note that the assumption that v is not horizontal implies that there exists $k_0 \geq 1$ so that $\partial_t c_{k_0,t}|_{t=0} \neq 0$. Indeed, assume for a contradiction that $\partial_t c_{k,t}|_{t=0} = 0$ for all $k \geq 1$. Then $\partial_t c_{1,t}|_{t=0} = 0$ implies $v(c) = 0$, and, using $v(c_k) = \partial_t c_{k+1,t}|_{t=0} - f'(c_k)v(c_{k-1})$ for $k \geq 1$, we prove inductively that $v(c_k) = 0$ for all $k \geq 1$, which would imply that v is horizontal, a contradiction.

Let $\Sigma(t)$ be the symbolic critical itinerary for f_t, that is, $(\Sigma_1(t), \Sigma_2(t), \dots) \in \{L, C, R\}^{\mathbb{N}}$, with $\Sigma_j(t) = L$ if $f_t^j(c) < c$, $\Sigma_j(t) = C$ if $f_t^j(c) = c$, and $\Sigma_j(t) = R$ if $f_t^j(c) > c$. Put $\Theta(\Sigma, k_0) = \cap_{n\geq k_0} (f_t^n)^{-1}(I_{\Sigma_n})$, with $I_L = [a, c)$, $I_R = (c, b]$, $I_C = \{c\}$. The map $t \mapsto \Theta(\Sigma(t), k_0)$ is continuous from $(-\epsilon, \epsilon)$ to $\mathbb{R}$. It is easy to see that $\Theta(\Sigma(t), k_0) = c_{k_0,t}$, so that $\Theta(\Sigma(t), k_0)$ is not constant, and that is enough to end the proof. □

Proof of Theorem 7.1. The key property that we shall use is that, for each fixed $k \geq 1$, the limit

$$\beta_k := \lim_{t\to 0} \frac{c_{k,t} - c_k}{t} \tag{87}$$

exists and satisfies the twisted cohomological equation

$$X(c_{k+1}) = \beta_{k+1} - f'(c_k)\beta_k \,. \tag{88}$$

By definition, $\beta_1 = X(c_1)$, so that

$$\beta_k = \sum_{j=0}^{k-1} X(c_{k-j})(f^j)'(c_{k-j}). \tag{89}$$

In particular, if $\mathcal{J}(f, X) \neq 0$, i.e. if $\alpha_{(0)}(c_1) \neq X(c_1)$ (recall Lemma 4.2), we have $\beta_1 \neq \alpha_{(0)}(c_1)$. We shall next have to be a little more careful about the limiting process (87), and distinguish between the cases where γ is zero or strictly positive.

Note that

$$\beta_k \leq |(f^{k-1})'(c_1)| \sup |X|(1-\lambda)^{-1} \,,$$

(recall (4) for the definition of λ) and put

$$Y := \max\{\sup_t \left|\frac{f_t - f_0}{t} - v\right|_{L^\infty}, \sup_{x\neq c} |f''(x)|, \frac{\sup|X|}{1-\lambda^{-1}}, 1\} \,.$$

If $\gamma > 0$, for fixed t, we let $M(t) \in \mathbb{Z}$ be the largest integer so that

$$6Y^3|t||(f^M)'(c_1)| < \gamma/2 \,. \tag{90}$$

If $M(t) \geq 1$, it is not difficult to show inductively that for all $k \leq M(t)$ we have $d(c_{k,t}, c_k) < \gamma/2$ and

$$\left|\frac{c_{k,t} - c_k}{t} - \beta_k\right| \leq |t| 6Y^2 |(f^k)'(c_1)| \,. \tag{91}$$

Indeed, define $B_{k,t}$ by

$$tB_{k,t}(f^k)'(c_1) = (c_{k,t} - c_k)/t - \beta_k \,,$$

and use (88) to see that

$$\begin{aligned}\frac{c_{k+1,t} - c_{k+1}}{t} - \beta_{k+1} = v'(\tilde{w}_{k,t})(c_{k,t} - c_k) + tg_t(c_{k,t}) + tf'(c_k)B_{k,t} \\ + tf''(w_{k,t})(\beta_k + tB_{k,t})^2 \,,\end{aligned}$$

where $g_t = (f_t - f_0)/t - v$ and $w_{k,t}$ and $\tilde{w}_{k,t}$ are between c_k and $c_{k,t}$. Then it is easy to see that $\sup_{k,t} |B_{k,t}| \leq 6Y^2$ for $k \leq M(t)$ if $M(t) \geq 1$.

If $\gamma = 0$, we let $M(t) \in \mathbb{Z}$ be the largest integer so that

$$6Y^3|t||(f^M)'(c_1)| < 1 \,. \tag{92}$$

If $M(t) \geq 1$, our assumption that $\lim_{x\to c, x<c} f'(x) = \lim_{x\to c, x>c} f'(x)$ implies that (91) holds for all $k \leq M(t)$.

We next revisit the construction from Subsection 3.3 in order to allow comparison between different nonperiodic dynamics. For $\eta > 0$, consider the Banach space

$(\widehat{\mathcal{B}}_\infty, \|\cdot\|)$ of pairs $\phi = (\phi_{reg}, \phi_{sal})$ with ϕ_{reg} continuous and of bounded variation, and $\phi_{sal} = (u_k)_{k=1,\ldots,\infty}$, normed by

$$\|\phi\| = \|\phi_{reg}\|_{BV} + |\phi_{sal}|_\eta \text{ with } |\phi_{sal}|_\eta = \sup_{1\le k\le\infty} (1+\eta)^k |u_k| \,, \tag{93}$$

and so that, in addition, $\phi_{reg}(x) = \sum_{k=1}^\infty u_k$ for all $x < a$. Recall the space $\widehat{\mathcal{B}}_t$ associated to f_t in Subsection 3.3. If the postcritical orbit of f_t is infinite then $\widehat{\mathcal{B}}_t = \widehat{\mathcal{B}}_\infty$, and we set $\mathcal{E}_t = \mathcal{F}_t$ to be the identity on $\widehat{\mathcal{B}}_\infty$. If the orbit of c is finite (but not periodic) for f_t, letting $n_{0,t}$ and $n_{1,t}$ be minimal so that $c_{n_{0,t},t}$ is periodic of prime period $n_{1,t}$, we introduce $\mathcal{E}_t : \widehat{\mathcal{B}}_t \to \widehat{\mathcal{B}}_\infty$, which maps a finite vector $(w_j, 1 \le j \le n_{0,t} + n_{1,t} - 1)$ to an infinite vector v_ℓ according to

$$\begin{aligned} v_\ell &= w_\ell \,, \qquad \ell \le n_{0,t} - 1 \,, \\ v_{n_{0,t}+j+\ell n_{1,t}} &= w_{n_{0,t}+j} \big((f_t^{n_{1,t}})'(c_{n_{0,t}+j,t})\big)^\ell \big(1 - \big((f_t^{n_{1,t}})'(c_{n_{0,t}+j,t})\big)^{-1}\big) \,, \\ & \qquad 0 \le j \le n_{1,t} - 1 \,, \ell \ge 0 \,, \end{aligned}$$

and $\mathcal{F}_t : \widehat{\mathcal{B}}_\infty \to \widehat{\mathcal{B}}_t$ defined by

$$\begin{aligned} w_\ell &= v_\ell \,, \qquad \ell \le n_{0,t} - 1 \,, \\ w_{n_{0,t}+j} &= \sum_{\ell \ge 0} v_{n_{0,t}+j+\ell n_{1,t}} \,, \qquad\qquad 0 \le j \le n_{1,t} - 1 \,. \end{aligned}$$

It is not difficult to see that $\mathcal{E}_t$ and $\mathcal{F}_t$ are bounded, uniformly in small t, and that $\mathcal{F}_t \circ \mathcal{E}_t$ is the identity on $\widehat{\mathcal{B}}_t$.

This ends the preliminaries, and we now move on to the proof, considering $\varphi \in C^\infty(I)$ so that $\int \varphi \, d\rho_0 = 0$ (this does not restrict generality). □

Proof if the orbit of c is infinite but not dense. Assume that the closure of $\{f^j(c) \mid j \ge 0\}$ is an infinite set which does not coincide with $[c_2, c_1]$. Since the orbit of c is not dense in $[c_2, c_1]$, there exists a C^∞ function φ with $\int \varphi \, d\mu_0 = 0$ and $\varphi(c_j) = 1$ for all $j \ge 1$.

Since $\mathcal{J}(f, X) \ne 0$, Lemma 7.2 gives a sequence $t_n \to 0$ so that c is not periodic for f_{t_n}. For $t = 0$ or $t = t_n$ for some n, put

$$\mathcal{G}_t = \Gamma_0 \circ \mathcal{F}_0 \circ \mathcal{E}_t \circ \Gamma_t^{-1} : \mathcal{B}_t \to \mathcal{B}_0 \,, \widetilde{\mathcal{G}}_t = \Gamma_t \circ \mathcal{F}_t \circ \mathcal{E}_0 \circ \Gamma_0^{-1} : \mathcal{B}_0 \to \mathcal{B}_t \,, \tag{94}$$

(the above operators are bounded uniformly in t) and redefine $\mathcal{P}_t$ as

$$\mathcal{P}_t = \mathcal{G}_t \circ \mathcal{L}_{1,t} \circ \widetilde{\mathcal{G}}_t : \mathcal{B}_0 \to \mathcal{B}_0 \,.$$

Since $\mathcal{E}_0 = \mathcal{F}_0$ is the identity, we find $\widetilde{\mathcal{G}}_t \circ \mathcal{G}_t = \mathrm{id}$, and the spectral decomposition $\mathcal{L}_{1,t}^k(\varphi) = \rho_t \int \varphi \, dx + \mathcal{R}_t^k(\varphi)$, with $\|\mathcal{R}_t^k\|_{\mathcal{B}_t} \le C\tau^k$, gives a spectral decomposition

$$\mathcal{P}_t^k(\phi) = \mathcal{G}_t(\rho_t) \int \widetilde{\mathcal{G}}_t(\phi) \, dx + \mathcal{G}_t(\mathcal{R}_t^k(\widetilde{\mathcal{G}}_t(\phi))) \,.$$

Using this new definition of $\mathcal{P}_t$, we revisit the proof of Theorem 5.1, and we study

$$\rho_{tn} - \rho_0 = (\mathcal{G}_{t_n}(\rho_{t_n}) - \rho_0) + (\rho_{t_n} - \mathcal{G}_{t_n}(\rho_{t_n})) \,. \tag{95}$$

Assume first that $\gamma > 0$.

Let us consider the first term in the right hand side of (95). Step 1 of the proof of Theorem 5.1 until (62) uses the fact that f_t and f_0 are conjugate only (but essentially) to evaluate the second term of (95). Step 3 of the proof of Theorem 5.1 does not use the fact that f_0 and f_t are conjugate, so that (56) and (58) hold.

Step 2 of the proof of Theorem 5.1 appears to use the conjugacies h_t, but a careful look reveals that what is crucial there are properties (91) and (88) of β_k. More precisely, taking $M(t_n)$ from (90) and replacing in Step 2 the number $\alpha(c_k)$ by β_k, we use

$$\frac{1}{|(f^{M+1})'(c_1)|} < \frac{12Y^3|t_n|}{\gamma}$$

to handle the truncated terms for $\ell > M(t_n)$, and deduce that there is C depending only on f and on X so that

$$\|\mathcal{P}_{t_n}(\rho_0) - \rho_0\|_{\mathcal{B}_0} \le C|t_n|, \forall |t_n| < \delta, t_n \text{ not periodic.}$$

(Note that $C = O(\gamma^{-1})$.) The above considerations imply that there is $\widetilde{C} = O(\gamma^{-1})$ and $\delta > 0$ so that for all $|t_n| < \delta$ with c not periodic

$$|\mathcal{G}_{t_n}(\rho_{t_n}) - \rho_0|_{Radon} \le \widetilde{C}|t_n|\,.$$

Note that δ depends only on the constants in the Lasota-Yorke inequality, on λ, and on the spectral gap $\tau < 1$ of the transfer operator.

We now consider the first term in (95), that is

$$\sum_{k\ge 1} \frac{s_{1,t_n}}{(f_{t_n}^{k-1})'(c_1)} (H_{c_{k,t_n}} - H_{c_k})\,.$$

The terms for $k > M(t_n)$ give a contribution which is $\le \bar{C}|t_n|$ for $\bar{C} = O(\gamma^{-1})$, so that we may restrict to $k \le M(t_n)$.

Then for $k \ge 1$

$$\lim_{t_n \to 0} \frac{s_{1,t_n}}{(f_t^{k-1})'(c_1)} \int \varphi \frac{H_{c_{k,t}} - H_{c_k}}{t_n} dx = 0\,.$$

It is easy to see that there exists $N = N(f)$ so that $|s_1 \sum_{j=1}^k \frac{X(c_j)}{(f^{j-1})'(c_1)}| \ge \mathcal{J}(f, X)/2$ for all $k \ge N$. Note that N depends only on λ and $\sup|X|$. The properties of β_k give $|C_{k,n}| \le \widehat{C}$ and $|C_n| \le \widehat{C}$, uniformly in n and k, so that for all t_n small enough so that $M(t_n) > N$

$$\begin{aligned}
\left| \sum_{k \le M(t_n)} \frac{s_{1,t_n}}{(f_t^{k-1})'(c_1)} \int \varphi \frac{H_{c_{k,t_n}} - H_{c_k}}{t_n} dx \right| &= \left| \sum_{k\le M(t_n)} (\beta_k + C_{k,n}) \frac{s_1}{(f^{k-1})'(c_1)} \varphi(c_k) \right| \\
&= \left| C_n + s_1 \sum_{k \le M(t_n)} \varphi(c_k) \sum_{j=1}^k \frac{X(c_j)}{(f^{j-1})'(c_1)} \right| \\
&\ge (M(t_n) - N)|\frac{|\mathcal{J}(f,X)|}{2} - \widehat{C}\,.
\end{aligned} \tag{96}$$

Since $M(t_n) = \Theta(\ln(t_n))$, we have proved the theorem in the case where the postcritical orbit is infinite but $\gamma > 0$.

If the postcritical orbit is infinite and not dense, but $\gamma = 0$ then we should use definition (92) for $M(t)$. (We still have $M(t_n) = \Theta(\ln(t_n))$.) Our additional assumption then yields constants $\widetilde{C}$ and $\bar{C}$ independent of γ. $\square$

Proof if c is preperiodic for f. Assume that c_{n_0} ($n_0 > 0$ minimal for this property, note that then $n_0 \geq 2$) is periodic of prime period $n_1 \geq 1$, in particular $\gamma > 0$. Take a C^∞ observable with $\int \varphi \, d\mu_0 = 0$ and

$$\varphi(c_j) = 1, \forall j \geq 1 \, .$$

By Lemma 7.2, there is a sequence $t_n \to 0$ so that c has an infinite forward orbit under f_{t_n}. For $t = 0$ or $t = t_n$, recalling (94), consider $\mathcal{M}_t = \widetilde{\mathcal{G}}_t \circ \mathcal{L}_{1,0} \circ \mathcal{G}_t$ acting on $\mathcal{B}_t$. Since $\mathcal{G}_t \circ \widetilde{\mathcal{G}}_t = \mathrm{id}$, we have the spectral decomposition

$$\mathcal{M}_t^k = \widetilde{\mathcal{G}}_t(\rho_0) \int \mathcal{G}_t(\varphi) \, dx + \widetilde{\mathcal{G}}_t(\mathcal{R}_0^k(\mathcal{G}_t(\varphi))) \, .$$

We consider

$$\rho_{tn} - \rho_0 = (\rho_{t_n}) - \widetilde{\mathcal{G}}_t(\rho_0)) + (\widetilde{\mathcal{G}}_t(\rho_0) - \rho_0) \, . \tag{97}$$

Revisiting the proof of Theorem 5.1 once more, using $\mathcal{B}_t$ instead of $\mathcal{B}_0$, we can treat this case in a manner analogous to that of the infinite postcritical orbit with $\gamma > 0$. □

Proof if the orbit of c is dense. We have $\mathcal{E}_0 = \mathcal{F}_0 = \mathrm{id}$ and, using (94), we can consider $\mathcal{P}_t$ as in the case when the orbit is infinite but not dense. The new difficulty resides in the choice of the observable.

We recall ([6, Thm 8.1]) the following central limit theorem with speed for f and μ_0. If $\int \varphi d\mu_0 = 0$ and if there is no $\tilde\varphi \in BV$ so that $\varphi = \tilde\varphi - \tilde\varphi \circ f$ in BV, i.e., except on an at most countable set (it is not difficult to see that such $\varphi \in C^\infty(I)$ exist) then

$$\sigma^2 := \lim_{n\to\infty} \int \left(\frac{\sum_{k=0}^{n-1} \varphi(f^k(x))}{\sqrt{n}} \right)^2 d\mu_0 > 0 \, ,$$

and there exists $C(\varphi)$ depending only on the C^1-norm of φ so that for any $y \in \mathbb{R}$

$$|\mathbb{P}(\{x \mid \sum_{k=0}^{n-1} \varphi(f^k(x)) \leq y\sigma\sqrt{n}\}) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-s^2/2} ds| \leq \frac{C(\varphi)}{\sqrt{n}} \, .$$

where $\mathbb{P}(E) = \int \chi_E d\mu_0$. Fix φ satisfying the above conditions, $y < 0$ small and let $N_1 = N_1(y)$ be so that $\frac{C(\varphi)}{\sqrt{N_1}} < \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-s^2/2} ds$. Then there exists $x_0 \in [c_2, c_1]$ so that

$$|\sum_{k=0}^{n-1} \varphi(f^k(x_0))| \geq |y|\sigma\sqrt{n}, \forall n \geq N_1 \, .$$

Since the postcritical orbit is dense, for any $\delta > 0$ there exists $j_0 \geq 1$ so that $d(c_{j_0}, x_0) < \delta$. Put $\Lambda_f = \sup |f'|$. If $\delta \Lambda_f^m \leq \delta |(f^m)'(c_{j_0})| < 1/2$ for some $m \geq N_1$ then for all $j_0 \leq n \leq j_0 + m$ we have

$$\begin{aligned} |\sum_{k=0}^{n-1} \varphi(c_{k+1}))| &\geq |\sum_{k=0}^{n-j_0-2} \varphi(f^k(x_0))| - 2\sup|\varphi'| - |\sum_{k=1}^{j_0-1} \varphi(c_k)| \\ &\geq |y|\sigma\sqrt{n-j_0} - 2\sup|\varphi'| - |\sum_{k=1}^{j_0-1} \varphi(c_k)| \, . \end{aligned} \tag{98}$$

Assume now for a contradiction that $|\int \varphi d\mu_t| \leq A|t|$ for some $A < \infty$ and all small enough t. Let $t_n \to 0$ be a sequence of parameters so that c is not periodic for f_{t_n}

(this exists by Lemma 7.2). Recall the argument in the case when the orbit of c is infinite but not dense. For $|t_n| < \delta_0$, let $\widetilde{C}$ be the Lipschitz constant corresponding to the first term of (95) and let $\bar{C}$ be the Lipschitz constant corresponding to the truncated terms for $k \ge M(t_n)$ (where $M(t_n)$ is defined by (92)) in the second term of (95).

For arbitrarily small t, taking N as in the preperiodic case, the chain of inequalities (96) becomes

$$\left|\sum_{k=1}^{M(t_n)} \frac{s_{1,t_n}}{(f_{t_n}^{k-1})'(c_1)} \int \varphi \frac{H_{c_{k,t_n}} - H_{c_k}}{t_n} dx\right| = \left|C_n + s_1 \sum_{k=1}^{M(t_n)} \varphi(c_k) \sum_{j=1}^{k} \frac{X(c_j)}{(f^{j-1})'(c_1)}\right|$$
$$\ge |\sum_{k=1}^{M(t_n)} \varphi(c_k)|(M(t_n) - N)\frac{|\mathcal{J}(f,X)|}{2} - \widehat{C}\,.$$

If

$$|\sum_{k=1}^{M(t_n)} \varphi(c_k)|(M(t_n) - N)\frac{|\mathcal{J}(f,X)|}{2} - \widehat{C} - \widetilde{C} - \bar{C} > A\,,$$

we have obtained our contradiction. Otherwise

$$|\sum_{k=1}^{M(t_n)} \varphi(c_k)| \le \frac{(A + \widetilde{C} + \bar{C} + \widehat{C})}{(M(t_n) - N)} \frac{2}{|\mathcal{J}(f,X)|}\,.$$

If the above held for all small enough t, then we would have proved that there is $\epsilon > 0$ and a constant $D(f_t, \varphi)$ with

$$|\sum_{k=1}^{M(t_n)} \varphi(c_k)| \le \frac{D}{M(t_n)}\,, \quad \forall |t_n| < \epsilon\,.$$

We shall end the proof by showing that the above estimate gives a contradiction.

Recall that φ, σ, $y < 0$ and $N_1(y)$ are fixed, and that we haven chosen a generic x_0 as above. Take $m \ge N_1$ and let $\delta > 0$ be so that $\delta\Lambda_f^m < 1/2$. Then take $j_0(\delta) \ge 1$ so that $d(c_{j_0}, x_0) < \delta$. If j_0 does not tend to infinity as $m \to \infty$, then, recalling (98), the following expression tends to infinity as $m \to \infty$

$$|\sum_{k=0}^{j_0+m-1} \varphi(c_{k+1})| \ge |y|\sigma\sqrt{m} - 2\sup|\varphi'| - |\sum_{k=1}^{j_0-1} \varphi(c_k)|\,,$$

and we have obtained a contradiction. Otherwise, up to taking large enough m, there exist s so that $|M(s) - j_0| \le 1$, and $|t_n| \le |s|$ so that $|M(t_n) - j_0 - m| \le 1$. Then, recalling (98)

$$|\sum_{k=0}^{j_0+m-1} \varphi(c_{k+1})| \ge |y|\sigma\sqrt{m} - 2\sup|\varphi'| - |\sum_{k=1}^{j_0-1} \varphi(c_k)| \ge |y|\sigma\sqrt{m} - 2\sup|\varphi'| - \frac{D}{j_0}$$
$$|\sum_{k=0}^{j_0+m-1} \varphi(c_{k+1})| \le \frac{D}{M(t_n)}\,.$$

The righmost lower bound in the first line clearly diverges as $m \to \infty$, giving the desired contradiction. □

Appendix A. An auxiliary lemma

Lemma A.1. *Let f and g be two piecewise expanding C^1 unimodal maps and assume that $c = 0$. If $\sup_x\{1/|f'(x)|,\ 1/|g'(x)|\} \le \theta$ and $\sup_x |f(x) - g(x)| \le \delta$, then for all points x_f and x_g such that*

$$f^k(x_f) \cdot g^k(x_g) \ge 0\,, \forall k \le n\,, \tag{99}$$

we have $|x_f - x_g| < \theta^n + \frac{\delta}{1-\theta}$.

Proof. We can extend the inverse branches of f and g, denoted ψ^f_σ, ψ^g_σ, for $\sigma \in \{+,-\}$, to C^1 diffeomorphisms defined on $f(I) \cup g(I)$, so that they also have derivatives bounded from above by θ and

$$\max_{\sigma=+,-} \sup_{y \in f(I)\cup g(I)} |\psi^f_\sigma(y) - \psi^g_\sigma(y)| < \delta\,.$$

Condition (99) implies that there exists a sequence $\sigma_k \in \{+,-\}$, $k \le n$, such that

$$\psi^f_{\sigma_1} \circ \cdots \circ \psi^f_{\sigma_n}(f^n(x_f)) = x_f \text{ and } \psi^g_{\sigma_1} \circ \cdots \circ \psi^g_{\sigma_n}(f^n(x_g)) = x_g\,.$$

The lemma then follows from

$$\begin{aligned}
&|f^k(x_f) - g^k(x_g)| = |\psi^f_{\sigma_{k+1}}(f^{k+1}(x_f)) - \psi^g_{\sigma_{k+1}}(g^{k+1}(x_g))| \\
&\le |\psi^f_{\sigma_{k+1}}(f^{k+1}(x_f)) - \psi^f_{\sigma_{k+1}}(g^{k+1}(x_g))| + |\psi^f_{\sigma_{k+1}}(g^{k+1}(x_g)) - \psi^g_{\sigma_{k+1}}(g^{k+1}(x_g))| \\
&\le \theta |f^{k+1}(x_f) - g^{k+1}(x_g)| + \delta\,.
\end{aligned}$$

□

Appendix B. Keller-Liverani bounds for sequences of weak norms

We explain how (55) and (56) imply that for each $\gamma > 0$, there exist $\epsilon_0 > 0$ and $K \ge 1$ so that

$$\|(z - \mathcal{P}_t)^{-1}\|_{\mathcal{B}_0} \le K\,, \quad \forall |t| < \epsilon_0\,, \text{ if } |z| \ge \tau \text{ and } |z-1| \ge \gamma\,, \tag{100}$$

by adapting the proof of [16, Theorem 1] of Keller and Liverani. Since we have

$$(\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t)(\varphi) = -\frac{1}{2i\pi}\oint \frac{1}{z-1}(z - \mathcal{P}_t)^{-1}(\varphi)\, dz\,, \quad \forall \varphi \in \mathcal{B}_0\,,$$

(on any contour $|z - 1| = \gamma$ with $\gamma \in (0, 1-\tau)$), the bound (100) implies that $\|(\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t)\|_{\mathcal{B}_0}$ is bounded uniformly in $|t| < \epsilon_0$, i.e., (57).

Fix $\lambda < \tau < 1$ as after (20). The first remark is that [16, Lemma 1] is replaced by the claim that there exist ϵ_1, n_1 and C_1, depending only on C from (55) and on τ, so that for any $|z| \ge \tau$, all $\varphi \in \mathcal{B}_0$, all $|t| \le \epsilon_1$

$$\|\varphi\|_{\mathcal{B}_0} \le C_1 \|\widehat{Q}_t(z)\varphi\|_{\mathcal{B}_0} + C_1 |\varphi|_{weak,n_1}\,. \tag{101}$$

Now, the beginning of the proof of [16, Theorem 1] gives that that (55) and (56) imply for all $m \ge 0$, $n \ge 0$ and all $|z| \ge \tau$, we have (see [16, (12)])

$$\begin{aligned}
|\widehat{\mathcal{Q}}_t(z)^{-1}\varphi|_{weak,m} \le &\Bigg(\|\widehat{\mathcal{Q}}_0^{-1}(z)\|_{\mathcal{B}_0} C(2C + |z|) \left(\frac{\lambda}{\tau}\right)^n \\
&+ \big(\|\widehat{\mathcal{Q}}_0^{-1}(z)\|_{\mathcal{B}_0} C + \frac{C}{1-\tau}\big)(C\delta_{m+n}(t)) \left(\frac{1}{\tau}\right)^n \Bigg) \|\varphi\|_{\mathcal{B}_0} \\
&+ \big(\|\widehat{\mathcal{Q}}_0^{-1}(z)\|_{\mathcal{B}_0} C + \frac{C}{1-\tau}\big) \left(\frac{1}{\tau}\right)^n |\varphi|_{weak,m+n}\,.
\end{aligned} \tag{102}$$

Fix $\gamma > 0$, write $H = \sup_{|z| \ge \tau, |z-1| > \gamma} \|\widehat{\mathcal{Q}}_0^{-1}(z)\|_{\mathcal{B}_0}$, and take

$$n_2 = \left[\frac{\ln(4C_1 HC(2C+2))}{\ln(\tau/\lambda)} \right] .$$

Then two applications of (101) as in the proof of [16, (15)] (taking $m = n_1$, $n = n_2$ in (102)) show that, taking,

$$\epsilon_0 = \sup \left\{ |t| \mid \delta_{n_1+n_2}(t) \left(HC + \frac{C}{1-\tau} \right) \left(\frac{1}{\tau} \right)^{n_2} \le \frac{1}{4C_1} \right\} ,$$

we have

$$\begin{aligned} \|\varphi\|_{\mathcal{B}_0} &\le 2C_1 \|\widehat{\mathcal{Q}}_t(z)(\varphi)\|_{\mathcal{B}_0} + \frac{1}{2\delta_{n_1+n_2}(\epsilon_2)} |\widehat{\mathcal{Q}}_t(z)(\varphi)|_{weak, n_1+n_2} \\ &\le K \|\widehat{\mathcal{Q}}_t(z)(\varphi)\|_{\mathcal{B}_0} , \end{aligned}$$

for all $|t| \le \epsilon_0$, and any $|z| \in [\tau, 2]$ with $|z-1| > \gamma$, proving (100).

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