CORRIGENDUM TO: LINEAR AND FRACTIONAL RESPONSE FOR THE SRB MEASURE OF SMOOTH HYPERBOLIC ATTRACTORS AND DISCONTINUOUS OBSERVABLES

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ABSTRACT. The first main result of [1] is modified as follows: For any θ in the Sobolev space $H^r_p(M)$, with $1 < p < \infty$ and $0 < r < 1/p$, the map $t \mapsto \int \theta \, d\rho_t$ is α-Hölder continuous for all $\boldsymbol{\alpha < r - |\log \mathcal{J}|/(p|\log \nu_s|)}$ **where** $\boldsymbol{\mathcal{J} \le 1}$ **is the strongest volume contraction and** $\boldsymbol{\nu_s < 1}$ **is the weakest contraction.** This applies to $\theta(x) = h(x)\Theta(g(x) - a)$ (for all $\boldsymbol{\alpha < 1 - |\log \mathcal{J}|/|\log \nu_s|}$) for h and g smooth and Θ the Heaviside function, if a is not a critical value of g."

1. Correcting [1, Theorem 2.1] on fractional response

The essential spectral radius of the operator $\mathcal{L}_t$ defined by [1, (2.1)] acting on $W^{u,s}_{p',\dagger}(f_t, V)$ for $s < 0 < u$ is only guaranteed to be strictly smaller than 1 by [2] if

$$\lim_{m\to\infty} \sup_{\Lambda_t} [|\det D_x f_t^m|^{-1/p} \max\{(\nu_s(f_t^m))^{|s|}, (\nu_u(f_t^m))^{-u}\}]^{1/m} < 1 \,, \tag{1}$$

where $\nu_s(x, f_t^m) = \sup_{v \in E^s(x)\setminus\{0\}} \frac{\|D_x f_t^m(v)\|}{\|v\|}$, $\nu_u(x, f_t^m) = \inf_{v \in E^u(x)\setminus\{0\}} \frac{\|D_x f_t^m(v)\|}{\|v\|}$ for $x \in \Lambda_t$ and $m \in \mathbb{Z}_+$. Because of this, [1, Theorem 2.1] must be modified:

Theorem 1.1 (Fractional response)**.** *Fix $\beta \in (0, 1)$. Let $t \mapsto f_t$, for $t \in [-\epsilon_0, \epsilon_0]$, be a $C^{2+\beta}$ family of C^3 diffeomorphisms f_t on a smooth Riemann manifold M, so that f_t has a transitive compact hyperbolic attractor $\Lambda_t \subset M$. Let ρ_t be the (unique) SRB measure of f_t on Λ_t.* **Assume that, for some $\boldsymbol{1 < p < \infty}$ and all $\boldsymbol{t}$,**[1]

$$\boldsymbol{\nu_{u,t}^{-\beta} := \lim_{m\to\infty} \sup_{\Lambda_t} [\nu_u(x, f_t^m)]^{-\frac{\beta}{m}} < \mathcal{J}_t^{\frac{1}{p}} := \lim_{m\to\infty} \inf_{\Lambda_t} |\det Df_t^m(x)|^{\frac{1}{pm}}} \,. \tag{2}$$

Let $\theta : M \to \mathbb{C}$, be so that[2] $\boldsymbol{\theta \in H^r_p(\bar{V})}$ *for some $0 < r < 1/p$. Then there exists $\epsilon_1 \in (0, \epsilon_0]$ so that $t \mapsto \int_M \theta \, d\rho_t$ is α-Hölder continuous on $[-\epsilon_1, \epsilon_1]$ for any*

$$\boldsymbol{0 < \alpha < r - \sup_{|t|<\epsilon_1} \frac{1}{p} \frac{|\log \mathcal{J}_t|}{|\log \nu_{s,t}|}} \,. \tag{3}$$

Date: April 22, 2017.

We thank S. Galatolo for pointing out the counter-example to the initial statement of Theorem 2.1. We are grateful to S. Gouëzel, P.-A. Guihéneuf, and Benoît Saussol for useful comments.

[1]**If each $\boldsymbol{f_t}$ is $\boldsymbol{C^{N+1}}$ and the family is** $C^{N+\beta}$ **for $\boldsymbol{N \ge 3}$, then** (2) **can be replaced by the condition** $\boldsymbol{\nu_{u,t}^{-(N+\beta-2)} < \mathcal{J}_t^{\frac{1}{p}}}$**.**

[2]$\boldsymbol{H^r_p(\bar{V})}$ **is the closure of $\boldsymbol{C^\infty(\bar{V})}$ for the norm of** $H^r_p(M)$.

If r is close enough to $1/p$, the r.h.s. of (3) is > 0 if, for[3] all t

$$\boldsymbol{\nu_{s,t} := \lim_{m\to\infty} \sup_{\Lambda_t} [\nu_s(x, f_t^m)]^{\frac{1}{m}} < \mathcal{J}_t \,.} \tag{4}$$

Remark 2.3 from [1] should be modified correspondingly: If[4] (2) and (4) hold for all $p > 1$, and if h is supported in V, we get that $t \mapsto \int h(x)\Theta(g(x) - a)\, d\rho_t$ is α-Hölder for any

$$\alpha < \frac{1}{p}\big(1 - \sup_t \frac{|\log \mathcal{J}_t|}{|\log \nu_{s,t}|}\big) < 1 - \sup_t \frac{|\log \mathcal{J}_t|}{|\log \nu_{s,t}|} > 0\,.$$

(Except if $|\det Df_t| \equiv 1$ for all t, this is weaker than the claim in [1, Remark 2.3].)

Note that $\mathcal{J}_t \in [(\nu_{u,t})^{d_u}(\bar\nu_{s,t})^{d_s}, 1]$, with $\bar\nu_{s,t}$ defined as $\nu_{s,t}$, replacing the sup over $v \in E^s \setminus \{0\}$ by an inf. Thus, $\sup_t \frac{|\log \mathcal{J}_t|}{|\log \nu_{s,t}|} \le \frac{|d_u \log \nu_u + d_s \log \bar\nu_s|}{|\log \nu_s|} < 1$ if

$$d_u \log \nu_u > d_s |\log \bar\nu_s| - |\log \nu_s|\,, \tag{5}$$

where $\nu_u = \inf_t \nu_{u,t} > 1$ is the weakest expansion, $\nu_s = \sup_t \nu_{s,t} < 1$ is the weakest contraction, and $\bar\nu_s = \inf_t \bar\nu_{s,t} \le \nu_s$ is the strongest contraction.

We do not claim that the sufficient conditions (5) or (4) are necessary, but some assumption is required to get fractional response: Stefano Galatolo pointed out at a conference in Warwick in April 2017 that, for a well-chosen perturbation f_t of the standard solenoidal attractor f_0, the map $t \mapsto \int_M \theta\, d\rho_t$ is discontinuous at $t = 0$ for certain $\theta \in H^r_p$, since the SRB measure has atoms in the stable direction. (In this example, $d_u = 1$ and $d_s = 2$, with $\mathcal{J}_0 = 1/2$ and $\nu_{s,0} = 1/2$, so that (4) is violated.)

Changes in the proof of [1, Theorem 2.1] are limited to the following:

•Each $H^\sigma_q(M)$, for $1 < q < \infty$ and $\sigma \in \mathbb{R}$ must be replaced by $H^\sigma_q(\bar V)$.

•6 lines below (2.3), p. 1210: "for all real numbers $1 < p' < \infty$ and $u - 2 < s < 0 < u < 2$" must be replaced by "for all real numbers $1 < p' < \infty$ and all $u - 2 < s < 0 < u < 2$ so that (1) holds."

•3 lines above (2.4), p. 1210: "enjoys the Perron-Frobenius spectral properties described above" must be replaced by "enjoys the Perron-Frobenius spectral properties described above, if (1) holds when replacing s by $s - 1$ and u by $u - 1$."

•Lines 1-2 of p. 1212: "(taking $s' < s < 0$ close enough to zero)" must be replaced by "(taking $-r < s' < s < 0$)."

•2 lines after (2.8), p 1212: "Fix $|s| < r$ small" must be replaced by "Fix $s_0 < |s| < r$ close to $s_0 := \sup_t \frac{|\log \mathcal{J}_t|}{p|\log \nu_{s,t}|} < 1$."

•Line after (2.11), p. 1212: "For each $0 < \alpha < r$" must be replaced by "For each α satisfying (3)". Last line of p. 1212, "small enough $\tilde r > |s| > 0$" must be replaced by "$\tilde r > |s| > s_0$ close enough to s_0."

References

[1] V. Baladi, T. Kuna, and V. Lucarini, *Linear and fractional response for the SRB measure of smooth hyperbolic attractors and discontinuous observables,* Nonlinearity **30** (2017) 1204–1220, DOI: 10.1088/1361-6544/aa5b13

[2] V. Baladi and M. Tsujii, *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms,* Ann. Institut Fourier **57** (2007) 127–154.

[3] **Use semi-continuity.**

[4] **By Footnote 1, condition (2) can be waived if g, h, f_t are smooth enough.**