FIXING TYPOS IN *FRACTIONAL SUSCEPTIBILITY FUNCTIONS FOR THE QUADRATIC FAMILY: MISIUREWICZ–THURSTON PARAMETERS*, BY BALADI AND SMANIA, COMM. MATH. PHYS. 385 (2021) 1957–2007.

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1. Formula (50)

The use of w_0, w_1 in [1, (50)] is incorrect. This formula must be replaced by[1]

$$\begin{aligned}\rho_t(x) = \psi_0(x) &+ \sum_{k=1}^{\infty} C_k^{(0)} \frac{\mathbf{1}_{\upsilon_{k-1}<s_{k-1}(x-c_k)<0}}{\sqrt{|x-c_k|}} \\ &+ \sum_{k=1}^{\infty} C_k^{(1)} \cdot \mathbf{1}_{\upsilon_{k-1}<s_{k-1}(x-c_k)<0} \cdot \sqrt{|x-c_k|} \\ &+ \sum_{k=1}^{\infty} C_k^{(2)} \cdot \mathbf{1}_{\upsilon_{k-1}<s_{k-1}(x-c_k)<0} \cdot \sqrt{|x-c_k|^{3/2}}\,,\end{aligned} \tag{1.1}$$

where the υ_k are such that $s_k(\upsilon_k + c_k)$ is preperiodic in $\{u_1, u_2, v_1, v_2\}$, for suitable

$$c_2 < u_1 < v_1 < c < v_2 < u_2 < c_1 \text{ with } f(u_2) = u_1\,,\ f(v_2) = f(v_1) = u_2\,, \tag{1.2}$$

and where there exists U_t such that

$$C_k^{(0)} = \frac{\rho_t(0)}{|Df_t^{k-1}(c_1)|^{1/2}}\,,\ |C_k^{(1)}| \le \frac{U_t}{|Df_t^{k-1}(c_1)|^{1/2}}\,, |C_k^{(2)}| \le \frac{U_t}{|Df_t^{k-1}(c_1)|^{3/2}}\,,\ \forall k \ge 1\,.$$

Note also that ψ_0 is not only C^1, but in addition supported in $[c_2, c_1]$, with $\psi_0' \in C^0 \cap BV$, and $\psi_0(c_1) = \psi_0(c_2) = \psi_0'(c_1) = \psi_0'(c_2) = 0$.

This does not affect the other statements of [1] in view of §2.

We give an example, for the reader's convenience: Let t be such that $f_t(c_3) = c_3$ is repelling. Then, setting $f = f_t$ and $c_k = c_{k,t}$, the construction in [2, §2, §10] gives u_1, u_2, v_1, v_2 as in (1.2) and an invariant Cantor set

$$H(u_1) := \{x \le c_1 \mid f^n(x) \ge u_1\,, \forall n \ge 0\} \subset [u_1, v_1] \cup [v_2, u_2] \text{ with } c_3 \in H(u_1)\,.$$

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[1]The expression [3, (14)] must be corrected similarly. This does not seem to affect the results in [3] in view of the use of the parameter called A there.

The sequence $w_n \in \{u_1, u_2, v_1, v_2\}$ constructed in [2, §10] is[2] $w_0 = u_2$, $w_1 = u_1$, $w_{2k} = v_2$ and $w_{2k+1} = u_2$ for all $k \geq 1$. The decomposition of [2, Thm 9] is

$$\begin{aligned}\rho(x) = \phi(x) + \phi(c) \cdot \Bigg[&\frac{u_2 - x}{u_2 - c_1} \frac{1}{\sqrt{c_1 - x}} \mathbf{1}_{x \in [u_2, c_1)} \\ &+ \frac{1}{|f'(c_1)|^{1/2}} \frac{u_1 - x}{u_1 - c_2} \frac{1}{\sqrt{x - c_2}} \mathbf{1}_{x \in (c_2, u_1]} \\ &+ \sum_{k \geq 1} \Big[\frac{1}{|(f^{2k})'(c_1)|^{1/2}} \frac{v_2 - x}{v_2 - c_3} \frac{1}{\sqrt{c_3 - x}} \mathbf{1}_{x \in [v_2, c_3)} \\ &\quad + \frac{1}{|(f^{2k+1})'(c_1)|^{1/2}} \frac{u_2 - x}{u_2 - c_3} \frac{1}{\sqrt{x - c_3}} \mathbf{1}_{x \in (c_3, u_2]} \Big] \Bigg],\end{aligned}$$

with ϕ a C^0 function supported in $[c_2, c_1]$ (in particular $\phi(c_2) = \phi(c_1) = 0$ and $\phi(c) = \rho(c)$). Clearly, there exist real numbers γ_1, γ_2, and $\gamma_{3,\pm}$ such that, setting

$$\begin{aligned}\psi(x) = \gamma_1 \sqrt{c_1 - x} \cdot \mathbf{1}_{x \in [u_2, c_1)} + \gamma_2 \sqrt{x - c_2} \cdot \mathbf{1}_{x \in (c_2, u_1]} \\ + \gamma_{3,+} \sqrt{c_3 - x} \cdot \mathbf{1}_{x \in [v_2, c_3)} + \gamma_{3,-} \sqrt{x - c_3} \cdot \mathbf{1}_{x \in (c_3, u_2]}\end{aligned}$$

we have

$$\begin{aligned}\rho(x) = \phi(x) + \psi(x) + \phi(c) \cdot \Bigg[\frac{1}{\sqrt{c_1 - x}} \mathbf{1}_{x \in [u_2, c_1)} + \frac{1}{|f'(c_1)|^{1/2}} \frac{1}{\sqrt{x - c_2}} \mathbf{1}_{x \in (c_2, u_1]} \\ + \sum_{k \geq 1} \Big[\frac{1}{|(f^{2k})'(c_1)|^{1/2}} \frac{1}{\sqrt{c_3 - x}} \mathbf{1}_{x \in [v_2, c_3)} + \frac{1}{|(f^{2k+1})'(c_1)|^{1/2}} \frac{1}{\sqrt{x - c_3}} \mathbf{1}_{x \in (c_3, u_2]} \Big] \Bigg].\end{aligned}$$

For example, if $u_2 \leq x < c_1$, then

$$\frac{u_2 - x}{u_2 - c_1} \frac{1}{\sqrt{c_1 - x}} = \frac{1}{\sqrt{c_1 - x}} + \frac{\sqrt{c_1 - x}}{u_2 - c_1}.$$

Finally, [2, Remark 16 a), Appendix A] gives (1.1), applying a similar procedure to the square root singularities in $\phi(x)$ from [2, Thm 9]. (This produces terms $|x - c_k|^{3/2}$.)

2. Truncated spikes and square roots

The claims on truncated spikes $\phi_{x_0,\sigma,\mathcal{Z}}$ and square roots $\bar{\phi}_{x_0,\sigma,\mathcal{Z}}$ in [1, Lemma 3.2, Lemmas 4.4–4.5] require $\mathcal{Z} > 1$. This is not enough to handle the characteristic functions in the corrected expression (1.1) for [1, (50)] when proving [1, Thm C] in [1, §5.3]. Fortunately, the requirement that $\mathcal{Z} > 1$ can be replaced by $\mathcal{Z} > 0$ in [1, Lemma 3.2, Lemmas 4.4–4.5] (indeed, the four power series below [1, (70) in App. A] converge absolutely as soon as $y/\mathcal{Z} < 1$). In particular, the supremum over $\mathcal{Z} > 1$ in the last claims of [1, Lemma 3.2, Lemmas 4.4, Lemma 4.5] can be replaced by a supremum over $\mathcal{Z} > \epsilon$ for any fixed $\epsilon > 0$. Finally, truncated expressions $|x - c_k|^{3/2}$ may be handled similarly.

[2] For more general Misiurewicz–Thurston parameters, w_n is preperiodic.

3. Formula (5) and Lemma E, Proposition F

Formula (5) only holds for almost every t in Ω. As a consequence, Lemma E and Proposition F should be amended as follows (this does not affect the rest of the paper):

-The last statement of Lemma E is only for almost all t in Ω (the Hölder exponent is claimed to be uniform in t).

-Both statements of Proposition F are only for almost all t in Ω.

Finally, in the proof of Lemma E, "for any $\beta > 2$" should be "for any $\beta < 2$."

4. Other remarks

Four lines below, the dual of $\mathcal{L}_t$ fixes Lebesgue measure on $\mathbb{R}$, there is no need to restrict to I_t as mentioned in [1, §1.1]. This simplifies some arguments, as observed by Sedro [3]. In the third line of the statement of Theorem C, $\tilde{\psi}(\ell + p)$ should be $\tilde{\psi}(\ell + P)$.

References

[1] V. Baladi and D. Smania, *Fractional susceptibility functions for the quadratic family: Misiurewicz–Thurston parameters,* Comm. Math. Phys. **385** (2021) 1957–2007.

[2] D. Ruelle, *Structure and f-dependence of the A.C.I.M. for a unimodal map f of Misiurewicz type,* Comm. Math. Phys. **287** (2009) 1039–1070.

[3] J. Sedro, *Pre-threshold fractional susceptibility functions at Misiurewicz parameters,* Nonlinearity **34** (2021) 7174–7184.

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