Corrections and complements to:
The quest for the ultimate anisotropic Banach space

VIVIANE BALADI

We provide corrections and complements to [1, §4]: Formulas (29) and, especially, (31) must be amended, as explained in §3 and §4 below. An argument to replace the C^r norm of the weight by a supremum was missing at the end of the proof of Theorem 4.1 (see §2 below). §1, §5, and §6 below give minor clarifications. None of the main statements are changed, except that:

• In Lemma 4.2 and Theorem 4.1, the condition $-(r-1) < s < -t < 0$ must be replaced by $t-(r-1) < s < -t < 0$. (See §4 below.)

• The bound (25) on the essential spectral radius in Theorem 4.1 is only proved up to an (arbitrarily small) $\epsilon > 0$ and for a "zoomed" version of the space $\mathcal{U}_1^{t,s}(R)$ for some large $R(\epsilon)$. (See (1) in §2 below.)

1. Details for the leafwise Young inequality (38)

To prove (38) on p. 542, notice that, for $\ell \geq 1$ and $x \in \mathbb{R}^{d_s}$, Fubini implies

$$(\psi_\ell^{(d_s)})^{Op}\big[(\varphi * \hat\psi) \circ \pi_\Gamma^{-1}\big](x) = [(\psi_\ell^{(d_s)})^{Op} \int \hat\psi(z)\varphi(\pi_\Gamma^{-1}(\cdot) - z)\mathrm{d}z](x)$$

$$= [(\psi_\ell^{(d_s)})^{Op} \int \hat\psi(z)\varphi(\pi_{\Gamma - z}^{-1}(\cdot))\mathrm{d}z](x) = \int_{\mathbb{R}^n} \hat\psi(z)(\psi_\ell^{(d_s)})^{Op}\big[\varphi \circ \pi_{\Gamma-z}^{-1}\big](x)\mathrm{d}z\,.$$

Since $\| \int_{\mathbb{R}^n} \hat\psi(z)\Phi_z(\cdot)\mathrm{d}z\|_{L_p(\mathbb{R}^{d_s})} \leq \|\hat\psi\|_{L_1(\mathbb{R}^d)} \sup_z \|\Phi_z(\cdot)\|_{L_p(\mathbb{R}^{d_s})}$ by the Minkowski integral inequality, we find,

$$\Big\|(\psi_\ell^{(d_s)})^{Op}\big[(\varphi * \hat\psi) \circ \pi_\Gamma^{-1}\big]\Big\|_{L_p(\mathbb{R}^{d_s})} \leqslant \|\hat\psi\|_{L_1(\mathbb{R}^d)} \sup_{\tilde\Gamma \in \mathcal{F}} \big\|(\psi_\ell^{(d_s)})^{Op}[\varphi \circ \pi_{\tilde\Gamma}^{-1}]\big\|_{L_p(\mathbb{R}^{d_s})}\,.$$

2. Proof of Theorem 4.1

The bound (43) is correct as stated, but to prove Theorem 4.1 we would need it for weights $\tilde f_m(x) = \prod_{j=0}^{m-1}(\theta_{\vec\omega}^{(m)} g(T^{-j}(x)))$, where $\sum_{\vec\omega} \theta_{\vec\omega}^{(m)} \equiv 1$ is a smooth partition of unity adapted to T^m (independently of Γ), taking into account the fact that minimal sub-covers are needed for thermodynamic estimates. The bound (43) does not seem to always hold for such $\tilde f_m$. Hence, Theorem 4.1 is only proved if we add (arbitrarily small) $\epsilon > 0$ to the expression (25), and if we replace $\mathcal{U}_1^{t,s}$ by its R-zoomed version $\mathcal{U}_1^{t,s}(R)$(as in [3, §2.3]), for R depending on ϵ:

We recall the construction in [3, §2.3]. Our charts are $\kappa_\omega : U_\omega \to V_\omega \subset M$ with $U_\omega \in \mathbb{R}^d$ and $\omega \in \Omega$. Fix a C^∞ function $\alpha : \mathbb{R}^d \to [0,1]$ with $\alpha(z) = 0$ if $|z| \geq d$, and $\sum_{m\in\mathbb{Z}^d} \alpha(z-m) = 1$. For $R \geq 1$ and

$$(\omega, m) \in \mathcal{Z}(R) := \{(\omega, m) \mid \omega \in \Omega\,,\, m \in (R \cdot U_\omega) \cap \mathbb{Z}^d\}\,,$$

define $\hat\alpha_{\omega,m}^R : M \to [0,1]$ by $\hat\alpha_{\omega,m}^R(x) = 0$ if $x \notin V_\omega$, and

$$\hat\alpha_{\omega,m}^R(x) = \alpha[R \cdot (\kappa_\omega^{-1}(x)) - m]\,, \quad \forall x \in V_\omega\,.$$

Date: January 21, 2018.
I thank M. Jézéquel, who found several unclear points in [1] and helped to clarify them.

This gives a partition of unity in the sense that $\sum_{m\in(R\cdot U_\omega)\cap\mathbb{Z}^d}\hat\alpha^R_{\omega,m}(x)=1$ for all $x\in V_\omega$. (The intersection multiplicity of this partition of unity is bounded, uniformly in R.) Finally, fixing $s,t\in\mathbb{R}$ and $R\ge 1$, we write $\kappa^R_\omega(z)=\kappa_\omega(z/R)$, and we set for $\varphi\in L^\infty(M)$,[1]

$$\|\varphi\|_{\mathcal{U}^{t,s}_1(R)}=\sum_{(\omega,m)\in\mathcal{Z}(R)}\|(\hat\alpha^R_{\omega,m}\cdot\varphi)\circ\kappa^R_\omega\|_{\mathcal{U}^{t,s}_1}\le\infty\,. \tag{1}$$

The space $\mathcal{U}^{t,s}_p(R)$ is the closure of $L^\infty(M)$ for the norm $\|\varphi\|_{\mathcal{U}^{t,s}_p(R)}$.

Since $\|\tilde f^R_m\|_{C^r}\le 2\sup|\tilde f^R_m|$, for $\tilde f_m(z)=\tilde f_m(z/R)$, if $R\ge 1$ is large enough (depending on m), we may then prove Theorem 4.1 up to adding $\epsilon>0$ to (25) and replacing $\mathcal{U}^{t,s}_1$ by $\mathcal{U}^{t,s}_1(R)$ for suitable $R(\epsilon)$.

Finally, the last sentence in the proof of Theorem 4.1 on p. 543 must be shortened to: "We just mention here that, in the present case, the "fragmentation lemma" (used to expand along a partition of unity) is just the triangle inequality, while the "reconstitution lemma" (used to regroup the terms from a partition of unity) is the trivial inequality $\sum|a_ke_k|\le(\sum|a_k|)\sup|e_k|$." Footnote 18. on the same page must be suppressed.

3. Details for (40) in Sublemma 4.4 and correcting (29)

We explain how to bound $\|f(\varphi\circ F)\|^s_{p,\Gamma}$, by duality, giving the proof of (40) on p. 543: Since $B^s_{p,\infty}$ is the dual of $b^{|s|}_{p',1}$ (with $1/p'=1-1/p$), setting $F_\Gamma=\pi_{F(\Gamma)}\circ F\circ\pi_\Gamma^{-1}$, it suffices to estimate $\|((fh)\circ F_\Gamma^{-1})|\det(DF_\Gamma)^{-1}|\|_{B^{|s|}_{p',1}(\mathbb{R}^{d_s})}$ for C^∞ functions h. First note that $\|((fh)\circ F_\Gamma^{-1})|\det(DF_\Gamma)^{-1}|\|_{L_{p'}(\mathbb{R}^{d_s})}\le\sup\frac{|f|}{|\det DF_\Gamma|^{1/p}}$. The $B^{|s|}_{p',1}(\mathbb{R}^{d_s})$ norm of v is equivalent to[2]

$$\|v\|_{W^{\lceil|s|-1\rceil}_{p'}(\mathbb{R}^{d_s})}+\sum_{|\beta|=\lceil|s|-1\rceil}\int_{\mathbb{R}^{d_s}}\frac{1}{|y|^{d_s}}\frac{\|Z(D^\beta v,\cdot,y)\|_{L_{p'}(\mathbb{R}^{d_s})}}{|y|^{|s|-\lceil|s|-1\rceil}}\mathrm{d}y\,,$$

where $\|v\|_{W^k_{p'}}=\sum_{0\le|\beta|\le k}\|D^\beta v\|_{L_{p'}}$ and $Z(w,x,y)=w(x+y)+w(x-y)-2w(x)$. Thus, since $\inf|DF_\Gamma|\ge\|F\|_-\ge 1$, and using the"Zygmund derivation" in[3] [4, §2]

$$\begin{aligned}Z(fh,x,y)&=f(x)Z(h,x,y)+h(x)Z(f,x,y)\\&\qquad+\cdot\Delta_+(f,x,y)\Delta_+(h,x,y)+\cdot\Delta_-(f,x,y)\Delta_-(h,x,y)\,,\end{aligned}$$

where $\Delta_+(v,x,y)=(v(x+y)-v(x))$ and $\Delta_-(v,x,y)=(v(x)-v(x-y))$, and recalling that for any noninteger $\sigma>0$ [6, Prop 2.1.2, Prop 2.2.1]

$$\|v\|_{\mathcal{W}^\sigma_{p',p'}}\le C(p',\sigma)\|v\|_{B^\sigma_{p',p'}}\le C^2(p',|s|)\|v\|_{B^\sigma_{p',1}}\,,$$

---

[1] Note that for fixed R, the sum in (1) involves a uniformly bounded number of terms.

[2] See e.g. [6, §2.1], with $\lceil x\rceil$ the smallest integer which is $\ge x$.

[3] See also [4, (2.6)–(2.8)], writing $|f_i'|_\delta=|f_i'||f_i'|_\delta/|f_i'|$ in [4, (2.5)], and noting that $(|(F_\Gamma^{-1})'|_\delta=|F'_\Gamma|_\delta/|F'_\Gamma|^2$ so that $|(F_\Gamma^{-1})'|_\delta/|(F_\Gamma^{-1})'|=|F'_\Gamma|_\delta/|F'_\Gamma|$.

(with $\mathcal{W}^\sigma_{p',p'}$ the Slobodeckij norm), we find for any $\epsilon > 0$ constants $C(\mathcal{F},)$ and $C(\mathcal{F}, \epsilon)$ so that

$$
\begin{aligned}
(\widetilde{40})\ & \|((fh)\circ F_\Gamma^{-1})|\det(DF_\Gamma)^{-1}|\|_{B^{|s|}_{p',1}(\mathbb{R}^{d_s})} \\
& \le C(\mathcal{F}) \sum_{j=0}^{\lceil |s|-1\rceil} \sum_{\ell=0}^{j} \frac{1}{\|F\|_-^\ell} \frac{1}{|\det DF_\Gamma|^{1/p}} \|h\|_{W^\ell_{p'}} \\
& \qquad \times \sum_{i=0}^{j-\ell} \|f\circ F_\Gamma^{-1}\|_{C^i} \||\det(DF_\Gamma^{-1})|\|_{C^{j-\ell-i}} \|DF_\Gamma^{-1}\|_{C^{j-\ell-i}} \\
& \quad + C(\mathcal{F}) \sum_{\ell=1}^{\lceil |s|-1\rceil} \frac{1}{\|F\|_-^\ell} \frac{1}{|\det DF_\Gamma|^{1/p}} \|h\|_{B^\ell_{p',1}} \\
& \qquad \times \sum_{i=0}^{\lceil |s|-1\rceil-\ell} \|f\circ F_\Gamma^{-1}\|_{C^i} \||\det(DF_\Gamma^{-1})|\|_{C^{\lceil |s|-1\rceil-\ell-i}} \|DF_\Gamma^{-1}\|_{C^{\lceil |s|-1\rceil-\ell-i}} \\
& \quad + C(\mathcal{F},\epsilon) \frac{\|DF_\Gamma^{-1}\|_{C^\epsilon}}{\|F\|_-^{|s|-2\epsilon}} \frac{\||\det(DF_\Gamma^{-1})|\|_{C^\epsilon}}{|\det DF_\Gamma|^{1/p}} \|h\|_{B^{|s|-\epsilon}_{p',1}} \|f\circ F_\Gamma^{-1}\|_{C^\epsilon} \\
& \quad + C(\mathcal{F},\epsilon) \frac{1}{\|F\|_-^{|s|}} \frac{\sup |f|}{|\det DF_\Gamma|^{1/p}} \|h\|_{B^{|s|}_{p',1}} .
\end{aligned}
$$

Finally, using $\inf|\det(DF|_{(C'_+)^\perp})| \ge C|\det(D(F|_\Gamma)|$, we get (40), up to slightly amending (29) as follows (d_s and p are fixed):

$$(29^*)\ C(F,\Gamma,s) = C'(\mathcal{F})|s|\|D(F|_\Gamma)^{-1}\|_{C^{r-1}}\||\det(DF_\Gamma^{-1})|\|_{C^{r-1}} .$$

Similarly, in (42) one should replace $\sup_\Gamma \|f\circ F^{-1}\|_{C^{r-1}(F(\Gamma))}$ by

$$\sup_\Gamma \|f\circ F^{-1}\|_{C^{r-1}(F(\Gamma))}\||\det(D(F_{(C'_+)^\perp})^{-1})|\|_{C^{r-1}(F(\Gamma))} .$$

4. Bounding $\|H^{\ell,\tau}_{n,\sigma}(v)\|^s_{p,\Gamma}$ (proof of Lemma 4.2) — Correcting (31)

Since $\|\cdot\|^s_{p,\Gamma}$ is not an L_p norm, (38) does not suffice to deduce from (54) a bound on $\|H^{\ell,\tau}_{n,\sigma}(v)\|^s_{p,\Gamma}$. For any compact $K\subset\mathbb{R}^d$ and any $\delta>0$, there exists $C_0\ge 2$ so that for all $C'_0 \ge C_0$ there exists $\widetilde{C}_0$ so that for all v supported in K,

$$
\begin{aligned}
\|(\psi^{Op}_\ell v)\circ\pi^{-1}_{\widetilde\Gamma}\|_{L_p(\mathbb{R}^{d_s})} &\le \widetilde{C}_0 2^{\ell(-s+\delta)} \sum_{j=0}^{\ell+[C'_0]} 2^{j(s-\delta)} \|(\psi^{d_s}_j)^{Op}((\psi^{Op}_\ell v)\circ\pi^{-1}_{\widetilde\Gamma})\|_{L_p(\mathbb{R}^{d_s})} \\
&\quad + C_0 \sum_{j=\ell+[C'_0]+1}^{\infty} 2^{-jr} \sum_{m=\ell-2}^{\ell+2} \sup_{\widehat\Gamma} \|(\psi^{Op}_m v)\circ\pi^{-1}_{\widehat\Gamma}\|_{L_p(\mathbb{R}^{d_s})} ,\ \forall\ell, \forall\widetilde\Gamma .
\end{aligned}
$$

(This is clear if $\widetilde\Gamma$ is[4] affine, otherwise, proceed as in [2, Lemma 3.5], using the L_p version of the leafwise Young inequality [5, Lemma 4.2], to obtain the above estimate in the sum over $j>\ell+C'_0$, after decomposing $v=\sum_m \psi^{Op}_m v$ and using

[4]The second line is then not needed.

almost orthogonality.) Therefore, since $|s| < r$ and $a \le b + \epsilon a$ implies $a \le (1-\epsilon)^{-1} b$ if $a > 0$, $b > 0$, and $\epsilon < 1$, for each $\delta > 0$ there is C so that

$$(0*) \quad 2^{\ell t} \|(\psi_\ell^{Op} v) \circ \pi_{\widetilde{\Gamma}}^{-1}\|_{L_p(\mathbb{R}^{d_s})} \le C 2^{\ell(-s+\delta)} \|v\|_{\mathcal{U}_p^{t,s}} \quad \forall \widetilde{\Gamma}\,, \forall \ell\,.$$

Then, applying $\|\phi\|_{B^s_{p,\infty}} \le C \|\phi\|_{L_p}$ to $\phi = (H^{\ell,\tau}_{n,\sigma}(\sum_{i=-2}^{2} \psi^{Op}_{\ell+i} v)) \circ \pi_\Gamma^{-1}$, and using (54) and the L_p version of [5, Lemma 4.2], one obtains

$$2^{\ell t} \|H^{\ell,\tau}_{n,\sigma}(v)\|^s_{p,\Gamma} = 2^{\ell t} \|H^{\ell,\tau}_{n,\sigma}(\sum_{i=-2}^{2} \psi^{Op}_{\ell+i} v)\|^s_{p,\Gamma} \le C_{F,f} 2^{-(r-1)\max\{n,\ell\}} 2^{(-s+\delta)\ell} \|v\|_{\mathcal{U}_p^{t,s}}\,.$$

This replaces the stronger bound stated two lines above (52) on p. 546 and gives the following weakening of (31):

$$(31^*) \quad \|(\phi - \mathcal{R}_{n_0}) \mathcal{M}_c \varphi\|_{\mathcal{U}_p^{\mathbf{C}'_\pm, t, s}} \le C_{F,f,\delta} 2^{-(r-1-2\delta-(t-s))n_0}\,.$$

Therefore, one must replace $-(r-1) < s < -t < 0$ by $t - (r-1) < s < -t < 0$ in Lemma 4.2, and thus in[5] Theorem 4.1.

5. Fixing the end of the proof of Sublemma 4.4

The formulas for some kernels in the proof of Sublemma 4.4 (p. 548) are garbled. The corrections are detailed below. The statement of the sublemma is unchanged.

Lines 9–14 and the footnote of p. 548 must be replaced by "Recalling the functions b_m from (53), we claim that there exists a constant $C_0 > 1$ depending only on $C_{\mathcal{F}}$ and $\mathbf{C}_\pm$ so that, for any $\Gamma \in \mathcal{F}(\mathbf{C}_+)$ and all n, n_s, the kernels $V^{n,-}_{n_s,\Gamma}(w,y)$ defined for $w \in \Gamma$ and $y \in \mathbb{R}^d$ by

$$\int_{\mathbb{R}^d} \mathbb{F}^{-1}(\psi_{\Theta',n,-})(-x)(\phi \cdot \psi^{Op(\Gamma+x)}_{n_s} \tilde{\varphi})(w+x) \mathrm{d}x = \frac{1}{(2\pi)^{d+d_s}} \int_{\mathbb{R}^d} V^{n,-}_{n_s,\Gamma}(w,y) \tilde{\varphi}(y)\, \mathrm{d}y$$

satisfy,[21] (60) $\quad |V^{n,-}_{n_s,\Gamma}(w,y)| \le C_0 2^{-(r-1)n} b_{n_s}(w-y)$ if $C_0 2^{n_s} \le 2^n$ or $2^{n_s} \ge C_0 2^n$."

Replace lines 15–19 of p. 548 by: "To prove (60), recall (16) and note that

$$V^{n,-}_{n_s,\Gamma}(w,y) = \int_{\eta \in \mathbb{R}^d,\, x_-, \eta_s \in \mathbb{R}^{d_s}} e^{-ix(y,x_-)\eta} e^{i(\pi_{\Gamma+x(y,x_-)}(w+x(y,x_-)) - z(y,x_-))\eta_s}$$
$$\times \frac{\phi(w+x(y,x_-))}{|\det D\mathcal{Y}_{\Gamma,x_-}(\mathcal{Y}^{-1}_{\Gamma,x_-}(y))|} \psi^{(d_s)}_{n_s}(\eta_s) \psi_{\Theta',n,-}(\eta) \mathrm{d}\eta \mathrm{d}\eta_s \mathrm{d}x_-\,,$$

using for each $x_- \in \mathbb{R}^{d_s}$ the C^r change of variable $y = \mathcal{Y}_{\Gamma,x_-}(z,x_+) := \pi^{-1}_{\Gamma+(x_-,x_+)}(z)$ in $\mathbb{R}^d$, with $z \in \mathbb{R}^{d_s}$ and $x_+ \in \mathbb{R}^{d_u}$, setting also

$$x(y,x_-) = (x_-, \Pi_+(\mathcal{Y}^{-1}_{\Gamma,x_-}(y)))\,, \quad z(y,x_-) = \pi_{\Gamma+x(y,x_-)}(y)\,,$$

where $\Pi_+ : \mathbb{R}^{d_s+d_u} \to \mathbb{R}^{d_u}$ is defined by $\Pi_+(x_-,x_+) = x_+$. Next just like in [10, 11] (see also [Lemma 2.34, 2]), using that $\pi_{\Gamma+x}(w+x) = \pi_\Gamma(w) + x_-$ if $x = (x_-,x_+) \in \mathbb{R}^{d_s} \times \mathbb{R}^{d_u}$ and that $\Gamma \in \mathcal{F}$, first integrate by parts (see Appendix 3) $(r-1)$ times with respect to $x_- \in \mathbb{R}^{d_s}$ in the formula for $V^{n,-}_{n_s,\Gamma}(w,y)$, and second, noticing that $\|y-w\| > \epsilon$ implies that either $\|\pi_\Gamma(w) - \pi_{\Gamma+x(y,x_-)}(y)\| > \epsilon/(2C_0)$ or

---

[5] The condition on s and t was not explicited in Theorem 4.1.

[21] For the kernels $V^{n,+}_{n_s,\Gamma+x}(w,y)$ defined by replacing $\psi_{\Theta',n,-}$ with $\psi_{\Theta,n,+}$, we only get $C_0 > 1$ so that $|V^{n,+}_{n_s,\Gamma}(w,y)| \le C_0 2^{-(r-1)n} b_{n_s}(w-y)$ if $C_0 2^{n_s} \ge 2^n$. In particular, $V^{n,+}_{n_s,\Gamma}$ need not be small if n is big and n_s small.

$\|\Pi_+(\mathcal{Y}^{-1}_{\Gamma,x_-}(y))\| > \epsilon/(2C_0)$, integrate by parts with respect the other variables as many times as necessary. It is an enlightening exercise to prove (60) for affine Γ."

There is a minor typo in the left-hand side of (64) on p. 549, which should read:

$$\| \int \mathbb{F}^{-1}(\psi_{\Theta',n,-})(-x)\cdot(\mathcal{R}_{\tilde n_s,\Gamma+x})(\tilde\varphi)(\cdot+x)\mathrm{d}x\|^s_{p,\Gamma} \le \sup_x C_1 2^{-(r-1)m_0}\|\tilde\varphi\|^s_{p,\Gamma+x}\,. \tag{64}$$

Finally, (64) follows from (60) and the leafwise Young inequality (38). □

6. Typos

On p. 537, the condition $\mathbb{R}^{d_s}\times\{0\}\subset \mathbf{C}_-$ must be replaced by "$\mathbb{R}^{d_s}\times\{0\}$ is included in $(\mathbb{R}^d\setminus\mathbf{C}_+)\cup\{0\}$" (thrice, including Defs 3.2–3.3). Also, the assumptions ensure that Π_Γ is surjective. Same page, 6 lines after (17), the norms are equivalent uniformly in Γ, not equal, due to the Jacobian. In Lemma 4.2, one must assume that F can be extended by a bilipschitz regular cone hyperbolic diffeomorphism $\tilde F$ of $\mathbb{R}^d$, with $\|\tilde F\|_+$, $1/\|\tilde F\|_-$, $1/\|\tilde F\|_{--}$ and $1/|\det(D\tilde F|_{(\mathbf{C}'_+)^\perp})|$ controlled by twice the corresponding constants for F. In lines 2–3 of p. 555, the sum is over all $\ell\ge 0$, and in line 2, one of the $(1+\|x_-\|)^{Q_1}$ must be replaced by $(1+\|x_+\|)^{Q_2}$, while $\psi^{Op}_\ell \upsilon(x)$ should be replaced by $\|\psi^{Op}_\ell\upsilon\|_{L_\infty}$.

References

[1] V. Baladi, *The quest for the ultimate anisotropic Banach space,* J. Stat. Phys. **166** (2017) 525–557 (in honour of Ruelle and Sinai). DOI: 10.1007/s10955-016-1663-0.

[2] V. Baladi, *Characteristic functions as bounded multipliers on anisotropic spaces,* arXiv:1704.00157.

[3] V. Baladi and S. Gouëzel, *Banach spaces for piecewise cone hyperbolic maps,* J. Modern Dynam. **4** (2010) 91–135.

[4] V. Baladi, Y. Jiang, and O.E. Lanford, *Transfer operators acting on Zygmund functions,* Trans. Amer. Math. Soc. **348** (1996) 1599–1615.

[5] V. Baladi and M. Tsujii, *Dynamical determinants and spectrum for hyperbolic diffeomorphisms,* In Probabilistic and Geometric Structures in Dynamics, pp. 29–68, Contemp. Math., **469**, Amer. Math. Soc., Providence, RI (2008).

[6] T. Runst and W. Sickel, Sobolev spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, Walter de Gruyter & Co., Berlin (1996).