

# Fonctions zêta dynamiques (Dynamical Zeta Functions)

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## 1. INTRODUCTION

Dynamical zeta functions or dynamical determinants are power series  $\zeta(z)$ , respectively  $d(z)$ , which are constructed from (weighted) periodic orbit data arising from a, say, discrete-time dynamical system  $f : M \rightarrow M$  and a function  $g : M \rightarrow \mathbb{C}$ , and which play the part of a (generalised) Fredholm determinant for the transfer operator  $\mathcal{L}$  (on a suitable Banach space) associated to  $f$  and the weight  $g$ , in the sense that they define a meromorphic, respectively holomorphic, function in some domain where their poles (respectively zeroes) are in bijection with the inverse eigenvalues of  $\mathcal{L}$  in this domain. Although we shall not explain this here, the spectral properties of  $\mathcal{L}$  are often closely related to the statistical properties of the dynamical system.

*Exercise 0.* Let  $L$  be a finite matrix with complex coefficients. Check that

$$\det(\text{Id} - zL) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \text{Tr } L^n .$$

In this introduction, we shall study the case of a one-dimensional dynamical system (i.e. a transformation of a compact interval) and see how far Exercise 0 can take us. The course will then be devoted to a presentation of more sophisticated arguments, inspired initially by the work of Milnor and Thurston, which will allow us to treat completely the one-dimensional situation. We shall then discuss much more recent results in higher dimensions.

### 1.1 The one-dimensional setting – the transfer operator $\mathcal{L}$ .

Let  $I = [0, 1]$  be the unit interval (one could take any other compact interval) and let  $f : I \rightarrow I$  be a continuous map which is piecewise monotone and piecewise  $C^1$  with inverse branches having a derivative of bounded variation. This means that we

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assume that there is a partition of  $I$  into  $N$  nontrivial subintervals  $I_j = [a_j, a_{j+1}]$ ,  $j = 0, \dots, N-1$  (here we only consider the case of finite  $N$ ) such that:

- (1) the restriction of  $f$  to  $I_j = [a_j, a_{j+1}]$  is strictly monotone for each  $j = 0, \dots, N-1$ ;
- (2) the restriction of  $f$  to  $I_j$  extends to a strictly monotone  $C^1$  map to a small neighbourhood of  $I_j$ , and the absolute value of the derivative of the inverse  $\psi_j$  of this map

$$g_j := \chi_{f(I_j)} \cdot \psi_j' = \chi_{f(I_j)} \cdot \frac{1}{|f' \circ f|_{I_j}^{-1}}$$

is of bounded variation (on the closure of  $f(I_j)$ ). Here, we are slightly abusing notation and we have set  $f'(a_i) = \lim_{x < a_i, x \rightarrow a_i} f'(x) = f'(a_i-)$  for  $i = 1, \dots, N$ , and  $f'(a_0) = f'(a_0+)$ .

We shall *not* assume that the intervals  $I_j$  are maximal for the monotonicity property (1).

We recall for the convenience of the reader that a function  $g : \mathbb{R} \rightarrow \mathbb{C}$  is of bounded variation, noted  $g \in BV$  if

$$\text{var}_{\mathbb{R}} g = \sup_{m, t_i} \left\{ \sum_{i=0}^m |g(t_i) - g(t_{i+1})|, t_0 < t_1 < \dots < t_{m+1} \right\} < \infty.$$

If  $J \subset \mathbb{R}$  then  $g$  is of bounded variation on  $J$ , noted  $g \in BV(J)$  if  $\text{var}_J(g) < \infty$  where  $\text{var}_J(g)$  is the above supremum restricting the partitions to  $t_i \in J$ .

*Exercise 1.* Check that assumption (1) implies that  $\sup_j g_j$  is finite.

Recall that  $f_*(\mu)$ , if  $\mu$  is a finite complex Borel measure, is defined by  $f_*(\mu)(E) = \mu(f^{-1}(E))$  for all Borel sets  $E$ . Exercise 1 can be used to prove that the *transfer operator* (also called Ruelle operator, or, in this specific context, Perron-Frobenius or density transformer operator) defined as an operator on  $L^1 = L^1(I, \text{Leb})$  by

$$\int \psi(\mathcal{L}\varphi) d\text{Leb} = \int \varphi(\psi \circ f) d\text{Leb}, \forall \varphi \in L^1, \psi \in L^\infty(I, \text{Leb}),$$

or, equivalently

$$(\mathcal{L}\varphi) d\text{Leb} = f_*(\varphi d\text{Leb}),$$

or finally

$$\mathcal{L}\varphi(x) = \sum_{f y = x} \frac{\varphi(y)}{|f'(y)|} = \sum_{j=0}^{N-1} g_j(x) \varphi \circ \psi_j(x),$$

is bounded, and that

$$\left| \int_I \mathcal{L}\varphi d\text{Leb} \right| \leq \int_I |\varphi| d\text{Leb},$$

i.e. the norm of  $\mathcal{L}$  on the Banach space  $L^1$  is (at most) 1.

*Exercise 2.* Find an example of  $f$  such that  $\mathcal{L}$  does not preserve the Banach space  $C^0(I)$  of continuous functions on  $I$ .

*Exercise 3.* Show that if there is a nonnegative  $\varphi_0 \in L^1$  with  $\mathcal{L}\varphi_0 = \varphi_0$  (and  $\int \varphi_0 dLeb > 0$ ) then the measure  $\mu_0 = \varphi_0 dLeb / \int \varphi_0 dLeb$  is an (absolutely continuous)  $f$ -invariant (probability) measure, i.e.  $f_*(\mu_0) = \mu_0$ .

Absolutely continuous invariant measures  $\mu_0$  are especially interesting when they are ergodic. Indeed, the Birkhoff ergodic theorem then says that for a set of *positive Lebesgue measure* of initial conditions  $x$  we have ( $\delta_y$  denotes the Dirac mass at  $y$  and the convergence is in the weak\* topology):

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x} = \mu_0.$$

This is often interpreted as an indication that such a measure  $\mu_0$  is a natural, or physical, measure for  $f$ .

We shall not discuss this here, but one can prove for example if  $\sup_j \inf g_j < 1$  that:

- (1) 1 is indeed an eigenvalue (of finite multiplicity) of  $L^1$ , and the spectrum of  $L^1$  on the unit circle consists in roots of unity  $e^{2ik\pi/k_0}$ ,  $k = 0, \dots, k_0 - 1$  for some integer  $k_0 \geq 1$ ; in fact, 1 is a simple eigenvalue if and only if  $\mu_0$  is ergodic and in this case  $k_0 = 1$  if and only if  $\mu_0$  is mixing;
- (2) every point in the open unit disc is an eigenvalue of infinite multiplicity of  $\mathcal{L}$  acting on  $L^1$  (the spectrum of  $\mathcal{L}$  is therefore the entire closed unit disc).

It appears that  $L^1$  is not very suitable to obtain spectral information reflecting finer statistical properties (stability of the absolutely continuous invariant measure under small deterministic or probabilistic perturbations, exponential decay of correlations for suitable observables, central limit theorem, etc.) of the dynamics. In some sense,  $L^1$  is too big a Banach space and we should find a smaller invariant Banach space. (Note that the Hilbert space  $L^2$  suffers from the same “problems” as  $L^1$ .) In our one-dimensional framework, the most natural candidate is the Banach space  $BV = BV(I)$  of functions of bounded variation on  $I$ . (If we had assumed that the partition satisfies a Markov property – see below – it would be possible to consider other choices.)

## 1.2 The transfer operator acting on $BV$ : quasicompactness.

Let us consider the Banach space

$$BV = BV(I) = \{\varphi : I \rightarrow \mathbb{C}, \text{var}_I \varphi < \infty\},$$

endowed with the norm

$$\|\varphi\|_{BV(I)} = \text{var}_I \varphi + \sup_I |\varphi|.$$

(The supremum term is here to distinguish constant functions on  $I$ , it could be replaced e.g. by the  $L^1$  norm, or also by substituting  $\text{var}_I$  by  $\text{var}_{\mathbb{R}}$ .)

In fact, it will be more convenient to consider the quotient

$$\mathcal{B} = BV/\mathcal{N}, \|\varphi\| = \|\varphi\|_{\mathcal{B}} = \inf\{\|\phi\|_{BV(I)} \mid \phi - \varphi \in \mathcal{N}\},$$

where  $\mathcal{N}$  is the space of complex-valued functions on  $I$  which vanish except on an at most countable set.

*Exercise 4.* Show that  $BV(I)$  and  $\mathcal{B}$  are indeed Banach spaces (i.e. they are complete) and that  $\mathcal{L}$  maps  $BV(I)$  into  $BV(I)$  boundedly.

We are now ready to state and prove our first result:

**Theorem 1 (Quasicompactness of  $\mathcal{L}$  on  $\mathcal{B}$ ).** *Let  $f$  and  $\mathcal{L}$  be as above. Then  $\mathcal{L}$  is a bounded operator on  $\mathcal{B} = BV(I)/\mathcal{N}$ , and, outside of the closed disc of radius*

$$\widehat{R} := \limsup_{n \rightarrow \infty} \sup_x \left( \frac{1}{|(f^n)'(x)|} \right)^{1/n},$$

*the spectrum of  $\mathcal{L}$  consists in isolated eigenvalues of finite multiplicity.*

One can prove additionally that the spectrum of  $f$  on  $BV(I)$  and on  $\mathcal{B}$  coincide outside of the disc of radius  $\widehat{R}$ , we shall not do this here.

*Proof of theorem 1.* Since the transfer operator is a sum of operators of composition and multiplication, and since the variation is essentially the  $L^1$  norm of the distributional derivative, the ingredients are the Leibniz formula (derivative of a product) and the chain rule (derivative of a composition). We present a conceptual proof due to Ruelle, the starting point of which is to replace elements of  $\mathcal{B}$  by Radon measures:

**Lemma 0 (Bounded variation and Radon measures).** *The Banach spaces*

$$\mathcal{B}' = \{\varphi \in \mathcal{B} \mid \varphi(0+) = 0\}$$

*and*

$$C^0(I)^* = \{\nu : C^0(I) \rightarrow \mathbb{C} \mid \text{linear and continuous}\}$$

*are isomorphic, the Banach space isomorphism being given by the distributional derivative (Stieltjes measure associated to a function of bounded variation)*

$$d\varphi(c, d] = \varphi(d+) - \varphi(c+), (c, d] \subset I, \quad \varphi \in BV.$$

*The inverse of  $d$  will be denoted by  $\mathcal{S}$ , and satisfies  $\mathcal{S}\mu(x) = \mu([0, x])$ .*

Recall that if  $\nu$  is a Radon measure and  $\varphi$  a bounded function the Radon measure  $\varphi\nu$  is defined by  $\varphi\nu(\psi) = \nu(\varphi\psi)$  where we use that  $C^0(I)^*$  is the space of bounded complex Borel measures on the compact metric space  $I$ .

**Lemma 1 (Leibniz formula in BV/integration by parts).** Let  $\varphi_1, \varphi_2 \in BV(I)$ . In the quotient  $\mathcal{B} = BV(I)/\mathcal{N}$  we may take representatives which are continuous at  $a_0 = 0$  and which only have regular discontinuities  $2\varphi_i(x) = \varphi_i(x+) + \varphi_i(x-)$ . Then, for these representatives

$$d(\varphi_1\varphi_2) = \varphi_1d(\varphi_2) + \varphi_2d\varphi_1,$$

in the sense of Radon measures.

**Lemma 2 (Change of variables).** Let  $J \subset I$  be an interval,  $\psi : J \rightarrow \psi(J)$  be a homeomorphism, and  $\varphi \in BV(I)$ . Then

$$\chi_J d(\varphi \circ \psi) = (\epsilon\chi_J)(\psi^{-1})_*(d\varphi),$$

where  $\epsilon = +1$  if  $\psi$  preserves the orientation and  $-1$  if  $\psi$  reverses the orientation.

The sign in Lemma 2 comes from the fact that if  $(c, d] \subset J$ , e.g. then

$$d(\varphi \circ J)(c, d] = \varphi(\psi(d+)) - \varphi(\psi(c+))$$

while

$$(\psi^{-1})_*(d\varphi)(c, d] = (d\varphi)\psi(c, d] = \begin{cases} (d\varphi)(\psi c, \psi d] = \varphi(\psi(d+)) - \varphi(\psi(c+)) & \text{if } \epsilon > 0, \\ (d\varphi)[\psi d, \psi c) = \varphi(\psi(c+)) - \varphi(\psi(d+)) & \text{if } \epsilon < 0. \end{cases}$$

The complete proofs of Lemmas 0–2 are to be found, e.g., in [DS1] or in [Ba2].

*Back to the proof of Theorem 1.* Our first step is to replace the transfer operator  $\mathcal{L}$  on  $\mathcal{B}$  by  $\mathcal{M}$ , the rank-one perturbation of  $\mathcal{L}$  given by

$$\mathcal{M}\varphi = \mathcal{L}\varphi - \mathcal{L}\varphi(0+).$$

The arguments below will show that if we can prove the claim on  $\mathcal{M}$  acting on  $\mathcal{B}$ , then it will be automatically satisfied for  $\mathcal{L}$  acting on  $\mathcal{B}$ . The next observation is that  $\mathcal{M}$  maps  $\mathcal{B}$  into  $\mathcal{B}'$ , so that it suffices to analyse the spectrum of  $\mathcal{M}$  on  $\mathcal{B}'$  (recall the definition of the nonzero spectrum and write  $(\text{Id} - z\mathcal{M})^{-1} = (\text{Id} - z\mathcal{M})^{-1}z\mathcal{M} + \text{Id}$ ).

The operator on  $C^0(I)^*$  conjugated to  $\mathcal{M} : \mathcal{B}' \rightarrow \mathcal{B}'$  by the isomorphisms of Lemma 0 is  $d \circ \mathcal{M} \circ \mathcal{S}$ . Applying Lemmas 1 and 2, it is not difficult to see that  $d\mathcal{M}\mathcal{S}$  can be decomposed as

$$d\mathcal{M}\mathcal{S} = \widehat{\mathcal{M}} + \widehat{\mathcal{N}}\mathcal{S},$$

where, setting  $\epsilon_i = +1$  if  $\psi_i$  is increasing and  $\epsilon_i = -1$  if  $\psi_i$  is decreasing

$$\begin{aligned} \widehat{\mathcal{M}}(\mu)(\varphi) &= \sum_{i=0}^{N-1} \chi_{(a_i, a_{i+1}]} \epsilon_i ((g_i\varphi) \circ \psi_i^{-1}) d\mu \\ &= \int \epsilon_f \frac{\varphi \circ f}{|f'|} d\mu, \end{aligned}$$

where  $\epsilon_f = \epsilon_i$  on  $(a_i, a_i + 1)$  and 0 on the  $a_i$ s, or, introducing the more compact notation  $g|_{(a_i, a_{i+1}]} := g_i \circ \psi_i^{-1}$  (and  $g(a_0) = g(a_0 +)$ ),

$$\widehat{\mathcal{M}}(\mu) = (\epsilon_f \cdot g) f_*(\mu), \quad \widehat{\mathcal{M}} : C^0(I)^* \rightarrow C^0(I)^*,$$

and

$$\widehat{\mathcal{N}}(\varphi) = \sum_{i=0}^{N-1} d(g_i) \varphi \circ \psi_i, \quad \widehat{\mathcal{N}} : \mathcal{B}' \rightarrow C^0(I)^*.$$

(We shall not need this but note that  $\widehat{\mathcal{N}}$  can be viewed as  $d\widehat{\mathcal{M}}_0 - \widehat{\mathcal{M}}d$  where  $\widehat{\mathcal{M}}_0$  is  $\widehat{\mathcal{M}}$  “acting on functions.”)

It is not difficult to check that the spectral radius of  $\widehat{\mathcal{M}}$  on  $C^0(I)^*$  is not larger than  $\widehat{\mathcal{R}}$ . The next important step is encapsulated in a last sublemma:

**Lemma 3.** *Let  $\mu_0$  be a fixed Radon measure. Then the linear operator on  $C^0(I)^*$*

$$\mu \mapsto (\mathcal{S}\mu) \cdot \mu_0$$

*is compact*

*Sketch of the proof of Lemma 3.* Since  $\mu \mapsto \mathcal{S}\mu$  is bounded from  $C^0(I)^*$  to  $\mathcal{B}'$  it suffices to see that the operator  $\mathcal{K}$  from  $\mathcal{B}'$  to  $C^0(I)^*$  defined by

$$\mathcal{K}(\varphi) = \varphi \cdot \mu_0$$

is compact. For this, we need to prove that any sequence  $\mathcal{K}(\varphi_n)$  with  $\text{var } \varphi_n \leq 1$  admits a convergent subsequence. This can be deduced from the fact that  $\mathcal{K}$  can be approached by a sequence of finite-rank operators. To construct this sequence, note that for any  $\delta > 0$  there is a finite partition  $c_0 < \dots < c_i < \dots < c_M$  of  $I$  such that the norm of the Radon measure  $\chi_{(c_i, c_{i+1})} \mu_0$  is smaller than  $\delta$  for each  $0 \leq i < M$ . Setting

$$\Pi_\delta \varphi|_{(c_i, c_{i+1})} = \varphi(c_i +)$$

for all  $0 \leq i < M$ , it is clear that  $\Pi_\delta : \mathcal{B}' \rightarrow \mathcal{B}'$  is a finite-rank (rank  $M$ , in fact, projection), so that the operator  $\mathcal{K}_\delta := \mathcal{K}\Pi_\delta$  is also finite-rank from  $\mathcal{B}'$  to  $C^0(I)^*$ . Finally, it is not difficult to show that

$$\|(\mathcal{K}\varphi - \mathcal{K}_\delta\varphi)\| = \|\varphi\mu_0 - \Pi_\delta\varphi\mu_0\| \leq \delta \text{var}_I \varphi.$$

(A full proof can be found, e.g., in [Ba2].)

*End of the proof of Theorem 1.* Applying Lemma 3 to the finite set of measures of the form  $\mu_0 = d(g_i)$ , we see that  $\widehat{\mathcal{N}}\mathcal{S}$  is a compact operator. If the reader knows that a perturbation of a bounded operator of spectral radius  $\rho$  by a compact operator can only add isolated eigenvalues of finite multiplicity outside of the disc of radius  $\rho$  (“compact

perturbations do not change the essential spectral radius”), he or she should be satisfied. Otherwise, let us proceed with the proof, decomposing the resolvent as

$$(\lambda \text{Id} - d\mathcal{MS})^{-1} = (\lambda \text{Id} - (\widehat{\mathcal{M}} + \widehat{\mathcal{N}}\mathcal{S}))^{-1} = (\lambda \text{Id} - \widehat{\mathcal{M}})^{-1}(\text{Id} - \widehat{\mathcal{N}}\mathcal{S}(\lambda \text{Id} - \widehat{\mathcal{M}})^{-1})^{-1}.$$

If  $|\lambda| > \widehat{R}$ , the resolvent  $(\lambda \text{Id} - \widehat{\mathcal{M}})^{-1}$  of  $\widehat{\mathcal{M}}$  is a bounded operator (depending holomorphically on  $\lambda$  in the domain  $C_{\widehat{R}} = \{|\lambda| > \widehat{R}\}$ ). Therefore, the operator

$$\mathcal{Q}(\lambda) = \widehat{\mathcal{N}}\mathcal{S}(\lambda \text{Id} - \widehat{\mathcal{M}})^{-1},$$

being the composition of a bounded operator and a compact operator, is compact. It also depends holomorphically on  $\lambda$  in  $C_{\widehat{R}}$ . Our aim is therefore to study the set

$$\text{Sing} = \{\lambda \in C_{\widehat{R}} \mid 1 \text{ is an eigenvalue of } \mathcal{Q}(\lambda)\}.$$

Our first remark is that  $\text{Sing}$  is a discrete subset of (a bounded subset of) the complex plane. Indeed, as  $|\lambda| \rightarrow \infty$ , the spectral radius of  $\mathcal{Q}(\lambda)$  goes to zero, so that  $\mathcal{Q}(\lambda)$  cannot have an eigenvalue 1 for all  $\lambda$  in the connected domain  $C_{\widehat{R}}$ . Since the nonzero eigenvalues of a family of compact operators depending analytically on a parameter  $\lambda$  are either constant or take any fixed value on a discrete set, we are done. (The last claim is analogous to the corresponding result for finite matrices – simple eigenvalues depend analytically on analytic perturbations, multiple eigenvalues can have at worse algebraic (roots) singularities — and both statements can be found in [Ka, II.§1 and VII.§1], e.g.)

It remains to be seen that any point in the discrete set  $\text{Sing}$  is an eigenvalue of finite multiplicity of  $\widehat{\mathcal{M}} + \widehat{\mathcal{N}}\mathcal{S}$ . If  $\lambda \in \text{Sing}$ , it is not difficult to associate to the fixed function  $\varphi_\lambda$  of  $\mathcal{Q}(\lambda)$  an eigenfunction of  $\widehat{\mathcal{M}} + \widehat{\mathcal{N}}\mathcal{S}$  for the eigenvalue  $\lambda$ . (This is left as an exercise to the reader.) This does not completely end our task, since this eigenvalue could in principle have infinite multiplicity. In order to finish the proof, we present a few reminders about the theory of spectral projectors associated to isolated points in the spectrum of an operator (see, e.g., [Ka]).

So let  $\lambda_0$  be an isolated point in the spectrum of a bounded linear operator  $L$  on a Banach space (we are not assuming that  $\lambda_0$  is an eigenvalue). This implies that there is a nontrivial complex disc  $D_\gamma(\lambda_0)$  centered at  $\lambda_0$  which does not intersect any other point in the spectrum of  $L$ . Letting  $\gamma = \gamma(\lambda_0)$  be the path corresponding to going along once the circle bounding this disc counterclockwise, we define a bounded operator on our Banach space by

$$P_{\lambda_0}^L = \frac{1}{2i\pi} \oint_\gamma (\lambda \text{Id} - L)^{-1} d\lambda.$$

(To check that the sign is correct, consider  $L \equiv 0$  and  $\lambda_0 = 0$  and verify that  $P_0^L = \text{Id}$ .) Let us verify that  $P_{\lambda_0}^L$  is a projector, i.e.  $(P_{\lambda_0}^L)^2 = P_{\lambda_0}^L$ . For this, the first remark is that if  $\gamma'$  is the circle centered at  $\lambda_0$  and of radius one-half the radius of  $\gamma$  (for example) then

since  $(\lambda \text{Id} - L)^{-1}$  is holomorphic in the annulus bounded by the two circles we can also write

$$P_{\lambda_0}^L = \frac{1}{2i\pi} \oint_{\gamma'} (\lambda' \text{Id} - L)^{-1} d\lambda'.$$

Therefore, using the easily checked “resolvent identity”

$$(\lambda \text{Id} - L)^{-1} - (\lambda' \text{Id} - L)^{-1} = (\lambda - \lambda')(\lambda \text{Id} - L)^{-1}(\lambda' \text{Id} - L)^{-1},$$

we find

$$P_{\lambda_0}^L P_{\lambda_0}^L = \frac{1}{(2i\pi)^2} \int_{\gamma} \int_{\gamma'} (\lambda - \lambda')^{-1} [(\lambda \text{Id} - L)^{-1} - (\lambda' \text{Id} - L)^{-1}] d\lambda' d\lambda.$$

We finish by observing that

$$\frac{1}{2i\pi} \int_{\gamma'} (\lambda - \lambda')^{-1} d\lambda' = 0,$$

and

$$\frac{1}{2i\pi} \int_{\gamma} (\lambda - \lambda')^{-1} d\lambda = 1.$$

$P_{\lambda_0}^L$  being a projector, it follows that  $\text{Id} - P_{\lambda_0}^L$  is also a projector, and clearly the two projectors are orthogonal (i.e.  $P_{\lambda_0}^L(\text{Id} - P_{\lambda_0}^L) = 0 = (\text{Id} - P_{\lambda_0}^L)P_{\lambda_0}^L$ ). Also, one easily checks that the definition implies  $LP_{\lambda_0}^L = P_{\lambda_0}^L L$ , and, of course, similarly for the other projector. Finally, one can show that the nonzero spectrum of  $LP_{\lambda_0}^L$  consists in the single point  $\lambda_0$ , while the spectrum of  $L(\text{Id} - P_{\lambda_0}^L)$  does not intersect the closed disc  $D_{\gamma}$  centered at  $\lambda_0$ . Now, in the case when  $P_{\lambda_0}^L$  is finite rank, the operator  $LP_{\lambda_0}^L$  acting on the finite-dimensional Banach space  $\text{Im } P_{\lambda_0}^L$  is of course finite rank, so that its spectrum (which we already know is  $\{\lambda_0\}$ ) must be an eigenvalue of finite multiplicity. (Note that  $\text{Im } P_{\lambda_0}^L$  is the generalised eigenspace, i.e. it does not always contain only eigenfunctions but also generalised eigenfunctions  $\varphi$  such that  $(\lambda_0 \text{Id} - L)^k \varphi = 0$  for some  $k \geq 2$  but  $\neq 0$  for  $k = 1$ . In particular, the dimension of  $\text{Im } P_{\lambda_0}^L$  is the algebraic multiplicity of  $\lambda_0$ .)

Let us return to our specific problem, i.e. showing that a point  $\lambda_0$ , such that  $1 = \rho(\lambda_0)$  is an eigenvalue of  $\mathcal{Q}(\lambda_0)$ , is an eigenvalue of finite multiplicity of  $d\mathcal{MS}$ . Since the nonzero spectrum of a compact operator consists in isolated points, and since the small perturbations  $\mathcal{Q}(\lambda)$  in operator norm of  $\mathcal{Q}(\lambda_0)$  produce small perturbations  $\rho(\lambda)$  of our isolated eigenvalue 1, up to taking a smaller isolating disc  $D_{\gamma}(\lambda_0)$  for  $d\mathcal{MS}$  and the spectral point  $\lambda_0$ , we may find a disc centered at 1, bounded by a curve  $\Gamma = \Gamma(\lambda_0, \gamma)$ , such that for each  $\lambda$  in  $D(\lambda_0)$ , the curve  $\Gamma(\lambda_0, \gamma)$  does not intersect the spectrum of  $\mathcal{Q}(\lambda)$ . In particular, the spectral projectors

$$P_{\rho(\lambda)}^{\mathcal{Q}(\lambda)} = \frac{1}{2i\pi} \int_{\Gamma} (\rho \text{Id} - \mathcal{Q}(\lambda))^{-1} d\rho,$$

will have constant rank equal to the multiplicity of 1 for  $\mathcal{Q}_{\lambda_0}$  for all  $\lambda \in \gamma = \gamma(\lambda_0)$ .

It will follow that the spectral projector

$$P_{\lambda_0}^{d\mathcal{M}\mathcal{S}} = \frac{1}{2i\pi} \int_{\gamma} (\lambda \text{Id} - d\mathcal{M}\mathcal{S})^{-1} d\lambda,$$

can be written as a path integral of a finite-rank operator. Indeed, we may use the finite rank spectral projectors  $P_{\rho(\lambda)}^{\mathcal{Q}(\lambda)}$  associated to the perturbation of the eigenvalue 1 for  $\mathcal{Q}_{\lambda_0}$  to refine our previous decomposition of the resolvent of  $d\mathcal{M}\mathcal{S}$  as:

$$\begin{aligned} (\lambda \text{Id} - d\mathcal{M}\mathcal{S})^{-1} &= (\lambda \text{Id} - \widehat{\mathcal{M}})^{-1} (\text{Id} - \widehat{\mathcal{N}}\mathcal{S}(\lambda \text{Id} - \widehat{\mathcal{M}})^{-1})^{-1} \\ &= (\lambda \text{Id} - \widehat{\mathcal{M}})^{-1} (\text{Id} - \mathcal{Q}(\lambda) P_{\rho(\lambda)}^{\mathcal{Q}(\lambda)})^{-1} P_{\rho(\lambda)}^{\mathcal{Q}(\lambda)} \\ &\quad + (\lambda \text{Id} - \widehat{\mathcal{M}})^{-1} (\text{Id} - \mathcal{Q}(\lambda))^{-1} (\text{Id} - P_{\rho(\lambda)}^{\mathcal{Q}(\lambda)}). \end{aligned}$$

The second term in the above decomposition being holomorphic in the disc bounded by  $\gamma$ , the corresponding path integral vanishes. The first term is the composition of a bounded operator and a finite-rank operator, it is thus finite-rank. It also depends holomorphically on  $\lambda$  on  $\gamma$  (and meromorphically on  $\lambda$  in the disc bounded by  $\gamma$ ).

To finish, it suffices to note that the path integral of finite-rank operators being compact, the projector  $P_{\lambda_0}^{d\mathcal{M}\mathcal{S}}$  is compact and therefore necessarily finite-rank.  $\square$

*Exercise 6.* Note that since the dual of  $\mathcal{L}$  acting on the dual of  $\mathcal{B}$  preserves Lebesgue measure, the operator  $\mathcal{L}^*$  has a fixed point in  $\widehat{\mathcal{B}}^*$ . Assume that  $\widehat{R} < 1$ . Since the spectrum of  $\mathcal{L}$  on  $\mathcal{B}$  outside of the disc of radius  $\widehat{R}$  consists only of isolated eigenvalues of finite multiplicity, show that any eigenvalue of  $\mathcal{L}^*$  in this domain must be an eigenvalue of  $\mathcal{L}$ . It follows that  $\mathcal{L}$  has a fixed point in  $\mathcal{B}$ .

### 1.3 The dynamical zeta function in the Markov expanding affine case.

We shall now combine Theorem 1 and Exercise 0 to obtain a result on the dynamical zeta function of  $f$ , but only under three additional assumptions which are quite restrictive. In some sense, the purpose of the course is to show how one can get rid of these assumptions. Here are the first two:

- (1) We suppose that  $f'|_{(a_i, a_{i+1})}$  is constant for each  $i$ ;
- (2) and that the partition of  $I$  into  $N$  intervals  $I_j$  satisfies the Markov property: if  $f(I_j) \cap \text{Int } I_\ell \neq \emptyset$  then  $\text{Int } I_\ell \subset f(I_j)$ . (In other words, each  $\overline{f(I_j)}$  can be written exactly as a union of  $\overline{I_\ell}$ s.)

Let us consider the  $N$ -dimensional vector subspace of  $\mathcal{B}$  defined by

$$\mathcal{V} = \{ \varphi \in \mathcal{B} \mid \varphi|_{(a_i, a_{i+1})} \text{ is constant} \}.$$

It is not difficult to check that  $\mathcal{L}$  maps  $\mathcal{V}$  into itself. Also, introducing the  $N \times N$  Markov (or transition) matrix associated to  $f$ :

$$A_{jk} = 1 \text{ if } f(I_j) \cap \text{Int } I_k \neq \emptyset, \quad A_{jk} = 0 \text{ if } f(I_j) \cap \text{Int } I_k = \emptyset,$$

it is easy to verify that the matrix of  $\mathcal{L}|_{\mathcal{V}}$  in the standard basis is given by the matrix  $A_g$  defined by

$$(A_g)_{jk} = A_{jk}g_j.$$

The spectrum of  $\mathcal{L}$  on  $\mathcal{V}$  is thus a well defined set of eigenvalues (of finite multiplicity). Note that we do not claim that this set intersects the complement of the disc of radius  $\widehat{R}$  (but see Exercise 6 and Lemma 4).

Our first result is:

**Lemma 4.** *Assume (1–2). Outside of the closed disc of radius  $\widehat{R}$ , the spectrum of  $\mathcal{L}$  on  $\mathcal{B}$  coincides with the spectrum of  $\mathcal{L}$  on  $\mathcal{V}$ .*

*Proof of Lemma 4.* Since  $\mathcal{V} \subset \mathcal{B}$ , if  $\varphi \in \mathcal{V}$  is an eigenfunction for  $\lambda$  and  $\mathcal{L}$  then  $\lambda$  is also an eigenvalue for  $\mathcal{L}$  acting on  $\mathcal{B}$ . (Note that this is also true if  $|\lambda| \leq \widehat{R}$ .) To show the reverse inclusion, let us suppose that there is  $\lambda$  with  $|\lambda| > \widehat{R}$  and a nonzero  $\varphi \in \mathcal{B}$  with  $\mathcal{L}\varphi = \lambda\varphi$  (we know by Theorem 1 that in this domain the spectrum of  $\mathcal{L}$  on  $\mathcal{B}$  consists in eigenvalues). Then, using the operator  $\mathcal{M}$  from the proof of Theorem 1, we have for all  $k \in \mathbb{N}$

$$\varphi - \frac{\varphi(0+)}{\lambda} = \lambda^{-k} \mathcal{M}^k \varphi.$$

Let us rewrite the right-hand-side of the above equality in the coordinates given by Lemma 0, using the notation there

$$\lambda^{-k} \widehat{\mathcal{M}}^k d\varphi + \lambda^{-k} \sum_{m_i, n_i \in \{0,1\}, \sum m_i + n_i = k} \prod_{i=1}^k \widehat{\mathcal{M}}^{m_i} (\widehat{\mathcal{N}}\mathcal{S})^{n_i} (d\varphi).$$

For any  $\widetilde{R} > \widehat{R}$  there is  $C > 0$  so that  $\|\widehat{\mathcal{M}}^k(d\varphi)\| \leq C\widetilde{R}^k \|\varphi\|$ , so that, taking  $\widetilde{R} > |\lambda| > \widehat{R}$ , we see that the first term in the above decomposition goes to zero as  $k$  goes to infinity. Let us then concentrate on the second term. Our Markov and piecewise affine assumptions imply that each of the  $N$  measures  $d(g_i)$  is a (positive) linear combination of the two Dirac masses at  $f(a_i) = a_{u(i)}$  and  $f(a_{i+1}) = a_{v(i)}$  (by construction, the weight of  $\delta_{a_0}$  vanishes). Multiplying  $d(g_i)$  by an arbitrary element  $\psi$  of  $\mathcal{B}$  amounts to changing the coefficients of this linear combination (using  $\psi(a_{u(i)\pm})$  and  $\psi(a_{v(i)\pm})$ ). Therefore, it suffices to analyse the action of  $\widehat{\mathcal{M}}^\ell$  ( $\ell \geq 1$ ) on a linear combination of Dirac masses at the  $a_j$ s: Using again the Markov assumption, we see that  $\widehat{\mathcal{M}}^\ell$  is still a linear combination at the endpoints  $a_j$ . Putting everything together, we see that  $d(\varphi - \varphi(0+)/\lambda)$  must be a linear combination of Dirac masses at the  $a_j$ s, so that  $\varphi$  belongs to  $\mathcal{V}$ , as claimed.  $\square$

From now on until the end of the introduction, we make our third and final additional assumption:

$$(3) \quad \widehat{R} < 1.$$

We shall use the notation  $\text{Fix } f$  for the set

$$\text{Fix } f = \{x \in I \mid f(x) = x\},$$

and similarly for  $\text{Fix } f^n$  for each nonnegative integer.

*Exercise 7.* Prove that assumption (3) implies that for each  $n$  the set  $\text{Fix } f^n$  is finite. (We shall obtain more precise information on this set soon.)

Exercise 7 allows us to define the weighted dynamical zeta function of  $f$  by the following formal power series (recall the notation  $g$  from the proof of Theorem 1):

$$\zeta(z) = \zeta_{f,g}(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \text{Fix } f^n} \prod_{k=0}^{n-1} g(f^k(x)).$$

Here is the final result of the introduction:

**Theorem 2 (Zeta function and spectrum in the Markov expanding affine case).** *Assume (1–3) from Subsection 1.1 and (1–3) from Subsection 1.3. Then the weighted zeta function  $\zeta(z)$  is meromorphic in the open disc of radius  $\widehat{R}^{-1} > 1$ , where its poles are exactly the inverse eigenvalues (outside of the closed disc of radius  $\widehat{R}$ ) of  $\mathcal{L}$  acting on  $\mathcal{B}$ . The order of each such pole  $z$  coincides with the algebraic multiplicity of the corresponding eigenvalue  $1/z$ .*

*Proof of Theorem 4.* The Markov assumption allows us to construct a symbolic model for  $f : I \rightarrow I$  which is a subshift of finite type (SFT). Let us define this SFT, recalling the transition matrix  $A$ . Consider the set  $\Sigma$  of one-sided sequences with coefficients in the finite alphabet  $\{0, \dots, N-1\}$ , and the subset  $\Sigma_A \subset \Sigma$  of sequences with  $A$ -admissible transitions, i.e.

$$\Sigma_A = \{t \in \Sigma \mid A_{t_i t_{i+1}} = 1, \forall i \in \mathbb{Z}\}.$$

(This is a compact set for the product topology arising from the discrete topology on our finite alphabet.) The one-sided shift to the left  $(\sigma(t))_i = t_{i+1}$  leaves  $\Sigma_A$  invariant. We next construct a semi-conjugacy between  $\sigma|_{\Sigma_A}$  on  $\Sigma_A$  and  $f$  on  $I$ , i.e. a surjective map  $\pi : \Sigma_A \rightarrow I$  with  $f \circ \pi = \pi \circ \sigma$  on  $\Sigma_A$ . For this one first observes that for each  $t \in \Sigma_A$  the set  $\bigcap_{i=0}^{\infty} f^{-i} I_{t_i}$  is a single point in  $I$ . (This set is nonempty because the sequence is admissible, and it has zero diameter because we assumed  $\widehat{R} < 1$ .) Setting

$$\pi(t) = \bigcap_{i=0}^{\infty} f^{-i} I_{t_i},$$

we obtained the desired conjugacy. To check surjectiveness, note that each trajectory of each point  $x \in I$ , i.e. each admissible sequence of symbols  $t_i$  with  $f^i(x) \in \overline{I_{t_i}}$  gives  $t \in \Sigma_A$  with  $\pi(t) = x$ . There may be an ambiguity if  $f^i(x) = a_j$  for some  $i, j$ , so that the map  $\pi$  is not injective in general.

In order to obtain a conjugacy (i.e. a bijection making the diagram commute), it is convenient to slightly modify the original interval map  $f$  on  $I$  by “doubling” the  $N-1$  points  $a_1, \dots, a_{N-1}$  and all their countably many preimages  $f^{-k} a_i$ ,  $k \geq 1$ . Between each such pair of doubled points we introduce a small interval of length, say,  $\epsilon/(N^k k^2)$ , in such a way that the total added length is finite. This allows us to embed our Cantor

set  $\widehat{I}$  into a compact interval of the real line. Abusing slightly notation,  $I \subset \widehat{I}$  and the closed intervals  $\overline{I}_j$  are disjoint in  $\widehat{I}$ . We may extend  $f$  to  $\widehat{I}$ , the new map  $\widehat{f}$  being just  $f$  in the interior of each  $I_j$ s and being set to  $f(a_i+)$ , respectively  $f(a_i-)$  in the new right or left boundaries. Similarly, we extend the weight  $g$  to  $\widehat{I}$  by taking the appropriate left or right limit. It should be clear that  $\pi$  is now a bijection between the Cantor sets  $\Sigma_A$  and  $\widehat{I}$ , such that  $\widehat{f}\pi = \pi\sigma$  on  $\Sigma_A$ .

Let us next analyse the weighted zeta function of  $\sigma$  and the weight  $\widehat{g} \circ \pi$ , i.e.

$$\zeta_{\sigma, \widehat{g}\pi}(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{t \in \text{Fix } \sigma^n} \prod_{k=0}^{n-1} \widehat{g}(\pi(\sigma^k(t))).$$

We claim that

$$\sum_{t \in \text{Fix } \sigma^n} \prod_{k=0}^{n-1} \widehat{g}(\pi(\sigma^k(t))) = \text{Tr } A_g^n.$$

Indeed, the above equality is obvious for  $n = 1$ . More generally we have

$$(A_g^n)_{ii} = \sum_{i=i_0, \dots, i_{n-1}, (\vec{i})^\infty \in \Sigma_A} \prod_{k=0}^{n-1} g_{i_k},$$

where  $(\vec{i})^\infty$  means the finite (length- $n$ ) sequence  $\vec{i}$  repeated infinitely many times. All fixed points of  $\sigma^n$  are obtained that way, and we have  $g(\pi(\sigma^k(\vec{i}^\infty))) = g_{i_k}$ .

Applying Exercise 0 to the finite matrix  $A_g$ , it follows that

$$\zeta_\sigma(z) = (\det(\text{Id} - zA_g))^{-1}.$$

By Lemma 4, we know that the eigenvalues of  $A_g$  outside of the disc of radius  $\widehat{R}$  are in bijection with the spectrum of  $\mathcal{L}$ , outside the disc of radius  $\widehat{R}$ , acting on  $\mathcal{B}$ .

By construction

$$\zeta_{\widehat{f}, \widehat{g}}(z) = \zeta_{\sigma, \widehat{g}\pi}(z),$$

so that, it suffices to understand the relation between  $\zeta_{f,g}(z)$  and  $\zeta_{\widehat{f}, \widehat{g}}(z)$  to end the proof. More precisely it suffices to show that their ratio is a nonzero holomorphic function in the disc of radius  $\widehat{R}^{-1}$ . Clearly, the periodic points of  $f$  whose orbits do not meet any of the  $a_i$ s are in bijection with the periodic points of  $\widehat{f}$  whose orbits do not meet  $a_0$ ,  $a_N$ , and any of the twins  $a_{i\pm}$ ,  $i = 1, \dots, N-1$ ; also the contributions of these “good” periodic points to the respective zeta functions coincide. (Recall the Markov assumption.) So let us consider one of the finitely many possible periodic points  $a_i = f^{p(i)}(a_i)$  (assuming that  $p(i) \geq 1$  is minimal for the fixed point property). There are three possibilities: if  $a_i$  is a local extremum for  $f^{p(i)}$  then either  $a_i+$  (if the extremum is a minimum) or  $a_i-$  (maximum), but not both, will be a periodic point for  $\widehat{f}$ , with minimal period equal to  $p(i)$ . If  $f^{p(i)}$  is increasing in a neighbourhood of  $a_i$  (or if  $i = 0$ ),

then both  $a_{i+}$  and  $a_{i-}$  are periodic of minimal period  $p(i)$  for  $\hat{f}$ . If  $f^{p(i)}$  is decreasing in a neighbourhood of  $a_i$ , then both  $a_{i+}$  and  $a_{i-}$  will be periodic points for  $\hat{f}$ , but their minimal period will be  $2p(i)$ . The analysis just made also describes the (finitely many) periodic points of  $\hat{f}$  whose orbits meet  $a_0$ ,  $a_N$ , or any of the twins  $a_{i\pm}$ ,  $i = 1, \dots, N-1$ .

Let us consider one of these finitely many “bad” periodic orbits  $x = f^p x$ , or  $\hat{x} = \hat{f}^{\hat{p}} \hat{x}$ , with  $p, \hat{p} \geq 1$  its minimal period and  $\lambda = \prod_{k=0}^{p-1} g(f^k x)$  respectively  $\hat{\lambda} = \prod_{k=0}^{\hat{p}-1} \hat{g}(\hat{f}^k \hat{x})$  the associated weight. Clearly, the corresponding contribution to the weighted zeta function is:

$$\exp \sum_{m=1}^{\infty} \frac{z^{mp}}{mp} \lambda^m = (1 - z^p \lambda)^{-1/p}, \text{ or } \exp \sum_{m=1}^{\infty} \frac{z^{m\hat{p}}}{m\hat{p}} \hat{\lambda}^m = (1 - z^{\hat{p}} \hat{\lambda})^{-1/\hat{p}}.$$

To finish, it suffices to observe that  $\lambda \geq \hat{R}^p$ ,  $\hat{\lambda} \geq \hat{R}^{\hat{p}}$ . (This follows from the definition for  $\lambda$ , while a short argument is required for  $\hat{\lambda}$ .)  $\square$

## 2. KNEADING THEORY IN DIMENSION ONE

### 2.1 Introduction.

Although the ideas at the basis of the kneading theory in these notes are present in the very classical paper of Milnor and Thurston [MT], which was written in the seventies, they were only applied to weighted zeta functions and the analysis of the spectra of transfer operators in the nineties. Before that, other methods had been developed (in dimensions one and higher, and under various assumptions of expansion, hyperbolicity, and/or regularity) in the continuation of the Markov approach of Section 1. In this introductory section, we first give a very brief and incomplete presentation of some of the results obtained by these older methods between 1976 and now, referring to [Ba1, Ba3] for more general surveys; we then give a very brief presentation of the key result of Milnor and Thurston which inspired the new kneading approach.

#### The Markov approach for piecewise monotone maps

This approach consists in viewing the situation of the subshift of finite type and locally constant weight as the paradigm, and trying to make more general weighted dynamical systems fit into this model.

The first generalisation of Theorem 2 in Section 1 consists in maintaining the assumption that the piecewise monotone interval map  $f$  is piecewise affine (or more generally consider a locally constant weight  $g_i$ , which could be unrelated to the derivative), but relaxing the Markov assumption. (Historically, the case of Markov maps with non locally constant weights was studied earlier by Mayer e.g., since the important Gauss map  $x \mapsto \{1/x\}$  fits in this framework – see also the discussion about analytic systems below.) A helpful tool here is the Markov extension devised by Hofbauer in the seventies. This associates to any piecewise monotone interval map  $f$  a semiconjugated map  $\tilde{f}$  which possesses a countable Markov partition into intervals (the so-called Hofbauer tower). Using this tool, Hofbauer and Keller [HK] proved in 1984 that Theorem 2 from

Section 1 holds even if the initial interval map  $f$  does not admit a finite Markov partition into intervals where it is monotone.

The next generalisation consists in allowing  $1/|f'|$  (or a more general weight  $g$ ) to be of bounded variation (sometimes, an additional assumption of continuity is used). The idea here is of course to approach  $g$  by a locally constant weight, where locally constant changes meaning as the initial partition is refined by the dynamics by considering finite intersections  $\cap_{i=0}^M f^{-k} I_{t_k}$ . (Here, instead of assuming that  $\widehat{R} < 1$  it is enough to suppose that the initial partition is generating, i.e. that the diameter of  $\cap_{i=0}^{\infty} f^{-k} I_{t_k}$  is zero.) The corresponding version of Theorem 2 was published in 1990 by Baladi and Keller, a different proof (based on a slightly different “Markov” philosophy) is contained in the book of Ruelle [Ru2].

Piecewise injective (and piecewise expanding) maps have been studied also in higher dimensions. The spectral theory of the transfer operator is more technical (for example it is not obvious which Banach space to use!). The survey [Ba3] contains references to the results of Saussol, Buzzi, Tsujii, and others. A version of the Hofbauer tower can be constructed, and Buzzi and Keller [BK] recently used it to prove an analogue of Theorem 2 from Section 1 in the case when  $f$  is piecewise affine, piecewise expanding (in higher dimensions), and not necessarily Markov.

The Hofbauer tower, or variants of it, has also been successful to prove a version of Theorem 2 for one-dimensional (quadratic e.g.) maps with critical points (Keller-Nowicki) i.e.  $c$  so that  $f'(c) = 0$ .

It must be noted that the approach we just described is a little heavy to implement.

#### **The Markov approach for smooth “hyperbolic” maps**

A different class of problems is given by  $C^r$  ( $r > 1$ ) maps on compact manifolds which are assumed to be uniformly expanding or uniformly hyperbolic. There, classical results of Bowen, Sinai, Ruelle, and others, guarantee the existence of a finite Markov partition. (The definition of Markov is slightly more involved in the hyperbolic case, it also guarantees semiconjugacy with a SFT.) So the “only” problem here is to approach nonconstant weights  $g$  (such as  $|\det Df|^{-1}$ ) by locally constant weights. For Holder  $g$ , this was done by Ruelle, Pollicott, and Haydn who proved a version of Theorem 2 from Section 1 for hyperbolic diffeomorphisms. In the case of higher smoothness  $r \geq 2$  and  $g$  at least  $C^1$ , the natural object is in fact not the weighted zeta function, but a weighted dynamical determinant of the type

$$d(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \text{Fix } f^n} \frac{\prod_{k=0}^{n-1} g(f^k(x))}{|\det(Df^n(x) - \text{Id})|}.$$

It is possible to express the dynamical zeta function as an alternated product of dynamical determinants. If  $f$  and  $g$  are  $C^\omega$  (real-analytic) then results from Cauchy and Grothendieck can be applied, and Ruelle [Ru0] showed in 1976 that  $d(z)$  is entire, and  $\zeta_{f,g}(z)$  is meromorphic if  $f$  is additionally assumed to be (locally) uniformly expanding. The hyperbolic case produces serious additional difficulties, and the fact that  $\zeta(z)$  is meromorphic in the whole complex plane was proved much more recently by Rugh [Rug]

in dimension 2 and Fried [Fr] in general. The case of  $C^r$  (non analytic) data was treated by Ruelle [Ru1] for (locally) expanding maps. The hyperbolic  $C^r$  case is still partially mysterious, despite an important breakthrough by Kitaev [Kit].

Similar results exist for continuous-time dynamics (flows and semi-flows).

### The original Milnor and Thurston formula

Let us now return to the situation of a continuous piecewise monotone transformation  $f$  of a compact interval. Milnor and Thurston do not make any additional assumptions in [MT], so that the sets  $\text{Fix } f^n$  may have infinite (even uncountable) cardinality. However, if  $f^n$  is decreasing on an interval  $J$ , it may have at most one fixed point in  $J$ . Each set

$$\text{Fix}^- f^n = \{x \in I \mid f^n(x) = x, f^n \text{ is decreasing in a neighbourhood of } x\}$$

therefore has finite cardinality, and it makes sense to define a “negative zeta function”

$$\zeta^-(z) = \exp \sum_{n=1}^{\infty} 2 \cdot \#\text{Fix}^-(f^n)$$

(the naive idea is that doubling “negative” fixed points makes up for the “forgotten” “positive” fixed points – a more precise interpretation, making use of Lefschetz signs, is presented in the 1996 paper of Ruelle quoted in [BaR]).

Milnor and Thurston then introduce the *kneading matrix*. If  $f$  has  $N$  maximal intervals of monotonicity, it is an  $(N-1) \times (N-1)$  matrix with coefficients power series in  $z$ , the coefficients of which belong to  $\{-1, 0, +1\}$ . The  $i$ th line of this matrix is

$$\frac{\theta(a_{i+})(z) - \theta(a_i-)(z)}{2}, \quad i = 1, \dots, n-1,$$

where (we use the notation  $\epsilon_f$  from Section 1) the kneading coordinate  $\theta(x)(z)$  is the power series

$$\theta(x)(z) = \sum_{k=0}^{\infty} z^k \prod_{j=0}^{k-1} \epsilon_f(f^j(x)) \cdot \alpha(f^k(x)),$$

where  $\alpha(y)$  is the  $N-1$ -tuple

$$(\text{sgn}(y - a_j), j = 0, \dots, N),$$

with

$$\text{sgn}(\xi) = \begin{cases} -1 & \xi < 0, \\ 0 & \xi = 0, \\ 1 & \xi > 0. \end{cases}$$

One of the key results in [MT] is the following remarkable equality:

**Theorem 0 (Milnor-Thurston identity).**

$$\zeta^-(z) = \frac{1 - z(\epsilon(a_0+) + \epsilon(a_N-)/2)}{\det(1 + D(z))}.$$

As an immediate consequence, the negative zeta function is meromorphic in the unit disc.

The above result is extremely beautiful, but (for the moment) a bit mysterious. The proof of Milnor and Thurston (a homotopy argument involving the bifurcations of a path  $f_t$  of piecewise monotone maps between  $f = f_1$  and a “trivial” map  $f_0$  having the same intervals of monotonicity as  $f$  and whose graph is strictly under the diagonal) does not give any insight on “why” Theorem 1 holds. Note also that it is not clear how to introduce weights in the negative zeta functions, and that there is no spectral interpretation of the zeroes or poles of the dynamical zeta function. The purpose of the remainder of Section 2 is to describe the one-dimensional kneading theory which addresses these points.

*Exercise 0.* Check that  $\frac{\theta(a_i+)(0) - \theta(a_i-)(0)}{2}$  is indeed the vector  $(0, \dots, 0, 1, 0, \dots, 0)$  where the 1 is at the  $i$ th position. Prove the Milnor-Thurston identity for  $f_0$ .

## 2.2 The setting – Essential spectral radius.

It will turn out to be more convenient to consider a slightly more general setting, allowing the  $\psi_i$  to be “independent” local homeomorphisms (in particular, with the possibility that  $\text{Im } \psi_i \cap \text{Im } \psi_j \neq \emptyset$  for  $i \neq j$ ) instead of the local inverse branches of a piecewise monotone interval map  $f$  as was the case until now.

### The data

We fix a compact interval  $I \subset \mathbb{R}$  and a finite set  $\Omega$  of indices. (The restriction to finite  $\Omega$  is mostly for convenience and countable or even uncountable index-sets endowed with a positive, not necessarily finite, measure can also be used. See [Ru3] and the exercises and remarks below.)

For each  $\omega \in \Omega$ , we take a nonempty open subinterval  $I_\omega$  of  $I$  and a (local) homeomorphism

$$\psi_\omega : I_\omega \rightarrow \psi_\omega(I_\omega),$$

assuming  $\psi_\omega(I_\omega) \subset I$  and setting  $\epsilon_\omega = +1$  if  $\psi_\omega$  preserves orientation and  $\epsilon_\omega = -1$  otherwise. We also consider a function  $g_\omega : \mathbb{R} \rightarrow \mathbb{C}$  satisfying:

- (1)  $g_\omega$  is supported in  $I_\omega$ ,
- (2)  $\text{var } g_\omega < \infty$ ,
- (3)  $g_\omega$  is continuous (on  $I$ , say).

The third assumption allows us to use the Leibniz formula. Taking the  $\psi_\omega = \psi_j$  to be the local inverse branches of a piecewise monotone  $f$  (with  $I_j = f(a_j, a_{j+1})$ ), and  $g_j = \chi_{I_j} |f' \circ \psi_j|^{-1}$  this assumption is not satisfied by all examples in Section 1, but it can be essentially weakened, see [Go].

*The dual system (Exercise 1).* Given  $\Omega$ ,  $\psi_\omega$  and  $g_\omega$  as above we may introduce a new “dual” system by setting  $\widehat{I}_\omega = \psi_\omega(I_\omega)$ ,  $\widehat{\psi}_\omega = \psi_\omega^{-1}$ , and  $\widehat{g}_\omega = \chi_{\widehat{I}_\omega} \epsilon_\omega g_\omega \circ \psi_\omega^{-1}$ . Check that this new system satisfies all of the conditions (in particular  $\widehat{g}_\omega$  is continuous contrary to what its expression might suggest). Abusing notation, we shall write  $\widehat{g}_\omega = \epsilon_\omega g_\omega \circ \psi_\omega^{-1}$ .

### The transfer operators

**Definition of the transfer operators.** We associate to our data two transfer operators acting either on the Banach space  $L^\infty$  of bounded functions (modulo functions which vanish except on an at most countable set), or on the Banach space  $\mathcal{B} = BV(I)/\mathcal{N}$ :

$$\begin{aligned}\mathcal{M}\varphi &= \sum_{\omega} g_\omega \cdot \varphi \circ \psi_\omega, \\ \widehat{\mathcal{M}}\varphi &= \sum_{\omega} \widehat{g}_\omega \cdot \varphi \circ \widehat{\psi}_\omega = \sum_{\omega} \epsilon_\omega g_\omega \circ \psi_\omega^{-1} \cdot \varphi \circ \psi_\omega^{-1}.\end{aligned}$$

We shall use the notation

$$\widehat{\mathcal{R}} = \limsup_{n \rightarrow \infty} \sup_{\varphi, \sup|\varphi| \leq 1} \sup_I |\widehat{\mathcal{M}}^n \varphi|^{1/n}.$$

*Exercise 2.* Show that both  $\mathcal{M}$  and  $\widehat{\mathcal{M}}$  are bounded on both Banach spaces considered. Show that the lim sup defining  $\widehat{\mathcal{R}}$  is in fact a limit and is the spectral radius of  $\widehat{\mathcal{M}}$  on  $L^\infty$ . If  $\Omega$  is countable, find sufficient assumptions on the  $\psi_\omega$  and  $g_\omega$  which imply that both operators are bounded on both Banach spaces.

( $\widehat{\mathcal{M}}$  as the dual of  $\mathcal{M}$ ). A priori,  $\widehat{\mathcal{M}}$  depends on the data  $I$ ,  $\Omega$ ,  $\psi_\omega$ ,  $g_\omega$ . (Using partitions of unity, it is easy to obtain different data giving rise to the same operator.) It is possible to show [BaRu] that in fact it only depends on  $\mathcal{M}$  as an operator (on  $\mathcal{B}$ , say) and not on the representation of  $\mathcal{M}$  given by the  $g_\omega$  and  $\psi_\omega$ . For the moment we shall not need this fact (we just have to be aware that a preferred representation must always be given, at least implicitly) but we slightly abuse terminology by viewing  $\widehat{\mathcal{M}}$  as a dual of  $\mathcal{M}$  (see also the following exercise).

*Exercise 3.* Check (using each time the “obvious” representation of the transfer operator) that  $\widehat{\widehat{\mathcal{M}}} = \mathcal{M}$  and that  $\widehat{\mathcal{M}_1 \mathcal{M}_2} = \widehat{\mathcal{M}_2} \widehat{\mathcal{M}_1}$  for all transfer operators  $\mathcal{M}$ ,  $\mathcal{M}_1$ ,  $\mathcal{M}_2$ .

*Exercise 4.* If the  $\psi_\omega$  are the local inverse branches of a piecewise monotone interval map and  $g_\omega = \chi_{I_\omega} g \circ \psi_\omega$  for a single function  $g$ , find a simpler expression for  $\widehat{\mathcal{M}}$  (and check that it is compatible with the notation used in Section 1).

For convenience, let us now introduce terminology that we have avoided until now (see [Ba2] for more):

**Definition of essential spectral radius.** Let  $L : B \rightarrow B$  be a bounded linear operator on a Banach space  $B$ . The *essential spectral radius*  $\rho_{\text{ess}}(L)$  of  $L$  is

$$\rho_{\text{ess}}(L) = \inf\{\rho > 0 \mid \text{if } \lambda \in \text{sp}(L) \text{ and } |\lambda| > \rho \implies \lambda \text{ is an isolated eigenvalue of finite multiplicity}\}.$$

In other words, outside of the disc of radius  $\rho_{\text{ess}}$  the spectrum is just like the spectrum of a compact operator. If one can prove that the essential spectral radius is strictly smaller than the spectral radius, one often says that the operator is quasicompact.

**Theorem 1 (Bound on  $\rho_{\text{ess}}(\mathcal{M})$ ).** *Let  $\mathcal{M}$  be as above. Then the essential spectral radius of  $\mathcal{M}$  acting  $\mathcal{B} = BV/\mathcal{N}$  is at most equal to  $\widehat{R}$ .*

*Remark.* It is possible to show that the spectral radius of  $\mathcal{M}$  on  $\mathcal{B}$  is at most  $\max(R, \widehat{R})$  where  $R$  is the spectral radius of  $\mathcal{M}$  acting on  $L^\infty$ . The situation discussed at the end of Section 1 involved the quasicompact case where  $\widehat{R} < R$ . One can easily construct examples of data so that  $R < \widehat{R}$  (just take  $\Omega$  a singleton and  $\psi_\omega$  a linear expansion) or  $\widehat{R} = R$  (in the case where  $\psi_\omega$  is the identity, e.g.). Note that it is possible to obtain *lower bounds* on the essential spectral radius, but this is much more tricky, since we have to control sums of nonnecessarily positive numbers. (See [Go].)

*Proof of Theorem 1.* This can be proved just like Theorem 1 of Section 1. Checking it is a good exercise.  $\square$

### 2.3 Sharp traces and sharp determinants.

We next define a (formal) trace and its associated (formal) determinant for transfer operators, the “sharp trace”  $\text{Tr}^\# \mathcal{M}$  and “sharp determinant”  $\text{Det}^\#(1 - z\mathcal{M})$ . We shall immediately prove some of their basic properties, but it is only in the next section that we shall introduce the so-called “kneading operators” which will allow us to show (in § 2.5) that the zeroes of  $\text{Det}^\#(1 - z\mathcal{M})$  in a suitable disc describe some of the inverse eigenvalues of  $\mathcal{M}$  on  $\mathcal{B}$ .

It is possible to show [BaRu] that the following definition only depends on  $\mathcal{M}$  as an operator on  $\mathcal{B}$  (instead, we always assume that a preferred representation is given):

**Definition (Sharp trace and sharp determinant).** Let  $\mathcal{M}$  be a transfer operator associated to data  $\psi_\omega, g_\omega$  as in § 2.2. Then we write:

$$\text{Tr}^\# \mathcal{M} = \sum_\omega \int \frac{1}{2} \text{sgn}(\psi_\omega(x) - x) dg_\omega(x),$$

and (as a formal power series in  $z$ )

$$\text{Det}^\#(1 - z\mathcal{M}) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \text{Tr}^\# \mathcal{M}^n.$$

The expression for the trace is well-defined since each  $dg_\omega$  is a complex measure and the integrand is a bounded function. Note for future use that  $\text{sgn}(\psi_\omega x - x)$  is a function of bounded variation.

*Exercise 5.* Show that  $d\text{sgn}$  is twice the Dirac mass at 0. (See also the proof of Lemma 2 in § 1). Rewrite the expression for  $\text{Tr}^\# \mathcal{M}$  if each  $g_\omega$  is  $C^1$ .

Let us first prove an easy but very useful result:

**Lemma 1 (“Functional relation”).**  $\text{Tr}^\# \widehat{\mathcal{M}} = -\text{Tr}^\# \mathcal{M}$ . In particular

$$\text{Det}^\#(1 - z\widehat{\mathcal{M}}) = \frac{1}{\text{Det}^\#(1 - z\mathcal{M})}.$$

*Proof of Lemma 1.* The proof is based on the change of variable formula (Lemma 2 from Section 1):

$$\begin{aligned} \text{Tr}^\# \widehat{\mathcal{M}} &= \sum_\omega \int \epsilon_\omega \frac{1}{2} \text{sgn}(\psi_\omega^{-1}(x) - x) d(g_\omega \circ \psi_\omega^{-1}(x)) \\ &= \sum_\omega \int \frac{1}{2} \text{sgn}(y - \psi_\omega(y)) d(g_\omega(y)) \\ &= -\text{Tr}^\# \mathcal{M}. \end{aligned}$$

The second claim is left as an exercise.  $\square$

The next lemma requires more effort, but it gives legitimacy to the “trace” terminology (not yet in the sense that the trace is related to the eigenvalues, however):

**Lemma 2 (Trace property).**

$$\text{Tr}^\#(\mathcal{M}_1 \mathcal{M}_2) = \text{Tr}^\#(\mathcal{M}_2 \mathcal{M}_1).$$

*Exercise 6.* Show that Lemma 2 implies that we may perform the usual algebraic manipulations on the sharp determinants i.e.:

$$\begin{aligned} \text{Det}^\#(1 - z\mathcal{M}_1 \mathcal{M}_2) &= \text{Det}^\#(1 - z\mathcal{M}_2 \mathcal{M}_1) \\ \text{Det}^\#(1 - z\mathcal{M}_1) \text{Det}^\#(1 - z\mathcal{M}_2) &= \text{Det}^\#(1 - z(\mathcal{M}_1 + \mathcal{M}_2 + z\mathcal{M}_1 \mathcal{M}_2)). \end{aligned}$$

*Proof of Lemma 2.* By linearity of the sharp trace, it is enough to prove the lemma in the case where  $\Omega_1$  and  $\Omega_2$  are both singletons i.e.

$$\mathcal{M}_1 \varphi = g_1 \cdot \varphi \circ \psi_1, \quad \mathcal{M}_2 \varphi = g_2 \varphi \circ \psi_2.$$

First assume that  $\psi_2 \psi_1$  is increasing, and let  $\epsilon = \pm 1$  depending on whether  $\psi_1$  and  $\psi_2$  are increasing or decreasing. Since  $\psi_1$  and  $\psi_2$  are continuous, the set  $\{x : \psi_2 \psi_1 x \neq x\}$

is the union of at most countably many open intervals  $(c_i, d_i)$ . Correspondingly,  $\{y : \psi_1\psi_2y \neq y\}$  is the union of intervals  $(c'_i, d'_i)$  where

$$c'_i = \psi_1c_i = \psi_2^{-1}c_i, \quad d'_i = \psi_1d_i = \psi_2^{-1}d_i,$$

if  $\epsilon = 1$  and

$$c'_i = \psi_1d_i = \psi_2^{-1}d_i, \quad d'_i = \psi_1c_i = \psi_2^{-1}c_i,$$

if  $\epsilon = -1$ . If  $\sigma_i$  is the sign of  $\psi_2\psi_1x - x$  on  $(c_i, d_i)$ , then  $\sigma'_i = \epsilon\sigma_i$  is the sign of  $\psi_1\psi_2y - y$  on  $(c'_i, d'_i)$ .

We have

$$\begin{aligned} \mathrm{Tr} \# \mathcal{L}_1 \mathcal{L}_2 &= \int d(g_1(x)g_2(\psi_1(x))) \frac{1}{2} \mathrm{sgn}(\psi_2\psi_1x - x) \\ &= \frac{1}{2} \sum_i \int_{c_i}^{d_i} d(g_1(x)g_2(\psi_1(x))) \sigma_i \\ &= \frac{1}{2} \sum_i \sigma_i [g_1(d_i)g_2(\psi_1d_i) - g_1(c_i)g_2(\psi_1c_i)] \\ &= \frac{1}{2} \sum_i \sigma_i \epsilon [g_1(\psi_2d'_i)g_2(d'_i) - g_1(\psi_2c'_i)g_2(c'_i)] \\ &= \frac{1}{2} \sum_i \sigma'_i [g_2(d'_i)g_1(\psi_2d'_i) - g_2(c'_i)g_1(\psi_2c'_i)] \\ &= \int d(g_2(y)g_1(\psi_2(y))) \frac{1}{2} \mathrm{sgn}(\psi_1\psi_2y - y) \\ &= \mathrm{Tr} \# \mathcal{L}_2 \mathcal{L}_1. \end{aligned}$$

If  $\psi_2\psi_1$  is decreasing, either it has no fixed point and  $\psi_1\psi_2$  has no fixed point either, or it has a unique fixed point  $c$  and

$$c' = \psi_1c = \psi_2^{-1}c$$

is the unique fixed point of  $\psi_1\psi_2$ . Then

$$\begin{aligned} \mathrm{Tr} \# \mathcal{L}_1 \mathcal{L}_2 &= g_1(c)g_2(\psi_1c) \\ &= g_2(c')g_1(\psi_2c') \\ &= \mathrm{Tr} \# \mathcal{L}_2 \mathcal{L}_1 \end{aligned}$$

concluding the proof.  $\square$

*Exercise 7 (Sharp determinant as a Lefschetz weighted zeta function).* Let  $\mathcal{M}$  be given by data such that for each  $n \geq 1$  and each  $\omega_1, \dots, \omega_n$  with

$$g_{\omega_n} \circ \psi_{\omega_{n-1}} \circ \dots \circ \psi_{\omega_1} \cdots g_{\omega_2} \circ \psi_{\omega_1} \cdot g_{\omega_1}$$

well-defined and nonzero, the set of points fixed by  $\psi_{\omega_n} \circ \dots \circ \psi_{\omega_1}$  is finite. Assume furthermore that each  $\psi_\omega$  is a local diffeo. Show that

$$\mathrm{Tr} \# \mathcal{M} = \sum_{\omega} \sum_{x: \psi_\omega(x)=x} g_\omega(x) \mathrm{sgn}(1 - \psi'_\omega(x)).$$

Write the analogous expressions for  $\mathrm{Tr} \# \mathcal{M}^n$  and  $\mathrm{Det} \#(1 - z\mathcal{M})$ .

#### 2.4 Kneading operators and the key equality.

We consider data  $\psi_\omega$  and  $g_\omega$  as in § 2.2. We finally introduce the weighted analogues of the Milnor and Thurston kneading matrices and prove the corresponding “weighted Milnor-Thurston” identity between determinants.

Let us first consider a slightly *simpler case* where the notation is less heavy: assume additionally that the  $g_\omega$  are  $C^1$  and the  $\psi_\omega$  are local diffeomorphisms. We then introduce two auxiliary transfer operators (we shall see later where they act, we can take e.g.  $\mathcal{B}$ ,  $L^\infty$  or  $L^2(\mathrm{Leb})$ ):

$$\begin{aligned} \mathcal{N}\varphi &= \sum_{\omega} g'_\omega \cdot \varphi \circ \psi_\omega, \\ \widehat{\mathcal{N}}\varphi &= \sum_{\omega} \epsilon_\omega g'_\omega \circ \psi_\omega^{-1} \cdot (\psi_\omega^{-1})' \cdot \varphi \circ \psi_\omega^{-1}. \end{aligned}$$

We also introduce a convolution operator (mapping e.g.  $L^\infty$  functions with compact support to  $L^\infty$  functions):

$$\begin{aligned} \mathcal{S}\varphi(x) &= \int \frac{1}{2} \mathrm{sgn}(x-y) \varphi(y) dy \\ &= \int \frac{1}{2} \mathrm{sgn}(y) \varphi(x-y) dy. \end{aligned}$$

Finally we define the kneading operators  $\mathcal{D}(z)$  and  $\widehat{\mathcal{D}}(z)$  in the sense of formal power series with operator coefficients (using the shorthand notation  $(\mathrm{Id} - z\mathcal{M})^{-1}$  for the formal power series  $\sum_{k \geq 0} z^k \mathcal{M}^k$ ):

$$\mathcal{D}(z) = z\mathcal{N}(\mathrm{Id} - z\mathcal{M})^{-1}\mathcal{S}, \quad \widehat{\mathcal{D}}(z) = z\widehat{\mathcal{N}}(\mathrm{Id} - z\widehat{\mathcal{M}})^{-1}\mathcal{S}. \quad (1)$$

The description of  $\mathcal{D}(z)$  and  $\widehat{\mathcal{D}}(z)$  below as kernel operators will show that the series above define bounded operators on  $L^2(d\mu)$  whenever  $1/z \notin \mathrm{sp} \mathcal{M}$  (acting on  $\mathcal{B}$ ) respectively  $1/z \notin \mathrm{sp} \widehat{\mathcal{M}}$  (acting on  $L^\infty$ ).

**Although  $\mathcal{S}$  can be viewed as a bounded operator from  $L^2(\mathrm{Leb}, I)$  to itself, it does not map  $L^2(\mathrm{Leb})$  into itself boundedly, and  $(\mathrm{Id} - z\mathcal{M})^{-1}$  is not necessarily bounded on  $L^2(\mathrm{Leb})$  or  $L^2(\mathrm{Leb}, I)$  even if  $1/z \notin \mathrm{sp}(\mathcal{M}|\mathcal{B})$ .**

*Exercise 8.* The notation  $\mathcal{S}$  is not strictly compatible with the notation from Section 1. Show that the operator  $\mathcal{S}$  from Lemma 0 in Section 1, when acting on (the densities

of) Radon measures which are absolutely continuous with respect to Lebesgue, can be written as

$$\mathcal{S}\varphi(x) = \int \frac{1}{2}(\operatorname{sgn}(y) + \operatorname{sgn}(x - y)) \varphi(y) dy.$$

In the *general case* (i.e.  $g_\omega$  is continuous and of bounded variation while  $\psi_\omega$  is a local homeomorphism), it is useful to associate a finite nonnegative measure to our data:

$$\mu = \sum_{\omega} |dg_\omega| + \sum_{\omega} |d(g_\omega \circ \psi_\omega^{-1})|.$$

This measure is constructed in such a way as to guarantee that the Radon-Nikodym derivatives  $dg_\omega/d\mu$  and  $d(g_\omega \circ \psi_\omega^{-1})/d\mu$  exist and are bounded. We can now redefine the auxiliary transfer operators:

$$\begin{aligned} \mathcal{N}\varphi &= \sum_{\omega} \frac{dg_\omega}{d\mu} \cdot \varphi \circ \psi_\omega, \\ \widehat{\mathcal{N}}\varphi &= \sum_{\omega} \epsilon_\omega \frac{d(g_\omega \circ \psi_\omega^{-1})}{d\mu} \cdot \varphi \circ \psi_\omega^{-1}. \end{aligned}$$

Similarly, we redefine the convolution operator as:

$$\mathcal{S}\varphi(x) = \int \frac{1}{2} \operatorname{sgn}(x - y) \varphi(y) d\mu.$$

Finally,  $\mathcal{D}(z)$  and  $\widehat{\mathcal{D}}(z)$  are defined as in equation (1) above. Note that  $\mathcal{S}$  is bounded from  $L^2(d\mu)$  to  $L^2(d\mu)$  (restricting  $x$  to the compact support of  $\mu$  in the left-hand-side), but this will not be very useful since the resolvent  $(\operatorname{Id} - z\mathcal{M})^{-1}$  is not bounded on  $L^2(d\mu)$  for all  $1/z \notin \operatorname{sp}(\mathcal{M})$ . What we will do next is notice that  $\mathcal{D}(z)$  and  $\widehat{\mathcal{D}}(z)$  are kernel operators and examine their kernels for appropriate  $z$ , i.e. write

$$\mathcal{D}(z)\varphi(x) = \int_I K^z(x, y) \varphi(y) d\mu(y).$$

This is straightforward: we have, formally

$$\begin{aligned} \mathcal{D}(z)\varphi(x) &= z\mathcal{N}(\operatorname{Id} - z\mathcal{M})^{-1}\mathcal{S}\varphi(x) \\ &= \int [z\mathcal{N}_x(\operatorname{Id} - z\mathcal{M})_x^{-1} \frac{1}{2} \operatorname{sgn}(\cdot - y)](x) \varphi(y) d\mu(y) \\ &= \int \sum_{\omega} z \frac{dg_\omega}{d\mu}(x) (\operatorname{Id} - z\mathcal{M})_x^{-1} \frac{1}{2} \operatorname{sgn}(\cdot - y)(\psi_\omega x) \varphi(y) d\mu(y). \end{aligned}$$

(The index in  $\mathcal{N}_x$  or  $(\operatorname{Id} - z\mathcal{M})_x^{-1}$  is here to emphasize on which variable the transfer operator is acting.) Since  $\operatorname{sgn}(\cdot - y)/2$  is of bounded variation (uniformly) for each  $y$ , it

is clear from the above expression that if  $1/z \notin \text{sp}(\mathcal{M})$  (on  $\mathcal{B}$ ) then the kernel  $K^z(\cdot, y)$  is bounded uniformly in  $y$ , therefore a bounded function of  $x \in I$  and  $y$  in the support of  $\mu$  and since  $L^\infty(I \times I) \subset L^2(d\mu \times d\mu)$  the kernel is in  $L^2(d\mu \times d\mu)$ . Similarly

$$\begin{aligned} \widehat{\mathcal{D}}(z)\varphi(x) &= z\widehat{\mathcal{N}}(\text{Id} - z\widehat{\mathcal{M}})^{-1}\mathcal{S}\varphi(x) \\ &= \int \sum_{\omega} z \frac{d\widehat{g}_{\omega}}{d\mu}(x) (\text{Id} - z\widehat{\mathcal{M}})_x^{-1} \frac{1}{2} \text{sgn}(\cdot - y) [(\widehat{\psi}_{\omega}x) \varphi(y) d\mu(y), \end{aligned}$$

so that the kernel of  $\widehat{\mathcal{D}}(z)$  is in  $L^2(d\mu \times d\mu)$  for all  $1/z \notin \text{sp}\widehat{\mathcal{M}}$ , where we consider  $\widehat{\mathcal{M}}$  acting on bounded functions. It follows that

**Lemma 3.** *For all  $1/z \notin \text{sp}(\mathcal{M})$  (on  $\mathcal{B}$ ) the operator  $\mathcal{D}(z)$  is a compact operator when acting on  $L^2(d\mu)$ , it is in fact Hilbert-Schmidt. For all  $1/z \notin \text{sp}(\widehat{\mathcal{M}})$  (on  $L^\infty$ ) the operator  $\widehat{\mathcal{D}}(z)$  is a compact, in fact Hilbert-Schmidt operator when acting on  $L^2(d\mu)$ .*

In particular, the spectra of both kneading operators for  $|z| < 1/\widehat{R}$  and  $1/z \notin \text{sp}(\mathcal{M}|\mathcal{B})$  consist in eigenvalues of finite multiplicity which can only accumulate at 0, and which are the zeroes of entire functions, the regularised determinants (of order two)

$$\lambda \mapsto \text{Det}_2(1 - \lambda\mathcal{D}(z)), \quad \lambda \mapsto \text{Det}_2(1 - \lambda\widehat{\mathcal{D}}(z)).$$

We refer to Appendix A for the part of the theory of Hilbert-Schmidt operators that we use.

Before finally stating the weighted equivalent of the Milnor and Thurston formula, let us propose an exercise which makes the link between the notation used here and the original definitions in [BaRu].

*Exercise 9.* If  $\mu$  is a finite nonnegative measure on  $I$  and  $K \in L^2(d\mu \times d\mu)$ , we associate to an operator

$$D\varphi(x) = \int K(x, y) \varphi(y) d\mu(y),$$

acting on  $L^2(d\mu)$  another operator on  $L^2(d\mu)$ , noted  $D^*$ , by setting

$$D^*\varphi(x) = \int K(y, x) \varphi(y) d\mu(y).$$

- (1) Show that for every  $m \geq 2$ , the kernel  $K_m(x, y)$  of  $D^m$  coincides with the kernel  $K_m^*(x, y)$  of  $(D^*)^m$  on the diagonal  $x = y$ . (If one has a preferred representant for  $K(x, y)$ , then the statement also makes sense for  $m = 1$ , otherwise the fact that the diagonal has zero measure in the square  $I \times I$  causes problem: understanding this is necessary to do the exercise! See also Lemma 6 in Section 2.)
- (2) Check that the kernel of  $\mathcal{D}(z)^*$  is

$$\sum_{\omega} z \frac{dg_{\omega}}{d\mu}(y) [(\text{Id} - z\mathcal{M})_y^{-1} \frac{1}{2} \text{sgn}(\cdot - x)] \psi_{\omega}(y).$$

**The weighted Milnor and Thurston identity**

Since  $\mathcal{D}(z)$  and  $\widehat{\mathcal{D}}(z)$  are Hilbert-Schmidt on  $L^2(d\mu)$  for each  $z < 1/\widehat{R}$  and  $1/z \notin \text{sp}(\mathcal{M})$  (on  $\mathcal{B}$ ), the theory in Appendix A allows us to introduce formal determinants:

$$\begin{aligned} \text{Det}_*(\text{Id} + \mathcal{D}(z)) &= \exp\left(+ \int_I K^z(x, x) d\mu(x)\right) \cdot \text{Det}_2(\text{Id} + \mathcal{D}(z)) \\ \text{Det}_*(\text{Id} + \widehat{\mathcal{D}}(z)) &= \exp\left(+ \int_I \widehat{K}^z(x, x) d\mu(x)\right) \cdot \text{Det}_2(\text{Id} + \widehat{\mathcal{D}}(z)), \end{aligned}$$

where we also used the fact that the kernels  $K^z(x, y)$  and  $\widehat{K}^z(x, y)$  of  $\mathcal{D}(z)$  and  $\widehat{\mathcal{D}}(z)$  are bounded functions well-defined almost everywhere in the diagonal  $x = y$ .

The first important result of this section is:

**Theorem 4 (Weighted Milnor-Thurston identity).** *In the sense of formal power series, we have:*

$$\text{Det}^\#(1 - z\mathcal{M}) = \text{Det}_*(1 + \widehat{\mathcal{D}}(z)) = \frac{1}{\text{Det}_*(1 + \mathcal{D}(z))}.$$

To prove Theorem 4, we shall express  $\text{Det}_*(\text{Id} + \mathcal{D}(z))$  as an exponential of a sum of “Fredholm-type” traces, i.e. averages of the kernels of  $\mathcal{D}^n(z)$  on the diagonal of  $I \times I$ . For this, the following lemma is essential:

**Lemma 5 (Trace and Fredholm trace for  $L^2$  kernels).** *Let  $\mu$  be a finite non-negative Borel measure on  $I$ . Let  $A : L^2(d\mu) \rightarrow L^2(d\mu)$  be a Hilbert-Schmidt operator described by (an  $L^2(d\mu \times d\mu)$  kernel  $K : I \times I \rightarrow \mathbb{C}$ :*

$$A\varphi(x) = \int K(x, y)\varphi(y) d\mu(y).$$

(In particular,  $A^2 \in \mathcal{S}^1$ .) Then  $\text{Tr} A^2 = \int K(x, y)K(y, x) d\mu(y)$ .

*Proof of Lemma 5.* Combine the Lidskii theorem in the Appendix with exercise 49 of Chapter XI in [DS2].  $\square$

*Consequence of Lemma 5.*

If  $|z| < 1/\widehat{R}$  and  $1/z \notin \text{sp}(\mathcal{M})$  (on  $\mathcal{B}$ ) then, writing  $K^{z,n}(x, y)$  for the kernel of  $\mathcal{D}(z)^n$  acting on  $L^2(d\mu)$ , we have

$$\text{Det}_*(1 + \mathcal{D}(z)) = \exp - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_I K^{z,n}(x, x) d\mu(x).$$

In the proof of Theorem 4, we shall view the above expression only in the sense of formal power series, and it will be convenient to use the decomposition  $\mathcal{D}(z) =$

$z\mathcal{N} \sum_{k=0}^{\infty} z^k \mathcal{M}^k \mathcal{S}$  and the corresponding expressions for the  $K^{z,n}(x,y)$  as formal power series.

Our final ingredient for the proof of Theorem 4 is the following purely algebraic exercise on formal traces and formal determinants:

*Exercise 10 (Properties of formal determinants).* Let  $\mathcal{A}$  be a vector space over  $\mathbb{C}$  which is a subset of an algebra. We write  $\mathcal{A}^{\infty}$  for the set  $\{\mathcal{K} \in \mathcal{A} \mid \mathcal{K}^n \in \mathcal{A}, \forall n \geq 1\}$ . To any function (the *formal trace*)

$$\tilde{\text{tr}} : \mathcal{A} \rightarrow \mathbb{C},$$

we associate a *formal determinant*

$$\widetilde{\text{det}}(\text{Id} + \lambda \cdot) : \mathcal{A}^{\infty} \rightarrow \mathbb{C}[[\lambda]],$$

by setting

$$\widetilde{\text{det}}(\text{Id} + \lambda \mathcal{K}) = \exp - \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n} \tilde{\text{tr}} \mathcal{K}^n.$$

Show that:

- (1) If  $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{A}$  are such that  $\mathcal{K}_1 \mathcal{K}_2 \in \mathcal{A}^{\infty}$  and  $\mathcal{K}_2 \mathcal{K}_1 \in \mathcal{A}^{\infty}$ , and

$$\tilde{\text{tr}}((\mathcal{K}_1 \mathcal{K}_2)^n) = \tilde{\text{tr}}((\mathcal{K}_2 \mathcal{K}_1)^n)$$

for all integer  $n \geq 1$ , then

$$\widetilde{\text{det}}(\text{Id} + \lambda \mathcal{K}_1 \mathcal{K}_2) = \widetilde{\text{det}}(\text{Id} + \lambda \mathcal{K}_2 \mathcal{K}_1).$$

- (2) If, additionally,  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are such that, for each integer  $m \geq 1$  (say, even) and every  $m$ -tuple  $j_1, \dots, j_m$  of nonnegative integers, we have

$$\mathcal{K}_j^m = \mathcal{K}_1^{j_1} \mathcal{K}_2^{j_2} \dots \mathcal{K}_1^{j_{m-1}} \mathcal{K}_2^{j_m} \in \mathcal{A}$$

and  $\tilde{\text{tr}} \mathcal{K}_j^m$  is well-defined and invariant under circular permutations of the  $j_\ell$ , then

$$\widetilde{\text{det}}(\text{Id} + \lambda \mathcal{K}_1) \cdot \widetilde{\text{det}}(\text{Id} + \lambda \mathcal{K}_2) = \widetilde{\text{det}}((\text{Id} + \lambda \mathcal{K}_1)(\text{Id} + \lambda \mathcal{K}_2)).$$

(Note that both conditions hold if the first one is true for  $m = 1$  and all  $\mathcal{K}_1, \mathcal{K}_2$ , if  $\mathcal{A}$  is an algebra. In our application we will not have an algebra strictly speaking.)

*Proof of Theorem 4.* We shall use the notation

$$\sigma(t) = \frac{\text{sgn}(t)}{25}.$$

In order to apply Exercise 10, we must formalise the operator spaces which appear and show that the commuting property holds. For this, we set  $\mathcal{A}^S$  to be the vector space generated by finite products of operators  $\mathcal{M}$ ,  $\mathcal{N}$ , and  $\mathcal{S}$  such that there is at least one factor  $\mathcal{S}$ , at least an  $\mathcal{M}$  or an  $\mathcal{N}$  between any two  $\mathcal{S}$  factors, and at least one factor  $\mathcal{S}$  between any two  $\mathcal{N}$  factors. By definition  $\mathcal{D}(z) \in \mathcal{A}^S[[z]]$ .

The following two properties will allow us to invoke Exercise 10:

**(I) Properties of the kernel**

If  $\mathcal{K} \in \mathcal{A}^S$  then there exists  $K \in L^2(\mu \times \mu)$  so that (on  $L^2(d\mu)$ , say)

$$\mathcal{K}\varphi(x) = \int K(x, y)\varphi(y) d\mu(y),$$

and, additionally,  $K(x, y)$  is a (finite) linear combination of expressions

$$h(x) \cdot \tilde{h}(y) \cdot \sigma(\psi(x) - \tilde{\psi}(y)),$$

where

- (1)  $\psi$  and  $\tilde{\psi}$  are homeomorphisms, or local homeomorphism the support of which contain the supports of  $h$ , respectively  $\tilde{h}$ ;
- (2)  $h$  is a linear combination of continuous functions of bounded variations, which may be multiplied by a factor  $(dg_\omega/d\mu) \circ \psi'$  with  $\psi'$  a (local) homeomorphism (with good support) if there is a factor  $\mathcal{N}$  which is not followed by a (post)-composition with an  $\mathcal{S}$ ; its support is compact if the leftmost factor of  $\mathcal{K}$  is not  $\mathcal{S}$ ;
- (3)  $\tilde{h}$  is bounded; its support is compact if the rightmost factor of  $\mathcal{K}$  is not  $\mathcal{S}$ ;

The kernel  $K$  is not uniquely defined, but our proof of its existence is constructive so that there is no ambiguity once the data  $g_\omega$ ,  $\psi_\omega$  is given. Note that property (I) says in particular that the trace

$$\mathrm{Tr}^* \mathcal{K} = \int K(x, x) d\mu(x),$$

is well defined for each element of  $\mathcal{A}^S$ . Renaming  $\mathrm{Tr}^*$  (for the sake of uniform notation)  $\mathrm{Tr}^\#$  on this vector space, and setting  $\mathcal{X}$  to be the algebra generated by (powers of)  $\mathcal{M}$ , we may extend  $\mathrm{Tr}^\#$  by linearity to  $\mathcal{A} = \mathcal{X}[[z]] \oplus \mathcal{A}^S[[z]]$ . All expressions appearing in the proof of Theorem 4 will be in  $\mathcal{A}$ .

**(II) Commutations**

- (1) If  $\mathcal{K} \in \mathcal{A}^S$  then  $\mathrm{Tr}^\#(\mathcal{M}\mathcal{K}) = \mathrm{Tr}^\#(\mathcal{K}\mathcal{M})$  and  $\mathrm{Tr}^\#(\mathcal{N}\mathcal{K}) = \mathrm{Tr}^\#(\mathcal{K}\mathcal{N})$ .
- (2) If  $\mathcal{K} \in \mathcal{A}^S$  and neither the leftmost nor the rightmost factor of  $\mathcal{K}$  is  $\mathcal{S}$  then  $\mathrm{Tr}^\#(\mathcal{S}\mathcal{K}) = \mathrm{Tr}^\#(\mathcal{K}\mathcal{S})$ .
- (3)  $\mathrm{Tr}^\#(\mathcal{M}\mathcal{N}) = \mathrm{Tr}^\#(\mathcal{N}\mathcal{M})$  (in fact, we do not use this in the proof; recall also Lemma 2).

The third property will be crucial in the proof:

**(III) Naturality**

For each  $m \geq 1$

$$\mathrm{Tr}^\#((\mathcal{M} - \mathcal{S}\mathcal{N})^m) = 0.$$

We next show how Theorem 4 follows from Exercise 10 and (I-II-III). It suffices to show that  $\mathrm{Det}^\#(\mathrm{Id} + \mathcal{D}(z))\mathrm{Det}^\#(\mathrm{Id} - z\mathcal{M}) \equiv 1$ :

$$\begin{aligned} \mathrm{Det}^\#(\mathrm{Id} + \mathcal{D}(z))\mathrm{Det}^\#(\mathrm{Id} - z\mathcal{M}) &= \mathrm{Det}^\#(\mathrm{Id} + z\mathcal{N}(\mathrm{Id} - z\mathcal{M})^{-1}\mathcal{S})\mathrm{Det}^\#(\mathrm{Id} - z\mathcal{M}) \\ &= \mathrm{Det}^\#(\mathrm{Id} + z\mathcal{S}\mathcal{N}(\mathrm{Id} - z\mathcal{M})^{-1})\mathrm{Det}^\#(\mathrm{Id} - z\mathcal{M}) \\ &= \mathrm{Det}^\#(\mathrm{Id} - z\mathcal{M} + z\mathcal{S}\mathcal{N}) = 1, \end{aligned}$$

where we used the definition of  $\mathcal{D}(z)$  in the first equality, (I) and (II (2)) (together with Exercise 10) in the second one, and (I) and (II (1, 2)) (with Exercise 10 again) in the third, and (III) in the last.

It remains to check (I,II, III).

We shall prove (I) by induction on the number of factors, multiplying to the left. The claim is obvious for  $\mathcal{K} = \mathcal{S}$ . For  $\mathcal{S}\mathcal{M}$  and  $\mathcal{M}\mathcal{S}$ , we compute:

$$\begin{aligned} \mathcal{S}\mathcal{M}\varphi(x) &= \int \sigma(x-y) \sum_{\omega} g_{\omega}(y)\varphi(\psi_{\omega}(y)) d\mu(y) \\ &= \int \sum_{\omega} \epsilon_{\omega} \chi_{\psi_{\omega}(I_{\omega})} \sigma(x - \psi_{\omega}^{-1}(z)) g_{\omega}(\psi_{\omega}^{-1}(z)) \varphi(z) d\mu(z); \end{aligned}$$

and

$$\mathcal{M}\mathcal{S}\varphi(x) = \int \sum_{\omega} g_{\omega}(x)\sigma(\psi_{\omega}(x) - y)\varphi(y) d\mu(y).$$

The above computations also show that  $\mathcal{S}\mathcal{N}$  and  $\mathcal{N}\mathcal{S}$  have kernels with the desired properties.

Next, assuming that the kernel of  $\mathcal{K}$  in  $\mathcal{A}$  has the desired properties, we consider  $\mathcal{M}\mathcal{K}$ ,  $\mathcal{N}\mathcal{K}$  (if there is at least an  $\mathcal{S}$  postcomposed with the last factor  $\mathcal{N}$  in  $\mathcal{K}$ ),

$$\begin{aligned} \mathcal{M}\mathcal{K}\varphi(x) &= \int \sum_{\omega} g_{\omega}(x)K(\psi_{\omega}(x), y)\varphi(y) d\mu(y), \\ \mathcal{N}\mathcal{K}\varphi(x) &= \int \sum_{\omega} \frac{dg_{\omega}}{d\mu}(x)K(\psi_{\omega}(x), y)\varphi(y) d\mu(y), \end{aligned}$$

for which it is obvious that the induction hypotheses suffice.

Finally, we consider  $\mathcal{S}\mathcal{K}$  (if the leftmost factor of  $\mathcal{K}$  is not  $\mathcal{S}$ ):

$$\begin{aligned} \mathcal{S}\mathcal{K}\varphi(x) &= \int \int \sigma(x-y)K(y, z)\varphi(z) d\mu(z) d\mu(y) \\ &= \int \int \sigma(x-y)K(y, z) d\mu(y) \varphi(z) d\mu(z) \\ &= \int \int \sigma(x-y)h(y)\tilde{h}(z)\sigma(\psi(y) - \tilde{\psi}(z)) d\mu(y) \varphi(z) d\mu(z). \end{aligned}$$

By induction, the support of  $h$  is compact. We must study

$$\int \sigma(x-y)h(y)\sigma(\psi(y)-u) d\mu(y),$$

where we wrote  $u = \tilde{\psi}(z)$  for simplicity. If  $h$  is a linear combination of continuous functions of bounded variation, we use the easily proved fact that  $\mathcal{S}_{1 \rightarrow 0}(hd\mu)$  is a continuous function of bounded variation, where  $\mathcal{S}_{1 \rightarrow 0}$  is the isomorphism between Radon measures and  $\mathcal{B}$  from Lemma 0 in Section 1 (the isomorphism was called  $\mathcal{S}$  here, but the notation  $\mathcal{S}$  in the present Section 2 represents the convolution operator  $\mathcal{S} = \mathcal{S}_{0 \rightarrow 0}$ ). If  $\mathcal{K}$  contains an  $\mathcal{N}$  factor not postcomposed by any  $\mathcal{S}$ , then  $h$  may contain terms of the form  $h'(y)(dg_\omega/d\mu)(\psi'y)$  with  $h'$  continuous and of bounded variation, and

$$\mathcal{S}_{1 \rightarrow 0}(h' \frac{dg_\omega}{d\mu} \circ \psi' d\mu) = \mathcal{S}_{1 \rightarrow 0}(\epsilon_{\psi'} \chi_{\psi'} h' \circ (\psi')^{-1} dg_\omega)$$

is again a continuous function of bounded variation. Therefore, we may use the Leibniz formula (Lemma 1 from Section 1) in both cases:

$$\begin{aligned} \int \sigma(x-y)\sigma(\psi(y)-u)h(y) d\mu(y) &= \int \sigma(x-y)\sigma(\psi(y)-u) d\mathcal{S}_{1 \rightarrow 0}(hd\mu)(y) \\ &= \sigma(\psi(x)-u)\mathcal{S}_{1 \rightarrow 0}(hd\mu)(x) \\ &\quad - \epsilon_\psi \chi_\psi \sigma(x-\psi^{-1}(u))\mathcal{S}_{1 \rightarrow 0}(hd\mu)(\psi^{-1}(u)), \end{aligned}$$

where we used  $d\sigma = \delta_0$  again (the assumptions on the support of  $h$  imply that there is no boundary term). Inspecting the above expression, we see that we have performed the inductive step successfully.

Let us prove II(1). Using the expression obtained previously for  $\mathcal{MK}$ , and a change of variables, we get

$$\begin{aligned} \text{Tr} \# \mathcal{MK} &= \int \sum_\omega K(\psi_\omega(y), y) g_\omega(y) d\mu(y) \\ &= \int \sum_\omega \epsilon_\omega \chi_{\psi_\omega(I_\omega)} K(x, \psi_\omega^{-1}(x)) g_\omega(\psi_\omega^{-1}(x)) d\mu(x). \end{aligned}$$

Similarly, we have:

$$\begin{aligned} \mathcal{KM}\varphi(y) &= \int K(y, z) \mathcal{M}\varphi(z) d\mu(z) \\ &= \int K(y, z) \sum_\omega g_\omega(z) \varphi(\psi_\omega(z)) d\mu(z) \\ &= \int \sum_\omega \epsilon_\omega \chi_{\psi_\omega(I_\omega)} K(y, \psi_\omega^{-1}(x)) g_\omega(\psi_\omega^{-1}(x)) \varphi(x) d\mu(x), \end{aligned}$$

which gives  $\text{Tr}^\# \mathcal{M}\mathcal{K} = \text{Tr}^\# \mathcal{K}\mathcal{M}$ . Since we did not use integration by parts, the same computation yields  $\text{Tr}^\# \mathcal{N}\mathcal{K} = \text{Tr}^\# \mathcal{K}\mathcal{N}$ .

To show II(2), we first use the expression for  $\mathcal{S}\mathcal{K}$  to see that

$$\text{Tr}^\# \mathcal{S}\mathcal{K} = \int \int \sigma(x-y)K(y,x) d\mu(y) d\mu(x).$$

On the other hand

$$\begin{aligned} \mathcal{K}\mathcal{S}\varphi(x) &= \int K(x,y)\mathcal{S}\varphi(y) d\mu(y) \\ &= \int \int K(x,y)\sigma(y-z)\varphi(z) d\mu(z) d\mu(y), \end{aligned}$$

so that

$$\text{Tr}^\# \mathcal{K}\mathcal{S} = \int \int K(x,y)\sigma(y-x) d\mu(x) d\mu(y),$$

proving the claim.

The proof of II(3) goes along the lines of the proof of Lemma 2 (the Leibniz formula is all right since there is only a single factor  $\mathcal{N}$ ).

Finally, we check (III) by induction on  $m$ . For  $m = 1$ , using our formula for  $\text{Tr}^\# \mathcal{K}\mathcal{N}$  in the case  $\mathcal{K} = \mathcal{S}$  (so that  $K(x,y) = \sigma(x-y)$ ), we find by a double change of variable

$$\begin{aligned} \text{Tr}^\# \mathcal{S}\mathcal{N} &= \int \sum_{\omega} \sigma(x - \psi_{\omega}^{-1}(x)) \frac{dg_{\omega}}{d\mu}(\psi_{\omega}^{-1}(x)) d\mu(x) \\ &= \int \sum_{\omega} \sigma(x - \psi_{\omega}^{-1}(x)) dg_{\omega}(\psi_{\omega}^{-1}(x)) \frac{d\mu}{d\mu \circ \psi_{\omega}^{-1}}(x) \\ &= \int \sum_{\omega} \sigma(\psi_{\omega}(y) - y) dg_{\omega}(y) \\ &= \text{Tr}^\# \mathcal{M}. \end{aligned}$$

Next, we set

$$\widetilde{\mathcal{M}} = \mathcal{M} - \mathcal{S}\mathcal{N}.$$

We have just seen that  $\text{Tr}^\# \widetilde{\mathcal{M}} = 0$  and we want to show that  $\text{Tr}^\# (\widetilde{\mathcal{M}})^m = 0$  for all  $m \geq 1$ . For this, it suffices to show that

$$(\widetilde{\mathcal{M}})^m = \widetilde{\mathcal{M}}^m$$

(where the notations are self-explanatory i.e.  $\widetilde{\mathcal{M}}^m = \mathcal{M}^m - \mathcal{N}_m$  where  $\mathcal{N}_m$  is associated to  $\mathcal{M}^m$  via an appropriate auxiliary measure  $\mu_m$ ). Indeed, the case  $m = 1$  applied to  $\mathcal{M}^m$  would give the claim. To prove the above naturality statement, it is enough (by

density of  $\mathcal{B}$  in  $L^2(d\mu)$  to show that  $(\widetilde{\mathcal{M}})^m\varphi = \widetilde{\mathcal{M}}^m\varphi$  for each  $\varphi$  of bounded variation. Let us rewrite  $\mathcal{SN}$  on  $\mathcal{B}$ , integrating by parts:

$$\begin{aligned}\mathcal{SN}\varphi(x) &= \int \sum_{\omega} \sigma(x-y)\varphi(\psi_{\omega}(y)) dg_{\omega}(y) \\ &= \sum_{\omega} g_{\omega}(x)\varphi(\psi_{\omega}(x)) \\ &\quad - \int \sum_{\omega} \sigma(x-y)g_{\omega}(y)d(\varphi \circ \psi_{\omega})(y) \\ &= \mathcal{M}\varphi(x) - \int \sum_{\omega} \epsilon_{\omega}\chi_{\psi_{\omega}(I_{\omega})}\sigma(x - \psi_{\omega}^{-1}(z))g_{\omega}(\psi_{\omega}^{-1}(z))d\varphi(z) \\ &= \mathcal{M}\varphi(x) - \mathcal{N}_{1 \rightarrow 0}(d\varphi)(x),\end{aligned}$$

where  $\mathcal{N}_{1 \rightarrow 0}$  is bounded from Radon measures to functions of bounded variation. In other words,

$$\widetilde{\mathcal{M}} = \mathcal{N}_{1 \rightarrow 0}d.$$

Similarly, we may decompose

$$\widetilde{\mathcal{M}}^m = \mathcal{N}_{m,1 \rightarrow 0}d,$$

and it is easy to see that

$$\mathcal{N}_{m,1 \rightarrow 0}d\mathcal{N}_{1 \rightarrow 0} = \mathcal{N}_{m+1,1 \rightarrow 0}.$$

(Just use that  $d\sigma(x - \cdot)$  is the dirac at  $x$ .) To finish,

$$\widetilde{\mathcal{M}}^m\widetilde{\mathcal{M}} = \mathcal{N}_{m,1 \rightarrow 0}d\mathcal{N}_{1 \rightarrow 0}d = \mathcal{N}_{m+1,1 \rightarrow 0}d = \widetilde{\mathcal{M}}^{m+1}. \quad \square$$

## 2.5 $\text{Det}^{\#}(\text{Id} - z\mathcal{M})$ and the spectrum of $\mathcal{M}$ .

In this final section, we exploit the Milnor-Thurston identity to prove:

**Theorem 6.** *Det<sup>#</sup>(Id - zM) is holomorphic in the disc  $|z| < 1/\widehat{R}$  and its zeroes in this disc are the inverses of the eigenvalues of modulus larger than  $\widehat{R}$  of  $\mathcal{M}$  acting on  $\mathcal{B}$ . The order of the zero coincides with the algebraic multiplicity of the eigenvalue.*

*Proof of Theorem 6.* Combining Lemma 3 with the first equality in Theorem 4

$$\text{Det}^{\#}(\text{Id} - z\mathcal{M}) = \text{Det}_{*}(\text{Id} + \widehat{\mathcal{D}}(z))$$

and the fact that the spectral radius of  $\widehat{\mathcal{M}}$  on bounded functions is not larger than  $\widehat{R}$ , we get the holomorphy claim ( $\sigma(\cdot - y)$  is a bounded function for all  $y$ ).

Using the second equality we see that if  $z_0$  with  $|z_0| < \widehat{R}$  is a zero of  $\text{Det}^{\#}(\text{Id} - z\mathcal{M}) = \text{Det}_{*}(\text{Id} + \widehat{\mathcal{D}}(z))^{-1}$ , then  $1/z_0$  must be an eigenvalue of  $\mathcal{M}$  acting on  $\mathcal{B}$  ( $\sigma(\cdot - y) \in \mathcal{B}$  for all  $y$ ).

Let us now prove that if  $\lambda_0 = 1/z_0$  is a simple eigenvalue, then the order of the pole of  $\text{Det}_*(\text{Id} + \mathcal{D}(z_0))$  is at most one. Writing  $\mathcal{P} : \mathcal{B} \rightarrow \mathcal{B}$  for the rank-one spectral projector associated to  $\mathcal{M}$  and  $\lambda_0$ , we may decompose

$$(\text{Id} - z\mathcal{M})^{-1} = \frac{1}{1 - \lambda_0 z} \mathcal{P} + (\text{Id} - z\mathcal{M})^{-1}(\text{Id} - \mathcal{P}),$$

the second term being holomorphic in a neighbourhood of  $z = z_0$ . We wish to use the above decomposition via multilinearity of the determinants. For this it is useful to use a Plemelj-Smithies formula

$$\text{Det}_*(\text{Id} + \mathcal{D}(z)) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \Phi_n(\mathcal{D}(z)),$$

where, writing  $\mathcal{K}^z(x, y)$  for the kernel of  $\mathcal{D}(z)$ , we set for  $n \geq 1$

$$\Phi_n(\mathcal{D}(z)) = \int_{I^n} \det_{n \times n} \left( \mathcal{K}^z(x_i, x_j) \right) d\mu(x_1) \cdots d\mu(x_n).$$

The Plemelj-Smithies formula claimed above can be obtained by combining the consequence of Lemmas 5–6 and the Plemelj-Smithies formula for  $\text{Det}_2(\text{Id} + \mathcal{D}(z))$  (see Corollary of Theorem 2 in the Appendix). Using our decomposition of the resolvent and the definition of  $\mathcal{K}^z(x, y)$ , we find

$$\begin{aligned} \mathcal{K}^z(x, y) &= \sum_{\omega} z \frac{dg_{\omega}}{d\mu}(x) [(\text{Id} - z\mathcal{M})^{-1} \sigma(\cdot - y)] \psi_{\omega}(x) \\ &= \frac{z}{1 - \lambda_0 z} \alpha(x) \beta(y) + B^z(x, y), \end{aligned}$$

where  $\alpha$  and  $\beta$  are independent of  $z$  and are bounded on  $I$ , while  $B^z(x, y)$  is bounded on  $I \times I$  and depends holomorphically on  $z$  in a neighbourhood of  $z_0$ . We next develop each

$$\det_{n \times n} \left( \frac{z}{1 - \lambda_0 z} \alpha(x_i) \beta(x_j) + \mathcal{B}^z(x_i, x_j) \right)$$

by multilinearity. The Hadamard inequality (used in the classical Fredholm theory) gives

$$\det_{n \times n} \left( \mathcal{B}^z(x_i, x_j) \right) \leq C_0^n n^{n/2},$$

for some finite constant  $C_0$ , in a neighbourhood of  $z_0$ . The other terms have one or several columns of the form  $\frac{z}{1 - \lambda_0 z} \alpha(x_i) \beta(x_j)$ . If there is a single such column, the Hadamard inequality gives that in a neighbourhood of  $z_0$  the determinant is at most

$$\frac{C_1}{|1 - \lambda_0 z|} C_0^n n^{n/2},$$

where  $C_1$  is another finite constant. If there are two or more such columns, they are proportional so that the corresponding determinant vanishes. Finally, we get by summing all terms and integrating over our finite measure  $\mu^n$ :

$$|\Phi_n(\mathcal{D}(z))| \leq \frac{C_2^{n+1}}{|1 - \lambda_0 z|} n^{n/2+1}.$$

Putting this estimate back into the Plemelj-Smithies formula, we see that the order of the pole at  $z_0 = 1/\lambda_0$  is at most one, as claimed. Note that the argument may be adapted if the algebraic multiplicity is larger than one, showing that the order of the pole is at most the algebraic multiplicity of the eigenvalue (we shall not need this).

To finish the proof, we show (using again the first equality in Theorem 4) that if  $\lambda_0$  with  $|\lambda_0| > \widehat{R}$  is an eigenvalue of  $\mathcal{M}$  acting on  $\mathcal{B}$ , then  $z_0 = 1/\lambda_0$  is a zero of  $\text{Det}^\#(1 - z\mathcal{M}) = \text{Det}_*(1 + \widehat{\mathcal{D}}(z))$  of order the algebraic multiplicity of the eigenvalue. Since  $\text{Det}_*(1 + \widehat{\mathcal{D}}(z)) = \text{Det}_*(1 + \widehat{\mathcal{D}}(z)^*)$ , it is enough to show that  $-1$  is an eigenvalue of  $\widehat{\mathcal{D}}(z_0)^*$  acting on  $L^2(d\mu)$ , with the correct multiplicity.

Let then  $\varphi \in \mathcal{B}$  be an eigenfunction for  $\mathcal{M}$  and the eigenvalue  $\lambda_0$ . We can assume that  $\varphi$  has only regular discontinuities, and the eigenfunction equation implies that  $\varphi$  is supported in  $\cup_\omega I_\omega$ . In particular,  $\varphi \in L^2(d\mu)$ . We next show that  $\widehat{\mathcal{D}}(z_0)^*\varphi = -\varphi$ . First recall that

$$\begin{aligned} \widehat{\mathcal{D}}(z_0)^*\varphi(y) &= \int \sum_\omega z\epsilon_\omega \frac{d(g_\omega \circ \psi_\omega^{-1})}{d\mu}(x) [(\text{Id} - z\widehat{\mathcal{M}}_x)^{-1}\sigma(\cdot - y)]\psi_\omega^{-1}(x)\varphi(x) d\mu(x) \\ &= \int \sum_\omega z\epsilon_\omega [(\text{Id} - z\widehat{\mathcal{M}}_x)^{-1}\sigma(\cdot - y)]\psi_\omega^{-1}(x)\varphi(x) d(g_\omega \circ \psi_\omega^{-1})(x). \end{aligned}$$

Using

$$\varphi d(g_\omega \circ \psi_\omega^{-1}) = d(\varphi(g_\omega \circ \psi_\omega^{-1})) - (d\varphi)(g_\omega \circ \psi_\omega^{-1}),$$

and the fact that  $\varphi$  only has regular discontinuities, we get

$$\begin{aligned} (1 + \widehat{\mathcal{D}}(z_0)^*)\varphi(y) &= - \int d\varphi(x) \left( \sigma(x - y) \right. \\ &\quad \left. + \sum_\omega z_0\epsilon_\omega d(g_\omega \circ \psi_\omega^{-1})(x) [(\text{Id} - z_0\widehat{\mathcal{M}}_x)^{-1}\sigma(\cdot - y)]\psi_\omega^{-1}(x) \right) \\ &\quad + \int \sum_\omega d(z_0\epsilon_\omega\varphi(g_\omega \circ \psi_\omega^{-1}))(x) [(\text{Id} - z_0\widehat{\mathcal{M}}_x)^{-1}\sigma(\cdot - y)]\psi_\omega^{-1}(x) \\ &= \int -d\varphi(x) (\text{Id} + z_0\widehat{\mathcal{M}}_x (\text{Id} - z_0\widehat{\mathcal{M}}_x)^{-1})\sigma(x - y) \\ &\quad + \int \sum_\omega d(z_0(\varphi \circ \psi_\omega)g_\omega)(u) (\text{Id} - z_0\widehat{\mathcal{M}}_u)^{-1}\sigma(u - y) \\ &= - \int d(\varphi - z_0\mathcal{M}\varphi)(x) (\text{Id} - z_0\widehat{\mathcal{M}}_x)^{-1}\sigma(x - y) = 0. \end{aligned}$$

If  $\lambda_0$  is a simple eigenvalue then we are done. (More generally, the above computation shows that the order of the zero is at least the geometric multiplicity of the eigenvalue. However, contrarily to the claims in the end of the proof of Theorem 4.4.5 [Go], it is not clear how to relate directly the order of the zero and the algebraic multiplicity.) Otherwise, letting  $m_0$  be the algebraic multiplicity of  $\lambda_0$ , we may find a small perturbation  $\mathcal{M}_\delta$  of  $\mathcal{M}$  (within the class of transfer operators associated to data  $\psi_\omega, g_\omega$ ) such that  $\lambda_0$  is replaced by  $m_0$  simple eigenvalues for  $\mathcal{M}_\delta$ . (The details are left to the reader. We use that a small perturbation in operator norm does not change the spectrum away from a neighbourhood of  $\lambda_0$  too much and cannot increase the algebraic multiplicity of any perturbation  $\lambda_{0,\delta}$  of  $\lambda_0$ , and also that the possibility of choosing independent supports for the  $\psi_{\omega,\delta}$  gives us enough “degrees of freedom,” to ensure that  $\text{Ker}(\text{Id} - \lambda\mathcal{M}_\delta)^{m_0}$  is one-dimensional for  $\lambda$  in a neighbourhood of  $\lambda_0$ .) Then the arguments already given show that  $\text{Det}^\#(1 - z\mathcal{M}_\delta)$  has  $m_0$  simple zeroes in a neighbourhood of  $z_0$ , and that the holomorphic functions  $\text{Det}^\#(1 - z\mathcal{M}_\delta)$  converge uniformly to  $\text{Det}^\#(1 - z\mathcal{M})$  in compact sets.  $\square$

*Exercise 11.* In fact, any eigenfunction  $\varphi$  in  $\mathcal{B}$  for  $\mathcal{M}$  and an eigenvalue  $\lambda$  of modulus larger than  $\widehat{R}$  has a continuous representative. (The proof of this fact is analogous to the proof of Lemma 4 in Chapter 1. The starting point is to introduce  $\tilde{\varphi}(x) = \varphi(x+) - \varphi(x-)$ , noting that  $\mathcal{M}\tilde{\varphi} = \lambda\tilde{\varphi}$ , and writing  $(\Phi, \tilde{\varphi}) = \sum_x \Phi(x)\tilde{\varphi}(x)$  for any bounded function  $\Phi$ . Then, one can show that for each bounded  $\Phi$  we have  $(\Phi, \tilde{\varphi}) = \lambda^{-1}(\widehat{\mathcal{M}}\Phi, \tilde{\varphi})$  and end by iterating.)

*Exercise 12.* In Chapter 1 we used the decomposition

$$d_{0 \rightarrow 1}\mathcal{M}\mathcal{S}_{1 \rightarrow 0} = \widehat{\mathcal{M}}_{1 \rightarrow 1} + \widehat{\mathcal{N}}_{0 \rightarrow 1}\mathcal{S}_{1 \rightarrow 0}$$

to prove Theorem 1, with  $\widehat{\mathcal{N}}_{0 \rightarrow 1}(\varphi) = \sum_\omega dg_\omega \varphi \circ \psi_\omega$ . We can also write

$$\mathcal{M} = \mathcal{S}_{1 \rightarrow 0}\widehat{\mathcal{M}}_{1 \rightarrow 1}d_{0 \rightarrow 1} + \mathcal{S}_{1 \rightarrow 0}\widehat{\mathcal{N}}_{0 \rightarrow 1},$$

with  $\mathcal{S}_{1 \rightarrow 0}\widehat{\mathcal{N}}_{0 \rightarrow 1}$  compact on  $\mathcal{B}$ . Show that

$$\mathcal{S}_{1 \rightarrow 0}\widehat{\mathcal{M}}_{1 \rightarrow 1}d_{0 \rightarrow 1} = \widehat{\mathcal{M}}_{0 \rightarrow 0},$$

and (applying  $d_{0 \rightarrow 1}$  to both sides) that

$$\mathcal{S}_{1 \rightarrow 0}\widehat{\mathcal{N}}_{0 \rightarrow 1} = \mathcal{S}_{0 \rightarrow 0}\widehat{\mathcal{N}}_{0 \rightarrow 0},$$

with  $\widehat{\mathcal{N}}_{0 \rightarrow 0}\varphi = \sum_\omega \epsilon_\omega \frac{dg_\omega \circ \psi_\omega^{-1}}{d\mu} \varphi \circ \psi_\omega^{-1}$  and  $\mathcal{S}_{0 \rightarrow 0}\varphi(x) = \int \sigma(x - y)\varphi(y) d\mu(y)$ . This implies that (in the notations of Chapter 2)

$$(\text{Id} - z\mathcal{M})^{-1} = (\text{Id} - z\widehat{\mathcal{M}})^{-1}(\text{Id} - z\mathcal{S}\widehat{\mathcal{N}}(\text{Id} - z\widehat{\mathcal{M}})^{-1})^{-1},$$

Finally, recall from the proof of Theorem 4 that

$$\text{Det}_*(\text{Id} + \widehat{\mathcal{D}}(z)) = \text{Det}_*(\text{Id} + z\mathcal{S}\widehat{\mathcal{N}}(\text{Id} - z\widehat{\mathcal{M}})^{-1}).$$

### 3. KNEADING THEORY IN HIGHER DIMENSIONS

In this last chapter, we shall discuss partial extensions of the results of Chapter 2 to higher dimensions. In fact we shall only present a higher-dimensional version of the Milnor and Thurston identity (Ph.D. of Baillif [Bai] based on an unpublished idea of Kitaev), under a transversality assumption.

#### 3.1 Setting – The higher-dimensional Milnor-Thurston formula.

In this chapter  $\Omega$  is as before a finite index-set, and  $n \geq 2$  denotes the dimension, i.e., we are going to work in a compact subset  $K$  of  $\mathbb{R}^n$ . We also fix an integer order of differentiability  $r \geq 1$ . To each  $\omega \in \Omega$  we associate an nonempty open set  $U_\omega \subset \mathbb{R}^n$  and a (local)  $C^r$  diffeomorphism

$$\psi_\omega : U_\omega \rightarrow \psi_\omega(U_\omega),$$

assuming that  $\bigcup_\omega U_\omega \cup \bigcup_\omega \psi_\omega(U_\omega) \subset K$ . We also consider a function  $g_\omega : \mathbb{R}^n \rightarrow \mathbb{C}$  satisfying (these are not the weakest possible requirements)

- (1)  $g_\omega$  is supported in  $U_\omega$ ,
- (2)  $g_\omega$  is  $C^r$ .

Additionally, we make a *transversality* assumption on the dynamics  $\{\psi_\omega, | \omega \in \Omega\}$ : For each  $x \in \mathbb{R}^n$  such that there exist  $m \geq 1$  and  $\vec{\omega} = \omega_m \cdots \omega_1$  with  $\psi_{\vec{\omega}}^m(x) = \psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1}(x) = x$  (in particular,  $x$  lies in the compact set  $K$ ), we have

$$1 \notin \text{sp}(D_x \psi_{\vec{\omega}}^m).$$

(In other words,  $\psi_{\vec{\omega}}^m$  is a transversal diffeomorphism.)

Note that in dimension one no such assumption was present. We expect that this transversality requirement (which is weaker than hyperbolicity) will be eventually suppressed, but this will require some additional work.

Here are two consequences of transversality: First, since the fixed points of a transversal diffeomorphism are isolated, and since  $K$  is compact, there are only finitely many fixed points of  $\psi_{\vec{\omega}}^m$  for each fixed  $m \geq 1$ . Also, the Lefschetz number  $L(x, \psi_{\vec{\omega}}^m)$  does not vanish and can be written

$$L(x, \psi_{\vec{\omega}}^m) = \text{sgn}(\det(\text{Id} - D_x \psi_{\vec{\omega}}^m)) \in \{+1, -1\}.$$

Before we introduce the kneading operators and state the higher-dimensional version of the Milnor and Thurston formula, we need to recall some notations and definitions. Our  $C^r$  assumption will allow us to use Lebesgue measure  $dx$  as a reference measure and  $L^q$  will always denote  $L^q(dx)$ .

We will be working not only with functions, but more generally with  $k$ -forms. For  $k = 0, \dots, n$  and  $1 < q < \infty$ , we write  $A_k$  and  $A_{k,K}$  for the vector spaces of  $k$ -forms (with  $C^\infty$  coefficients), respectively  $k$ -forms supported in  $K$ . Also, we will write  $\mathcal{A}_{k,L^q}$  and  $\mathcal{A}_{k,L^q(K)}$  for the vector spaces of  $k$ -forms on  $\mathbb{R}^n$  with  $L^q$  coefficients, respectively  $L^q$  coefficients supported in  $K$ . (We refer to [Sp] for the basic theory of differential forms.)

This vector space inherits a Banach norm from the norms of the coefficient functions. Note however that the corresponding Banach space is not convenient for the spectral theory of the transfer operators  $\mathcal{M}_k$  to be introduced in a moment. It is however useful for intermediate steps, in particular when considering the kneading operators, also to be introduced below.

We shall denote the exterior derivative from  $A_{k,(K)}$  to  $A_{k+1,(K)}$  by  $d_k$  (or simply  $d$  when there is no ambiguity). Recall that if  $\phi = \sum_{\vec{j} \in I(k)} \phi_{\vec{j}} dx_{\vec{j}}$  (where  $I(k)$  denotes the set of ordered  $k$ -tuples in  $\{1, \dots, n\}$  and  $dx_{\vec{j}} = dx_{j_1} \cdots dx_{j_k}$ ), then  $d\phi = \sum_{i=1}^n \frac{\partial}{\partial x_i} \phi_{\vec{j}} dx_{\vec{j}} \wedge dx_i$  and that  $d_{k+1}d_k = 0$ . Sometimes  $d_k$  will be considered on forms whose coefficients are not  $C^\infty$ . We shall work with the pull-back  $\psi_\omega^*$  on  $\mathcal{A}_k$  (or  $\mathcal{A}_{k,K}$ , or  $\mathcal{A}_{k,L^q}$ ,  $\mathcal{A}_{k,L^q(K)}$ ) of  $\psi_\omega$ .

We may now associate an  $(n+1)$ -tuple of transfer operators  $\mathcal{M} = (\mathcal{M}_k, k = 0, \dots, n)$  to the data  $g_\omega, \psi_\omega$ , where  $\mathcal{M}_k$  acts on  $\mathcal{A}_{k,L^q(K)}$  (for example) by setting

$$\mathcal{M}_k \phi = \sum_{\omega} g_\omega (\psi_\omega^* \phi).$$

For  $k = 0$ , we recover the previous definition:

$$\mathcal{M}_0 \phi(x) = \sum_{\omega} g_\omega(x) (\phi \circ \psi_\omega)(x).$$

For  $m \geq 1$  we write  $\mathcal{M}^m$  for the  $(n+1)$ -tuple  $(\mathcal{M}_k^m, k = 0, \dots, n)$ . Clearly  $\mathcal{M}^m$  is associated to the data  $\Omega^m, \psi_{\vec{\omega}}^m$  (when the domain of definition of the composition is not empty) and

$$g_{\omega_m}(\psi_{\vec{\omega}}^{m-1}) \cdots g_{\omega_2}(\psi_{\omega_1}) g_{\omega_1}.$$

(Note that we do not claim, nor shall we need, that the  $g_\omega, \psi_\omega$  are unambiguously determined by the operators  $\mathcal{M}_k$ .)

Let us now define the sharp trace and the sharp determinant:

**Definition 1.** Let  $\mathcal{M}$  be associated to data  $\{\psi_\omega, g_\omega, \omega \in \Omega\}$  as above, then the sharp determinant of  $\mathcal{M}$  is defined by

$$\text{Det}^\#(1 - z\mathcal{M}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \text{Tr}^\# \mathcal{M}^m,$$

where

$$\text{Tr}^\# \mathcal{M} = \sum_{\omega \in \Omega} \sum_{x \in \text{Fix } \psi_\omega} g_\omega(x) L(x, \psi_\omega).$$

The formula in the following exercise will play a part in the proof of the main theorem of the present section:

*Exercise 1.* Define (here this is only a notation) for each  $k = 0, \dots, n$

$$\mathrm{Tr}^b \mathcal{M}_k = \sum_{\omega} \sum_{x \in \mathrm{Fix} \psi_{\omega}} g_{\omega}(x) \frac{\mathrm{Tr} \Lambda^k(D_x \psi_{\omega})}{|\det(\mathrm{Id} - D_x \psi_{\omega})|},$$

and

$$\mathrm{Det}^b(\mathrm{Id} - z\mathcal{M}_k) = \exp - \sum_{m=0}^{\infty} \frac{z^m}{m} \mathrm{Tr}^b \mathcal{M}_k^m.$$

Show that

$$\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = \prod_{k=0}^n (-1)^k \mathrm{Det}^b(\mathrm{Id} - z\mathcal{M}_k).$$

(Hint: use that for a finite matrix  $A$  we have  $\mathrm{Det}(\mathrm{Id} - A) = \sum_{k=0}^n (-1)^k \mathrm{Tr} \Lambda^k A$ .)

### The Milnor-Thurston formula

We are going to define homotopy operators

$$\mathcal{S}_k : \mathcal{A}_{k+1, C^{r(-1)}(K)} \rightarrow \mathcal{A}_{k, C^{r(-1)}}, k = -1, \dots, n-1,$$

(it is in fact possible to see that  $\mathcal{S}_k(\mathcal{A}_{k+1, L^q}) \subset \mathcal{A}_{k, L^q}$  for each  $1 < q < \infty$ ) with the property that, on compactly supported  $k$ -forms,

$$d_{k-1} \mathcal{S}_{k-1} + \mathcal{S}_k d_k = \mathrm{Id}.$$

The above homotopy equation can be solved because we are considering forms in  $\mathbb{R}^n$ . In order to apply the techniques presented in this chapter to dynamical systems on compact manifolds, one should first embed the manifold in  $\mathbb{R}^n$  for suitable  $n$  and then extend the dynamics in a tubular neighbourhood of the manifold. (See [Bai].) It will be clear from the construction below that the kernel  $\sigma_k(x, y)$  of  $\mathcal{S}_k$  is smooth except on the diagonal  $x = y$  in  $\mathbb{R}^n$  where its singularities are of the type  $x_j / \|x\|^n$ .

We shall also introduce auxiliary transfer-type operators

$$\mathcal{N}_k : \mathcal{A}_{k, L^q} \rightarrow \mathcal{A}_{k+1, L^q}, k = 0, \dots, n-1, 1 < q < \infty,$$

defined by

$$\mathcal{N}_k \phi = (d_k \mathcal{M}_k - \mathcal{M}_{k+1} d_k) \phi = \sum_{\omega} dg_{\omega} \wedge (\psi_{\omega}^* \phi)$$

(we used the Leibniz formula). The operators  $\mathcal{N}_k$  also map  $\mathcal{A}_{k, C^{r-1}} \rightarrow \mathcal{A}_{k+1, C^{r-1}(K)}$ .

Finally, the kneading operators are defined, for the moment as formal power series with coefficients bounded operators from  $\mathcal{A}_{k+1, C^{r-1}(K)}$  to  $\mathcal{A}_{k+1, C^{r-1}(K)}$  (with  $k = 0, \dots, n-1$ ) by

$$\mathcal{D}_k(z) = z \mathcal{N}_k (\mathrm{Id} - z \mathcal{M}_k)^{-1} \mathcal{S}_k.$$

Writing  $\mathcal{D}_k(z)$  as a kernel operator with kernel  $\mathcal{K}_k^z(x, y) = \sum_{j=0}^{\infty} \kappa_{k,j}(x, y)z^j$  we shall prove (using the transversality assumption, see Lemma 4) that  $\kappa_{k,j}(x, x) \in L^1(\mathbb{R}^n)$  and define the formal trace  $\mathrm{Tr}_*(\mathcal{D}_k(z))$  to be the power series

$$\mathrm{Tr}_*(\mathcal{D}_k(z)) = \sum_{j=0}^{\infty} z^j (-1)^{(n+1)k} \int_{\mathbb{R}^n} \kappa_{k,j}(x, x) dx.$$

(Note that the sign factor is not present e.g. in odd dimensions.) Proceeding similarly for iterates of  $\mathcal{D}_k(z)$  we can define a formal determinant from the formal trace as usual:

$$\mathrm{Det}_*(\mathrm{Id} + \mathcal{D}_k(z)) = \exp - \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell} \mathrm{Tr}_*(\mathcal{D}_k(z)^\ell).$$

We shall then prove the following theorem:

**Theorem 1 (Milnor-Thurston-Kitaev-Baillif formula [Bai]).** *In the sense of formal power series:*

$$\mathrm{Det}^\#(\mathrm{Id} - z\mathcal{M}) = \prod_{k=0}^{n-1} \mathrm{Det}_*(\mathrm{Id} + \mathcal{D}_k(z))^{(-1)^{k+1}}.$$

*Remarks on Theorem 1.*

- (1) We shall see that in fact for small enough  $|z|$  the operator  $\mathcal{D}_k(z)$  is bounded on  $\mathcal{A}_{k+1, L^q}$  for  $1 < q < \infty$  and that  $\mathcal{D}_k(z)^{[n/2]+1}$  is Hilbert-Schmidt on  $L^2(K)$ . This additional information allows us to express  $\mathrm{det}_*(\mathrm{Id} + \mathcal{D}_k(z))$  as the product of a regularised determinant of order  $2[n/2] + 2$  and a holomorphic non-vanishing function. This is useful to show that  $\mathrm{Det}_*(\mathrm{Id} + \mathcal{D}_k(z))$  has a positive radius of convergence (see [Bai]) and to extract spectral information from its zeroes [BB].
- (2) By using the transversality assumption it can be shown [Bai] that if the data  $\psi_\omega$  and  $g_\omega$  are  $C^\infty$  then  $\mathrm{Det}_*(\mathrm{Id} + \mathcal{D}_k(z))$  is in fact the flat-determinant [AB1, AB2] of  $\mathcal{D}_k(z)$ . Properties of the flat determinants can be used to give a short proof of Lemma 6 below. One has to use an approximation argument in case the original data is just  $C^r$  for finite  $r$  (see [Bai]). We shall thus use from now on the notation (which is slightly abusive if  $r \neq \infty$ ):

$$\mathrm{Det}^b(\mathrm{Id} + \mathcal{D}_k(z)) = \mathrm{Det}_*(\mathrm{Id} + \mathcal{D}_k(z)), \quad \mathrm{Tr}^b \mathcal{D}_k(z) = \mathrm{Tr}_* \mathcal{D}_k(z).$$

*Exercise 2.* For  $n \geq 2$ , show that the functions  $x_j/\|x\|^n$  are in  $L^q(K')$  for any compact subset  $K'$  of  $\mathbb{R}^n$  and all  $1 \leq q < n/(n-1)$ .

### 3.2 Definition of the homotopy operators $\mathcal{S}_k$ .

Let us now proceed with the definition of the homotopy operators  $\mathcal{S}_k$ . The starting point is to find an inverse to the Laplacian acting on (compactly supported)  $k$ -forms i.e. a solution  $G_k$  (for  $k = 0, \dots, n$ ) to

$$\Delta G_k = \text{Id}, \quad G_k \Delta = \text{Id}$$

on  $\mathcal{A}_{k, C^\infty(K)}$ , where  $\Delta = \Delta_k$  is the Laplacian operator acting on  $k$ -forms:

$$\Delta_k \left( \sum_{\vec{j} \in I(k)} \phi_{\vec{j}} dx_{\vec{j}} \right) = - \sum_{\vec{j} \in I(k)} \left( \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \phi_{\vec{j}} \right) dx_{\vec{j}}.$$

**Lemma 2.** Let  $E \in \mathcal{A}_{n, L^1(K)}$  be the “Green kernel”

$$E(x) = e(x) dx_1 \wedge \dots \wedge dx_n = \begin{cases} \frac{\Gamma(n/2)}{(n-2)2\pi^{n/2}} \frac{1}{\|x\|^{n-2}} dx_1 \wedge \dots \wedge dx_n & n \geq 3 \\ \frac{1}{2\pi} \log(\|x\|) dx_1 \wedge dx_2 & n = 2, \end{cases}$$

where  $\Gamma$  is Euler’s gamma-function. For  $k = 0, \dots, n$  define a  $k$ -form in  $x$  and an  $n - k$ -form in  $y$  (with coefficients in  $L^1(K)$ )  $E_k(x, y)$  by

$$E(x - y) = \sum_{k=0}^n (-1)^{n(k+1)} E_k(x, y).$$

Then the operator on compactly supported  $k$ -forms with  $C^t$  coefficients ( $t \geq 0$ ) defined by

$$G_k \phi(x) = \int_{\mathbb{R}^n} E_k(x, y) \wedge \phi(y)$$

is an inverse for the Laplacian  $\Delta_k$ .

*Proof of Lemma 2.* The function  $e(x)$  in the Green’s kernel has the property that (as a distribution on compactly supported  $C^\infty$  functions)

$$\Delta e(x) = \delta_0,$$

where the right-hand-side is the dirac mass at 0. This property can be proved by using Green’s formula – see [Sch, p. 46] for details of this classical and elementary computation. From this, it is not difficult to deduce that  $\Delta G_k = \text{Id}$  by noting first that

$$E_k(x, y) = \sum_{\vec{j} \in I(k)} s(\vec{j}') e(x - y) dx_{\vec{j}} \wedge dy_{\vec{j}'}$$

where  $\vec{j}' \subset I(k)$  is the ordered complementary of  $\vec{j}$  in  $\{1, \dots, n\}$  and  $s(\vec{j}') \in \{-1, +1\}$  is the sign of the permutation reordering  $(\vec{j}', \vec{j})$ ; so that

$$\Delta_x E_k(x, y) = \delta_{0,x}(x - y) \left( \sum_{\vec{j}} s(\vec{j}') dx_{\vec{j}} \wedge dy_{\vec{j}'} \right).$$

Indeed, it follows that for any  $k$ -form  $\phi = \phi_{\vec{\ell}} dx_{\vec{\ell}}$

$$\begin{aligned} \Delta G_k \phi(x) &= \int_{\mathbb{R}^n} \Delta_x E_k(x, y) \wedge \phi(y) \\ &= \int_{\mathbb{R}^n} \Delta_y E_k(x, y) \wedge \phi_{\vec{\ell}}(y) \wedge dy_{\vec{\ell}} \\ &= s(\vec{\ell}') \int_{\mathbb{R}^n} \delta_{0, y}(x - y) \phi_{\vec{\ell}}(y) dx_{\vec{\ell}} \wedge dy_{\vec{\ell}'} \wedge dy_{\vec{\ell}}. \end{aligned}$$

Integration by parts and one more use of  $\frac{\partial^2}{dy_i^2} E(x - y) = \frac{\partial^2}{dx_i^2} E(x - y)$  then implies  $G_k \Delta = \text{Id}$ .  $\square$

Recall now the classical identity

$$\Delta = \Delta_k = d_{k+1}^* d_k + d_{k-1} d_k^*,$$

where  $d_{k+1}^* : \mathcal{A}_{k+1} \rightarrow \mathcal{A}_k$  may be defined by duality

$$\langle d_{k+1}^* \phi, \psi \rangle = \langle \phi, d_k \psi \rangle,$$

where, for any two  $\ell$ -forms  $\varphi_1, \varphi_2$  we set

$$\langle \varphi_1, \varphi_2 \rangle = \begin{cases} \int \varphi_{1, \vec{j}} \varphi_{2, \vec{j}} dx_1 \wedge \dots \wedge dx_n & \text{if } \varphi_1, \varphi_2 \text{ have the same support } \vec{j} \in I(\ell) \\ 0 & \text{otherwise.} \end{cases}$$

Note that if  $\phi$  is a  $C^1$  function then

$$d_k^* \phi(x) dx_1 \wedge \dots \wedge dx_k = \sum_{j=1}^k (-1)^{j+1} \frac{\partial}{\partial x_j} \phi(x) dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_k,$$

where  $\widehat{dx_j}$  means that the factor  $dx_j$  has been suppressed.

Use of the following homotopy operators was first suggested by Kitaev, the expression given in the definition below was remarked by Ruelle but the operators are the same as those appearing in [Bai]:

**Homotopy operators.** For  $k = -1, \dots, n$  (and  $t \geq 0$ ) we set

$$\mathcal{S}_k = d_{k+1}^* G_{k+1} : \mathcal{A}_{k+1, C^t(K)} \rightarrow \mathcal{A}_{k, C^t}.$$

*Exercise 2.* Show (formally) that  $\mathcal{S}_{k-1} \mathcal{S}_k \equiv 0$ .

**Lemma 3.**

- (1) The homotopy operator  $\mathcal{S}_k$  on  $\mathcal{A}_{k,C^{r(-)}(K)}$  admits an expression in kernel form as

$$\mathcal{S}_k \phi(x) = \int \sigma_k(x, y) \wedge \phi(y)$$

where  $\sigma_k(x, y)$  is a  $k$ -form in  $x$  and an  $n - k$  form in  $y$  obtained from the  $n - 1$ -form

$$\sigma(x) := d^* E(x) = \sum_{i=1}^n (-1)^{i+1} \frac{x_i}{\|x\|^n} dx_1 \wedge \cdots \wedge \widehat{dx}_i \wedge \cdots \wedge dx_n,$$

by using the decomposition

$$\sigma(x - y) = \sum_{k=0}^n (-1)^{nk} \sigma_k(x, y)$$

- (2)  $d_{k-1} \mathcal{S}_{k-1} + \mathcal{S}_k d_k = Id$  on  $\mathcal{A}_{k,C^{r(-)}(K)}$ .

*Proof of Lemma 3.* In view of the definitions, the proof of (1) consists in checking that the signs match, and this is left as an exercise to the reader. Let us prove (2), i.e. verify that  $dd^* G_{k+1} + d^* G_{k+2} d = Id$ . But this is an easy consequence of the following identity

$$dd^* G_{k+1} + d^* G_{k+2} d = \Delta G_{k+1} - d^*(dG_{k+1} - G_{k+2}d),$$

since  $dG_{k+1} - G_{k+2}d = 0$  (integrating by parts).  $\square$

### 3.3 Properties of the kneading operators and other kernel operators.

In order to prove Theorem 1, we shall make use of the transversality assumption to prove that the kneading operators, and also some other related operators, are such that either their kernel can be integrated along the diagonal in the sense of an  $L^1$  function (Lemma 4), or (Lemma 5) that their generalised (Schwartz) kernel (which a priori is only a current over  $\mathbb{R}^{2n}$ ) can be restricted to the ( $n$ -dimensional) diagonal where it gives rise to a distribution, which can then be evaluated over the constant function 1 (say). In fact, it is convenient for part of the computations to assume that  $r = \infty$ . If the original data only enjoys finite smoothness, an approximation argument can be used (thanks to transversality). We refer to [Bai] for this, and will only present the proof of Theorem 1 in the case  $r = \infty$ .

To proceed, we introduce two vector spaces of operators corresponding to the two cases just discussed. The definitions will ensure that  $\mathcal{D}_k(z)$  is a power series with coefficients operators in the first space  $\mathcal{K}_{k+1}$ . In the case  $r = \infty$ , all auxiliary operators which will be introduced in the proof of Theorem 1 will be power series with coefficients operators in the second space  $\mathcal{K}_k^d$ .

**Definition (The spaces  $\mathcal{K}_k$  and  $\mathcal{K}_k^d$ ).** Let  $\psi_\omega, g_\omega$  be as in Section 3.1 for some  $r \geq 1$ , let  $\mathcal{M}_j, \mathcal{N}_\ell$  and  $\mathcal{S}_m$  be the operators defined above (acting on locally supported forms with coefficients in  $C^{r-1}(K)$ ). We say that a (finite) composition of operators  $\mathcal{M}_j, \mathcal{N}_\ell$  and  $\mathcal{S}_m$  is admissible if the degrees of the forms match. Fix an integer  $0 \leq k \leq n$ .

- (1) We write  $\mathcal{K}_k$  for the vector space of bounded operators from  $\mathcal{A}_{k, C^{r-1}(K)}$  to  $\mathcal{A}_{k, C^{r-1}}$  generated by admissible compositions of  $\mathcal{M}_j, \mathcal{N}_\ell$  and  $\mathcal{S}_m$ , with at least one  $\mathcal{S}$  factor, no two immediately successive  $\mathcal{S}$  factors, and the first or last factor of type  $\mathcal{M}$  or  $\mathcal{N}$ .
- (2) If  $r = r - 1 = \infty$ , we write  $\mathcal{K}_k^d$  for the vector space of bounded operators from  $\mathcal{A}_{k, C^\infty(K)}$  to  $\mathcal{A}_{k, C^\infty}$  generated by admissible compositions of  $\mathcal{M}_j, \mathcal{N}_\ell, \mathcal{S}_m$ , and  $d_q$  with at least one  $\mathcal{S}$  factor, no two immediately successive  $\mathcal{S}$  factors, at least one  $\mathcal{M}_j$  or  $\mathcal{N}_\ell$  between two  $d$ s, and the first or last factor of type  $\mathcal{M}$  or  $\mathcal{N}$ .

*Exercise 3 ( $\mathcal{K}_k$  and  $\mathcal{K}_k^d$ ).* Check that  $\mathcal{D}_k(z) \in \mathcal{K}_{k+1}[[z]]$  and that  $\mathcal{M}_k \in \mathcal{K}_k^d$ . (Hint: use  $d\mathcal{S} + \mathcal{S}d = \text{Id}$ .)

**Lemma 4.** If  $\mathcal{Q} \in \mathcal{K}_k$  then  $\mathcal{Q}$  is a linear combination of kernel operators  $\mathcal{Q}_i$  with

$$\begin{aligned} \mathcal{Q}_i : \mathcal{A}_{k, C^{r-1}(K)} &\rightarrow \mathcal{A}_{k, C^{r-1}(K)} \\ \mathcal{Q}_i \varphi(x) &= \int h(x) K(x, y) \tilde{h}(y) \wedge \phi(y) \end{aligned}$$

where

- (1) there are  $\hat{s}(i) \geq 1$  and  $\Psi = \psi_{\tilde{\omega}}^{\hat{s}}$  so that  $K(x, y)$  is a  $k$ -form in  $x$  and an  $n-k$ -form in  $y$ , which is  $C^r$  except on  $\Psi(x) = y$ ;
- (2)  $h$  and  $\tilde{h}$  are  $C^{r-1}$  functions on  $\mathbb{R}^n$ ,  $h$  is supported in  $K$  if the leftwards factor in  $\mathcal{Q}_i$  is not  $\mathcal{S}_k$  while  $\tilde{h}$  is supported in  $K$  if the rightwards factor is not  $\mathcal{S}_{k-1}$ ;
- (3)  $\chi_K K(x, x) \in L^p(\mathbb{R}^n)$  for all  $1 \leq p \leq n/(n-1)$ .

Lemma 4 allows us to define the flat-trace of an element of  $\mathcal{K}_k$  or of  $\mathcal{K}_k[[z]]$  (in particular,  $\text{Tr}^b(\mathcal{D}(z))$  is now well-defined):

**Flat trace of kernel operators.** If  $\mathcal{Q} \in \mathcal{K}_k$  then, using the notation from Lemma 4, we define  $\text{Tr}^b \mathcal{Q} \in \mathbb{C}$  by

$$\text{Tr}^b \mathcal{Q} = (-1)^{(n+1)k} \int_{\mathbb{R}^n} h(x) K(x, x) \tilde{h}(x) dx.$$

If  $\mathcal{Q}(z) = \sum_{j=0}^{\infty} z^j \mathcal{Q}_j \in \mathcal{K}_k[[z]]$  (convergent or not), then we set  $\text{Tr}^b \mathcal{Q}(z) \in \mathbb{C}[[z]]$

$$\text{Tr}^b \mathcal{Q}(z) = \sum_{j=0}^{\infty} z^j (-1)^{(n+1)k} \int_{\mathbb{R}^n} h_j(x) K_j(x, x) \tilde{h}_j(x) dx.$$

*Remark on the flat trace on  $K_d$ .* Let  $\mathcal{Q} \in \mathcal{K}_k$ . We shall not need the following facts:

- (1) If  $r = \infty$ , it is possible to show that  $\text{Tr}^b \mathcal{Q}$  coincides with the classical Atiyah-Bott flat trace [AB1, AB2]. See [Bai].

(2) One can prove that  $\text{Tr}^b \mathcal{Q}$  only depends on the values of  $\mathcal{Q}$  on  $C^\infty$  locally supported  $k$ -forms.

*Sketch of proof of Lemma 4.* The first two claims can be proved by induction on the number of factors, with  $s(i)$  being the number of non- $\mathcal{S}$  factors. (It is convenient in the proof to introduce a unified notation for the operators  $\mathcal{M}$  and  $\mathcal{N}$  by writing

$$T\varphi(x) = \sum_{\omega} \eta_{\omega}(x) \wedge \psi_{\omega}^* \varphi(x)$$

with  $\eta_{\omega}$  in  $\mathcal{A}_{\ell-k, C^{r-1}(K)}$  for  $\ell = k$  or  $k + 1$ .)

We concentrate on the proof of (3). Our starting point is the following easily proved expression for  $K(x, y)$  (use Lemma 3):

$$K(x, y) = \int_{(\mathbb{R}^n)^s} G(x, x^{(1)}, \dots, x^{(s)}, y) H(x, x^{(1)}, \dots, x^{(s)}, y) dx^{(1)} \wedge \dots \wedge dx^{(s)},$$

where  $s \leq \hat{s}(i)$ ,  $G(x, x^{(1)}, \dots, x^{(s)}, y)$  is  $C^{r-1}$  and compactly supported on  $\{x\} \times (\mathbb{R}^n)^s \times \{y\}$ , while, setting  $x^{(0)} = x$ ,  $x^{(s+1)} = y$ ,  $H(x^{(0)}, x^{(1)}, \dots, x^{(s)}, x^{(s+1)})$  can be written as a linear combination of expressions

$$H_{\vec{j}}(x^{(0)}, x^{(1)}, \dots, x^{(s)}, x^{(s+1)}) = \prod_{t=0}^s \frac{\psi_t(x^{(t)})_{j_t} - x_{j_t}^{(t+1)}}{\|\psi_t(x^{(t)}) - x^{(t+1)}\|^n}$$

for suitable  $1 \leq j_1, \dots, j_s \leq n$ . (Each  $\psi_t$  is a composition of finitely many  $\psi_{\omega}$ s.) It thus suffices to show that each

$$\left| \prod_{t=0}^s \chi_J(x^{(t)}) H_{\vec{j}}(x^{(0)}, x^{(1)}, \dots, x^{(s)}, x^{(0)}) \right|^p$$

belongs to  $L^1(\mathbb{R}^n)^{(s+1)}$ . The singularities of  $H_{\vec{j}}$  are isolated (by transversality), there are thus finitely many of them in a compact set. We shall content ourselves with proving local integrability in the neighbourhood of the “worse” possible singularities  $\hat{x}$ , i.e.

$$\begin{cases} \psi_t(\hat{x}^{(t)}) = \hat{x}^{(t+1)}, \forall t = 0, \dots, s-1, \\ \psi_s(\hat{x}^{(s)}) = \hat{x}^{(0)}. \end{cases}$$

(The task of checking that the singularities corresponding to the vanishing of some, but not all, of the  $s + 1$  factors in the denominator of  $H_{\vec{j}}$  are also locally integrable is left to the reader.) Let us perform the change of variables

$$\begin{cases} w^{(t)} = \psi_t(x^{(t)}) - x^{(t+1)}, \forall t = 0, \dots, s-1, \\ w^{(s)} = \psi_s(x^{(s)}) - x^{(0)}. \end{cases}$$

We shall check later (using transversality) that the Jacobian  $J(w) = |\det(\frac{d}{dx}w(x))|$  of the above change of variables (which is obviously  $C^{r-1}$ ) does not vanish at  $\hat{w} = w(\hat{x}) = 0$ . In the new coordinates, we have (with  $\delta(\epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$ )

$$\begin{aligned} & \int_{\|x-\hat{x}\|\leq\epsilon} |H(x^{(0)}, \dots, x^{(s)}, x^{(0)})|^p dx^{(0)} \dots dx^{(s)} \\ & \leq \int_{w \in \mathbb{R}^{n(s+1)}, \|\hat{w}\|\leq\delta} \frac{1}{J(w)} \left| \prod_{t=0}^s \frac{w_{j_t}^{(t)}}{\|w^{(t)}\|^n} \right|^p dw^{(0)} \dots dw^{(s)} \\ & \leq C \prod_{t=0}^s \int_{y \in \mathbb{R}^n, \|y\|\leq\delta} \left( \frac{|y_{j_t}|}{\|y\|^n} \right)^p dy < \infty, \end{aligned}$$

since  $p < n/(n-1)$ .

It remains to check that  $J(0) \neq 0$ . For this, we observe that  $J = \det D$  with  $D(w)$  the  $n \times n$  matrix with entries

$$D(w)_{tu} = \begin{cases} D(\psi_t)_{x(w)} & t = u, \\ -1 & u = t + 1 \leq s \text{ or } t = s, u = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If  $\det D(0) = 0$ , then there would exist a nonzero vector  $v_t$ ,  $t = 0, \dots, s$ , with  $Dv = 0$ , i.e.,  $v_{t+1} = D(\psi_t)_{\hat{x}_t} v_t$  and  $v_0 = D(\psi_s)_{\hat{x}_s} v_s$ . But then,  $v_0 = D(\psi_t^{s+1})_{\hat{x}} v_0$ , contradicting transversality at the periodic point  $\hat{x}$ .  $\square$

We shall not give the proof of the following claim, referring instead to [Bai]. It relies on transversality. ([Bai] uses of results of Guillemin and Sternberg and the wave-front-set, and he notes that the flat-trace in  $\mathcal{K}_k^d$  coincides with that of Atiyah-Bott [AB1, AB2].)

**Lemma-Definition 5.** *Let  $\mathcal{Q} \in \mathcal{K}_k^d$  and let  $K_{\mathcal{Q}}(x, y)$  be its Schwartz kernel [Sch], which is a  $k$ -current in  $x$ , and an  $n - k$ -current in  $y$ , with coefficients distributions of finite order. Then  $\delta(x - y)K_{\mathcal{Q}}(x, y)$  is a compactly supported distribution on  $\mathbb{R}^{2n}$ . It can thus be evaluated on the constant function 1, giving a meaning to the following definition:*

$$\text{Tr}^b \mathcal{Q} = (-1)^{(n+1)k} \int_{\mathbb{R}^n} K_{\mathcal{Q}}(x, x) dx.$$

*Exercise 4.* Give an expression for the Schwartz kernel of  $\mathcal{M}_k$ . Check that the flat trace of  $\mathcal{M}_k$  as defined in Lemma-Definition 5 coincides with the formal definition of Exercise 1.

The proof of Lemma 5 in [Bai] shows that it is legitimate to invoke the Fubini theorem when manipulating the Schwartz kernels of elements of  $\mathcal{K}_k^d$ . (This is not obvious since these are not functional kernels.) As a consequence, he proves:

**Lemma 6.** *If  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  are finite compositions of  $\mathcal{S}_m$ ,  $\mathcal{N}_p$  and  $\mathcal{M}_q$  so that  $\mathcal{Q}_1\mathcal{Q}_2 \in \mathcal{K}_k^d$  and  $\mathcal{Q}_2\mathcal{Q}_1 \in \mathcal{K}_\ell^d$ , then*

$$\mathrm{Tr}^b \mathcal{Q}_1\mathcal{Q}_2 = \mathrm{Tr}^b \mathcal{Q}_2\mathcal{Q}_1.$$

As an immediate consequence, we get

**Corollary of Lemma 6.** *Under the assumptions of Lemma 6:*

$$\mathrm{Det}^b(\mathrm{Id} - z\mathcal{Q}_1\mathcal{Q}_2) = \mathrm{Det}^b(\mathrm{Id} - z\mathcal{Q}_2\mathcal{Q}_1).$$

*If, additionally,  $k = \ell$  and  $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{K}_k^d$  then:*

$$\mathrm{Det}^b(\mathrm{Id} - z\mathcal{Q}_1 - z\mathcal{Q}_2 + z^2\mathcal{Q}_1\mathcal{Q}_2) = \mathrm{Det}^b(\mathrm{Id} - z\mathcal{Q}_1) \mathrm{Det}^b(\mathrm{Id} - z\mathcal{Q}_2).$$

*Exercise 5.* Formulate Lemma 6 and its Corollary for elements of  $\mathcal{K}_k^d[[z]]$ , such as  $(1 - z\mathcal{M}_k)^{-1}$ .

### 3.4 Proof of the Milnor–Thurston formula in the $C^\infty$ case.

Let us exploit Lemmas 4–6 from § 3.3 to sketch a proof of Theorem 1 under the additional assumption that  $r = \infty$ . (We refer to [Bai] for the general case which uses an approximation argument due to Kaloshin.) We start by rewriting  $\mathrm{Det}^b(\mathrm{Id} + \mathcal{D}_k(z))$ :

$$\begin{aligned} \mathrm{Det}^b(\mathrm{Id} + \mathcal{D}_k(z)) &= \mathrm{Det}^b(\mathrm{Id} + z\mathcal{N}_k(\mathrm{Id} - \mathcal{M}_k)^{-1}\mathcal{S}_k) \\ &= \mathrm{Det}^b(\mathrm{Id} + z\mathcal{S}_k\mathcal{N}_k(\mathrm{Id} - \mathcal{M}_k)^{-1}) \\ &= \mathrm{Det}^b(\mathrm{Id} - z(\mathcal{M}_k - \mathcal{S}_k\mathcal{N}_k)) \mathrm{Det}^b((\mathrm{Id} - \mathcal{M}_k)^{-1}) \\ &= \mathrm{Det}^b(\mathrm{Id} - z(\mathcal{M}_k - \mathcal{S}_k\mathcal{N}_k)) (\mathrm{Det}^b(\mathrm{Id} - \mathcal{M}_k))^{-1}. \end{aligned}$$

By Exercise 1, it thus suffices to check that

$$\prod_{k=0}^{n-1} \mathrm{Det}^b(\mathrm{Id} - z(\mathcal{M}_k - \mathcal{S}_k\mathcal{N}_k))^{(-1)^k} = \mathrm{Det}^b(\mathrm{Id} - z\mathcal{M}_n)^{(-1)^{n-1}}.$$

But this follows from

$$\begin{aligned} \mathrm{Det}^b(\mathrm{Id} - z(\mathcal{M}_k - \mathcal{S}_k\mathcal{N}_k)) &= \mathrm{Det}^b(\mathrm{Id} - z(\mathcal{M}_k - \mathcal{S}_k d\mathcal{M}_k + \mathcal{S}_k\mathcal{M}_{k+1}d)) \\ &= \mathrm{Det}^b(\mathrm{Id} - z(d\mathcal{S}_{k-1}\mathcal{M}_k + \mathcal{S}_k\mathcal{M}_{k+1}d)) \\ &= \mathrm{Det}^b(\mathrm{Id} - zd\mathcal{S}_{k-1}\mathcal{M}_k) \mathrm{Det}^b(\mathrm{Id} - z\mathcal{S}_k\mathcal{M}_{k+1}d) \\ &= \mathrm{Det}^b(\mathrm{Id} - zd\mathcal{S}_{k-1}\mathcal{M}_k) \mathrm{Det}^b(\mathrm{Id} - zd\mathcal{S}_k\mathcal{M}_{k+1}) \end{aligned}$$

(in the third line we used  $d^2 = 0$ ). Indeed, it is clear that the factors in the alternated product cancel, except for

$$\mathrm{Det}^b(\mathrm{Id} - zd\mathcal{S}_{-1}\mathcal{M}_0) = 1 \text{ and } \mathrm{Det}^b(\mathrm{Id} - zd\mathcal{S}_{n-1}\mathcal{M}_n) = \mathrm{Det}^b(\mathrm{Id} - z\mathcal{M}_n). \quad \square$$

**A. Hilbert-Schmidt operators and their regularised determinants.**

Let  $H$  be a separable Hilbert space. We recall here the results that we need, referring to [GGK] for proofs and for more statements.

**Definition (Hilbert-Schmidt operator).** A compact linear operator  $A : H \rightarrow H$  is called Hilbert-Schmidt, noted  $A \in \mathcal{S}_2$  if  $B = A^*A$  is a trace-class operator, noted  $B \in \mathcal{S}_1$ . A compact linear operator  $B$  on  $H$  is called trace-class if  $\sum_{j=1}^{\infty} s_j(B) < \infty$ , where the  $j$ th singular number of  $B$  is defined by  $s_j(B) := \sqrt{\lambda_j(B^*B)}$ , with

$$\lambda_1(B^*B) \geq \lambda_2(B^*B) \geq \dots \geq \lambda_j(B^*B) \geq \dots > 0,$$

the sequence of nonzero eigenvalues of  $B^*B$ , repeated according to multiplicity.

*Equivalent definition.* A compact linear operator  $A$  on  $H$  is Hilbert-Schmidt if and only if there is an orthonormal basis  $\{\varphi_j\}$  of  $H$  for which  $\sum_j \|A\varphi_j\|^2 < \infty$ . (The sum then converges for every orthonormal basis of  $H$ .)

We refer e.g. to [DS1] for a proof of the very classical result:

**Proposition (Hilbert-Schmidt operators on  $L^2(d\mu)$ ).** Let  $\mu$  be a nonnegative measure on a  $\sigma$ -algebra of a set  $I$ . Let  $K(x, y)$  be a measurable function on  $I \times I$ . Then the (kernel) operator on the Hilbert space  $H = L^2(d\mu)$  associated to  $K$  by

$$A\varphi(x) = \int_I K(x, y) \varphi(y) d\mu(y)$$

is Hilbert-Schmidt if and only if  $K \in L^2(d\mu \times d\mu)$ , i.e.

$$\int_{I \times I} |K(x, y)|^2 d\mu(x) d\mu(y) < \infty.$$

We now return to our abstract separable Hilbert space and discuss traces and determinants. Norms on the so-called Schatten classes  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are introduced in the following exercise:

*Exercise 0.* Let  $H$  be a separable Hilbert space and write  $L(H)$  for the algebra of bounded linear operators on  $H$ . Show that the expressions

$$\|A\|_2 := \sqrt{\sum_j s_j^2(A)}, \quad \|B\|_1 := \sum_j s_j(B)$$

define norms on  $\mathcal{S}_2$ , respectively  $\mathcal{S}_1$ , that  $\mathcal{S}_1$  is a complete subalgebra of  $L(H)$  for this norm:

$$\|BB'\|_1 \leq \|B\|_1 \|B'\|_1,$$

and that  $\mathcal{S}_1 \subset \mathcal{S}_2$  continuously.

**Lemma 0** ( $\mathcal{S}_2\mathcal{S}_2 \subset \mathcal{S}_1$ ). If  $A, A'$  belong to  $\mathcal{S}_2$  then  $AA' \in \mathcal{S}_1$  and

$$\|AA'\|_1 \leq \|A\|_2 \|A'\|_2.$$

*Sketch of proof of the lemma.* For each  $k \geq 1$  one can easily show that

$$\sum_{j=1}^k s_j(AA') \leq \sum_{j=1}^k s_j(A)s_j(A'),$$

to finish, one applies the Cauchy-Schwarz inequality.

### The algebra of trace-class operators $\mathcal{S}_1$

We already noted in Exercise 0 that  $\mathcal{S}_1$  is a subalgebra of  $L(H)$ . This algebra is in fact continuously embedded in  $L(H)$ , i.e. for each  $B \in \mathcal{S}_1$  the operator norm is bounded by the norm in  $\mathcal{S}_1$ :

$$\|B\|_{L(H)} \leq \|B\|_1.$$

This embedded subalgebra has the approximation property that the space of finite rank operators  $\mathcal{F}$  on  $H$  is dense in  $\mathcal{S}_1$  (for the  $\mathcal{S}_1$  norm  $\|\cdot\|_1$ ). We are thus in a position to apply the following extension theorem (see e.g. [GGK, Chapter II.2] for a proof) to  $\tilde{\mathcal{S}} = \mathcal{S}_1$ :

**Theorem 1 (Extending the trace and determinant).** Let  $\tilde{\mathcal{S}}$  be a continuously embedded subalgebra of  $L(H)$  with the approximation property. The following properties are equivalent:

- (1) The trace  $F \mapsto \text{Tr} F$  on  $\mathcal{F} \cap \tilde{\mathcal{S}}$  is a bounded functional for the norm  $\|\cdot\|_{\tilde{\mathcal{S}}}$ .
- (2) The trace  $\text{Tr} F$  and the determinant  $F \mapsto \text{Det}(Id + F)$  admit continuous extensions from  $\mathcal{F} \cap \tilde{\mathcal{S}}$  to  $\tilde{\mathcal{S}}$ . The continuity is in the sense of the  $\tilde{\mathcal{S}}$  norm and we have for any sequence  $F_n \in \mathcal{F} \cap \tilde{\mathcal{S}}$  converging to  $A \in \tilde{\mathcal{S}}$ :

$$\text{Tr}(A) = \lim_{n \rightarrow \infty} \text{Tr} F_n, \quad \text{Det}(Id + A) = \lim_{n \rightarrow \infty} \text{Det}(Id + F_n).$$

We shall make use of the following properties of the extended determinants (the proofs are to be found in [GGK, II.3 and II.6]):

**Theorem 2 (Properties of the extended determinant).** Assume that we are in the equivalent conditions of the previous theorem. Then for each compact  $A \in \tilde{\mathcal{S}}$ :

- (1) The function  $\lambda \mapsto \text{Det}(Id - \lambda A)$  is an entire function. (There is a formula for the coefficients of the Taylor series at zero, called the Plemelj-Smithies formula.)
- (2)  $\text{Det}(Id - \lambda A) = \exp - \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \text{Tr} A^n$ .
- (3)  $\text{Det}(Id - \lambda_0 A) = 0$  with order  $m_0 \geq 1$  if and only if  $1/\lambda_0$  is an eigenvalue of  $A$  of algebraic multiplicity  $m_0$ .

The following property is essential to our application:

*Exercise 1 (Analyticity of the extended trace and determinant).* Let  $z \mapsto A(z)$  be an analytic map at  $z_0 \in \mathbb{C}$  with each  $A(z)$  in a Banach algebra  $\tilde{\mathcal{S}}$  satisfying the equivalent conditions of the extension Theorem 1. Then both maps  $z \mapsto \text{Tr } A(z)$  and  $z \mapsto \text{Det}(\text{Id} + A(z))$  are analytic at  $z_0$ .

Finally, we have:

**Lidskii Trace Theorem ([GGK, IV.6]).** For  $A \in \mathcal{S}_1$ , writing  $\lambda_j(A)$  for the eigenvalues of  $A$  repeated with multiplicity, we have

$$\text{Tr } A = \sum_j \lambda_j(A), \quad \text{Det}(\text{Id} - A) = \prod_j (1 - \lambda_j(A)).$$

### Hilbert-Schmidt operators and their regularised determinants

If  $A$  is a Hilbert-Schmidt operator on a separable Hilbert space  $H$ , then the following operator is trace-class:

$$R_A := \text{Id} - (\text{Id} - A) \exp(A).$$

Indeed

$$\begin{aligned} R_A &= \text{Id} - \sum_{j=0}^{\infty} \frac{A^j}{j!} + \sum_{j=0}^{\infty} \frac{A^{j+1}}{j!} \\ &= \sum_{j=2}^{\infty} \frac{A^j(1-j)}{j!}, \end{aligned}$$

so that  $R_A$  is an absolutely convergent sum of operators in  $\mathcal{S}_1$  (use Lemma 1).

**Definition (regularised determinant).** To  $A \in \mathcal{S}_2$  we associate a regularised determinant (of order two) by setting:

$$\text{Det}_2(\text{Id} - A) = \text{Det}(\text{Id} - R_A) = \text{Det}((\text{Id} - A) \exp(A)).$$

Note that there exists a theory of regularised determinants of order  $p \geq 2$  for the Schatten classes  $\mathcal{S}_p$  which have the property that  $A \in \mathcal{S}_p$  implies  $A^p \in \mathcal{S}_1$ . (We refer to [GGK].)

The regularised determinant immediately inherits several properties from the determinant in  $\mathcal{S}_1$ :

**Corollary of Theorem 2 (Properties of the regularised determinant).** For each  $A \in \mathcal{S}_2$ :

- (1) If  $A \in \mathcal{S}_1$  (for example  $A \in \mathcal{F}$ ) then  $\text{Det}_2(\text{Id} - A) = \text{Det}(\text{Id} - A) \cdot \exp(\text{Tr } A)$ .
- (2) The function  $\lambda \mapsto \text{Det}_2(\text{Id} - \lambda A)$  is an entire function. (There is a formula for the coefficients of the Taylor series at zero, also called the Plemelj-Smithies formula [GGK, IX.3].)
- (3)  $\text{Det}_2(\text{Id} - \lambda A) = \exp - \sum_{n=2}^{\infty} \frac{\lambda^n}{n} \text{Tr } A^n$ . [GGK, IX.3]
- (4)  $\text{Det}_2(\text{Id} - \lambda_0 A) = 0$  with order  $m_0 \geq 1$  if and only if  $1/\lambda_0$  is an eigenvalue of  $A$  of algebraic multiplicity  $m_0$ .

As a consequence e.g. of (1) we see that the regularised determinant is *not multiplicative*. In applications, it is often necessary to complete it by a factor “replacing” the missing  $\exp -\text{Tr}$ .

*Proof of the Corollary.* We only prove (4), leaving the other claims as exercises. (In particular, (3) follows from (1) and the fact that  $\text{Det}_2(\text{Id} - A) = \lim \text{Det}_2(\text{Id} - F_n)$  where  $F_n$  is a sequence of finite-rank operators converging to  $A \in \mathcal{S}_2$  in the  $\|\cdot\|_2$  norm, noting that  $\mathcal{S}_2$  has the approximation property.)

Let then  $\lambda_0 \in \mathbb{C}$  be such that  $\text{Det}_2(\text{Id} - \lambda_0 A) = 0$  with order  $m_0 \geq 1$ . For simplicity, we assume that  $\lambda_0 = 1$ . Our assumption is equivalent to the fact that 1 is an eigenvalue of algebraic multiplicity  $m_0$  for  $R_A = \text{Id} - (\text{Id} - A) \exp(A)$ . Let then  $\{\varphi_j, j = 1, \dots, m_0\}$  be a basis for the generalised eigenspace of  $R_A$  and the eigenvalue 1. If  $R_A \varphi_j = \varphi_j$  then

$$-(\text{Id} - A) \exp(A) \varphi_j = -\exp(A)(\text{Id} - A) \varphi_j = 0,$$

so that  $\psi_j := \varphi_j$  is a fixed point of  $A$ .

Now, if  $\varphi_j$  and  $\ell \geq 2$  are such that  $(\text{Id} - R_A)^\ell \varphi_j = 0$  but  $\varphi'_j = (\text{Id} - R_A)^{\ell-1} \varphi_j \neq 0$  then (since  $\text{Id} - A$  commutes with  $\exp(A)$ )

$$0 = \exp(A)(\text{Id} - A) \varphi'_j = \exp(\ell A)(\text{Id} - A)^\ell \varphi_j,$$

while  $\exp((\ell - 1)A)(\text{Id} - A)^{\ell-1} \varphi_j \neq 0$ , and thus, using commutativity again,

$$\exp(\ell A)(\text{Id} - A)^{\ell-1} \varphi_j \neq 0.$$

Taking  $\psi_j := \varphi_j$ , we complete our identification of the generalised basis of  $R_A$  and that of  $A$  for 1.  $\square$

**Corollary of Exercise 1.** *Let  $z \mapsto A(z)$  be an analytic map at  $z_0 \in \mathbb{C}$  with  $A(z) \in \mathcal{S}_2$ . Then the map  $z \mapsto \text{Det}_2(\text{Id} + A(z))$  is analytic at  $z_0$ .*

We also mention for the record:

**Corollary of the Lidskii Theorem.** *For  $A \in \mathcal{S}_2$ , writing  $\lambda_j(A)$  for the eigenvalues of  $A$  repeated with multiplicity, we have*

$$\text{Det}_2(\text{Id} - A) = \prod_j (1 - \lambda_j(A)) \exp(\lambda_j(A)).$$

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