Dynamics beyond uniform hyperbolicity: Linear response in the absence of structural stability

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In 1997, Ruelle[18] obtained a *linear response* formula, that is a formula for the derivative $\partial_t \mu_t$ of the SRB measure of f_t, in the case of a one-parameter family f_t of smooth uniformly hyperbolic mixing attractors. He[19] conjectured that a similar formula should hold beyond the smooth uniformly hyperbolic case. In this note, we explain how we discovered (see[2] and our joint work[3,4] with D. Smania) that an additional condition is necessary in the one-dimensional setting of piecewise expanding unimodal maps. The (codimension one) condition is that the family f_t be tangent to the topological class of f_0. When this holds, we obtain Ruelle's candidate for the derivative. We end by mentioning recent developments in the nonuniformly hyperbolic setting.

Keywords: SRB measures, linear response, piecewise expanding maps, transfer operator.

1. Introduction

The minisymposium "Global dynamics beyond uniform hyperbolicity — mixing and statistical properties" held at Equadiff 07 explored several avenues beyond the framework of smooth uniformly hyperbolic dynamics. Sometimes, the statistical features (such as exponential mixing or invariance principles, e.g.) remain the same, although new ideas are needed to tackle technical difficulties. In other cases, intrinsically different behaviour appears (bifurcations, phase transitions, or slow mixing, to name a few). We shall present recent results from the second category: D. Ruelle[18] proved that the average $\int \varphi d\mu_t$ of a smooth observable φ with respect to the SRB[23] measure μ_t of a smooth one-parameter family f_t of smooth hyperbolic mixing attractors depends differentiably on t, and he gave a formula for the derivative: the *linear response formula.* He conjectured[19] (see also[20]) that linear response should hold much more generally (possibly in the sense of

Whitney), giving a candidate for the derivative in the form of a possibly divergent series. (Finding the appropriate resummation is part of the question.) We considered the (non structurally stable) setting of piecewise expanding unimodal interval maps, and first[2] exhibited a sufficient condition for the series proposed by Ruelle to admit a resummation. Then, with D. Smania, we[3,4] showed that this condition is equivalent to *horizontality,* i.e., that the path f_t be *tangent to the topological class of* f_0. In addition, we[3] proved that μ_t is differentiable at zero, and that the derivative coincides with the resummation for Ruelle's series obtained in our previous paper[2] *if and only if* the condition holds. [a] This (codimension one) horizontality condition for linear response had not been expected by anyone.

Before stating our results more precisely in Section 3, we shall briefly review previous literature and discuss Ruelle's program in Section 2. In Section 4, we mention new results on nonuniformly hyperbolic interval maps.

2. Previous results – Ruelle's candidate for the derivative

2.1. *A toy model*

Fix $\delta > 0$ and consider a $C^{2+\delta}$ locally uniformly expanding circle endomorphism $f : S^1 \to S^1$. Then the SRB measure is an absolutely continuous invariant probability measure with a $C^{1+\delta}$ density ρ, which is the fixed point of the transfer operator

$$\mathcal{L}\varphi(x) = \sum_{y:f(y)=x} \frac{\varphi(y)}{|f'(y)|} \tag{1}$$

acting on the Banach space $C^{1+\delta}(S^1)$. The eigenvalue 1 for this operator is simple, and the rest of the spectrum lies in a disc of radius strictly smaller than 1. (These spectral properties persist if $\mathcal{L}$ is viewed as acting on $C^{\delta}(S^1)$.) If $t \mapsto f_t$ is a $C^{1+\delta}$ map, with $f = f_0$, then it is not very difficult to see that $t \mapsto \mathcal{L}_t$ is differentiable at $t = 0$, if $\mathcal{L}_t$ is viewed as acting from $C^{1+\delta}$ to C^{δ}. Let us make the assumption[b] that $v = \partial_t f_t|_{t=0}$ can be written as $v = X \circ f$, with $X \in C^{1+\delta}(S^1)$. Let ρ_t be the invariant density of f_t. Computing the formula for $\partial_t \mathcal{L}_t|_{t=0}$ and using standard techniques from perturbation theory[12] , one can prove

$$\partial_t \rho_t|_{t=0} = -(\mathrm{id} - \mathcal{L})^{-1}(X'\rho + X\rho')\,, \tag{2}$$

[a] At the time of Equadiff 07, only sufficiency of the condition had been proved.
[b] See [21, beginning of §17] for an example of how to reduce to this case.

where a prime denotes derivative with respect to $x \in S^1$, and the derivative with respect to t is taken viewing ρ_t as an element of $C^\delta(S^1)$. By the spectral properties of $\mathcal{L}$ on $C^\delta(S^1)$, and since $\int_{S^1}(X\rho)'\,dx = 0$ ensures that the residue of the pole of $(\mathrm{id} - z\mathcal{L})^{-1}((X\rho)')$ at $z = 1$ vanishes, the right-hand-side of (2) is well–defined. *We view (2) as a "meta-formula," for the linear response which, suitably interpreted, gives the derivative of the SRB measure in rather wide generality.*

In higher dimensions, one puts $(X\rho)' = \rho \operatorname{div} X + \langle \operatorname{grad} \rho, X\rangle$. In settings where the SRB measure is not absolutely continuous, ρ and $(X\rho)'$ should be viewed as distributions, e.g. in the spaces introduced in[6,7,10] via

$$\int (X\rho)'\varphi\,dx = -\int \langle \operatorname{grad} \varphi, \rho X\rangle\,dx = -\int \langle \operatorname{grad}\varphi, X\rangle \rho\,dx\,,$$

for φ of sufficiently high differentiability. The *real* difficulty of this problem is to interpret suitably the right-hand-side of (2) when $X\rho'$ does not belong to a Banach space on which the transfer operator has good spectral properties.

In view of comparing (2) with Ruelle's formula (5), note that $\mathcal{L}^*(dx) = dx$ and integration by parts yield (the sums converge exponentially)

$$\begin{aligned} -\int \varphi(\mathrm{id}-\mathcal{L})^{-1}(X'\rho + X\rho')\,dx &= -\sum_{j=0}^{\infty}\int \mathcal{L}^j\big((\varphi\circ f^j)(X\rho)'\big)\,dx \\ &= \sum_{j=0}^{\infty}\int (\varphi\circ f^j)' X\,\rho\,dx\,. \end{aligned} \tag{3}$$

2.2. *The smooth hyperbolic and partially hyperbolic case*

Ruelle[18] considered C^3 paths $t \mapsto f_t$ of mixing C^3 Axiom A attractors. Writing μ_t for the SRB measure of the diffeomorphism f_t (see[23] for a definition), he showed that for any C^2 observable φ the map $t \mapsto \int \varphi\,d\mu_t$ is differentiable[c], and, writing $X := \partial_t f_t|_{t=0} \circ f_0^{-1}$, he proved that

$$\partial_t \int \varphi\,d\mu_t|_{t=0} = \sum_{j=0}^{\infty}\int \langle \operatorname{grad}(\varphi\circ f_0^j), X\rangle\,d\mu_0\,, \tag{4}$$

and that the right-hand-side of the above equation is an exponentially decaying sum (using for this a decomposition of X as (X^s, X^u)).

[c]Differentiability — without the linear response formula — had been obtained previously for Anosov flows by Katok et al. [13, p. 595]. Note that there are several typos there, in particular f in Cor. 1 need only be assumed Hölder.

We expect that a simpler proof of Ruelle's result[18] can be obtained by using the spaces introduced in[6,7,10] as explained when discussing the metaformula (2) above. See Butterley–Liverani[8] , for an implementation of such modern techniques in the framework of hyperbolic flows.

More recently, Dolgopyat[9] considered a class of partially hyperbolic systems: Assuming that f_0 is a C^∞ diffeomorphism which is an Abelian element of an Anosov action, and that f_0 is rapidly mixing, he proved that for any C^∞ path $t \mapsto f_t$, any C^∞ observable φ, and any SRB measure μ_t (in the sense of[23]) for f_t, the map $t \mapsto \int \varphi \, d\mu_t$ is differentiable at $t = 0$ and (4) holds (for $X = \partial_t f_t|_{t=0} \circ f_0^{-1}$). The sum in the right-hand-side is convergent (not necessarily exponentially, see [9, P. 405]). Generically, the time-one map of an Anosov flow is rapidly mixing, giving an example of f_0 satisfying the required conditions. (Toral extensions of Anosov diffeomorphisms give other simple examples, see [9, §2.3].) Contrary to the Axiom A case, the maps f_0 considered by Dolgopyat are not structurally stable in general. However "most" [9, §2.4] orbits can be shadowed. In other words, the breakdown of structural stability is not too drastic (morally, it occurs only in neutral directions, but we are not able to formulate this precisely).

2.3. *Ruelle's candidate for the derivative of the SRB*

In several papers and lectures, Ruelle[19,20] suggested that the linear response formula (4), suitably interpreted, should hold in more general settings, such as parametrised familes f_t of smooth unimodal maps or Hénon-like maps. Ruelle gave two crucial indications on how to interpret (4). The first one is that, since "bad" parameters t (e.g. for which there is no SRB measure μ_t) will exist arbitrarily close to a "good" nonuniformly hyperbolic (e.g. Collet-Eckmann or Benedicks-Carleson) parameter $t = 0$, differentiability should be understood only in the sense of Whitney [d] over a set of good parameters. The second one is that, since (4) will not always be a convergent sum (indeed, it may be exponentially divergent), it could be understood as *some kind* of analytic continuation at $z = 1$ of the *susceptibility function* $\Psi(z)$ associated to f_0, X, and φ, via

$$\Psi(z) = \sum_{j=0}^{\infty} \int z^j \langle \mathrm{grad}\,(\varphi \circ f_0^j), X \rangle \, d\mu_0 \, . \qquad (5)$$

[d] A function $\mathcal{R}(t)$ is differentiable in the sense of Whitney at $0 \in \Lambda$, where Λ contains 0 as an accumulation point, if there exists r so that $\mathcal{R}(t) = \mathcal{R}(0) + rt + o(t)$ for all $t \in \Lambda$.

(In the same spirit as one may define $1 + 2 + 4 + 8 + 16 + ...$ to be -1, in virtue of $\sum_{j=0}^{\infty}(2z)^j = (1-2z)^{-1}$.) Replacing $(\mathrm{id} - \mathcal{L})^{-1}$ by the resolvent $(\mathrm{id} - z\mathcal{L})^{-1} = \sum_{j=0}^{\infty}(z\mathcal{L})^j$ in the right-hand-side of our metaformula (2) gives an essentially equivalent reformulation of Ruelle's candidate.

Ruelle, first in[20] and then with Jiang[11] , considered certain subhyperbolic analytic unimodal maps f_0 (i.e., with a critical point landing on a repelling periodic orbit after finitely many iterates [e]). If X is analytic and φ is smooth, they proved[11,20] that $\Psi(z)$ extends meromorphically to the entire complex plane and $z = 1$ is not a pole. At the time, this gave hope that $\Psi(1)$, defined by this meromorphic continuation, could be the (Whitney) derivative of $t \mapsto \int \varphi \, d\mu_t$, if f_t is a smooth path through $f = f_0$ and $\partial_t f_t|_{t=0} = X \circ f$. However, in view of Section 3, the analytic continuation result in these Markov cases may be a fluke (if not a red herring).

3. Susceptibility function and linear response for piecewise expanding unimodal maps

Set $I = [-1, 1]$, and let $f : I \to I$ be continuous, with $f(-1) = f(1) = -1$. Assume that $f|_{[-1,0]}$ and $f|_{[0,1]}$ both extend to C^3 and uniformly expanding maps. Setting $c = 0$ and $c_k = f^k(c)$ for $k \geq 1$, assume in addition that c is not periodic (see[3,4] for a weakening of this assumption), and that f is topologically mixing on $[c_2, c_1]$. We call such f *nonperiodic mixing piecewise expanding and* C^3 *unimodal maps.* Although not structurally stable, such maps f enjoy strong statistical properties: They admit a unique absolutely continuous invariant probability measure, which is mixing and has a density ρ of bounded variation (by classical results of Lasota–Yorke). If $t \mapsto f_t$ is a C^2 path of such maps then Keller[14] proved that $t \mapsto \rho_t \in L^1(dx)$ has a $t \ln t$ modulus of continuity, so that it is η-Hölder for every $\eta < 1$.

In September 2006, David Ruelle was already working on the nonrecurrent unimodal case (see Subsection 4.1), and he generously shared with me his strategy of considering Banach spaces of sums of smooth functions and functions with singularities along the postcritical orbit. This inspired[2] the decomposition $\rho = \rho_{reg} + \rho_{sal}$, for the invariant density of a nonperiodic mixing piecewise expanding and C^3 unimodal map, where ρ'_{reg} is of bounded variation and, writing H_u for the Heaviside function $H_u(x) = -1$

[e]Such maps have a finite Markov partition and an SRB measure μ_0.

for $x < u$, $H_u(u) = -1/2$ and $H_u(x) = 0$ for $x > u$,

$$\rho_{sal} = \sum_{k=1}^{\infty} \frac{s_1}{(f^{k-1})'(c_1)} H_{c_k} \, ,$$

where $s_1 = -\lim_{x<c_1, x\to c_1} \rho(x)$. Then, we[2] proved:

Proposition 3.1. *Let f_0 be a nonperiodic mixing piecewise expanding and C^3 unimodal map. Let $X \in C^2(I)$ satisfy $X(-1) = 0$. Let $\varphi \in C^1(I)$. Then $\Psi(z)$, defined by (5) for $\mu_0 = \rho\, dx$, is holomorphic in the disc $|z| < 1$, where*

$$\Psi(z) = -\sum_{j=1}^{\infty} \varphi(c_j) \sum_{k=1}^{j} \frac{z^{j-k} s_1 X(c_k)}{(f^{k-1})'(c_1)} - \int \varphi(\mathrm{id} - z\mathcal{L})^{-1}(X'\rho_{sal} + (X\rho_{reg})')dx, \tag{6}$$

where $\mathcal{L}$ is defined by (1). If $\mathcal{J}(f, X) := \sum_{k=1}^{\infty} \frac{s_1}{(f^{k-1})'(c_1)} X(c_k) = 0$, the right-hand-side of (6) at $z = 1$ is a well-defined number, denoted[f] *Ψ_1.*

Consider now a C^2 path f_t through a nonperiodic mixing piecewise expanding C^3 unimodal map $f_0 = f$ (we refer to[3] for precise definitions) and write ρ_t for the invariant density. Assume $\partial_t f_t|_{t=0} = X \circ f$. If $\mathcal{J}(f, X) \neq 0$, examples where $t \mapsto \mathcal{R}(t) = \int \varphi\, \rho_t\, dx$ is *not* Lipschitz (for $\varphi \in C^\infty(I)$ and Markov f_0) were obtained independently by Mazzolena[17] and in[2] .

The meaning of the condition $\mathcal{J}(f, X) = 0$ escaped me at the time of writing[2] . I had not noticed that $\mathcal{J}(f, X) = s_1 J(f, X \circ f)$ with

$$J(f, v) = \sum_{j=0}^{\infty} \frac{v(f^j(c))}{(f^j)'(c_1)} \, , \tag{7}$$

where $J(f, v) \neq 0$ is a well-known *transversality* condition for $v = \partial f_t|_{t=0}$ in smooth unimodal dynamics (see e.g. Tsujii[22]), while the so-called *horizontality* condition $J(f, v) = 0$ had also appeared in the literature (see e.g.[1] and references there to previous work of Lyubich).

The picture for piecewise expanding unimodal maps finally became clear in my joint work with D. Smania: A C^2 path f_t of nonperiodic mixing piecewise expanding C^3 unimodal maps is *tangent (at $t = 0$) to the topological class of $f_0 = f$* if there exists a path $\tilde{f}_t$ and homeomorphisms h_t so that $\tilde{f}_t - f_t = O(t^2)$ and $\tilde{f}_t \circ h_t = h_t \circ f$ for all small enough t. We prove in[3] and[4] that f_t is tangent to the topological class of f if and only if $v = \partial f_t|_{t=0}$ is

[f] See [3, Prop. 4.6] for a condition guaranteeing that Ψ_1 is the Abelian limit of $\Psi(z)$ as $z \to 1$ in $[0, 1)$.

horizontal (that is, $J(f,v)=0$). [g] In addition, we show in[3] that the maps $t \mapsto h_t$ are then differentiable and that $\alpha = \partial_t h_t|_{t=0}$ is the unique bounded solution to the twisted cohomological equation

$$v(x) = \alpha(f(x)) - f'(x)\alpha(x)\,, \quad x \neq c\,. \tag{8}$$

It turns out[3] that when $v = \partial f_t|_{t=0} = X \circ f$ is horizontal the first term in the right-hand-side of (6) for $z=1$ can be written as $-\alpha\rho'_{sal}$. The main result of[3] is that horizontality is sufficient to get the linear response formula:

Theorem 3.1. *Assume that f_t is tangent to the topological class of f_0. Then $t \mapsto \rho_t\, dx$ from $(-\epsilon, \epsilon)$ to Radon measures is differentiable at 0, and*

$$\partial_t(\rho_t\, dx)|_{t=0} = -\alpha\rho'_{sal} - (\mathrm{id} - \mathcal{L})^{-1}(X'\rho_{sal} + (X\rho_{reg})')\, dx\,.$$

In particular, for any $\varphi \in C^0(I)$, the map $\mathcal{R}(t) = \int \varphi\, \rho_t\, dx$ is differentiable at $t=0$, and $\mathcal{R}'(0) = \Psi_1$ from Proposition 3.1

Lemma 4.1 in[2] then implies that

$$(\mathrm{id} - f_*)\partial_t(\rho_t\, dx)|_{t=0} = -X\rho'_{sal} - X'\rho\, dx - X\rho'_{reg}\, dx\,.$$

The above identity may be viewed as an avatar of our metaformula (2).

Finally, we prove in[3] that the horizontality condition is necessary:

Theorem 3.2. *Assume v is not horizontal for $f_0 = f$. If $\inf d(f^j(c), c) = 0$, assume in addition that $\lim_{x\to c, x<c} f'(x) = -\lim_{x\to c, x>c} f'(x)$.*

If the postcritical orbit of f_0 is infinite, then there exists $\varphi \in C^\infty(I)$ so that for any sequence $t_n \to 0$ so that c is not periodic under f_{t_n} we have

$$\lim_{n\to\infty} \left| \frac{\int \varphi\, \rho_{t_n}\, dx - \int \varphi\, \rho_0\, dx}{t_n} \right| \to \infty\,. \tag{9}$$

If the postcritical orbit of f_0 is finite, then there exists $\varphi \in C^\infty(I)$ so that (9) holds for any sequence $t_n \to 0$ so that c is infinite under f_{t_n}.

4. Nonuniformly hyperbolic interval maps

4.1. *Nonrecurrent real analytic unimodal maps*

Ruelle (see Section 3) had let us know that he was studying the transfer operator of nonrecurrent smooth interval maps in view of analyzing the

[g] Along the way, we[4] build a full theory of smooth deformations in this setting, constructing many nontrivial smooth families $\tilde f_t$ in the topological class of f_0, and thus many nontrivial smooth families f_t tangent to the class of f_0.

susceptibility function. In March–April 2007 we communicated to him our progress in the twin works[3,4] , in particular the fact that our co-dimension one condition $\mathcal{J}(f, X) = 0$ amounted to horizontality, i.e., that the path be tangent to a topological class. A few weeks later, Ruelle wrote to us that the horizontality condition also appeared in the setting he was considering, and that he was going to consider paths f_t within a topological class. His preprint[21] , available shortly after the Equadiff conference, deals with real analytic maps $f : I \to I$ having a unique critical point c with $f''(c) < 0$, and satisfying a Misiurewicz-type hyperbolicity condition [h] implying that $\inf d(c_k, c) > 0$. Fixing such a map f, the first main achievement of the paper[21] is the construction of a Banach space of functions which contains the invariant density ρ of f, and on which the transfer operator $\mathcal{L}$ defined by (1) has a simple eigenvalue at 1, while the rest of the spectrum is contained in a disc of radius strictly smaller than 1. Elements of this Banach space are sums of a differentiable function on I with a sum over $k \geq 1$ of an explicit function having a singularity of type $\sqrt{x - c_k}^{-1}$ with an explicit function having a singularity of type $\sqrt{x - c_k}$. The second key result in[21] is that, if X is a real–analytic function on I so that $J(f, X \circ f) = 0$ (recall (7)), then there is $\xi_1 < 1$, and for any smooth enough φ on I there is $\xi_2 > 1$, so that a resummation $\Psi(X, z)$ of the susceptibility function $\Psi(z)$ associated via (5) to $f_0 = f$, $\mu_0 = \rho\, dx$, X, and φ, defines a holomorphic function in $\xi_1 < |z| < \xi_2$. Finally, Ruelle considers in[21] a smooth enough family $t \mapsto f_t$, so that f_t is topologically conjugated to $f_0 = f$ and so that $\partial_t f_t|_{t=0} = X \circ f$, with X real analytic ($X \circ f$ is then horizontal), and he proves that for any smooth enough φ on I the function $t \mapsto \int \varphi \rho_t\, dx$ (where $\rho_t\, dx$ denotes the unique absolutely continuous invariant probability measure of f_t) is differentiable at $t = 0$, where its derivative coincides with $\Psi(X, 1)$ from the second result.

4.2. *Complex analytic quadratic-like Collet-Eckmann maps*

Put $I = [-1, 1]$ and recall that a C^3 map $f : I \to I$ is a *Collet-Eckmann S-unimodal* map if it has $c = 0$ as unique critical point, if there exist $C > 0$ and $\lambda > 1$ so that $|(f^n)'(f(c))| \geq C\lambda^n$ for all $n \geq 1$, and if in addition it has nonpositive Schwarzian derivative (see [i] e.g.[15]).

We say that f_t is a *holomorphic family of quadratic-like maps in a neighbourhood of I,* if there exists a complex neigbourhood U of I so that $t \mapsto f_t$

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[h]See[21] for details

[i]Our other assumptions will ensure $f''(c) \neq 0$.

is a holomorphic [j] map from a complex neighbourhood of zero to the Banach space $B(U)$ of holomorphic functions on U extending continuously to $\overline{U}$ (with the supremum norm), such that the two following conditions hold: Firstly, for real t, the map f_t is real on the real part of U, with $f_t(I) \subset I$ and $f(-1) = f(1) = -1$. Secondly, there exist simply connected complex domains W and V, whose boundaries are analytic Jordan curves, with $I \subset W$, $I \subset V$, $\overline{V} \subset U$, $\overline{V} \subset W$, and so that $f_0 : V \mapsto W$ is a double-branched ramified covering, with $c = 0$ as a unique critical point.

After the Equadiff conference, we proved with D. Smania[5] :

Theorem 4.1. *Let $t \mapsto f_t$ be a holomorphic family of quadratic-like maps in a neighbourhood of I, with all periodic orbits repelling. Assume that for each small real t the map f_t restricted to I is a (real) Collet-Eckmann S-unimodal map. Then there exists $\epsilon > 0$ so that for each real analytic function φ on I, the map $t \mapsto \int \varphi\rho_t \, dx$, where ρ_t is the invariant density of f_t, is real analytic on $(-\epsilon, \epsilon)$.*

An important remark is that the assumptions of Theorem 4.1 and Mañé-Sad-Sullivan[16] imply that for small real t, each f_t is topologically conjugated to f_0 on I. The other main ingredient of our proof are the results and constructions of Keller and Nowicki[15] which allow us to exploit dynamical zeta functions, as in in [13, Proof of Thm. 1].

4.3. *Collet-Eckmann maps of finite differentiability*

The conjectures on smooth (non analytic) Collet–Eckmann maps stated as Conjectures A, A', and B in[2] and[3] are still open at this moment.

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[j] Holomorphic means complex analytic.

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