Ergod. Th. & Dynam. Sys. (2010), **30**, 1–20
doi:10.1017/S0143385708001077

Alternative proofs of linear response for piecewise expanding unimodal maps

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(*Received* 9 *April* 2008 *and accepted in revised form* 17 *November* 2008)

Abstract. We give two new proofs that the Sinai–Ruelle–Bowen (SRB) measure $t \mapsto \mu_t$ of a C^2 path f_t of unimodal piecewise expanding C^3 maps is differentiable at 0 if f_t is tangent to the topological class of f_0. The arguments are more conceptual than the original proof of Baladi and Smania [Linear response formula for piecewise expanding unimodal maps. *Nonlinearity* **21** (2008), 677–711], but require proving Hölder continuity of the infinitesimal conjugacy α (a new result, of independent interest) and using spaces of bounded p-variation. The first new proof gives differentiability of higher order of $\int \psi \, d\mu_t$ if f_t is smooth enough and stays in the topological class of f_0 and if ψ is smooth enough (a new result). In addition, this proof does not require any information on the decomposition of the SRB measure into regular and singular terms, making it potentially amenable to extensions to higher dimensions. The second new proof allows us to recover the linear response formula (i.e. the formula for the derivative at 0) obtained by Baladi and Smania, by an argument more conceptual than the 'brute force' cancellation mechanism used by Baladi and Smania.

1. *Introduction*

Many chaotic dynamical systems $f : M \to M$ on a Riemannian manifold M admit a Sinai–Ruelle–Bowen (SRB) measure μ (see e.g. [**23**]) which describes the statistical properties of a 'large' set of initial conditions in the sense of Lebesgue measure. (In dimension one, a SRB measure is simply an absolutely continuous ergodic invariant probability measure $\mu_t = \rho_t \, dx$, with a positive Lyapunov exponent.) It is of interest (in particular, in view of applications to statistical mechanics, see e.g. [**18, 20**]) to study the smoothness of $t \mapsto \mu_t$, when f_t is a smooth family of dynamical systems, each having a SRB measure μ_t. If $t \mapsto \mu_t$ is differentiable, we say that the *linear response* holds. Ruelle [**18**] obtained not only differentiability, but also a formula for the derivative (the *linear response formula*), in the case of smooth uniformly hyperbolic dynamical systems. (See [**19**] for the formulas for higher-order derivatives, without proofs. Differentiability of higher order in this framework was subsequently proved in [**9, 10**], using 'modern' Banach spaces.)

In [**4**], we proved that the SRB measure $t \mapsto \mu_t$ of a C^2 family of piecewise C^3 and piecewise expanding unimodal maps f_t, with f_0 mixing (see §2.1 for formal definitions), is differentiable at $t = 0$ (as a Radon measure) if and only if f_t is tangent to the topological class of f_0 at $t = 0$ (for the necessity of the tangency condition, we require an additional mild technical condition). Keller [**14**] proved a long time ago that ρ_t has a $|t||\ln |t||$ modulus of continuity, as an element of $L^1(dx)$, and examples in [**1, 4**] show that this can be attained as a lower bound for non-tangential families. We also obtained in [**4**] a linear response formula analogous to that in [**18**], using a resummation introduced in [**1**].

More recently, differentiability of the SRB measure (in the weak sense, that is, as an appropriate distribution) was obtained [**6, 21**] for smooth families f_t of analytic and non-uniformly expanding unimodal maps which stay in the topological class of f_0. The cases of families of smooth non-uniformly expanding interval maps only tangent to the topological class (where Whitney differentiability is expected on suitable subsets of parameters), as well as higher-dimensional dynamical systems such as piecewise expanding/hyperbolic maps or Hénon-like maps, are still open, and much more difficult, see [**2**] for a discussion. In particular, the arguments in [**4**] and [**21**] used detailed information about the structure of the SRB measure, decomposing it into a regular and a singular term. This type of information may be far less accessible in higher dimensions.

In this article, we give two new proofs of the fact [**4**, Theorem 5.1] that the SRB measure of a C^2 family of piecewise C^3 and piecewise expanding unimodal maps f_t, with f_0 mixing, is differentiable at $t = 0$ if f_t is tangent to the topological class of f_0 at $t = 0$.

Section 3 contains our first new proof (see Corollary 3.2), more precisely, we obtain differentiability of $t \mapsto \int \psi \, d\mu_t$ for $\psi \in C^{1+\mathrm{Lip}}$ if f_t is a C^2 family of piecewise expanding C^3 unimodal maps tangent to the topological class of a mixing map f_0. The argument is based on thermodynamic formalism, using potentials $(s, t) \mapsto s(\psi \circ h_t) - \log |f_t' \circ h_t|$ (where h_t conjugates $\tilde{f}_t$ with f_0 and $|\tilde{f}_t - f_t| = O(t^2)$) and does not require any knowledge about the structure of μ_t. (Arguments for weak differentiability of Gibbs measures via thermodynamic formalism have been used previously in [**12, 13**], see [**13**, Corollary 1, p 595].) This argument may therefore be useful in more difficult situations (such as Hénon maps, see [**2**]). It requires the Hölder differentiability of the *infinitesimal conjugacy* α to be proved, a new result (Proposition 2.3) of independent interest. Also, this new proof gives that $t \mapsto \int \psi \, d\mu_t$ is a C^j function, if $\psi \in C^{j+\mathrm{Lip}}$ and f_t is a C^{j+1} family of piecewise expanding C^{j+2} maps in the topological class of f_0, for any $j \geq 1$ (this is a new result, Theorem 3.1). Note also that we do not require the assumption from [**4**] that there is a function X so that $\partial f_t|_{t=0} = X \circ f_0$.

The first new proof requires $\psi \in C^{1+\mathrm{Lip}}$ (instead of $\psi \in C^0$ as in [**4**]) and does not furnish the linear response formula. Section 4 contains our second new proof (Theorem 4.1), which uses spectral perturbation theory for transfer operators associated with the dynamics f_0 and the weight $1/|f_t' \circ h_t|$. This other proof gives differentiability of $\int \psi \, d\mu_t$ for $\psi \in C^0$ and, using the assumption that $\partial f_t|_{t=0} = X \circ f_0$, allows us to recover the linear response formula from [**4**]. (This second proof also uses the Hölder regularity of α from Proposition 2.3.) Note, however, that this second proof requires information on the structure of μ_t from [**1**, Proposition 3.3].

Putting together Theorems 3.1 and 4.1 (or Theorem 3.1 and [**4**, Theorem 5.1]), we obtain the following additional result (Corollary 4.4): if f_t is a C^2 family of piecewise expanding C^3 unimodal maps in the topological class of f_0, then $t \mapsto \mu_t$ is C^1 from a neighbourhood of zero to Radon measures.

We emphasize that neither new proof proves that the condition to be tangent to the topological class is necessary, in contrast to the argument in [**4**] (see Theorem 7.1 there). The proofs here are a bit shorter than that given in [**4**], although the present account requires some results from our previous papers (such as [**1**, Proposition 3.3], [**5**, Proposition 3.2 and Theorem 2], [**4**, Proposition 2.4, Lemma 2.6 and Proposition 3.3]).

2. *Definitions and notation: Hölder smoothness of the infinitesimal conjugacy*

2.1. *Formal definitions.* We denote $I = [-1, 1]$. For an integer $k \geq 1$, we define $\mathcal{B}^k$ to be the linear space of continuous functions $f : I \to \mathbb{R}$ such that f is C^k on the intervals $[-1, 0]$ and $[0, 1]$. Then $\mathcal{B}^k$ is a Banach space for the norm $\max\{|f|_{C^k([-1,0])}, |f|_{C^k([0,1])}\}$. For an integer $k \geq 1$, we define the set $\mathcal{U}^k$ of *piecewise expanding C^k unimodal maps* to be the set of $f \in \mathcal{B}^k$ such that† $f(-1) = f(1) = -1$, $\inf_{x\neq 0} |f'(x)| > 1$, and $f(0) \leq 1$ (so that $f(I) \subset I$). The point $c = 0$ is called the *critical point* of f.

A piecewise expanding C^k unimodal map f is *good* if either c is not periodic under f or $|(f^{q-1})'(f(c))| \min\{|f'_+(c)|, |f'_-(c)|\} > 2$, where $q \geq 2$ is the minimal period of c; it is *mixing* if f is topologically mixing on $[c_2, c_1]$, where $c_k = f^k(c)$.

For $1 \leq j \leq k$, a *C^j family of piecewise expanding C^k unimodal maps* is a C^j map f_t from $t \in (-\epsilon, \epsilon)$ to $\mathcal{U}^k$ for some $\epsilon > 0$. (In this paper, $k \geq 1$ is an integer and j is either an integer or $j = k - 1 + \mathrm{Lip}$ for $k \geq 2$, the notation $\mathcal{B}^{k+\mathrm{Lip}}$ and $\mathcal{U}^{k+\mathrm{Lip}}$ for integers $k \geq 1$ being self-explanatory. See also Remark 2.2.)

Remark 2.1. A C^j family f_t of piecewise expanding C^k unimodal maps is a $C^{j,k}$ perturbation of f_0 in the sense of [**4**] if $j = k \geq 2$.

Remark 2.2. Considering $\mathcal{B}^{k+\beta}$ and $\mathcal{U}^{k+\beta}$ for $k \geq 1$ integer and a Hölder exponent $0 < \beta < 1$ will perhaps allow us to avoid the loss of regularity from C^{k+1} to $C^{k+\mathrm{Lip}}$, e.g. in [**4**, Proposition 2.4] (this question was asked by J.-C. Yoccoz). However, since the spectral result of Wong [**22**] only holds on the space BV_p of functions of bounded p-variation if $1 \leq p < p_0$, for some $p_0 > 1$ depending on the dynamics, it may be necessary in this case to replace BV_p by spaces of generalized p-variation, as introduced by Keller [**15**]. (See also Remark 2.5.)

Assume that f_t is a C^j family of piecewise expanding C^k unimodal maps for $k \geq j > 1$. By classical results of Lasota–Yorke, each f_t has a unique absolutely continuous invariant probability measure $\mu_t = \rho_t\, dx$. This measure is ergodic and it is called the *SRB measure* of f_t. If f_t is mixing, then μ_t is mixing. If f_0 is good and mixing, then f_t is mixing for all small enough t (see [**14**] and references therein).

† A prime denotes differentiation with respect to $x \in I$, *a priori* in the sense of distributions.

We say that a piecewise expanding C^k unimodal map g is in the topological class of f if there is a homeomorphism $h : I \to I$ conjugating f and g, that is $h \circ f = g \circ h$. This implies that $h(c) = c$. We say that a C^j family f_t of piecewise expanding C^k unimodal maps is in the *topological class of* f_0 if there exist homeomorphisms $h_t : I \to I$ such that

$$h_t \circ f_0 = f_t \circ h_t \quad \text{for all } |t| < \epsilon. \tag{1}$$

(This implies that $h_t(c) = c$ for all c.) We proved in [**4**, Proposition 2.4] that $(x, t) \mapsto h_t(x)$ is continuous, and that for each x the map $t \mapsto h_t(x)$ is $C^{k+\text{Lip}-1}$. Differentiability of $t \mapsto h_t(x)$ will play an important role in our arguments below. We say that a C^j family f_t of piecewise expanding C^k unimodal maps ($k \geq j \geq 2$) is *tangent to the topological class* of f_0 if there exists a C^j family $\tilde{f}_t$ of piecewise expanding C^k unimodal maps in the topological class of f_0 so that $\tilde{f}_t = f_0$ and $\partial_t f_t|_{t=0} = \partial_t \tilde{f}_t|_{t=0}$. (Note that there is a typographical mistake in [**4**, p. 682, line 6], where '$C^{2,2}$ perturbation' should be replaced by '$C^{r_0,r}$ perturbation'.)

We say that a bounded function $v : I \to \mathbb{R}$ is *horizontal* for f, if $v(-1) = v(1) = 0$, and setting $M_f = q$ if c is periodic of minimal period q, and $M_f = +\infty$ otherwise,

$$J(f, v) = \sum_{j=0}^{M_f-1} \frac{v(c_j)}{(f^j)'(c_1)} = 0. \tag{2}$$

In [**4**, Corollary 2.6] we proved that if f_t is a C^2 family of piecewise expanding C^2 unimodal maps tangent to the topological class of f_0, then $v = \partial f_t|_{t=0}$ is horizontal for f_0. By [**4**, Theorem 2], if f_t is a C^2 family of piecewise expanding C^2 unimodal maps with f_0 good and $v = \partial f_t|_{t=0}$ is C^2 and horizontal for f_0, then there exists a $C^{1+\text{Lip}}$ family $\tilde{f}_t$ of piecewise expanding C^2 unimodal maps in the topological class of f_0 so that $\tilde{f}_t = f_0$ and $\partial_t f_t|_{t=0} = \partial_t \tilde{f}_t|_{t=0}$.

We proved in [**4**, Lemma 2.2] that if $v : I \to \mathbb{R}$ is bounded then the twisted cohomological equation (TCE)

$$v(x) = \alpha(f(x)) - f'(x)\alpha(x) \quad \text{for all } x \in I,\ x \neq c, \tag{3}$$

admits a unique bounded solution α satisfying $\alpha(c) = 0$. This solution is obtained as follows: if c is not in the forward orbit of x, set $M(x) = \infty$ and otherwise let $M(x)$ be the smallest integer $j \geq 0$ satisfying $f^j(x) = c$, then put

$$\alpha(x) = -\sum_{i=0}^{M(x)-1} \frac{v(f^i(x))}{(f^{i+1})'(x)}. \tag{4}$$

The function α is called the *infinitesimal conjugacy*. Note that if v is, in addition, C^2 *and horizontal*, then it follows from [**4**, Corollary 2.6, Theorem 2.8], see also [**5**, Theorem 2], that α is continuous. (Proposition 2.3 below states that α is continuous, in fact Hölder, if v is Hölder and horizontal. If v is C^0 and horizontal, then α should be continuous, approaching v by Hölder continuous functions.)

If $u : I \to \mathbb{R}$ is Hölder, we denote its Hölder norm by $|u|_\beta$. Slightly abusing notation, we sometimes write $\partial_t f_t$ for $\partial_s f_s|_{s=t}$, and similarly for other functions depending on t.

2.2. *Hölder smoothness of the infinitesimal conjugacy* α. A new result that we require throughout (see Lemmas 3.3 and 4.2) is the following.

PROPOSITION 2.3. (Smoothness of the infinitesimal conjugacy) *Let $f \in \mathcal{U}^2$ be such that c is not periodic. For any $\beta \in (0,1)$ there exist $C_\beta > 0$ and $\mathcal{V}_\beta$ a neighbourhood of f in $\mathcal{U}^2$ so that, for any $g \in \mathcal{V}_\beta$ and every β-Hölder $v : I \to \mathbb{R}$, with $v(-1) = v(1) = 0$ and $J(g, v) = 0$, the unique bounded function α (4) satisfying $\alpha(c) = 0$ and $v(x) = \alpha(g(x)) - g'(x)\alpha(x)$ for all $x \neq c$ is β-Hölder, with*

$$|\alpha|_\beta \leq C_\beta |v|_\beta.$$

If the critical point of $f \in \mathcal{U}^2$ is periodic, the statement holds up to taking (for appropriate $\xi(\beta) > 0$)

$$\mathcal{V}_\beta = \{g \mid \|g - f\|_{\mathcal{B}^2} < \xi(\beta), \exists \text{ homeomorphism } h : I \to I \text{ s.t. } g \circ h = h \circ f\}.$$

In particular, if $f \in \mathcal{U}^2$ and $J(f, v) = 0$ for some Lipschitz v with $v(-1) = v(1) = 0$, the function α solving (3) is β-Hölder for any $\beta < 1$.

Remark 2.4. Buzzi [**8**] showed us a simple proof that if h is a homeomorphism so that $h \circ f = g \circ h$, for two piecewise expanding C^1 unimodal maps f and g, then h is β-Hölder, for any $\beta < \log(\inf |g'|/2)/\log(2 \sup |f'|)$. This fact neither implies nor is implied by Proposition 2.3.

Proof.

Step I. For any $\beta < 1$, there exist a neighbourhood $\mathcal{V}_\beta$ of f in $\mathcal{U}^2$, $\ell \geq 1$ and $\eta > 0$ such that $\lambda = (\inf_{g \in \mathcal{V}_\beta} \inf_{x \neq c} |g'(x)|)^{-1} < 1$ and, for any $g \in \mathcal{V}_\beta$, letting $d_1 < d_2 < \cdots < d_p$ be the critical points of g^ℓ, putting $d_0 = -1$, $d_{p+1} = 1$ and setting

$$\theta = \max_{0 \leq i \leq p} \sup_{\substack{x, y \in (d_i, d_{i+1}) \\ |x - y| < \eta}} \frac{|(g^\ell)'(x)|^\beta}{|(g^\ell)'(y)|}, \tag{5}$$

we have $2\theta < 1$.

Put $\Delta_g = \min_{0 \leq i \leq p}\{d_{i+1} - d_i\}$. Then $\inf_{g \in \mathcal{V}_\beta} \Delta_g > 0$ if the critical point of f is not periodic. Otherwise we have $\inf_{g \in \mathcal{V}_\beta} \Delta_g > 0$, up to replacing $\mathcal{V}_\beta$ by a $\mathcal{B}^2$-neighbourhood of f in its topological class. In particular, we can assume that $\eta < \inf_{g \in \mathcal{V}_\beta} \Delta_g$. From now on, we fix $\mathcal{V}_\beta$, $\ell \geq 1$, and $\eta > 0$ as above.

Step II. We claim that it suffices to show the lemma for $g \in \mathcal{V}_\beta$ with a periodic critical point: indeed, if g has a non-periodic critical point, then we consider $g_t = g + tw$ with $g_t \in \mathcal{U}^2$, $w \in \mathcal{B}^2$, $w(-1) = w(1) = 0$, and $J(g, w) \neq 0$. By [**5**, Corollary 4.1], there exists a sequence $t_n \to 0$ such that each $g_n = g_{t_n}$ has a periodic critical point. In particular, g_n converges to g in the $\mathcal{U}^2$ topology. Then, by [**5**, Proposition 3.2] we have $\lim_{n\to\infty} J(g_n, v) = 0$. Let w_n be a β-Hölder function, with $w_n(-1) = w_n(1) = 0$ and $|w_n|_\beta \leq 1$, such that $J(g_n, w_n) = 1$. Set

$$v_n = v - J(g_n, v) w_n.$$

Then we have $J(g_n, v_n) = 0$ and $\lim_{n\to\infty} |v_n - v|_\beta = 0$. If the proposition holds for maps in $\mathcal{V}_\beta$ with a periodic turning point, the unique function α_n so that $\alpha_n(c) = 0$ and

$v_n(x) = \alpha_n(g_n(x)) - g_n'(x)\alpha_n(x)$ for all $x \neq c$, satisfies $|\alpha_n|_\beta \le C_\beta |v_n|_\beta$. We can choose a subsequence α_{n_i} converging in the sup norm to a function α. It follows from the uniform convergence of α_{n_i} that α satisfies the TCE (3) for g and v, and that $|\alpha|_\beta \le C_\beta |v|_\beta$.

Step III. We assume from now on that $g \in \mathcal{V}_\beta$ has a periodic turning point. The proof will be via an 'infinitesimal pull-back' argument.

First, since $J(g, v) = 0$, it is easy to see that there exists a β-Hölder function $\alpha_0 : I \to \mathbb{R}$ with $\alpha_0(-1) = \alpha_0(1) = \alpha_0(c) = 0$, $\alpha_0(g(c)) = v(c)$, and

$$v(x) = \alpha_0(g(x)) - g'(x)\alpha_0(x) \quad \text{for every } x \neq c \text{ in the (finite) forward orbit of } c.$$

Second, we define by induction continuous functions $\alpha_i : I \to \mathbb{R}$, for $i \ge 1$, such that $\alpha_i(-1) = \alpha_i(1) = \alpha_i(c) = 0$, $\alpha_i(g(c)) = v(g(c))$, that

$$v(x) = \alpha_i(g(x)) - g'(x)\alpha_i(x) \quad \text{for every } x \neq c \text{ in the (finite) forward orbit of } c, \qquad (6)$$

and, in addition,

$$v(x) = \alpha_{i-1}(g(x)) - g'(x)\alpha_i(x) \quad \text{for all } x \neq c. \qquad (7)$$

Indeed, suppose that we have defined α_i, for $0 \le i \le n$. Set $\alpha_{n+1}(c) = 0$, and

$$\alpha_{n+1}(x) = \frac{\alpha_n(g(x)) - v(x)}{g'(x)}, \quad x \neq c.$$

Clearly, $\alpha_{n+1}(-1) = \alpha_{n+1}(1) = 0$, and (7) holds for $i = n + 1$. Thus, since $v(x) = \alpha_n(g(x)) - g'(x)\alpha_n(x)$ for every $x \neq c$ in the forward orbit of c, we find $\alpha_n(x) = \alpha_{n+1}(x)$ for each $x \neq c$ in the forward orbit of c. Since $\alpha_n(c) = \alpha_{n+1}(c) = 0$, we conclude that (6) holds for $i = n + 1$, and $\alpha_{n+1}(g(x)) = v(x)$. Last, but not least, α_{n+1} is continuous on I because $\alpha_n(g(x)) = v(x)$.

Third, if x is not a critical point of g^j, we set $v_0(x) = 0$, and

$$v_j(x) = \sum_{i=0}^{j-1} (g^{j-1-i})'(g^{i+1}(x)) v(g^i(x)), \quad j \ge 1.$$

Recalling the notation ℓ, $\{d_i\}$, from Step I, it is easy to see that

$$v_\ell(x) = \alpha_{j\ell}(g^\ell(x)) - (g^\ell)'(x)\alpha_{(j+1)\ell}(x) \quad \text{for all } x \notin \{d_1, \ldots d_p\}, \text{ for all } j \ge 0. \qquad (8)$$

For $\ell \ge 2$ the function v_ℓ may have jump discontinuities at the critical points d_i of g^ℓ, but it is β-Hölder in the connected components of $I \setminus \{d_1, \ldots, d_p\}$.

Finally, we use the iterated TCE (8) to show that there exists $C_\beta < \infty$ so that, for all $g \in \mathcal{V}_\beta$ with a periodic turning point and all β-Hölder v with $J(g, v) = 0$ (and $v(-1) = v(1) = 0$), there exists $j_0 \ge 0$ so that $|\alpha_{j\ell}|_\beta \le C_\beta |v|_\beta$ for all $j \ge j_0$. In view of this, for $j \ge 0$, set $K_j^0 = \sup_I |\alpha_{j\ell}|$, $L^0 = \sup_I |v_\ell|$,

$$K_j^\beta = \sup_{x \neq y} \frac{|\alpha_{j\ell}(x) - \alpha_{j\ell}(y)|}{|x-y|^\beta}, \quad \widehat{K}_j^\beta = \max_{0 \le i \le p} \sup_{\substack{x \neq y, |x-y|<\eta \\ x,y \in (d_i, d_{i+1})}} \frac{|\alpha_{j\ell}(x) - \alpha_{j\ell}(y)|}{|x-y|^\beta},$$

and

$$L^\beta = \max_{0 \le i \le p} \sup_{\substack{x \neq y \\ x,y \in (d_i, d_{i+1})}} \frac{|v_\ell(x) - v_\ell(y)|}{|x-y|^\beta}, \quad D = \max_{0 \le i \le p} \sup_{x,y \in (d_i, d_{i+1})} \frac{|(g^\ell)''(x)|}{|(g^\ell)'(y)|^2}.$$

Clearly, $\max(L^0, L^\beta) \le \widetilde{C}_\beta |v|_\beta$ for all g and v under consideration, and we have

$$\frac{|\alpha_{(j+1)\ell}(x) - \alpha_{(j+1)\ell}(y)|}{|x-y|^\beta} \le 2\eta^{-\beta} K^0_{j+1} \quad \text{if } |x-y| \ge \eta, \text{ for all } j \ge 0. \tag{9}$$

Therefore, recalling the definition of λ and (5) from Step I, it suffices to show that

$$K^0_{j+1} \le \lambda^\ell (K^0_j + L^0) \quad \text{and} \quad \widehat{K}^\beta_{j+1} \le (L^0 + K^0_j) D + \lambda^\ell L^\beta + \theta K^\beta_j \quad \text{for all } j \ge 0. \tag{10}$$

Indeed, continuity of $\alpha_{(j+1)\ell}$ together with (9) (recall also that $\eta < \inf_g \Delta_g$, so that if $|x-y| < \eta$ then $[x, y]$ contains at most one point d_i) imply

$$K^\beta_{j+1} \le \max(2\eta^{-\beta} K^0_{j+1}, 2\widehat{K}^\beta_{j+1}).$$

The above bound together with (10) yield $E_\beta < \infty$ so that, for all $g \in \mathcal{V}_\beta$ with a periodic turning point, and all β-Hölder v with $J(g, v) = 0$, $v(-1) = v(1) = 0$, there exists $j_0 \ge 0$ so that

$$K^\beta_{j+1} \le E_\beta |v|_\beta + 2\theta K^\beta_j \quad \text{for all } j \ge j_0.$$

Since $2\theta < 1$, we conclude by a geometric series, taking larger j_0 if necessary.

It remains to show (10). We concentrate on the second bound (the first is easier and left to the reader). Let $x, y \in (d_i, d_{i+1})$ satisfy $|x-y| < \eta$. Then (8) implies (since $g \in \mathcal{U}^2$, the function $(g^\ell)'$ is C^1 in the intervals of monotonicity of g^ℓ)

$$\begin{aligned}
|\alpha_{(j+1)\ell}(x) - \alpha_{(j+1)\ell}(y)| &\le |v_\ell(x) + \alpha_{j\ell}(g^\ell(x))| \left| \frac{1}{(g^\ell)'(x)} - \frac{1}{(g^\ell)'(y)} \right| \\
&\quad + \frac{|v_\ell(x) - v_\ell(y)| + |\alpha_{j\ell}(g^\ell(x)) - \alpha_{j\ell}(g^\ell(y))|}{|(g^\ell)'(y)|} \\
&\le (L^0 + K^0_j) \frac{\sup_{[x,y]} |(g^\ell)''|}{\inf_{[x,y]} |(g^\ell)'|^2} |x-y| \\
&\quad + \frac{L^\beta + \sup_{[x,y]} |(g^\ell)'|^\beta K^\beta_j}{\inf_{[x,y]} |(g^\ell)'|} |x-y|^\beta \\
&\le (L^0 + K^0_j) D |x-y| + (\lambda^\ell L^\beta + \theta K^\beta_j) |x-y|^\beta.
\end{aligned}$$

Step IV. Defining $\tilde{\alpha}_n = (1/n) \sum_{j=0}^{n-1} \alpha_{j\ell}$, we can choose a subsequence $\tilde{\alpha}_{n_i}$ converging uniformly on I to a function $\tilde{\alpha}$ satisfying $|\tilde{\alpha}|_\beta \le C_\beta |v|_\beta$. By (8),

$$v_\ell(x) = \tilde{\alpha}(g^\ell(x)) - (g^\ell)'(x) \tilde{\alpha}(x) \quad \text{for all } x \notin \{d_1, \ldots, d_p\}. \tag{11}$$

Let $\alpha : I \to \mathbb{R}$ be the unique bounded solution vanishing at c to the TCE (3) for g and v, as in (4). Then

$$v_\ell(x) = \alpha(g^\ell(x)) - (g^\ell)'(x) \alpha(x) \quad \text{for all } x \notin \{d_1, \ldots, d_p\}. \tag{12}$$

Since $\tilde{\alpha}$ is continuous (11) and (12), imply $\alpha = \tilde{\alpha}$. We have proved that $|\alpha|_\beta \le C_\beta |v|_\beta$, for all $g \in \mathcal{V}_\beta$ with a periodic turning point, and thus the proposition. □

2.3. *Banach spaces of functions of bounded variation.* We consider the Banach space of functions of bounded variation

$$BV = \{\varphi : \mathbb{R} \to \mathbb{C} \mid \mathrm{var}(\varphi) < \infty,\ \mathrm{supp}(\varphi) \subset I\}/\sim,$$

endowed with the norm $\|\varphi\|_{BV} = \inf_{\psi\sim\varphi} \mathrm{var}(\psi)$, where var denotes total variation, and $\varphi_1 \sim \varphi_2$ if the bounded functions φ_1, φ_2 differ on an at most countable set. In addition, for $1 \le p < \infty$ we work with the Banach space of functions of bounded p-variation (used in interval dynamics by Wong [**22**])

$$BV_p = \{\varphi : \mathbb{R} \to \mathbb{C} \mid \mathrm{var}_p(\varphi) < \infty,\ \mathrm{supp}(\varphi) \subset I\}/\sim,$$

where

$$\mathrm{var}_p(\varphi) = \sup_{x_1<x_2<\cdots<x_n} \left(\sum_{i=1}^{n} |\varphi(x_{i+1}) - \varphi(x_i)|^p\right)^{1/p},$$

the supremum ranging over all ordered finite subsets of $\mathbb{R}$. Note that $\mathrm{var}_1 = \mathrm{var}$ and $BV = BV_1$. Wong [**22**] does not quotient by the equivalence relation $\varphi_1 \sim \varphi_2$, but his results remain unchanged if we consider elements in BV_p modulo $\sim$ (a function in BV_p is continuous except on an at most countable set, see also [**15**, Lemmas 1.4.a and 2.7] and [**7**]). Note that for each $p \ge 1$ there is $C \ge 1$ so that $|\varphi|_\infty \le C\|\varphi\|_{BV_p}$ for all φ, and if φ is $1/p$-Hölder, then $\|\varphi\|_{BV_p} \le |\varphi|_{1/p}$. In addition, we claim that for any $K > 0$ and $p > 1$, and for every φ with $\|\varphi\|_{C^1} \le K$

$$\|\varphi\|_{C^{1/p}} \le K^{1/(p-1)} \|\varphi\|_{BV_p}. \tag{13}$$

To prove (13), it suffices to show that if $\|\varphi\|_{C^{1/p}} = 1$, then $\|\varphi\|_{BV_p} \ge K^{-1/(p-1)}$. By the mean value theorem, there exist $x_i \le y_i \le x_{i+1}$ so that

$$1 = \|\varphi\|_{C^{1/p}} = |\varphi(x_{i+1}) - \varphi(x_i)||x_{i+1} - x_i|^{-1/p} = |\varphi'(y_i)||x_{i+1} - x_i|^{1-1/p},$$

and, in particular, $|x_{i+1} - x_i| \ge K^{-p/(p-1)}$. By definition, and the mean value theorem again,

$$\|\varphi\|_{BV_p} \ge |\varphi(x_{i+1}) - \varphi(x_i)| = |\varphi'(y_i)||x_{i+1} - x_i| \ge K^{-1/(p-1)}|\varphi'(y_i)||x_{i+1} - x_i|^{1-1/p}.$$

This ends the proof of (13). Finally,

$$\|\varphi_1\varphi_2\|_{BV_p} \le 2\|\varphi_1\|_{BV_p}\|\varphi_2\|_{BV_p} \quad \text{for all } p \ge 1, \tag{14}$$

and

$$\|\varphi \circ h\|_{BV_p} = \|\varphi\|_{BV_p} \quad \text{for any homeomorphism } h : I \to I \text{ and all } p \ge 1. \tag{15}$$

Remark 2.5. The reason we consider spaces BV_p for $p \ne 1$ is because we are concerned with differentiability in the t-parameter and we have to deal with derivatives $\partial_t(\psi \circ h_t)|_{t=0} = \psi' \cdot \partial_t h_t|_{t=0}$ or $\partial_t(f_t' \circ h_t)|_{t=0} = f_0''\partial_t h_t|_{t=0} + v'$, where $v' = \partial_t f_t'|_{t=0}$ is C^1, but $\partial_t h_t|_{t=0}$ does not belong to BV in general. We shall see, however, that Proposition 2.3 implies that $\alpha = \partial_t h_t|_{t=0}$ lies in BV_p for all $p > 1$.

3. *Weak differentiability of the SRB via the pressure*

The main result of this section (Theorem 3.1) says that for any $j \geq 1$, if f_t is a C^{j+1} family of piecewise expanding C^{j+2} unimodal maps in the topological class of f_0, then $R(t) = \int \psi \, d\mu_t$ is C^j if ψ is $C^{j+\mathrm{Lip}}$. Even if $j = 1$, this is a new result ([**4**, Theorem 5.1] only gives differentiability at $t = 0$). The argument is based on the topological pressure of the potential $(s, t) \mapsto -\log |f_t' \circ h_t| + s(\psi \circ h_t)$ for the map f_0. It is simple, but does not give the formula for $\partial_t R(t)|_{t=0}$ (or higher order derivatives). Using the linear response formula from [**4**, Theorem 5.1] or Theorem 4.1 below, Theorem 3.1 will imply Corollary 4.4.

THEOREM 3.1. *For any integer $j \geq 1$, if f_t is a C^{j+1} family of piecewise expanding C^{j+2} unimodal maps in the topological class of a mixing map f_0, then there is $\hat{\epsilon} > 0$ so that for any $C^{j+\mathrm{Lip}}$ function ψ the map $R(t) = \int \psi \rho_t \, dx$ is C^j in $(-\hat{\epsilon}, \hat{\epsilon})$.*

As an immediate corollary of Theorem 3.1 and Proposition A.1, we recover the first claim of [**4**, Theorem 5.1] if ψ is $C^{1+\mathrm{Lip}}$ (we do not need the assumption $\partial f_t|_{t=0} = X \circ f_0$ used in [**4**, Theorem 5.1]).

COROLLARY 3.2. *Assume that f_t is a C^2 family of piecewise expanding C^3 unimodal maps, where f_0 is a good mixing map. If f_t is tangent to the topological class of f_0, then for any $C^{1+\mathrm{Lip}}$ function $\psi : I \to \mathbb{C}$, the map $R(t) = \int \psi \, d\mu_t$ is differentiable at $t = 0$.*

Proof of Theorem 3.1. Fix $\psi \in C^{1+\mathrm{Lip}}$, recall the notation h_t from (1), put

$$g_{s,t}(y) = \frac{\exp(s\psi(h_t(y)))}{|f_t'(h_t(y))|}, \quad y \in I \setminus \{c\}, \tag{16}$$

and consider the transfer operators

$$\widetilde{\mathcal{L}}_{s,t}\varphi(x) = \sum_{f(y)=x} g_{s,t}(y)\varphi(y), \quad \mathcal{L}_{s,t}\varphi(x) = \sum_{f_t(y)=x} e^{s\psi(y)} \frac{\varphi(y)}{|f_t'(y)|}. \tag{17}$$

Then since $|f_t'|$ is the Jacobian of f_t with respect to Lebesgue measure dx the operator $\mathcal{L}_t = \mathcal{L}_{0,t}$ is just the usual transfer operator for f_t. In particular, the change of variable formula implies $\mathcal{L}_t^*(dx) = dx$ for all small t. Also, the main theorem of [**22**] applied to $\mathcal{L}_t$ gives $p_0 > 1$ (depending on f_0 through $\inf |f_0'|$ and $\sup |f_0'|$) so that for any $p \in [1, p_0)$ there exists $\epsilon_p > 0$ so that for all $|t| < \epsilon_p$ the operator $\mathcal{L}_t$ acting on BV_p has spectral radius 1, essential spectral radius < 1, and 1 is the only eigenvalue of modulus 1 and is simple (i.e. $\mathcal{L}_t$ has a spectral gap). Furthermore, the fixed vector ρ_t is strictly positive on $[c_2, c_1]$. The fixed vector ν_t of $\mathcal{L}_t^*$ is dx, and we normalize so that $\int \rho_t \, d\nu_t = 1$ and $\nu_t(I) = 1$. (Of course, $\mu_t = \rho_t \, dx$ is just the SRB measure of f_t.)

The transfer operator $\widetilde{\mathcal{L}}_{s,t}$ is conjugated to $\mathcal{L}_{s,t}$ via

$$\widetilde{\mathcal{L}}_{s,t}(\varphi \circ h_t) = \mathcal{L}_{s,t}(\varphi) \circ h_t. \tag{18}$$

Therefore, (15) (which says that h_t is an isometry of BV_p) implies that the spectra of $\widetilde{\mathcal{L}}_{s,t}$ and $\mathcal{L}_{s,t}$ on BV_p coincide. In particular, the operator $\widetilde{\mathcal{L}}_t = \widetilde{\mathcal{L}}_{0,t}$ on BV_p has a simple eigenvalue at 1, for the fixed point $\tilde{\rho}_t = \rho_t \circ h_t$, and the rest of its spectrum lies in a disc of strictly smaller radius. The fixed point of $\widetilde{\mathcal{L}}_t^*$ is the measure ν_t defined by

$$\int \varphi \, dx = \int \varphi \circ h_t \, d\nu_t. \tag{19}$$

(By definition, ν_t is a probability measure and $\int \tilde{\rho}_t \, d\nu_t = 1$.)

We view $\mathcal{L}_{s,t}$ as a perturbation of $\mathcal{L}_t$, writing

$$\mathcal{L}_{s,t}(\varphi) = \mathcal{L}_t(e^{s\psi(y)}\varphi). \tag{20}$$

Since ψ and the exponential are $C^{1+\mathrm{Lip}}$, the norm $\|e^{s\psi(y)} - 1\|_{BV_1}$ tends to zero as $s \to 0$ (uniformly in t). Therefore, applying classical perturbation theory [**11**], the operators $\mathcal{L}_{s,t}$ (or, equivalently, $\widetilde{\mathcal{L}}_{s,t}$) on BV_p have a real positive simple maximal† eigenvalue $\lambda_{s,t} > 0$ with a spectral gap, uniformly in (s, t) close enough to $(0, 0)$.

Consider first the case $j = 1$. Setting $\mathcal{P}_{s,t}(\varphi) = (g_{s,t}/g_{0,0})\varphi$, Lemma 3.3 below implies that the map $s \mapsto \mathcal{P}_{s,t}$ is C^1 from $\mathbb{R}$ to the Banach space of C^1 maps from $\{|t| < \epsilon\}$ to bounded operators on BV_p, and

$$\partial_s \mathcal{P}_{s,t}|_{s=u} = (\psi \circ h_t)\mathcal{P}_{u,t} \quad \text{for all } u \in \mathbb{R}. \tag{21}$$

Therefore, $s \mapsto \widetilde{\mathcal{L}}_{s,t} = \mathcal{L}_0 \circ \mathcal{P}_{s,t}$ is C^1 from $\mathbb{R}$ to the Banach space of C^1 maps from $\{|t| < \epsilon\}$ to bounded operators on BV_p, and

$$\partial_s \widetilde{\mathcal{L}}_{s,t}|_{s=u}(\varphi) = \widetilde{\mathcal{L}}_{u,t}((\psi \circ h_t)\varphi) \quad \text{for all } u \in \mathbb{R}.$$

We are thus in a position to apply classical perturbation theory of an isolated simple eigenvalue (see [**11**, Ch. VII.1.3] for the analytic case, see e.g. [**3**, Lemma 3.2] for the differentiable setting). It follows, on the one hand, that in a neighbourhood of (0, 0) the maximal eigenvalue $\lambda_{s,t} > 0$ of $\widetilde{\mathcal{L}}_{s,t}$ acting on BV_p is a C^1 function of s to the space of C^1 maps from $\{|t| < \epsilon\}$ to $\mathbb{R}$. On the other hand, by 'tedious but straightforward calculations' and [**11**, Ch. VII.1.5 and II.2.2], (to quote [**17**, (5.2)]), we have

$$\partial_s(\log \lambda_{s,t})|_{s=0} = \int \psi \circ h_t \tilde{\rho}_t \, d\nu_t = \int \psi \, d\mu_t$$

(use that $\tilde{\rho}_t$ and ν_t are the fixed eigenvectors of $\widetilde{\mathcal{L}}_{0,t}$ and its dual). Since $t \mapsto \partial_s(\log \lambda_{s,t})|_{s=0}$ is a C^1 function in a neighbourhood of zero, we have proved Theorem 3.1 in the case $j = 1$. If $j \geq 2$, apply Lemma 3.4 instead of Lemma 3.3. □

The following result is the key ingredient in the proof of Theorem 3.1, its proof hinges on Proposition 2.3 and [**4**, Proposition 2.4].

LEMMA 3.3. *Let f_t be a C^2 family of piecewise expanding C^3 unimodal maps in the topological class of f_0. For any $p > 1$ there exists $\epsilon_p > 0$ so that for any $\psi : I \to \mathbb{R}$ which is $C^{1+\mathrm{Lip}}$, the map $s \mapsto g_{s,t}$ defined by (16) is C^1 from $\mathbb{R}$ to the Banach space of C^1 maps from $\{|t| < \epsilon_p\}$ to BV_p. In addition, recalling the notation (1),*

$$\partial_s g_{s,t}|_{s=u} = (\psi \circ h_t) g_{u,t} \quad \text{for all } u \in \mathbb{R}. \tag{22}$$

In fact, s-analyticity holds in Lemma 3.3, but we do not need this.

Proof of Lemma 3.3. Fix $p > 1$. For every $x \neq c$, all small t, and all s_1, s_2 in $\mathbb{R}$, we have

$$g_{s_1,t}(x) - g_{s_2,t}(x) = g_{s_2,t}(x) \sum_{k=1}^{\infty} \frac{(s_1 - s_2)^k}{k!} (\psi(h_t(x)))^k. \tag{23}$$

† Of course, $\log \lambda_{s,t}$ is the topological pressure of $\log g_{s,t}$.

So, to prove both differentiability and (22), it suffices to see that the maps

$$t \mapsto \frac{1}{k!}(\psi \circ h_t)^k g_{s,t}, \quad k \geq 0,$$

are C^1 from a neighbourhood of 0 to BV_p, uniformly in k and in s in any compact set $K \subset \mathbb{R}$.

In view of this, we first study the maps $t \mapsto h_t(x)$. By [**4**, Proposition 2.4], there exists $\tilde{\epsilon} > 0$ so that the set of maps $\{t \mapsto h_t(x),\ x \in I\}$ is bounded in $C^{1+\mathrm{Lip}}([-\tilde{\epsilon}, \tilde{\epsilon}])$. Differentiating with respect to t the equation $h_t \circ f_0 = f_t \circ h_t$, and setting $\alpha_t = \partial_t h_t \circ h_t^{-1}$, we obtain

$$\alpha_t(f_t(c)) = \partial_t f_t(c), \quad \partial_t f_t(x) = \alpha_t(f_t(x)) - f_t'(x)\alpha_t(x) \quad \text{for all } x \neq c,\ |x| < \tilde{\epsilon}.$$

Since $\alpha_t(c) = 0$ this implies $J(f_t, \partial_t f_t) = 0$ for $|t| < \tilde{\epsilon}$ (recall (4)), so, for any fixed

$$\beta \in (1/p, 1) \quad \text{(we may and do assume also that } \beta < 1/\sqrt{p}),$$

Proposition 2.3 gives C and $\epsilon_p > 0$ so that

$$|\alpha_t|_\beta \leq C \quad \text{for all } |t| < \epsilon_p. \tag{24}$$

Let α_t^η be the η-regularization (in the variable x) of α_t, that is, the convolution $\alpha_t^\eta(x) = \int \alpha_t(y)\kappa_\eta(x-y)\,dy$ of α_t with a convolution kernel $\kappa_\eta(x) = \eta^{-1}\kappa(x/\eta)$, where the C^∞ function $\kappa : \mathbb{R} \to \mathbb{R}_+$ is supported in $[-1, 1]$, and $\int \kappa(x)\,dx = 1$. Note for further use that (24) gives $\widetilde{C}$ so that, for all $|t| < \epsilon_p$,

$$\|\alpha_t^\eta\|_{C^{1+1/p}} \leq \frac{\widetilde{C}}{\eta^{1+1/p-\beta}}, \quad \|\alpha_t^\eta\|_\beta \leq \widetilde{C}, \quad |\alpha_t^\eta - \alpha_t|_{1/p} \leq \widetilde{C}\eta^{\beta-1/p} \quad \text{for all } \eta \in (0, 1). \tag{25}$$

We now consider $t \mapsto g_{s,t}$. For $x \neq c$, we have

$$\partial_t g_{s,t}(x) = e^{s\psi(h_t(x))}\left[\frac{\psi'(h_t(x))\alpha_t(h_t(x))}{|f_t'(h_t(x))|} - \frac{\partial_t(|f_t'(h_t(x))|)}{|f_t'(h_t(x))|^2}\right], \tag{26}$$

where

$$\partial_t(|f_t'(h_t(x))|) = -\mathrm{sgn}(x)(f_t''(h_t(x))\alpha_t(h_t(x)) + \partial_t f_t'(h_t(x))). \tag{27}$$

We claim that the function $x \mapsto \partial_t g_{s,t}(x)$ has bounded $BV_{1/\beta}$ norm, uniformly in $s \in K$ and $|t| < \epsilon_p$. Indeed, decomposing

$$\partial_t g_{s,t} = b_{s,t} \circ h_t,$$

note that each $h_t : I \to I$ is a homeomorphism leaving both $[-1, c]$ and $[c, 1]$ invariant, while $b_{s,t}$ is β-Hölder on $[-1, c)$ and $(c, 1]$, uniformly in $s \in K$ and $|t| < \epsilon_p$ (because ψ' is C^β, f_t is a C^2 family of C^3 maps†, and α_t is β-Hölder, uniformly in $|t| < \epsilon_p$), and $\sup_{s\in K, |t|<\epsilon_p} |b_{s,t}(c_+) - b_{s,t}(c_-)| < \infty$ (using $\sup_{|t|<\epsilon_p} \|f_t\|_{\mathcal{B}^{2+\beta}} < \infty$).

To conclude, it suffices to prove that our candidate $b_{s,t} \circ h_t \in BV_p$ is really the t-derivative of $g_{s,t}$ (uniformly in s), that is,

$$\lim_{t_2 \to t_1} \sup_{s \in K} \left\| \frac{g_{s,t_2} - g_{s,t_1}}{t_2 - t_1} - b_{s,t_1} \circ h_{t_1} \right\|_{BV_p} = 0 \quad \text{for all } |t_1| < \epsilon_p, \tag{28}$$

† This implies, in particular, that $x \mapsto \partial_t f_t$ is C^2 in x, uniformly in t and $\partial_x \partial_t f_t = \partial_t f_t'$.

and that this derivative is continuous in t (uniformly in s), that is,

$$\lim_{\tilde{t}\to t_1} \sup_{t_2\in[t_1,\tilde{t}]} \sup_{s\in K} \|b_{s,t_2}\circ h_{t_2} - b_{s,t_1}\circ h_{t_1}\|_{BV_p} = 0 \quad \text{for all } |t_1| < \epsilon_p. \tag{29}$$

We first prove (29). Decomposing

$$b_{s,t_2}\circ h_{t_2} - b_{s,t_1}\circ h_{t_1} = (b_{s,t_2} - b_{s,t_1})\circ h_{t_2} + b_{s,t_1}\circ h_{t_2} - b_{s,t_1}\circ h_{t_1}, \tag{30}$$

we focus first on the second term in the right-hand side of (30). Let $\delta > 0$ be such that f_t', f_t'' and $\partial_t f_t'$ restricted to $[-1, c]$ and $[c, 1]$, respectively, extend to C^1 functions of x on $[-1-\delta, c+\delta]$ and $[c-\delta, 1+\delta]$, respectively, for all $|t| < \epsilon_p$. Denote by $b_{s,t}^{\eta,-}$ the function obtained from $b_{s,t}$ by substituting α_t with α_t^η, and also ψ', and the extensions to $[-1-\delta, c+\delta]$ of $f_t''|_{[-1,c]}$, $\partial_t f_t'|_{[-1,c]}$, with their x-convolutions with κ_η, for small $\eta > 0$ (to be determined later). Define $b_{s,t}^{\eta,+}$ similarly, using $[c-\delta, 1+\delta]$, and set $b_{s,t}^\eta(x) = b_{s,t}^{\eta,+}(x)$ if $x > c$ and $= b_{s,t}^{\eta,-}(x)$ if $x < c$. Since $\beta < 1$ and ψ' is Lipschitz, it is easy to see that there exists $\widehat{C} > 0$ so that for all $\eta \in (0, 1)$

$$\max\left(\sup_{s\in K} |(b_{s,t_1}^\eta|_{(-\infty,c)})'|_{1/p}, \sup_{s\in K} |(b_{s,t_1}^\eta|_{(c,\infty)})'|_{1/p}\right) \le \frac{\widehat{C}}{\eta^{1+1/p-\beta}} \quad \text{for all } |t_1| < \epsilon_p.$$

(Use the first two estimates of (25), and the analogous bounds for the regularizations of ψ' and f_t'', $\partial_t f_t'$.) Therefore, by the fundamental theorem of calculus and the Hölder (or Jensen) inequality, there exists $\bar{C} > 0$ so that for all $s \in K$, all $|t_1| < \epsilon_p$, $|t_2| < \epsilon_p$, all $\eta \in (0, 1)$, and any $x_0 < x_1 < \cdots < x_N \le c$,

$$\begin{aligned}
&\sum_{i=0}^{N-1} |b_{s,t_1}^\eta(h_{t_2}(x_i)) - b_{s,t_1}^\eta(h_{t_1}(x_i)) - b_{s,t_1}^\eta(h_{t_2}(x_{i+1})) + b_{s,t_1}^\eta(h_{t_1}(x_{i+1}))|^p \\
&\quad = \sum_i \left| \int_{t_1}^{t_2} \partial_t(b_{s,t_1}^\eta(h_t(x_i)))\,dt - \int_{t_1}^{t_2} \partial_t(b_{s,t_1}^\eta(h_t(x_{i+1})))\,dt \right|^p \\
&\quad \le \sum_i \int_{t_1}^{t_2} |(b_{s,t_1}^\eta)'(h_t(x_i))\alpha_t(h_t(x_i)) - (b_{s,t_1}^\eta)'(h_t(x_{i+1}))\alpha_t(h_t(x_{i+1}))|^p\,dt \\
&\quad = \int_{t_1}^{t_2} \sum_i |(b_{s,t_1}^\eta)'(h_t(x_i))\alpha(h_t(x_i)) - (b_{s,t_1}^\eta)'(h_t(x_{i+1}))\alpha_t(h_t(x_{i+1}))|^p\,dt \\
&\quad \le |t_2 - t_1| \left(\sup_{x\in(-\infty,c)} |(b_{s,t_1}^\eta)'(x)| \sup_t \|\alpha_t \circ h_t\|_{BV_p} + \sup_{x,t} |\alpha_t| |(b_{s,t_1}^\eta|_{(-\infty,c)})'|_{1/p} \right) \\
&\quad \le \bar{C} \frac{|t_2 - t_1|}{\eta^{1+1/p-\beta}}.
\end{aligned} \tag{31}$$

(We used (24) in the last inequality.) The same bounds hold for $c \le x_0 < x_1 < \cdots < x_N$, and it is easy to estimate the jump of $b_{s,t_1}^\eta \circ h_{t_2} - b_{s,t_1}^\eta \circ h_{t_1}$ at $x = c$ uniformly in s and t_1, t_2.

We next analyse the contribution of $b_{s,t_1} - b_{s,t_1}^\eta$ to the second term of (30). For this, observe that if h is an orientation-preserving homeomorphism fixing c and b is β-Hölder on $[-\infty, c]$ and $[c, \infty]$, then $\|b\circ h\|_{BV_p} \le |b|_{(-\infty,c)}|_\beta + |b|_{[c,\infty)}|_\beta + |b(c_+) - b(c_-)|$.

Then, the last bound of (25) and its analogue for the η-regularization of ψ', f_t'' and $\partial_t f_t'$ give a constant C' so that for all $|t_1| < \epsilon_p$, $|t_2| < \epsilon_p$ and $\eta \in (0, 1)$

$$\sup_{s\in K} \|(b_{s,t_1} - b^{\eta}_{s,t_1}) \circ h_{t_2} - (b_{s,t_1} - b^{\eta}_{s,t_1}) \circ h_{t_1}\|_{BV_p} \le 2 \sup_{s,t} \|(b_{s,t_1} - b^{\eta}_{s,t_1}) \circ h_t\|_{BV_p} \le C'\eta^{\beta-1/p}. \quad (32)$$

Taking $\xi \in (0, 1)$ and setting $\eta = (t_2 - t_1)^{\xi/(1+1/p-\beta)}$, we obtain from (31–32) that $\lim_{\tilde{t}\to t_1} \sup_{t_2\in[t_1,\tilde{t}]} \sup_{s\in K} \|b_{s,t_1} \circ h_{t_2} - b_{s,t_1} \circ h_{t_1}\|_{BV_p} = 0$.

To analyse the first term of (30), we start by noting that since $t \mapsto \partial_t h_t$ is Lipschitz, there exists a set $\mathcal{D}_p \subset (-\epsilon_p, \epsilon_p)$ of full Lebesgue measure so that $\partial_t h_t$ is differentiable at all t in $\mathcal{D}_p$. Differentiating twice $f_t \circ h_t(x) = h_t \circ f(x)$ with respect to t and† setting $\alpha_t^2 = \partial^2_{tt} h_t \circ h_t^{-1}$, we obtain for all $x \neq c$ and all $t \in \mathcal{D}_p$ that

$$f_t''(x)\alpha_t(x)^2 + 2\partial_t f_t'(x)\alpha_t(x) + \partial_{tt} f_t(x) = \alpha_t^2(f_t(x)) - f_t'(x)\alpha_t^2(x). \quad (33)$$

The left-hand side of the above TCE is β-Hölder in $[-1, c]$ and $[c, 1]$ and continuous in I, since $\alpha_t(c) = 0$ for every small t, so it is β-Hölder continuous. Therefore, by Proposition 2.3, there exist $\epsilon_p > 0$ and a constant C'' so that

$$|\alpha_t^2|_\beta \le C'' \quad \text{for all } t \in \mathcal{D}_p. \quad (34)$$

The fundamental theorem of calculus holds for the Lipschitz (and, therefore, almost everywhere differentiable) function $t \mapsto b_{s,t}$ and gives

$$(b_{s,t_2} - b_{s,t_1})h_{t_2}(x) = \int_{t_1}^{t_2} \partial_t b_{s,t}(h_{t_2}(x))\, dt \quad \text{for all } x \neq c. \quad (35)$$

The first term of (30) may then be estimated via the Hölder inequality and the fundamental theorem of calculus (35), as in (31), but exploiting (34) instead of using η-regularization. Details are left to the reader.

Finally, to show (28), start from

$$g_{s,\tilde{t}}(x) - g_{s,t_1}(x) - (\tilde{t} - t_1)b_{s,t_1}(h_{t_1}(x)) = \int_{t_1}^{\tilde{t}} (b_{s,t_2}(h_{t_2}(x)) - b_{s,t_1}(h_{t_1}(x)))\, dt_2,$$

for all $x \neq c$, and use the Hölder inequality and (29) (details are left to the reader).

The analysis of the maps $t \mapsto (\psi \circ h_t)^k g_{s,t}/k!$ for $k \ge 1$ goes along exactly the same lines. □

For the higher regularity statement in Theorem 3.1, we use the following result (again, analyticity in s holds).

LEMMA 3.4. *Let $j \ge 2$. Let f_t be a C^{j+1} family of piecewise expanding C^{j+2} unimodal maps in the topological class of f_0. For any $p > 1$ there exists $\epsilon_p > 0$ so that for any $\psi : I \to \mathbb{R}$ which is $C^{j+\mathrm{Lip}}$, the map $s \mapsto g_{s,t}$ defined by (16) is C^1 from $\mathbb{R}$ to the space of C^j maps from $\{|t| < \epsilon_p\}$ to BV_p, and, recalling (1), $\partial_s g_{s,t}|_{s=u} = (\psi \circ h_t)\, g_{u,t}$.*

† This is similar to the proof of [**4**, Proposition 2.4], but we make a more careful analysis of what was called F_i there.

Proof. Since the family f_t is C^{j+1}, the set $\{t \mapsto h_t(x),\, x \in I\}$ is bounded in $C^{j+\mathrm{Lip}}$ by [**4**, Proposition 2.4]. Let $\beta \in (1/p, 1)$ (with $\beta < 1/\sqrt{p}$, say). Assume first $j = 2$. Then, by (33), the function $\alpha_t^2 = \partial_{tt}^2 h_t \circ h_t^{-1}$, is well defined for all $|t| < \epsilon_p$ and there exists C so that $|\alpha_t^2|_\beta \le C$ for every $|t| < \epsilon_p$. For $j \ge 3$, a higher-order TCE similar to (33) gives that $\alpha_t^j = \partial_{t^j}^j h_t \circ h_t^{-1}$ is β-Hölder for all $|t| < \epsilon_p$. We put $\alpha_t^1 = \alpha_t$.

Then, computing $\partial_{t^j}^j g_{s,t}(x)$ at $x \ne c$ gives $b_{s,t}^{(j)}(h_t(x))$, where $b_{s,t}^{(j)}$ is an expression involving derivatives of order at most j of $\psi(x)$, functions α_t^ℓ, for $1 \le \ell \le j$, and derivatives (in x, t, or mixed) of total order at most j of $f_t'(x)$, in the numerator, and $|f_t'(x)|^m$ for $m \ge 1$ in the denominator. Our differentiability assumptions on ψ and the family f_t then allow us to proceed as in the proof of Lemma 3.3 (using Taylor series of higher order). □

4. *Recovering the linear response formula*

Here we give a slightly different proof of the differentiability of $R(t) = \int \psi \, d\mu_t$, where μ_t is the SRB measure of f_t, still relying heavily on Proposition 2.3 (via Lemma 4.2). The advantage with respect to Theorem 3.1 is that we recover the formula for $\partial_t R(t)|_{t=0}$, and we need only assume that ψ is C^0. (In particular, this gives a new proof of [**4**, Theorem 5.1].) We also obtain new information in Corollary 4.4 by combining Theorems 3.1 and 4.1.

We need notation. By [**1**, Proposition 3.3], we may decompose the invariant density of a piecewise expanding C^3 unimodal mixing map f_t as $\rho_t = \rho_{\mathrm{reg},t} + \rho_{\mathrm{sal},t}$, where $\rho_{\mathrm{reg},t} \in BV \cap C^0$, $\rho_{\mathrm{reg},t}' \in BV$, and

$$\rho_{\mathrm{sal},t} = \sum_{k=1}^{M_f} s_{k,t} H_{c_{k,t}}.$$

(Here, $H_u(x)$ denotes the Heaviside function $H_u(x) = -1$ if $x < u$, $H_u(x) = 0$ if $x > u$ and $H_u(u) = -1/2$.) If $M_f = \infty$, then it is not difficult to show that (see e.g. [**1, 4**], noting that if $c_{1,t}$ is preperiodic but not periodic our notation is slightly different than the notation there)

$$s_{k,t} = \frac{s_{1,t}}{(f^{k-1})'(c_{1,t})} \quad \text{for all } k \ge 1. \tag{36}$$

We simply write $\rho_0 = \rho = \rho_{\mathrm{reg}} + \rho_{\mathrm{sal}}$.

To compute the formula for the derivative, we assume, as in [**4**], that $v = \partial_t f_t|_{t=0}$ is of the form $v = X \circ f_0$ for a C^2 function† $X : I \to \mathbb{R}$.

THEOREM 4.1. *Let f_t be a C^2 family of piecewise expanding C^3 unimodal maps. Assume that f_0 is good and mixing, that f_t is tangent to the topological class of f_0, and that $v = \partial_t f_t|_{t=0} = X \circ f_0$ for a C^2 function X. Then, as Radon measures,*

$$\lim_{t\to 0} \frac{\mu_t - \mu_0}{t} = -\alpha \rho_{\mathrm{sal}}' - (\mathrm{id} - \mathcal{L}_0)^{-1}(X'\rho_{\mathrm{sal}} + (X\rho_{\mathrm{reg}})')\, dx, \tag{37}$$

where the function α is given by (4), and the operator $\mathcal{L}_0 = \widetilde{\mathcal{L}}_{0,0}$ is defined by (17). In addition, α is β-Hölder for any $\beta < 1$.

† See also the beginning of [**21**, §17].

Proof. Set $f = f_0$ for convenience. By Proposition A.1, we can assume that f_t lies in the topological class of f, denoting the conjugacies by h_t as usual. In the beginning of the proof of Theorem 3.1, we observed that the transfer operator $\widetilde{\mathcal{L}}_t = \widetilde{\mathcal{L}}_{0,t}$ on BV_p is conjugated to the transfer operator $\mathcal{L}_t$ on BV_p, and that there exists p_0 so that for any $p \in [1, p_0)$ there is $\epsilon_p > 0$ so that for each $|t| < \epsilon_p$ the operator $\widetilde{\mathcal{L}}_t$ acting on BV_p has a maximal eigenvalue equal to one which is simple, and the rest of the spectrum lies in a disc of strictly smaller radius. The fixed points of $\widetilde{\mathcal{L}}_t$ and its dual, $\widetilde{\rho}_t = \rho_t \circ h_t$ and ν_t from (19), were also introduced in the proof of Theorem 3.1.

From now on, we fix $p \in (1, p_0)$.

We next show that $t \mapsto \tilde{\rho}(t) \in BV_p$ and $t \mapsto \nu_t \in BV_p^*$ are differentiable at $t = 0$. By [**4**, Proposition 2.4 and Corollary 2.6] v is horizontal for f_0, $t \mapsto h_t(x)$ is differentiable, uniformly in $x \in I$, and $\alpha = \partial_t h_t|_{t=0}$ is continuous, with $\alpha(c) = 0$, $\alpha(c_1) = X(c)$, and α is the unique bounded solution (4) to the TCE (3). In addition, Proposition 2.3 gives that α is β-Hölder for arbitrary $\beta < 1$ (we take $\beta \in (1/p, 1/\sqrt{p})$).

Our assumptions on f_t then imply that v' is C^1 and the following operator is bounded on BV_p:

$$\mathcal{M}\varphi(x) = -\sum_{f(y)=x} \frac{f''(y)\alpha(y) + v'(y)}{|f'(y)|f'(y)} \varphi(y). \tag{38}$$

(Write $\mathcal{M}$ as $\mathcal{L}_0$ composed with a multiplication operator, as in (20), and use (14).) Lemma 4.2 below easily implies that $t \mapsto \tilde{\mathcal{L}}_t$ is differentiable as an operator on BV_p, and that

$$\partial_t \tilde{\mathcal{L}}_t|_{t=0} = \mathcal{M}. \tag{39}$$

As in the proof of Theorem 3.1, perturbation theory then gives that $t \mapsto \tilde{\rho}_t \in BV_p$ and $t \mapsto \nu_t \in BV_p^*$ are differentiable at $t = 0$. In particular, since $\tilde{\rho}_0 = \rho_0$,

$$\lim_{t \to 0} \|\tilde{\rho}_t - \rho_0\|_{BV_p} = 0. \tag{40}$$

We next show that $t \mapsto \mu_t = \rho_t\, dx$ is differentiable as a Radon measure, exploiting the formula for $\mathcal{M}$ to obtain the claimed formula for $\partial_t \mu_t|_{t=0}$. Fix $\psi : I \to \mathbb{C}$ continuous. Since $\tilde{\rho}_0 = \rho_0$, we can decompose

$$\int \psi \rho_t\, dx - \int \psi \rho_0\, dx = \int \psi \rho_t\, dx - \int \psi\, \tilde{\rho}_t\, dx + \int \psi\, \tilde{\rho}_t\, dx - \int \psi\, \tilde{\rho}_0\, dx. \tag{41}$$

We now see that

$$\lim_{t \to 0} \frac{\int \psi(\rho_t - \rho_t \circ h_t)\, dx}{t} = -\int \psi \alpha \rho_0'. \tag{42}$$

In view of (42), note first that $s_{k,t} \to s_k$ as $t \to 0$: Indeed, (40) gives (in BV_p)

$$\lim_{t \to 0} \tilde{\rho}_t = \lim_{t \to 0} \left(\rho_{\mathrm{reg},t} \circ h_t + \sum_{k=1}^{M_f} s_{k,t} H_{c_k} \right) = \tilde{\rho}_0 = \rho_{\mathrm{reg}} + \sum_{k=1}^{M_f} s_k H_{c_k}. \tag{43}$$

(We gave another proof of $\lim_{t \to 0} s_{k,t} = s_k$ in Step 1 of [**4**, Proof of Theorem 5.1].)

Decompose $\rho_t - \rho_t \circ h_t$ in (42) into $\rho_{\mathrm{sal},t} - \rho_{\mathrm{sal},t} \circ h_t + \rho_{\mathrm{reg},t} - \rho_{\mathrm{reg},t} \circ h_t$. For the singular term, we have in the sense of Radon measures:

$$\lim_{t \to 0} \frac{\rho_{\mathrm{sal},t} - \rho_{\mathrm{sal},t} \circ h_t}{t} = -\sum_{k=1}^{M_f} \alpha(c_k) s_k \mathrm{Dirac}_{c_k} = -\alpha \rho_{\mathrm{sal}}'. \tag{44}$$

(Just use that $s_{k,t} \to s_k$ and (36), which implies that the $s_{k,t}$ decay exponentially in k uniformly in t.)

We claim that the contribution of the regular term $\rho_{\mathrm{reg},t}$ in the decomposition of (42) is $-\int \psi\alpha\rho'_{\mathrm{reg}}\,dx$. In view of this, we first note that if $x \in [-1, c_1)$ is not along the postcritical orbit, we find, using $(\rho_{\mathrm{reg},t})'(y) = (\rho_t)'(y)$ if y is not on the postcritical orbit, that

$$(\rho_{\mathrm{reg},t})'(x) = (\rho_t)'(x) = (\mathcal{L}_t(\rho_t))'(x) = \sum_{f_t(y)=x} \frac{(\rho_{\mathrm{reg},t})'(y)}{|f'_t(y)|f'_t(y)} - \frac{\rho_t(y) f''_t(y)}{|f'_t(y)|(f'_t(y))^2}. \quad (45)$$

Next, by [**1**, Proposition 3.3], $\rho'_{\mathrm{reg},t} \in BV$. The proof of [**1**, Proposition 3.3] implies that the discontinuities of $\rho'_{\mathrm{reg},t}$ lie in the set $\{c_{k,t}\}$. In other words, we may decompose

$$\rho_{\mathrm{reg},t} = \rho_{\mathrm{regreg},t} + \rho_{\mathrm{regsal},t}, \quad (46)$$

with $\rho'_{\mathrm{regreg},t} = (\rho'_{\mathrm{reg},t})_{\mathrm{reg}}$ continuous (that is, $\rho_{\mathrm{regreg},t}$ is C^1), and

$$\rho'_{\mathrm{regsal},t} = (\rho'_{\mathrm{reg},t})_{\mathrm{sal}} = \sum_{k=1}^{M_f} s'_{k,t} H_{c_{k,t}}.$$

By the proof of [**1**, Proposition 3.3] $\rho_{\mathrm{regreg},t}$ is C^1 uniformly in t. We next show that the $s'_{k,t}$ decay exponentially uniformly in t. For this, introduce the notation†

$$\begin{aligned}
E_{1,t} &:= \left(-\frac{\rho_{\mathrm{reg},t}(c) f''_t(c_-)}{(f'_t(c_-))^3} + \frac{\rho_{\mathrm{reg},t}(c) f''_t(c_+)}{(f'_t(c_+))^3} \right) \\
&\quad + \sum_{k\ge 2, c_{k-1,t} > c} s_{k-1,t} \left(\frac{f''_t(c_-)}{(f'_t(c_-))^3} - \frac{f''_t(c_+)}{(f'_t(c_+))^3} \right) \\
E_{k,t} &:= \frac{s_{k-1,t} f''_t(c_{k-1,t})}{(f'_t(c_{k-1,t}))^3}, \quad k \ge 2, \\
E'_{1,t} &:= -\frac{(\rho_{\mathrm{reg},t})'(c)}{(f'_t(c_-))^2} + \frac{(\rho_{\mathrm{reg},t})'(c)}{(f'_t(c_+))^2} \\
E'_{k,t} &:= \frac{s'_{k-1,t}}{(f'_t(c_{k-1,t}))^2}, \quad k \ge 2.
\end{aligned}$$

Then equating the jump at $c_{k,t}$ in both sides of (45) implies that

$$s'_{k,t} = E'_{k,t} - E_{k,t},$$

and thus uniform exponential decay of the $s'_{k,t}$, which are determined by $s'_{1,t}$, $s_{1,t}$, and the postcritical first and second derivatives.

The argument above giving $s_{k,t} \to s_k$ also yields $s'_{k,t} \to s'_k$ (just differentiate once). Therefore, just like in (44), we have

$$\begin{aligned}
\lim_{t\to 0} \int \psi \frac{\rho_{\mathrm{regsal},t} - \rho_{\mathrm{regsal},t} \circ h_t}{t}\,dx &= -\sum_{k=1}^{M_f} \int \alpha(x) s'_k \psi(x) H_{c_k}(x)\,dx \\
&= \int \psi\alpha\rho'_{\mathrm{regsal}}\,dx. \quad (47)
\end{aligned}$$

† If c is periodic, then $(\rho_{\mathrm{reg},t})'(c)$ may be undefined, but $(\rho_{\mathrm{reg},t})'(c_\pm)$ are both defined.

In view of handling the term $\rho_{\text{regreg},t}$ from (46), observe that

$$\lim_{t\to 0} \|\varphi - \varphi \circ h_t\|_{BV} = 0 \quad \text{for all } \varphi \in C^1. \tag{48}$$

Indeed, for $\delta > 0$ and any partition $x_0 < \cdots < x_i < x_{i+1} < \cdots < x_n$ let $N \le n$ be so that $\min(x_N, \inf_t h_t(x_N)) > 1 - \delta$, and since $|h_t(y) - y| = O(t)$ uniformly in y, take t_0 so that $|x_i - h_t(x_i)| < \delta/N$ for all $i \le N$ and $|t| < t_0$. Then use

$$\begin{aligned}
&\sum_{i=0}^{n-1} |\varphi(x_i) - \varphi(x_{i+1}) - \varphi(h_t(x_i)) + \varphi(h_t(x_{i+1}))| \\
&\quad \le 2\sum_{i=0}^{N} |\varphi(x_i) - \varphi(h_t(x_i))| + \sum_{i=N+1}^{n-1} |\varphi(x_i) - \varphi(x_{i+1})| \\
&\qquad + \sum_{i=N+1}^{n-1} |\varphi(h_t(x_i)) - \varphi(h_t(x_{i+1}))|.
\end{aligned}$$

Since the C^1 norm of $\rho_{\text{regreg},t}$ is bounded uniformly in t, (48) and (40) together with $s'_{k,t} \to s'_k$ easily imply that

$$\begin{aligned}
&\lim_{t\to 0} \|\rho_{\text{regreg},t} - \rho_{\text{regreg}}\|_{BV_p} \\
&\quad = \lim_{t\to 0} \|\rho_{\text{regreg},t} - \rho_{\text{regreg},t} \circ h_t + \rho_{\text{regreg},t} \circ h_t - \rho_{\text{regreg}}\|_{BV_p} = 0.
\end{aligned} \tag{49}$$

(Note for the record that, since $s'_{k,t} \to s'_k$, with t-uniformly k-exponentially decaying $s'_{k,t}$, this implies $\lim_{t\to 0} \|\rho_{\text{reg},t} - \rho_{\text{reg}}\|_{BV_p} = 0$.) Then, by the mean value theorem and the x-uniform differentiability of $t \mapsto h_t(x)$

$$\begin{aligned}
&\lim_{t\to 0} \int \psi \frac{\rho_{\text{regreg},t} - \rho_{\text{regreg},t} \circ h_t}{t}\, dx \\
&\quad = \lim_{t\to 0} \int \psi(x) \frac{\rho_{\text{regreg},t}(x) - \rho_{\text{regreg},t}(h_t(x))}{x - h_t(x)} \frac{x - h_t(x)}{t}\, dx \\
&\quad = \lim_{t\to 0} \int \psi(x) \rho'_{\text{regreg},t}(x_t) \frac{x - h_t(x)}{t}\, dx \\
&\quad = -\lim_{t\to 0} \int \psi(x) \rho'_{\text{regreg},t}(x_t) \alpha(x)\, dx \\
&\quad = -\int \psi(x) \rho'_{\text{regreg},0}(x) \alpha(x)\, dx,
\end{aligned} \tag{50}$$

where x_t is in the interval between x and $h_t(x)$, and we used in the last line that $\rho'_{\text{regreg},t}$ is continuous on the compact interval I, uniformly in t, together with (49), Proposition 2.3, and (13). Putting (44), (47) and (50) together, we find (42).

We now turn to the estimation of the term $(\int \psi \tilde\rho_t\, dx - \int \psi \tilde\rho_0\, dx)/t$ from (41). In view of this, note that (39) implies that (as operators on BV_p)

$$\partial_t (z - \tilde{\mathcal{L}}_t)^{-1}|_{t=0} = (z - \mathcal{L}_0)^{-1} \mathcal{M} (z - \mathcal{L}_0)^{-1}.$$

Therefore, writing the spectral projectors as Cauchy integrals, we obtain by a simple residue computation, since $(z - \mathcal{L}_0)^{-1}\rho_0 = \rho_0/(z-1)$, that

$$\partial_t (\nu_t(\rho_0)\tilde\rho_t)|_{t=0} = (\text{id} - \mathcal{L}_0)^{-1} (\text{id} - \Pi_0) \mathcal{M} \rho_0, \tag{51}$$

where $\Pi_0(\varphi) = \rho_0 \int \varphi\, dx$.

Next, we claim that we have (in BV_p)

$$-\alpha\rho'_{\mathrm{reg}} + (\mathrm{id} - \mathcal{L}_0)^{-1}(\mathrm{id} - \Pi_0)(\mathcal{M}\rho_0) = -(\mathrm{id} - \mathcal{L}_0)^{-1}(\mathrm{id} - \Pi_0)(X'\rho_0 + X\rho'_{\mathrm{reg}}). \tag{52}$$

(Recall from [**1**, Proof of Proposition 4.4] that $\Pi_0(X'\rho_0 + X\rho'_{\mathrm{reg}}) = 0$.) Since the TCE (3) implies, using $v' = (X' \circ f) \cdot f'$, that

$$\mathcal{M}\rho_0(x) = (X(x) - \alpha(x)) \sum_{f(y)=x} \frac{f''(y)}{|f'(y)|f'(y)^2}\rho_0(y) - X'(x)\rho_0(x),$$

to prove (10), it suffices to show

$$-\alpha\rho'_{\mathrm{reg}} + (\mathrm{id} - \mathcal{L}_0)^{-1}(\mathrm{id} - \Pi_0)(X - \alpha)(\widetilde{\mathcal{M}}\rho_0) = -(\mathrm{id} - \mathcal{L}_0)^{-1}(\mathrm{id} - \Pi_0)(X\rho'_{\mathrm{reg}}),$$

where $\widetilde{\mathcal{M}}\varphi(x) = \sum_{f(y)=x}(f''(y)/(|f'(y)|f'(y)^2))\varphi(y)$. It follows from (45) that for any $x \in I$ which is not on the postcritical orbit

$$\widetilde{\mathcal{M}}(\rho_0)(x) = \sum_{f(y)=x} \frac{\rho'_{\mathrm{reg}}(y)}{|f'(y)|f'(y)} - \rho'_{\mathrm{reg}}(x).$$

In other words, we have (in BV_p)

$$\widetilde{\mathcal{M}}(\rho_0) = \bar{\mathcal{M}}(\rho'_{\mathrm{reg}}) - \rho'_{\mathrm{reg}},$$

where $\bar{\mathcal{M}}\varphi(x) = \sum_{f(y)=x}(\varphi(y)/(|f'(y)|f'(y)))$. So we have reduced the claim (52) to

$$-\alpha\rho'_{\mathrm{reg}} - (\mathrm{id} - \mathcal{L}_0)^{-1}(\mathrm{id} - \Pi_0)(\mathcal{L}_0 - \mathrm{id})(\alpha\rho'_{\mathrm{reg}}) = 0,$$

that is, using $\Pi_0\mathcal{L}_0 = \Pi_0$,

$$(\mathcal{L}_0 - \mathrm{id})(\alpha\rho'_{\mathrm{reg}}) = (\mathrm{id} - \Pi_0)(\mathcal{L}_0 - \mathrm{id})(\alpha\rho'_{\mathrm{reg}}) = (\mathcal{L}_0 - \mathrm{id})(\alpha\rho'_{\mathrm{reg}}).$$

Finally, since $\rho_0 \in BV_p$, and since $t \mapsto \nu_t$ and $t \mapsto \tilde{\rho}_t$ are differentiable in BV_p and BV_p^*, respectively, we have (in BV_p)

$$\partial_t(\nu_t(\rho_0)\tilde{\rho}_t)|_{t=0} = \partial_t(\nu_t(\rho_0))|_{t=0}\rho_0 + \partial_t(\tilde{\rho}_t)|_{t=0}. \tag{53}$$

Take the Lebesgue average of both sides of (53). Since $\partial_t \int \tilde{\rho}_t\, dx = 0$ (because each $\tilde{\rho}_t\, dt$ is a probability), and since

$$-\int (\mathrm{id} - \mathcal{L}_0)^{-1}(X'\rho_{\mathrm{sal}} + (X\rho_{\mathrm{reg}})')\, dx = 0$$

(again use $\Pi_0(X'\rho_0 + X\rho'_{\mathrm{reg}}) = 0$), we find that $\partial_t(\nu_t(\rho_0))|_{t=0} \int \rho_0\, dx = 0$. Therefore, $\partial_t(\nu_t(\rho_0))|_{t=0}$, and putting together (41), (42), (51), (52) and (53), we have proved the theorem. □

We have (a simplification of Lemma 3.3) the following result.

LEMMA 4.2. *Let f_t be a C^2 family of piecewise expanding C^3 unimodal maps in the topological class of f_0. Set $v = \partial_t f_t|_{t=0}$. For any $p > 1$ the map $t \mapsto g_t = 1/|f'_t \circ h_t| \in BV_p$ is C^1 in a neighbourhood of* 0, *and* $\partial_t g_t|_{t=0} = -(f''_0\alpha + v')/(|f'_0|f'_0)$.

Proof. Differentiability follows from Lemma 3.3 applied to $\psi \equiv 0$. The value of the derivative is given by (26) and (27) in the proof of that lemma, since $\partial_t f'_t|_{t=0} = v'$. □

Remark 4.3. We have the following strengthening of Lemma 4.2 if f_t is a C^3 family of piecewise expanding C^4 unimodal maps: for any $p > 1$ the map $t \mapsto g_t = 1/|f'_t \circ h_t| \in BV_p$ is C^2 in a neighbourhood of 0, and $|g_t - g_0 + t((f''_0 \alpha + v')/(|f'_0|f'_0))| = O(t)$. Recalling (38)–(39), this implies that $\|(\tilde{\mathcal{L}}_t - \mathcal{L}_0)/t - \mathcal{M}\|_{BV_p} = O(t)$.

We obtain the following (new) result as a corollary of Theorems 3.1 and 4.1.

COROLLARY 4.4. *If f_t is a C^2 family of piecewise expanding C^3 unimodal maps in the topological class of f_0, and if $\partial_t f_t|_{t=0} = X \circ f_0$ for a C^2 function X, then there exists $\epsilon > 0$ so that $t \mapsto \mu_t$ is C^1 from $(-\epsilon, \epsilon)$ to Radon measures.*

In particular, under the assumptions of Corollary 4.4, the Radon measure

$$-\alpha_t \rho'_{\mathrm{sal},t} - (\mathrm{id} - \mathcal{L}_t)^{-1}(X'_t \rho_{\mathrm{sal},t} + (X_t \rho_{\mathrm{reg},t})')\, dx$$

(recall (37)) is continuous as a function of t. (Here, α_t solves (3) for f_t and $v_t = \partial_s f_s|_{s=t}$, and $X_t \circ f_t = v_t$.) This fact is not clear *a priori* from the formula.

Remark 4.5. We expect that a careful analysis of the term (42) for C^1 functions ψ would allow us to bypass the reference to Theorem 3.1 in the proof of Corollary 4.4.

Proof of Corollary 4.4. We want to show that $t \mapsto \partial_u \mu_u|_{u=t} = \tilde{\mu}_t$ is continuous: we know that $\tilde{\mu}_t$ exists for all small t (as a Radon measure) by Theorem 4.1. Clearly, $|\int \psi\, d\tilde{\mu}_t| \le C \sup|\psi|$ for all continuous ψ and all small enough t.

Assume for a contradiction that $t \mapsto \tilde{\mu}_t$ is discontinuous at t_0. This means that there exist $\psi \in C^0$, with $\sup|\psi| = 1$, $\delta > 0$, and a sequence t_m with $|t_m - t_0| < 1/m$, so that $|\int \psi\, d\tilde{\mu}_{t_0} - \int \psi\, d\tilde{\mu}_{t_m}| > \delta$ for all m. Take $\tilde{\psi} \in C^1$ so that $\sup|\psi - \tilde{\psi}| < \delta/4$. Then $|\int \tilde{\psi}\, d\tilde{\mu}_{t_0} - \int \tilde{\psi}\, d\tilde{\mu}_{t_m}| > \delta/2$ for all m. However, Theorem 3.1 implies $|\int \tilde{\psi}\, d\tilde{\mu}_{t_0} - \int \tilde{\psi}\, d\tilde{\mu}_{t_m}| < \delta$ if m is large enough, a contradiction. □

A. *Appendix. A consequence of the Keller–Liverani bounds from* [**4**]

We state here for the record an immediate corollary of [**4**, Proposition 3.3] which was based on results in [**16**] (see Remark 2.1 and note that the assumptions below imply $\sup_I |f_t - \tilde{f}_t| = O(t^2)$).

PROPOSITION A.1. *Let f_t be a C^2 family of piecewise expanding C^2 unimodal maps. Assume that f_0 is mixing and good, and that f_t is tangent to the topological class of f_0, denoting by $\tilde{f}_t$ a family in the topological class of f_0 with $\tilde{f}_0 = f_0$ and $\partial f_t|_{t=0} = \partial \tilde{f}_t|_{t=0}$.*

Let $\mu_t = \rho_t\, dx$ and $\tilde{\mu}_t = \tilde{\rho}_t\, dx$ be the SRB measures of f_t and $\tilde{f}_t$, respectively. Then for any $\xi < 2$ there exists $C > 0$ so that for all small t

$$\|\rho_t - \tilde{\rho}_t\|_{L^1(\mathrm{Leb})} \le C|t|^{\xi}.$$

Acknowledgements. V.B. is partially supported by ANR-05-JCJC-0107-01. D.S. is partially supported by CNPq 470957/2006-9 and 310964/2006-7, FAPESP 2003/03107-9. D.S. is grateful to the DMA of École Normale Supérieure for their hospitality during a visit where a crucial part of this work was done. We thank the anonymous referee for comments which helped us improve the presentation.

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