“Long time semi-classical evolution of wave packets and Examples of non Quantum Unique Ergodicity with hyperbolic maps”

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- Quantum map on the torus as convenient models.
- Evolution of wave packets - Phase space distribution of eigenfunctions - Distribution of eigenvalues.

1.2 Linear hyperbolic map on the torus

- Classical and quantum dynamics on the torus. Quantum periods.
- Phase space representation of quantum states. Semi-classical measures and Snirelman quantum ergodic theorem.
- Evolution of wave packets up to twice the Ehrenfest time. Examples of non ergodic invariant semi-classical measures.
- Semi-classical interpretation in terms of constructive interferences along classical paths. How to generalize to non linear dynamics?

1.3 Non linear hyperbolic map on the torus

- Models of quantum non linear hyperbolic map on the torus.
- Some results about localization of semi-classical measures.
- Semi-classical expressions of the propagator and its trace for time beyond the Ehrenfest time: $\forall C$, $t \simeq C.\log(1/h)$.
- Some perspectives.

2 Summary

Classical Hamiltonian Mechanics can describe the behaviour of waves in the limit of short wave length, the so called *semi-classical limit.* For certain systems of waves, it may happen that in this limit, the classical dynamics be chaotic, with strong sensitivity to initial conditions. The wave behaviour is then strongly affected by this chaos, and is called *quantum chaos.* In physics, numerous examples of quantum chaos are known, for quantum systems, but also for acoustic, sismeic, or electromagnetic waves. In mathematics, the nice object to be studied is the wave equation (ou Laplacian spectrum) on Riemanian manifolds with negative curvature.

There is an important characteristic time in quantum chaos, the Erhenfest time, which is "very short", and corresponds to the time when details at the wave length scale are amplified to the macroscopic scale. Before that time, the evolution of a wave packet is well described by semi-classical theory. After that time, complex interferences appear, and a statistical approach seems more suitable. To these two range of time correspond respectively the periodic orbit theory, and the random matrix theory with its "universal" aspect.

Some important questions in quantum chaos concern the behaviour of waves during time evolution, the description of the complex interferences patterns which are rapidly produced. Some questions also concern the stationnary waves (eigenfunctions), their distribution over phase space, as well as the distribution of their frequencies (eigenvalues). An important result is the Quantum Erodicity theorem of Schnirelman which asserts that for an ergodic classical dynamics, in the semi-classical limit, *almost all* the eigenfunctions are equidistributed over phase space, i.e. their *semi-classical measure* is the Liouville measure, (but some exceptions could exist).

In the course, we will present some simple mathematical models of (quantum) chaos, the (quantum) maps on the torus. In particular, we will study the linear automorphism of the torus, which are hyperbolic map ("Quantum cat map"). First we will describe evolution of wave packets, introduce the characteristic Ehrenfest time (TE \sim log(1/hbar)/Lyap), and then we will able to construct some particular eigenfunctions, for which curiously, the associated semi-classical measure is one half the Liouville measure plus one half the Dirac measure on a periodic orbit. From the Quantum Ergodicity theorem of Schnirelman, this behaviour is exceptionnal. Using a semi-classical expression of the evolution operator, we will interpret their existence as a result of collective constructive interferences among multiple trajectories.

We will mention that is other cases, the Unique Quantum Ergodicity has been proven (i.e. every semi-classical measure is the Liouville measure). We will also discuss the generic case of a non linear hyperbolic map on the torus.

In the non linear case, we will show how to extend and control semi-classical formula (for the propagator and its trace) for time Cste*TE, i.e. any multiple of the Erhenfest time. This is a tentative to anderstand the behaviour of quantum state evolution for large time in the general case.