

Linear Response, Or Else

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Abstract. Consider a smooth one-parameter family $t \mapsto f_t$ of dynamical systems f_t , with $|t| < \epsilon$. Assume that for all t (or for many t close to $t = 0$) the map f_t admits a unique SRB invariant probability measure μ_t . We say that *linear response* holds if $t \mapsto \mu_t$ is differentiable at $t = 0$ (possibly in the sense of Whitney), and if its derivative can be expressed as a function of f_0 , μ_0 , and $\partial_t f_t|_{t=0}$. The goal of this note is to present to a general mathematical audience recent results and open problems in the theory of linear response for chaotic dynamical systems, possibly with bifurcations.

Mathematics Subject Classification (2010). Primary 37C40; Secondary 37D25, 37C30, 37E05.

Keywords. Linear response. Transfer operator. Ruelle operator. Physical measure. SRB measure. Bifurcations. Differentiable dynamical system. Unimodal maps. Expanding interval maps. Hyperbolic dynamical systems.

1. Introduction

A discrete-time dynamical system is a self-map $f : M \rightarrow M$ on a space M . To any point $x \in M$ is then associated its (future) orbit $\{f^n(x) \mid n \in \mathbb{Z}_+\}$ where $f^0(x) = x$, and $f^n(x) = f^{n-1}(f(x))$, for $n \geq 1$, represents the state of the system at time n , given the “initial condition” x . (If f is invertible, one can also consider the past orbit $\{f^{-n}(x) \mid n \in \mathbb{Z}_+\}$.) In this text, we shall always assume that M is a compact differentiable manifold (possibly with boundary), with the Borel σ -algebra, endowed with a Riemannian structure and thus normalised Lebesgue measure. Many natural dynamical systems are “chaotic” (in particular, a small error in the initial condition will grow exponentially with time) and best understood via ergodic theory. The ergodic approach often starts with finding a “natural” invariant probability measure μ (a probability measure is invariant if $\mu(f^{-1}(E)) = \mu(E)$ for every Borel set). Lebesgue measure is not always invariant, although there are important exceptions such as the angle-doubling map $x \mapsto 2x$ modulo 1 on the circle, hyperbolic linear toral automorphisms such as the “cat map” A_0 defined in (3.2) below, or symplectic diffeomorphisms. However, many interesting dynamical systems which do not preserve Lebesgue admit a “physical” invariant probability measure: The *ergodic basin* of an f -invariant probability measure μ is

*The toy model in Section 2 was presented at a minicourse at the Dynamical Systems Days in Antofagasta, Chile, December 2007.

the set of those initial conditions for which time averages converge to the space average for every continuous function $\varphi : M \rightarrow \mathbb{C}$, i.e., the set

$$\{x \in M \mid \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi d\mu, \forall \varphi \in C^0\}.$$

An invariant probability measure μ is called *physical* if its ergodic basin has positive Lebesgue measure.

If μ is f -invariant and absolutely continuous with respect to Lebesgue then, if it is in addition ergodic, it is a physical measure because of the Birkhoff ergodic theorem. It was one of the breakthrough discoveries of the 60's, by Anosov and others, that many natural dynamical systems (in particular smooth hyperbolic attractors) admit finitely many physical measures, while in general they do *not* admit any absolutely continuous invariant measure. Physical measures are sometimes called SRB¹ measures after Sinai, Ruelle, and Bowen, who studied them in the sixties [65].

Instead of a single discrete-time dynamical system f , let us now consider a one-parameter family $t \mapsto f_t$ of dynamical systems on the same space M , where $t \in [-\epsilon, \epsilon]$, for $\epsilon > 0$. We assume that the map $t \mapsto f_t$ is “smooth” (i.e., C^k for some $1 < k \leq \infty$), taking a suitable topology in the image, e.g., that of C^ℓ diffeomorphisms, or (piecewise) C^ℓ endomorphisms of M , for some $\ell > 1$. We can view f_t as a perturbation of the dynamics $f := f_0$. Let us *assume* that there exists a closed set Λ , containing 0 as an accumulation point, such that the map f_t admits a unique physical measure for every $t \in \Lambda$. (We shall give examples where this assumption holds below.) The question we are interested in is: *Does the map $t \mapsto \mu_t$ inherit any of the smoothness of $t \mapsto f_t$ at the point $t = 0$?* In particular, is $t \mapsto \mu_t$ differentiable at $t = 0$ (possibly by requiring k and ℓ large enough)?

As such, the question is not well defined, because we must be more precise regarding both the *domain* Λ and the *range* $\{\mu_t \mid t \in \Lambda\}$ of the map $t \mapsto \mu_t$. If Λ contains a neighbourhood U of 0, then differentiability is understood in the usual sense, and differentiability properties usually hold throughout U . However, if Λ does not contain² any neighbourhood of 0, “differentiability” of $t \mapsto \mu_t$ on Λ should be understood in the sense of the Whitney extension theorem, as was pointed out by Ruelle [49]. In other words, the map $t \mapsto \mu_t$ is called C^m at $0 \in \Lambda$ for a real number $m > 0$ if this map admits a C^m extension from Λ to an open neighbourhood of 0. If $0 \leq m < 1$ this is just continuity or Hölder continuity on a metric set. For $m = 1$, e.g., then “ μ_t is C^1 in the sense of Whitney on Λ at $t = 0$ ” means that there exists a continuous function $\mu_s^{(1)}$, defined for $s \in \Lambda$, so that

$$\mu_s = \mu_0 + s\mu_s^{(1)} + R_s, \text{ with } R_s = o(|s|), \quad \forall s \in \Lambda.$$

In order to give a precise meaning to $= o(|s|)$, we need to be more specific regarding the topology used in the *range*. Even if μ_t has a density with respect to Lebesgue,

¹The notions of SRB and physical measures do not always coincide, see [65]. In the present expository note, we shall ignore this fact.

²One could also decide to restrict Λ even if it originally contains a neighbourhood of 0.

the L^1 norm of this density can be too strong to get differentiability. What is often suitable is a distributional norm, i.e., the topology of the dual of C^r for some $r \geq 0$ ($r = 0$ corresponds to viewing μ_t as a Radon measure). In other words, the question is the differentiability of

$$t \mapsto \int \varphi d\mu_t.$$

where the “observable” φ belongs to $C^r(M)$. In some cases $(C^r(M))^*$ can be replaced by a space of anisotropic distributions (see §3.1).

We emphasize that considering a strict subset $\Lambda_0 \subset \Lambda$ containing 0 as an accumulation point may change the class of Whitney- C^m maps at 0: A given map μ_t defined on Λ could be (Whitney) C^m at $0 \in \Lambda_0$, but *not* (Whitney) C^m at $0 \in \Lambda$. It seems fair to take a “large enough” Λ , for example by requiring 0 to be a Lebesgue density point in Λ (i.e., $\lim_{r \rightarrow 0} m(\Lambda \cap [-r, r])/(2r) = 1$), or at least 0 *not* to be a point of dispersion in Λ (i.e., $\lim_{r \rightarrow 0} m(\Lambda \cap [-r, r])/(2r) > 0$).

We shall focus on $0 < m \leq 1$. (Higher differentiability results, including formulas, can be obtained [47] if one makes stronger smoothness assumptions on the individual dynamical systems $x \mapsto f_t(x)$ and on the map $t \mapsto f_t$.) If we can prove, under some assumptions on the family f_t , on the set Λ , and on k , ℓ , and r , that the map $t \mapsto \mu_t$ is differentiable at $0 \in \Lambda$, then it is natural to ask if there is a formula for

$$\partial_t \int \varphi d\mu_t|_{t=0}$$

in terms of f_0 , μ_0 , φ , and the vector field $v_0 := \partial_t f_t|_{t=0}$. If such a formula exists, it is called the *linear response formula* (it gives the response to first order of the system in terms of the first order of the perturbation). We shall assume that the perturbation takes place in the image point, i.e., there exists vector fields X_s so that

$$v_s := \partial_t f_t|_{t=s} = X_s \circ f_s, \quad \forall s, t \in [-\epsilon, \epsilon]. \quad (1.1)$$

(If each f_s is invertible, the above is just a definition of X_s .) The mathematical study of linear response has been initiated by Ruelle. In § 3.1, we shall present his pioneering result [44] on smooth hyperbolic systems (Axiom A attractors). Let us just mention now the key *linear response formula* he obtained in [44] for smooth hyperbolic attractors f_t and smooth observables φ :

$$\partial_t \int \varphi \rho_t dx|_{t=0} = \sum_{j=0}^{\infty} \int \langle X_0, \text{grad}(\varphi \circ f_0^j) \rangle d\mu_0, \quad (1.2)$$

where the sum is exponentially converging. In [46], Ruelle had shown how to derive (1.2) from heuristic arguments, which suggested to consider the following *susceptibility function* associated to f_t and φ :

$$\Phi_t(z) = \sum_{j=0}^{\infty} \int z^j \langle X_0, \text{grad}(\varphi \circ f_0^j) \rangle d\mu_0. \quad (1.3)$$

Under very weak assumptions, the power series $\Phi_t(z)$ (often denoted $\Phi_t(e^{i\omega})$) has a nonzero radius of convergence. If the radius of convergence is ≤ 1 and the series in the right-hand-side of (1.2) does not converge, Ruelle [48, (**)] suggested that the value at $z = 1$ could sometimes be obtained by analytic continuation, possibly giving the linear response formula. However, caution is necessary, as it was discovered since then (see Section 4.2 below) that linear response fails [7] in cases where a meromorphic continuation was known to exist [49], (see also the presentation of the results of [8] in Section 4.1.)

Before we sketch the contents of this note, we make two simple but essential remarks on (1.2). First note that the higher-dimensional version of the Leibniz expression $(X\rho)' = X'\rho + X\rho'$ reads

$$\rho \operatorname{div} X + \langle X, \operatorname{grad} \rho \rangle.$$

Second, defining the transfer operator associated to an invertible³ dynamical system f_t (acting, e.g., on L^∞ or L^1) by

$$\mathcal{L}_t \varphi(x) = \frac{\varphi(f_t^{-1}(x))}{|\det Df_t(f_t^{-1}(x))|},$$

we have $\int \mathcal{L}_t(\varphi) dx = \int \varphi dx$, for all φ (since the dual of \mathcal{L}_t preserves Lebesgue measure, this is the change of variable formula in an integral). If the transfer operator has a nonnegative fixed point $\mathcal{L}_t \rho_t = \rho_t \in L^1$, then $\mu_t = \rho_t dx$ is an absolutely continuous invariant probability measure for f_t and thus (if ergodic) a physical measure. In this case, if the eigenvalue 1 for \mathcal{L}_t is simple and isolated, Ruelle's formula (1.2) and integration by parts give,

$$\begin{aligned} \partial_t \int \varphi \rho_t dx|_{t=0} &= \sum_{j=0}^{\infty} \int \langle X_0, \operatorname{grad}(\varphi \circ f_0^j) \rangle \rho_0 dx \\ &= - \sum_{j=0}^{\infty} \int \varphi \circ f_0^j (\rho_0 \operatorname{div} X_0 + \langle X_0, \operatorname{grad} \rho_0 \rangle) dx \\ &= - \sum_{j=0}^{\infty} \int \varphi \mathcal{L}_0^j (\rho_0 \operatorname{div} X_0 + \langle X_0, \operatorname{grad} \rho_0 \rangle) dx \\ &= - \int \varphi (1 - \mathcal{L}_0)^{-1} (\rho_0 \operatorname{div} X_0 + \langle X_0, \operatorname{grad} \rho_0 \rangle) dx. \end{aligned} \quad (1.4)$$

Note that the residue of $(1 - z\mathcal{L}_0)^{-1} (\rho_0 \operatorname{div} X_0 + \langle X_0, \operatorname{grad} \rho_0 \rangle) dx$ at $z = 1$ vanishes, because Lebesgue measure is the fixed point of \mathcal{L}_0^* , and the manifold is boundaryless, so that $\int (\rho_0 \operatorname{div} X_0 + \langle X_0, \operatorname{grad} \rho_0 \rangle) dx = 0$, by integration by parts. The “metaformula” (1.4) for linear response in the last line can be guessed by applying perturbation theory to the fixed point ρ_t of the operator \mathcal{L}_t . We shall see in § 3.1 instances where the above is a rigorous argument, *even in cases where μ_t is not absolutely continuous with respect to Lebesgue* (then, μ_t is a distribution, enjoying

³See (2.1) for the noninvertible version.

smoothness along unstable directions), and in Section 4 instances where the computation above is invalid, even in cases where μ_t is in fact absolutely continuous with respect to Lebesgue. We emphasize that the tricky point is that the resolvent $(1 - z\mathcal{L}_0)^{-1}$ is evaluated at an expression involving differentiation of ρ_0 : While ρ_0 itself often belongs to a space on which \mathcal{L}_0 has nice spectral properties, this is not always true for its derivative.

The note is organised as follows: In § 2, we give a complete proof of linear response in the baby toy model of smooth locally expanding circle maps. Section 3 contains an account of two nontrivial occurrences of linear response in chaotic dynamics: The breakthrough [44] of Ruelle for smooth hyperbolic systems is presented in § 3.1, while Dolgopyat's result [20] in a (not necessarily structurally stable) partially hyperbolic case is stated in § 3.2. The next section, which contains both recent results and open problems, is devoted to situations where linear response is violated: We consider first the toy model of piecewise expanding interval maps, presenting in § 4.1 our results [9, 10] with Smania, and those with Marmi–Sauzin [8]. Then, we focus on the – more difficult – smooth, nonuniformly expanding, unimodal interval maps, discussing in § 4.2 the work of Ruelle [51], together with our work with Smania [11, 12], and our recent paper with Benedicks and Schnellmann [7]. Finally, § 4.3 contains a brief account of the techniques of proofs in [7].

The survey published by Nonlinearity in 2008 [6] contains a broad viewed account of the results, open problems, and conjectures at the time, with an emphasis on the role played by critical points (or more generally homoclinic tangencies) in the breakdown of linear response. That survey is thus complementary to the present more introductory presentation. (In view of the page limitation for this contribution, we sometimes do not give fully explicit statements and definitions, the reader is invited to consult the quoted references for clarification.)

We refer to Ruelle's articles [46, 48, 52] for motivation, applications to physics, and more conjectures. See also the interesting approach of Hairer and Majda [25], including references of applications to climate-change. In the present note, we do not discuss linear response for continuous time dynamics [50, 17], or for dynamical systems in infinite dimensions (such as coupled map lattices [27, 28]).

2. The toy model of expanding circle maps

In this section we present a proof of linear response in the (baby) toy model of smooth expanding circle maps. The result and proof are well known (and simpler than the analogous arguments in [24, 9]), but we are not aware of any reference.

Let $M = S^1$ be the unit circle, and let $f : S^1 \rightarrow S^1$ be a C^2 map which is *locally expanding*, i.e., there exists $\lambda > 1$ so that $|f'(x)| \geq \lambda$ for all x . It is known [38] that such an f admits a unique absolutely continuous invariant probability measure $\mu = \rho dx$. This measure is mixing and therefore ergodic. So a C^2 locally expanding map f admits a unique physical measure. In fact, ρ is C^1 , and it is

everywhere strictly positive. The transfer operator ⁴

$$\mathcal{L}\varphi(x) = \sum_{f_0(y)=x} \frac{\varphi(y)}{|f_0'(y)|} \quad (2.1)$$

is bounded on $C^1(S^1)$. It is known (see [4], e.g., for the relevant references to Ruelle and others) that ρ is a fixed point of \mathcal{L} , that the eigenvalue 1 of \mathcal{L} (acting on $C^1(S^1)$) has algebraic multiplicity equal to one, and that the rest of the spectrum of \mathcal{L} is contained in a disc of radius strictly smaller than one. (Thus, \mathcal{L} acting on $C^1(S^1)$ has a spectral gap.) Note that the eigenvector of \mathcal{L}^* for the eigenvalue 1 is just normalised Lebesgue measure (by the change of variable formula).

Fix $\lambda > 1$, and consider a C^2 path $t \mapsto f_t$ for $t \in (-\epsilon, \epsilon)$, where each f_t is now C^3 and locally λ -expanding (then, \mathcal{L}_t acts on C^2 , and $\rho_t \in C^2$). Assume that $\|f_t - f_s\|_{C^3(S^1, S^1)} = O(|t - s|)$. Then, using the fact that \mathcal{L}_t (acting on $C^2(S^1)$ or $C^1(S^1)$) satisfies the following *Lasota–Yorke* (or *Doebelin–Fortet*) ⁵ inequalities

$$\|\mathcal{L}_t^k \varphi\|_{C^j} \leq C \xi^k \|\varphi\|_{C^j} + C^k \|\varphi\|_{C^{j-1}}, \quad \forall \varphi, \forall k \geq 1, j = 1, 2, \quad (2.2)$$

(with uniform $0 < \xi < 1$ and $C \geq 1$), together with ⁶

$$\|(\mathcal{L}_t - \mathcal{L}_0)\varphi\|_{C^1} = O(|t|)\|\varphi\|_{C^2},$$

one obtains strong deterministic stability:

Theorem 2.1 (Strong deterministic stability, [14]). *There exists $C > 0$ so that*

$$\|\rho_t - \rho_s\|_{C^1} \leq C|t - s|, \quad \forall t, s \in (-\epsilon, \epsilon).$$

In addition, for any t there exists $\tau < 1$, so that, for all s close enough to t , the spectrum of \mathcal{L}_s , acting on $C^1(S^1)$ or $C^2(S^1)$, outside of the disc of radius τ consists exactly in the simple eigenvalue 1.

The above result implies that $t \mapsto \mu_t$ is Lipschitz, taking the C^1 topology of the density ρ_t of μ_t in the image.

Assume now further (this does not reduce much generality) that $v_t = \partial_s f_s|_{s=t}$ can be written as $v_t = X_t \circ f_t$ with $X_t \in C^2$. Then, we have linear response:

Theorem 2.2 (Linear response formula). *Viewing $\rho_t \in C^2$ as a C^1 function, the map $t \mapsto \rho_t$ is differentiable, and we have*

$$\partial_s \rho_s|_{s=t} = -(1 - \mathcal{L}_t)^{-1}((X_t \rho_t)'), \quad \forall t \in (-\epsilon, \epsilon).$$

Note that $X_t \rho_t$ is C^2 by assumption. Since integration by parts on the boundaryless manifold S^1 gives $\int (X_t \rho_t)' dx = 0$, the residue of the simple pole at $z = 1$ of the resolvent $(z - \mathcal{L}_t)^{-1}$ (acting on $C^1(S^1)$) vanishes at $(X_t \rho_t)'$.

We now prove Theorem 2.2, assuming Theorem 2.1:

⁴The number of terms in the sum is a constant finite integer ≥ 2 , the *degree* of the map.

⁵What is essential here is the compact embedding of C^j – the strong norm – in C^{j-1} – the weak norm.

⁶See Step 1 in the proof of Theorem 2.2 for a stronger claim.

Proof of Theorem 2.2. The proof consists in three steps, to be proved at the end:

Step 1: Considering \mathcal{L}_t as a bounded operator from $C^2(S^1)$ to $C^1(S^1)$, we claim that the map $t \mapsto \mathcal{L}_t$ is differentiable, and that, for every $t \in (-\epsilon, \epsilon)$, we have

$$\mathcal{M}_t(\varphi) := \partial_s \mathcal{L}_s(\varphi)|_{s=t} = -X'_t \mathcal{L}_t(\varphi) - X_t \mathcal{L}_t \left(\frac{\varphi'}{f'_t} \right) + X_t \mathcal{L}_t \left(\frac{\varphi f''_t}{(f'_t)^2} \right).$$

(This step will use $v_t = X_t \circ f_t$.)

Step 2: Let $\Pi_t(\varphi) = \rho_t \cdot \int \varphi dx$ be the rank one projector for the eigenvalue 1 of \mathcal{L}_t acting on $C^1(S^1)$. Then, for every $t \in (-\epsilon, \epsilon)$, we have

$$\partial_s \rho_s|_{s=t} = (1 - \mathcal{L}_t)^{-1} (1 - \Pi_t) \mathcal{M}_t(\rho_t).$$

(Note that $\rho_t \in C^2$, but \mathcal{M}_t is an operator from $C^2(S^1)$ to $C^1(S^1)$.)

Step 3: For every $t \in (-\epsilon, \epsilon)$, we have

$$(1 - \mathcal{L}_t)^{-1} [(1 - \Pi_t) \mathcal{M}_t(\rho_t)] = -(1 - \mathcal{L}_t)^{-1} ((X_t \rho_t)').$$

Theorem 2.2 follows from putting together Steps 2 and 3. To conclude, we justify the three steps:

Proof of Step 1: We must show that the operators defined for $s \neq t$ by

$$\mathcal{R}_{t,s} := \frac{\mathcal{L}_t - \mathcal{L}_s}{t - s} - \mathcal{M}_t$$

satisfy $\lim_{s \rightarrow t} \|\mathcal{R}_{t,s}\|_{C^2(S^1) \rightarrow C^1(S^1)} = 0$. We start by observing that the number of branches of f_s (which is just its degree) does not depend on s . So for any fixed t and any x , each inverse branch for $f_s^{-1}(x)$, for s close enough to t , can be paired with a well-defined nearby inverse branch $f_t^{-1}(x)$. For two such paired branches, we get, since $\varphi \in C^2$, each f_s is C^3 , and $t \mapsto f_t$ is C^2 , that

$$\begin{aligned} \frac{\varphi(f_t^{-1}(x))}{|f'_t(f_t^{-1}(x))|} - \frac{\varphi(f_s^{-1}(x))}{|f'_s(f_s^{-1}(x))|} &= O((t-s)^2) - (t-s) X'_t(x) \frac{\varphi(f_t^{-1}(x))}{|f'_t(f_t^{-1}(x))|} \\ &\quad - (t-s) X_t(x) \left[\frac{\varphi'(f_t^{-1}(x))}{|f'_t(f_t^{-1}(x))| |f'_t(f_t^{-1}(x))|} - \frac{\varphi(f_t^{-1}(x)) f''_t(f_t^{-1}(x))}{(f'_t(f_t^{-1}(x)))^2 |f'_t(f_t^{-1}(x))|} \right]. \end{aligned}$$

Proof of Step 2: Fix t . By Theorem 2.1, we can find a positively oriented closed curve γ in the complex plane so that, for any s close to t , the simple eigenvalue 1 of \mathcal{L}_s is contained in the domain bounded by γ , and no other element of the spectrum of \mathcal{L}_s acting on $C^2(S^1)$ lies in this domain. Step 2 then uses classical perturbation theory for isolated simple eigenvalues of bounded linear operators on Banach spaces (see [29], e.g., see also [36] for the use of similar ideas to get spectral stability), which tells us that, for any $\varphi \in C^2$ so that $\Pi_s(\varphi) = \int \varphi dx = 1$, we have

$$\rho_s = \frac{1}{2i\pi} \oint_{\gamma} (z - \mathcal{L}_s)^{-1} \varphi(z) dz. \quad (2.3)$$

(We used that $\int \rho_s dx = 1$ for all s and $\mathcal{L}_s^*(dx) = dx$.) Next, for $z \in \gamma$, we have the identity

$$(z - \mathcal{L}_t)^{-1} - (z - \mathcal{L}_s)^{-1} = (z - \mathcal{L}_t)^{-1}(\mathcal{L}_t - \mathcal{L}_s)(z - \mathcal{L}_s)^{-1},$$

where we view $(z - \mathcal{L}_s)^{-1}$ as acting on $C^2(S^1)$, the difference $(\mathcal{L}_t - \mathcal{L}_s)$ as an operator from $C^2(S^1)$ to C^1 , and $(z - \mathcal{L}_t)^{-1}$ as acting on $C^1(S^1)$. Letting s tend to t , and recalling Step 1, we have proved

$$\partial_s(z - \mathcal{L}_s)^{-1}|_{s=t} = (z - \mathcal{L}_t)^{-1}\mathcal{M}_t(z - \mathcal{L}_t)^{-1}.$$

Finally, taking (as we may) $\varphi = \rho_t \in C^2$ in (2.3),

$$\begin{aligned} \partial_s \rho_s|_{s=t} &= \frac{1}{2i\pi} \oint_{\gamma} (z - \mathcal{L}_t)^{-1} \mathcal{M}_t(z - \mathcal{L}_t)^{-1} \rho_t(z) dz \\ &= \frac{1}{2i\pi} \oint_{\gamma} (z - \mathcal{L}_t)^{-1} \frac{\mathcal{M}_t(\rho_t(z))}{z - 1} dz. \end{aligned}$$

An easy residue computation completes Step 2.

Proof of Step 3: It suffices to show $\mathcal{M}_t \rho_t - \Pi_t \mathcal{M}_t \rho_t = -(X_t \rho_t)'$. Step 1 implies

$$\mathcal{M}_t \rho_t = -X_t' \rho_t - X_t \mathcal{L}_t \left(\frac{\rho_t'}{f_t'} - \frac{\rho_t f_t''}{(f_t')^2} \right).$$

Now we use that $\rho_t' = (\mathcal{L}_t \rho_t)' \in C^1$ and

$$(\mathcal{L}_t \varphi)'(x) = \sum_{f_t(y)=x} \frac{\varphi'(y)}{|f_t'(y)|} \frac{1}{f_t'(y)} - \sum_{f_t(y)=x} \frac{\varphi(y) f_t''(y)}{|f_t'(y)| (f_t'(y))^2},$$

to see that

$$\mathcal{L}_t \left(\frac{\rho_t'}{f_t'} - \frac{\rho_t f_t''}{(f_t')^2} \right) = \rho_t'.$$

We have shown that $\mathcal{M}_t \rho_t = -(X_t \rho_t)'$, so that $\int \mathcal{M}_t \rho_t dx = 0$ and $\Pi_t \mathcal{M}_t \rho_t = 0$, ending the proof of Step 3, and thus of the theorem. \square

3. Linear response

3.1. Smooth hyperbolic dynamics (structural stability). A C^1 diffeomorphism $f : M \rightarrow M$ is called *Anosov* if there exist a Df -invariant continuous splitting $TM = E^u \oplus E^s$ of the tangent bundle and constants $C > 0$ and $\lambda > 1$ so that, for any $x \in M$, all $n \geq 1$, all $v \in E^s(x)$, and all $w \in E^u(x)$,

$$\|Df_x^n(v)\| \leq C\lambda^{-n}\|v\|, \quad \|Df_x^{-n}(w)\| \leq C\lambda^{-n}\|w\|. \quad (3.1)$$

Thus, Anosov diffeomorphisms are generalizations of the linear hyperbolic map

$$A_0 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad (3.2)$$

on the two-torus. Indeed (we refer to [30], e.g., for the basics of hyperbolic dynamics), a small smooth perturbation of A_0 is an Anosov diffeomorphism. Anosov diffeomorphisms f admit (finitely many) SRB measures as soon as they are $C^{1+\epsilon}$, and the SRB measure is unique if the diffeomorphism is transitive.⁷ For *Axiom A diffeomorphisms*, hyperbolicity (i.e., the existence of the continuous splitting $E^u \oplus E^s$) is assumed only at $T_x M$ for points x in the nonwandering set Ω ; in addition, periodic orbits are assumed to be dense in Ω . Smale’s horseshoe is a famous Axiom A diffeomorphism, but SRB measures exist in general only for Axiom A *attractors*, such as the solenoid. (Anosov diffeomorphisms are special cases of Axiom A attractors.) An important property of Axiom A diffeomorphisms is *structural stability*: If f_0 is an Axiom A attractor, and f_t is close to f_0 (in the C^1 topology), then f_t is also Axiom A, and, in addition f_0 is topologically conjugated to f_t , i.e., there is a one-parameter family⁸ of homeomorphisms h_t so that $f_t = h_t \circ f_0 \circ h_t^{-1}$.

Linear response holds for smooth hyperbolic systems: After pioneering results of de la Llave et al. [40] and Katok et al. [31], Ruelle proved the following landmark theorem ([44, 45], see also [26]):

Theorem 3.1 (Linear response for smooth hyperbolic systems). *Let M be a compact Riemann manifold. Let $t \mapsto f_t$ be a C^3 map from $(-\epsilon, \epsilon)$ to C^3 diffeomorphisms $f_t : M \rightarrow M$. Assume that each f_t is a topologically mixing Axiom A attractor, and let μ_t be its unique SRB probability measure. Then for any $\varphi \in C^2$, the map $t \mapsto \int \varphi d\mu_t$ is differentiable on $(-\epsilon, \epsilon)$. In addition, setting $X_t = \partial f_s|_{s=t} \circ f_t^{-1}$, we have*

$$\partial_s \int \varphi d\mu_s|_{s=t} = \sum_{j=0}^{\infty} \int \langle \text{grad}(\varphi \circ f_t^j), X_t \rangle d\mu_t, \quad (3.3)$$

where the series converges (exponentially).

In this situation, one shows that the susceptibility function (1.3) is holomorphic in a disc of radius strictly bigger than one.

Ruelle exploited symbolic dynamics in [44, 45]. For a more modern approach, using anisotropic Banach spaces, see the work of Gouëzel and Liverani ([23, Thm 2.8] for Anosov, and [24, Prop. 8.1] for Axiom A). The modern approach is much simpler, since the transfer operators \mathcal{L}_t of the diffeomorphisms f_t all have a uniform spectral gap on the same Banach space \mathcal{B} of anisotropic distributions, which contains, not only the SRB measure μ_t , but also its “derivative.” The “metaformula” (1.4) can then be easily justified rigorously.

3.2. Mild bifurcations. In § 4 we shall see examples where the breakdown of structural stability (the presence of bifurcations in the family f_t) is mirrored by a breakdown of linear response. However, structural stability is *not* necessary to obtain linear response – and neither is the spectral gap⁹ of the transfer operator

⁷Transitivity is automatic if f is volume preserving. It is conjectured that all Anosov diffeomorphisms on connected compact manifolds are transitive.

⁸The map $t \mapsto h_t$ is smooth and its derivative α_t solves the twisted cohomological equation (4.4), see also [6] and references therein.

⁹See the work of Hairer and Majda [25].

\mathcal{L}_t . We briefly describe a result of Dolgopyat [20] on a class of partially hyperbolic maps. We consider partially hyperbolic diffeomorphisms $f : M \rightarrow M$ on a smooth compact manifold M , i.e., we assume the tangent bundle is decomposed into invariant bundles $E^c \oplus E^u \oplus E^s$, where E^u and E^s are both nontrivial and enjoy (3.1). A partially hyperbolic diffeomorphism f is called an *Anosov element of a standard abelian Anosov action* if the central bundle E^c of f is tangent to the orbits of a C^∞ action g_t of \mathbb{R}^d so that $fg_t = g_tf$ (see [32, 33]). Assume further that f admits a unique physical (SRB) measure μ , whose basin has total Lebesgue measure. The action is called *rapidly mixing* if there exists and a (g_t -admissible) class of smooth functions \mathcal{F} , and, for any $m \geq 1$, there exists $C \geq 1$ so that, for all subsets S in a suitable class of unstable leaves of f , any $\varphi \in \mathcal{F}$, and for any smooth probability density ψ on S , we have

$$\left| \int_S (\varphi \circ f^n)(x) \psi(x) dx - \int \varphi d\mu \right| \leq C \|\varphi\|_{\mathcal{F}} \|\psi\| n^{-m}.$$

We refer to [20] for precise definitions of the objects above and of u -Gibbs states, we just recall here that SRB measures are u -Gibbs states. Dolgopyat's result follows:

Theorem 3.2 (Linear response for rapidly mixing abelian Anosov actions [20]). *Let f be a C^∞ Anosov element of a standard abelian Anosov action so that f has a unique SRB measure and is rapidly mixing. Then, for any C^∞ one-parameter family of diffeomorphisms $t \mapsto f_t$ through $f_0 = f$, choosing for each t a u -Gibbs state ν_t for f_t (which can be the SRB measure if it exists), we have that $\int \varphi d\nu_t$ is differentiable at $t = 0$ for any $\varphi \in C^\infty$, and the linear response formula (3.3) holds. (See [20, p. 405] for the linear response formula.)*

Besides giving a new proof in the Anosov case, applications of Theorem 3.2 include:

- time-one maps f of Anosov flows, which are generically rapidly mixing;
- toral extensions f of Anosov diffeomorphisms F defined by

$$f(x, y) = (F(x), y + \omega(x)), \quad x \in M, y \in \mathbb{T}^d, \omega \in C^\infty(M, \mathbb{T}^d),$$

which are generically rapidly mixing (under a diophantine condition).

It seems important here that structural stability may only break down in the central direction. This allows Dolgopyat to use rapid mixing to prove that most orbits can be shadowed, a key feature of his argument.

4. Or Else

The results stated in § 3.1 gave at the time some hope [49] that linear response could hold (at least in the sense of Whitney) for a variety of nonuniformly hyperbolic systems. In the present section we shall state some results obtained since 2007 which indicate that the situation is not so simple. We would like to mention that numerical experiments and physical arguments already gave a hint that something could go wrong (see [21], e.g., for fractal transport, see [35]).

4.1. Piecewise expanding interval maps. Piecewise expanding maps can be viewed as a toy model for the smooth unimodal maps to be discussed in § 4.2. The setting is the following: We let $I = [-1, 1]$ be a compact interval, and consider continuous maps $f : I \rightarrow I$ with $f(-1) = f(1) = -1$, and so that $f|_{[-1,0]}$ and $f|_{[0,1]}$ are C^2 , with $\inf_{x \neq c} |f'(x)| \geq \lambda > 1$. Such a map is called a *piecewise expanding unimodal map* (for λ). Lasota and Yorke [39] proved in the 70's that such a map possesses a unique absolutely continuous invariant probability measure $\mu = \rho dx$, which is always ergodic. In fact, the density ρ is of bounded variation. If μ is mixing, we have exponential decay of correlations for smooth observables, which can be proved by using the spectral gap of the transfer operator \mathcal{L}_t defined by (2.1) acting on the Banach space BV of functions of bounded variation, see e.g. [4]. We set $c = c_0 = 0$, and we put $c_k = f^k(c)$ for $k \geq 1$.

Consider now a C^1 path $t \mapsto f_t$, with each f_t a piecewise expanding unimodal map. Assume in addition that $f_0 = f$ is topologically mixing on $[c_2, c_1]$ (then $\mu = \mu_0$ is mixing), that $c_1 < 1$, and that c is not a periodic point of f_0 (this implies that f_0 is stably mixing, i.e., small perturbations of f_0 remain mixing). Then, applying [39], each f_t admits a unique SRB measure $\mu_t = \rho_t dx$ (and each transfer operator \mathcal{L}_t has a spectral gap on BV , the corresponding estimates are in fact uniform). Keller [34] proved that the map

$$t \mapsto \rho_t \in L^1(dx)$$

is Hölder for every exponent $\eta < 1$. In fact, Keller showed

$$\|\rho_t - \rho_s\|_{L^1} \leq C|t - s| |\log |t - s||. \tag{4.1}$$

From now on, we assume that each f_t is piecewise C^3 , that the map $t \mapsto f_t$ is C^2 , and that $v = \partial_t f_t|_{t=0} = X \circ f$. An example is given by taking the *tent maps*

$$\begin{aligned} f_t(x) &= a + t - (a + t + 1)x, & \text{if } x \in [0, 1], \\ f_t(x) &= a + t + (a + t + 1)x, & \text{if } x \in [-1, 0], \end{aligned} \tag{4.2}$$

choosing $0 < a < 1$ so that 0 is not periodic for f_a and so that f_a is mixing (note that $X_0(x) = (a + 1)^{-1}(x + 1)$). Observe that structural stability is strongly violated here: f_t is topologically conjugated to f_s only if $s = t$ [18]. In other words, the family f_t of tent maps undergoes strong bifurcations.

A piecewise expanding map is called *Markov* if c is preperiodic, that is, if there exists $j \geq 2$ so that c_j is a periodic point: $f^p(c_j) = c_j$ for some $p \geq 1$. (In this case, one can show that the invariant density is piecewise smooth, and the susceptibility function is meromorphic.) A Markov map is mixing if its transition matrix is aperiodic, stable mixing then allows to construct easily mixing tent maps.

It turns out that Keller's upper bound (4.1) is optimal, *linear response fails*:

Theorem 4.1 (Mazzolena [42], Baladi [5]). *There exist a Markov piecewise expanding interval map f_0 , a path f_t through f_0 , with a C^∞ observable φ , a constant $C > 0$, and a sequence $t_n \rightarrow 0$, so that*

$$\left| \int \varphi d\mu_{t_n} - \int \varphi d\mu_0 \right| \geq C|t_n| |\log |t_n||, \quad \forall n.$$

Setting $v = v_0 = \partial_t f_t|_{t=0}$, and assuming $v = X \circ f$, we introduce

$$\mathcal{J}(f, v) = \sum_{j=0}^{\infty} \frac{v(f^j(c))}{(f^j)'(c_1)} = \sum_{j=0}^{\infty} \frac{X(f^j(c_1))}{(f^j)'(c_1)}. \quad (4.3)$$

If $\mathcal{J}(f_0, v_0) = 0$ (a codimension-one condition on the perturbation v or X), we say that the path f_t is *horizontal* (at $t = 0$). This condition was first studied for smooth unimodal maps [60, 3]. In the setting of piecewise expanding unimodal maps, Smania and I proved the following result:

Theorem 4.2 (Horizontality and tangency to the topological class [9, 10]). *A path f_t is called tangent to the topological class of f_0 (at $t = 0$) if there exist a path \tilde{f}_t so that $f_t - \tilde{f}_t = O(t^2)$ and homeomorphisms h_t so that $\tilde{f}_t \circ h_t = h_t \circ f_0$. Then:*

- *The path f_t is horizontal (at $t = 0$) if and only if there is a continuous solution α to the twisted cohomological equation*

$$v(x) = X \circ f(x) = \alpha \circ f(x) - f'(x)\alpha(x), \quad x \neq c. \quad (4.4)$$

- *The path f_t is horizontal (at $t = 0$) if and only if it is tangent to the topological class of f_0 (at $t = 0$).*

Note that the family of tent maps given in (4.2) is *not* horizontal.

We already mentioned that $\rho_t \in BV$. Any function g of bounded variation can be decomposed as two functions of bounded variation $g = g^{sing} + g^{reg}$, where the regular component g^{reg} is a continuous function of bounded variation, while the singular component g^{sing} is an at most countable sum of jumps (i.e., Heaviside functions). In the particular case of the invariant density ρ_t of a piecewise expanding unimodal map, we proved [5] that $(\rho_t^{reg})'$ is of bounded variation, while the jumps of ρ_t^{sing} are located along the postcritical orbit c_k , with exponentially decaying weights, so that $(\rho_t^{sing})'$ is an exponentially decaying sum of Dirac masses along the postcritical orbit. *The fact that the derivative of ρ_0 does not belong to a space on which the transfer operator has a spectral gap is the glitch which disrupts the spectral perturbation mechanism described in Section 2* (in Section 3.1 the derivative of the distribution corresponding to the SRB measure *did* belong to a good space of anisotropic distributions). Note also that ρ_0^{sing} is intimately related to the postcritical orbit of f_0 , which is itself connected to the bifurcation structure of f_t at f_0 . (We refer also to [6].)

Our main result with Smania on piecewise expanding maps reads as follows:

Theorem 4.3 (Horizontality and linear response [9]).

- *If the path f_t is horizontal (at $t = 0$) then the map $t \mapsto \mu_t \in C(I)^*$ is differentiable at $t = 0$ (as a Radon measure), and we have the linear response formula:*

$$\partial_t \mu_t|_{t=0} = -\alpha(\rho^{sing})' - (1 - \mathcal{L}_0)^{-1}(X' \rho^{sing} + (X \rho^{reg})') dx. \quad (4.5)$$

- If the path f_t is not horizontal (at $t = 0$), then, if in addition $|f'(c_-)| = |f'(c_+)|$ or $\inf_j d(f^j(c), c) > 0$, we have:

If the postcritical orbit $\{c_k\}$ is not ¹⁰ dense in $[c_2, c_1]$, then there exist $\varphi \in C^\infty$ and $K > 0$ so that for any sequence $t_n \rightarrow 0$ so that the postcritical orbit of each f_{t_n} is infinite,

$$\left| \int \varphi d\mu_{t_n} - \int \varphi d\mu_0 \right| \geq K|t_n| |\log |t_n||, \quad \forall n. \quad (4.6)$$

If the postcritical orbit is dense in $[c_2, c_1]$, then there exist $\varphi \in C^\infty$ and sequences $t_n \rightarrow 0$ so that

$$\lim_{n \rightarrow \infty} \frac{\left| \int \varphi d\mu_{t_n} - \int \varphi d\mu_0 \right|}{|t_n|} = \infty. \quad (4.7)$$

We end this section with some of our results on the susceptibility function (recall (1.3))

$$\Psi_\varphi(z) = \sum_{j=0}^{\infty} \int z^j (\partial_x(\varphi \circ f_0^j)(x)) X_0(x) d\mu_0(x)$$

of piecewise expanding unimodal maps (for $\lambda > 1$), the most recent of which were obtained with Marmi and Sauzin (using work of Breuer and Simon [16]):

Theorem 4.4 ([5, 8]). *There exists a nonzero function $\mathcal{U}(z)$, holomorphic in $|z| > \lambda^{-1}$, and, for every non constant $\varphi \in C^0$ so that $\int \varphi d\mu_0 = 0$, there exists a nonzero function $\mathcal{V}_\varphi(z)$, holomorphic in $|z| > \lambda^{-1}$, so that the following holds: Put*

$$\sigma_\varphi(z) = \sum_{j=0}^{\infty} \varphi(c_{j+1}) z^j$$

(this function is holomorphic in the open unit disc), and set

$$\Psi^{hol}(z) = - \int \varphi(x) (1 - z\mathcal{L}_0)^{-1} (X' \rho^{sing} + (X \rho^{reg})')(x) dx.$$

Then:

- There exists $\tau \in (0, 1)$ so that $\Psi^{hol}(z)$ is holomorphic in the disc $|z| < \tau^{-1}$.
- The susceptibility function satisfies

$$\Psi_\varphi(z) = \sigma_\varphi(z) \mathcal{U}(z) + \mathcal{V}_\varphi(z) + \Psi^{hol}(z),$$

where the function $\mathcal{U}(z)$ vanishes at $z = 1$ if and only if $\mathcal{J}(f, v) = 0$, and in that case, we have

$$\partial_t \int \varphi d\mu_t|_{t=0} = \mathcal{V}_\varphi(1) + \Psi^{hol}(1).$$

¹⁰Generically the postcritical orbit is dense, see the references to Bruin in [56].

- If $\{c_k\}$ is dense in $[c_2, c_1]$ and $\varphi \neq 0$, then the unit circle is a (strong) natural boundary for $\sigma_\varphi(z)$ (and thus for $\Psi_\varphi(z)$). If ¹¹ $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \tilde{\varphi}(c_k) = \int \tilde{\varphi} d\mu_0$ for every $\tilde{\varphi} \in C^0$, then for every $\omega \in \mathbb{R}$

$$\lim_{z \xrightarrow{NT} e^{i\omega}} (z - e^{i\omega})\sigma_\varphi(z) = 0,$$

where $z \xrightarrow{NT} e^{i\omega}$ means that $|z| < 1$ tends to $e^{i\omega}$ nontangentially (e.g., radially).

In particular, if the path f_t is horizontal (at $t = 0$) and the postcritical orbit is generic, then

$$\partial_t \int \varphi d\mu_t|_{t=0} = \lim_{z \xrightarrow{NT} 1} \Psi_\varphi(z).$$

The law of the iterated logarithm (LIL), a property stronger than Birkhoff genericity, also holds generically for the postcritical orbit of piecewise expanding maps [57]. If the postcritical orbit satisfies (an $e^{i\omega}$ twisted upper bound version of) the LIL, then more can be said about σ_φ and Ψ_φ , see [8, Thm. 5].

Inspired by Breuer–Simon, we introduced in [8] *renacent right-limits*, a simple construction for candidates for a generalised (Borel monogenic [15], e.g.) continuation outside of the unit disc of power series having the unit circle as a natural boundary. In the case of Poincaré simple pole series, Sauzin and Tiozzo [55] showed that this construction gives the (unique) generalised continuation. However, for the susceptibility function of piecewise expanding maps, there are [8] uncountably many such candidates (even in the horizontal case). This may indicate that there is no reasonable way to extend $\Phi_\varphi(z)$ outside of the unit circle. The analogous problem is more delicate for smooth unimodal maps discussed in § 4.2 below, mainly because the natural boundary for the susceptibility function is expected to lie strictly inside the open unit disc — we refer to [8] for open questions and conjectures.

4.2. Smooth unimodal maps. We now consider the more difficult case of *differentiable* maps $f : I \rightarrow I$, where $I = [0, 1]$ is again a compact interval, and $c = 1/2$ is now a critical point in the usual sense: $f'(c) = 0$. The map f is still assumed unimodal, with $f(-1) = f(1) = -1$, and $f'(x) > 0$ for $-1 \leq x < c$, while $f'(x) < 0$ for $c < x \leq 1$. We denote $c_k = f^k(c)$ for $k \geq 1$ as before. For convenience, we assume that f is topologically mixing and C^3 , with negative Schwarzian derivative (see [18]). Finally, we suppose that $f''(c) < 0$. Of course, f is *not* uniformly expanding since $f'(c) = 0$. One way to guarantee enough (nonuniform) expansion is via the Collet–Eckmann condition: The map f is *Collet–Eckmann* (CE) if there exists $\lambda_c > 1$ and $H_0 \geq 1$ so that

$$|(f^k)'(c_1)| \geq \lambda_c^k, \quad \forall k \geq H_0.$$

If f is CE, then it admits a (unique) absolutely continuous (SRB) invariant probability measure $\mu = \rho dx$ (which is ergodic). We refer to [18] for more about the

¹¹This assumption of Birkhoff genericity of the postcritical orbit is generic [56].

CE condition, noting here only that the invariant density ρ is not bounded in the current setting — in fact, ρ contains a finite, or infinite exponentially decaying, sum of “spikes”

$$\sqrt{|x - f^k(c)|}^{-1}$$

along the postcritical orbit. Thus, $\rho \in L^p$ for all $1 \leq p < 2$, but $\rho \notin L^2$. If f is CE and topologically mixing on $[c_2, c_1]$, then Keller and Nowicki [37] and, independently, Young [63], proved that a spectral gap holds for a suitably defined transfer operator (acting on a “tower”), giving exponential decay of correlations.

We consider again a C^2 path $t \mapsto f_t$, $t \in (-\epsilon, \epsilon)$, say, of C^4 unimodal maps as above, through $f = f_{t_0}$ (with t_0 not necessarily equal to 0) which will be assumed to be (at least) CE. We let $v = v_{t_0} = \partial_t f_t|_{t=t_0}$ and assume that $v = X \circ f$. Noting that $\mathcal{J}(f, v)$ from (4.3) is well defined because of the CE condition, we say that the path f_t is horizontal at $t = t_0$ if $\mathcal{J}(f, v) = 0$.

The fully horizontal case (i.e., $\mathcal{J}(f_t, v_t) = 0$ for all t in a neighbourhood of t_0) happens when f_t is topologically conjugated to f_{t_0} for all t , so that f_t stays in the topological class of f_{t_0} . Then, if f_{t_0} is Collet–Eckmann, all the f_t are Collet–Eckmann (although it is not obvious from the definition, the CE property is a topological invariant [43]) and admit an SRB measure. In this fully horizontal case, viewing ρ_t as a distribution of sufficiently high order, first Ruelle [51] and then Smania and myself [11, 12] obtained linear response, with a linear response formula. (In [11], we even obtain analyticity of the SRB measure.) More precisely, Ruelle [51] considered the analytic case under the Misiurewicz¹² assumption that $\inf_k |f_{t_0}^k(c) - c| > 0$; Smania and myself considered on the one hand [11] a fully holomorphic setting (where the powerful machinery of Mañé–Sad–Sullivan [41] applies), and on the other hand [12] a finitely differentiable setting under a (generic) Benedicks–Carleson-type assumption of topological slow recurrence. The strategy in [12] involves proving the existence of a continuous solution α to the twisted cohomological equation (4.4) if f is Benedicks–Carleson and X corresponds to a horizontal path f_t .

Although the horizontal case is far from trivial (in the present nonuniformly expanding setting, one of the hurdles is to obtain uniform bounds on the constant $\lambda_c(t)$ for CE parameters t close to t_0), it is much more interesting to explore *transversal paths* $t \mapsto f_t$ (undergoing topological bifurcations). The archetypal such situation is given by the so-called *logistic* (or quadratic) family

$$f_t(x) = tx(1 - x).$$

A famous theorem of Jacobson says that the set of CE parameters in the logistic family has strictly positive Lebesgue measure (see [18], e.g.). Since the set Λ of CE parameters does not contain any interval, regularity of the map $t \mapsto \mu_t$ for t in Λ can be considered only in the sense of Whitney. Continuity of the map $t \mapsto \mu_t$, for t ranging in some appropriate subset of Λ (and for the weak $*$ topology in the image) was obtained by Tsujii [61] (see also Rychlik–Sorets [54]) in the 90’s.

¹²Misiurewicz is nongeneric. It implies Collet–Eckmann.

A map f is called *Misiurewicz–Thurston* if there exist $j \geq 2$ and $p \geq 1$ so that $f^p(c_j) = c_j$ and $|(f^p)'(c_j)| > 1$ (in other words, the critical point is *preperiodic*, towards a repelling periodic orbit, this implies that the map has a finite Markov partition). Clearly, Misiurewicz–Thurston implies Misiurewicz and thus Collet–Eckmann. There are only countably many Misiurewicz–Thurston parameters.

For the quadratic family, e.g., Thunberg proved [59, Thm C] that there are superstable parameters s_n of periods p_n , with $s_n \rightarrow t$, for a Collet–Eckmann parameter t , so that $\nu_{s_n} \rightarrow \nu$, where $\nu_{s_n} = \frac{1}{p_n} \sum_{k=0}^{p_n-1} \delta_{f_{s_n}^k(c)}$, and ν is the sum of atoms on a repelling periodic orbit of f_t . Other sequences $t_n \rightarrow t$ of superstable parameters have the property that $\nu_{t_n} \rightarrow \mu_t$, the absolutely continuous invariant measure of f_t . Starting from Thunberg’s result, Dobbs and Todd [19] have constructed sequences of both renormalisable and non-renormalisable Collet–Eckmann maps $f_{t'_n}$, converging to a Collet–Eckmann map f_t , but such that the SRB measures do not converge. Such counter-examples can be constructed while requiring that f_t and all maps $f_{t'_n}$ are Misiurewicz–Thurston. These examples show that continuity of the SRB measure cannot hold on the set of *all* Collet–Eckmann (or even Misiurewicz–Thurston) parameters: Some uniformity in the constants is needed (already when defining the “appropriate subsets” of [61]).

The main result of our joint work [7] with Benedicks and Schnellmann (which also contains parallel statements on more general transversal families of smooth unimodal maps) follows:

Theorem 4.5 (Hölder continuity of the SRB measure in the logistic family [7]). *Consider the quadratic family $f_t(x) = tx(1-x)$ on $I = [0, 1]$, and let $\Lambda \subset (2, 4]$ be the set of Collet–Eckmann parameters t .*

- *There exists $\Delta \subset \Lambda$, of full Lebesgue measure in Λ , so that for every $t_0 \in \Delta$, and for every $\Gamma > 4$, there exists $\Delta_{t_0} \subset \Delta$, with t_0 a Lebesgue density point of Δ_{t_0} , and there exists a constant C so that, for any $\varphi \in C^{1/2}(I)$, for any sequence $t_n \rightarrow t_0$, so that $t_n \in \Delta_{t_0}$ for all n , we have*

$$\left| \int \varphi(x) d\mu_{t_n} - \int \varphi(x) d\mu_{t_0} \right| \leq C \|\varphi\|_{C^{1/2}} |t_0 - t_n|^{1/2} |\log |t_0 - t_n||^\Gamma, \quad (4.8)$$

where $\|\varphi\|_{C^{1/2}}$ denotes the 1/2-Hölder norm of φ .

- *If f_{t_0} is Misiurewicz–Thurston, then there exists $\varphi \in C^\infty$, a constant $C > 1$, and a sequence $t_n \rightarrow t_0$, with $t_n \in \Lambda$ for all n , so that*

$$\frac{1}{C} |t_n - t_0|^{1/2} \leq \left| \int \varphi(x) d\mu_{t_n} - \int \varphi(x) d\mu_{t_0} \right| \leq C |t_n - t_0|^{1/2}. \quad (4.9)$$

The exponent 1/2 appearing in the theorem is directly related to the nondegeneracy assumption $f''(c) \neq 0$, which of course holds true for the quadratic family. Note also that using a C^∞ (instead of $C^{1/2}$) observable does not seem to allow better upper bounds in (4.8). It is unclear if the logarithmic factor in (4.8) is an artefact of the proof or can be discarded.

The proof of the claim (4.9) of the theorem gives a sequence t_n of Misiurewicz–Thurston parameters, but the continuity result of Tsujii [61] easily yields sequences of non Misiurewicz–Thurston (but CE) parameters t_n . We do *not know* whether t_0 is a Lebesgue density point of the set of sequences giving (4.9). Note that in the toy model from § 4.1, the first analogous construction of counter-examples (Theorem 4.1) was limited to a handful of preperiodic parameters (sequences of maps having preperiodic critical points converging to a map f_{t_0} with a preperiodic critical point), while the currently known set of examples (see (4.6) and (4.7)) are much more general, although not fully satisfactory yet. One important open problem is to describe precisely the set of sequences $t_n \rightarrow t_0$ giving rise to violation of linear response for the generic piecewise expanding unimodal maps with dense postcritical orbits in (4.7). This may give useful insight for smooth unimodal maps, both about the largest possible set of sequences giving (4.9), and about relaxing the Misiurewicz–Thurston assumption on f_{t_0} . (Note however that there is a quantitative difference with respect to the piecewise expanding case [9], where the modulus of continuity in the transversal case was $|\log |t - t_0|| |t - t_0|$, so that violation of linear response arose from the logarithmic factor alone.)

We suggested in [7] the following weakening of the linear response problem: Consider a one-parameter family f_t of (say, smooth unimodal maps) through f_{t_0} and, for each $\epsilon > 0$, a random perturbation of f_t with unique invariant measure μ_t^ϵ like in [58], e.g. Then for each positive ϵ , it should not be very difficult to see that the map $t \rightarrow \mu_t^\epsilon$ is differentiable at t_0 (for essentially any topology in the image). Taking a weak topology in the image, like Radon measures, or distributions of positive order, does the limit as $\epsilon \rightarrow 0$ of this derivative exist? How is it related with the perturbation? with the susceptibility function or some of its generalised continuations (e.g. in the sense of [8])?

More open questions are listed in [6] and [12, 7]. In particular, the results in [7] give hope that linear response or its breakdown (see [6] and [53]) can be studied for (the two-dimensional) Hénon family, which is transversal, and where continuity of the SRB measure in the weak-* topology was proved by Alves et al. [1, 2] in the sense of Whitney on suitable parameter sets. In [6, (17), (19)], we also give candidates for the notion of horizontality for piecewise expanding maps in higher dimensions and piecewise hyperbolic maps.

4.3. About the proofs. The main tool in the proof of Theorem 4.5 is a *tower construction*: We wish to compare the SRB measure of f_{t_0} to that of f_t for small $t - t_0$. Just like in [12], we use transfer operators $\widehat{\mathcal{L}}_t$ acting on towers, with a projection Π_t from the tower to $L^1(I)$ so that $\Pi_t \widehat{\mathcal{L}}_t = \mathcal{L}_t \Pi_t$, where \mathcal{L}_t is the usual transfer operator, and $\Pi_t \widehat{\rho}_t = \rho_t$ with $\mu_t = \rho_t dx$ (here, $\widehat{\rho}_t$ is the fixed point of $\widehat{\mathcal{L}}_t$, and ρ_t is the invariant density of f_t). In [12], we adapted the tower construction from [13] (introduced in [13] to study random perturbations, for which this version is better suited than the otherwise ubiquitous Young towers [64]). This construction allows in particular to work with Banach spaces of continuous functions. Another idea imported from [12] is the use of operators $\widehat{\mathcal{L}}_{t,M}$ acting on truncated towers, where the truncation level M must be chosen carefully depending

on $t - t_0$. Roughly speaking, the idea is that f_t is comparable to f_{t_0} for M iterates (corresponding to the M lowest levels of the respective towers), this is the notion of an *admissible pair* (M, t) . Denoting by $\hat{\rho}_{t,M}$ the maximal eigenvector of $\hat{\mathcal{L}}_{t,M}$, the starting point for both upper and lower bounds is (like in [12]) the decomposition

$$\begin{aligned} \rho_t - \rho_{t_0} = & [\Pi_t(\hat{\rho}_t - \hat{\rho}_{t,M}) + \Pi_{t_0}(\hat{\rho}_{t_0,M} - \hat{\rho}_{t_0})] \\ & + [\Pi_t(\hat{\rho}_{t,M} - \hat{\rho}_{t_0,M})] + [(\Pi_t - \Pi_{t_0})(\hat{\rho}_{t_0,M})], \end{aligned} \quad (4.10)$$

for admissible pairs. The idea is then to get upper bounds on the first two terms by using perturbation theory à la Keller–Liverani [36], and to control the last (dominant) term by explicit computations on $\Pi_t - \Pi$ (which represents the “spike displacement,” i.e., the effect of the replacement of $1/\sqrt{|x - f_{t_0}^k(c)|}$ by $1/\sqrt{|x - f_t^k(c)|}$ in the invariant density).

We now move to the differences between [12] and [7]: Using a tower with exponentially decaying levels as in [13] or [12] would provide at best an upper modulus of continuity $|t - t_0|^\eta$ for $\eta < 1/2$, and would not yield any lower bound. For this reason, we use instead “fat towers” with *polynomially decaying* sizes in [7], working with polynomially recurrent maps. In order to construct the corresponding parameter set, we use recent results of Gao and Shen [22].

Applying directly the results of Keller–Liverani [36] would only bound the contributions of the first and second terms of (4.10) by $|t - t_0|^\eta$ for $\eta < 1/2$. In order to estimate the second term, we prove that $\hat{\mathcal{L}}_{t,M} - \hat{\mathcal{L}}_{t_0,M}$ acting on the maximal eigenvector is $O(|\log |t - t_0||^\Gamma |t - t_0|^{1/2})$ in the strong¹³ norm; in the Misiurewicz–Thurston case we get a better $O(|t - t_0|^{1/2})$ control). It is usually not possible to obtain strong norm bounds when bifurcations are present [14, 36], and this remarkable feature here is due to our choice of admissible pairs (combined with the fact that the towers for f_t and f_{t_0} are identical up to level M). To estimate the first term, we enhance the Keller–Liverani argument, using again that it suffices to estimate the perturbation for the operators acting on the maximal eigenvector.

The changes just described are already needed to obtain the exponent $1/2$ in the upper bound (4.8). To get lower bound in (4.9), we use that the tower associated to a Misiurewicz–Thurston map f_{t_0} can be required to have levels with sizes bounded from below, and that the truncation level can be chosen to be slightly larger. Finally, working with Banach norms based on L^1 as in [12] would give that the first two terms in (4.10) are $\leq C|t - t_0|^{1/2}$, while the third is $\geq C^{-1}|t - t_0|^{1/2}$ for some large constant $C > 1$. However, *introducing Banach–Sobolev norms based on L^p for $p > 1$* instead, we are able to control the constants and show that the last term dominates the other two.

References

- [1] Alves, J.F., Carvalho, M., Freitas, J.M., Statistical stability for Hénon maps of the Benedicks–Carleson type, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), 595–

¹³The strong norm plays here the role of C^j in (2.2).

- 637.
- [2] Alves, J.F., Carvalho, M., Freitas, J.M., Statistical stability and continuity of SRB entropy for systems with Gibbs-Markov structures, *Comm. Math. Phys.* **296** (2010), 739–767.
 - [3] Avila, A., Lyubich, M., de Melo, W., Regular or stochastic dynamics in real analytic families of unimodal maps, *Invent. Math.* **154** (2003), 451–550.
 - [4] Baladi, V., *Positive Transfer Operators and Decay of Correlations*. Advanced Series in Nonlinear Dynamics 16, World Scientific, Singapore, 2000.
 - [5] Baladi, V., On the susceptibility function of piecewise expanding interval maps, *Comm. Math. Phys.* **275** (2007), 839–859.
 - [6] Baladi, V., Linear response despite critical points, *Nonlinearity* **21** (2008), T81–T90.
 - [7] Baladi, V., Benedicks, M., Schnellmann, D., Whitney–Hölder continuity of the SRB measure for transversal families of smooth unimodal maps, Preprint, arXiv:1302.3191 (2013).
 - [8] Baladi, V., Marmi, S., Sauzin, D., Natural boundary for the susceptibility function of generic piecewise expanding unimodal maps, *Ergodic Theory Dynam. Systems* **34** (2014), 777–800.
 - [9] Baladi, V., Smania, D., Linear response formula for piecewise expanding unimodal maps, *Nonlinearity* **21** (2008), 677–711. (Corrigendum, *Nonlinearity* **25** (2012), 2203–2205.)
 - [10] Baladi, V., Smania, D., Smooth deformations of piecewise expanding unimodal maps, *DCDS Series A* **23** (2009), 685–703.
 - [11] Baladi, V., Smania, D., Analyticity of the SRB measure for holomorphic families of quadratic-like Collet-Eckmann maps, *Proc. Amer. Math. Soc.* **137** (2009), 1431–1437.
 - [12] Baladi, V., Smania, D., Linear response for smooth deformations of generic nonuniformly hyperbolic unimodal maps, *Ann. Sci. Éc. Norm. Sup.* **45** (2012), 861–926.
 - [13] Baladi, V., Viana, M., Strong stochastic stability and rate of mixing for unimodal maps, *Ann. Sci. Éc. Norm. Sup.* **29** (1996), 483–517.
 - [14] Baladi, V., Young, L.-S., On the spectra of randomly perturbed expanding maps, *Comm. Math. Phys.* **156** (1993), 355–385.
 - [15] E. Borel, *Leçons sur les fonctions monogènes uniformes d’une variable complexe*. Gauthier-Villars, Paris, 1917.
 - [16] Breuer, J., Simon, B., Natural boundaries and spectral theory, *Adv. Math.* **226** (2011), 4902–4920.
 - [17] Butterley, O., Liverani, C., Smooth Anosov flows: correlation spectra and stability, *J. Mod. Dyn.* **1** (2007), 301–322.
 - [18] de Melo, W., van Strien, S., *One-Dimensional Dynamics*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1993.
 - [19] Dobbs, N., Todd, M., Entropy jumps up, Manuscript (2014)
 - [20] Dolgopyat, D., On differentiability of SRB states for partially hyperbolic systems, *Invent. Math.* **155** (2004), 389–449.
 - [21] Ershov, S.V., Is a perturbation theory for dynamical chaos possible? *Physics Letters A* **177** (1993), 180–185.

- [22] Gao, B., Shen, W., Summability implies Collet–Eckmann almost surely, *Ergodic Theory Dynam. Systems*, DOI: <http://dx.doi.org/10.1017/etds.2012.173>.
- [23] Gouëzel, S., Liverani, C., Banach spaces adapted to Anosov systems, *Ergodic Theory Dynam. Systems* **26** (2006), 189–217.
- [24] Gouëzel, S., Liverani, C., Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties, *J. Diff. Geom.* **79** (2008), 433–477.
- [25] Hairer, M., Majda, A.J., A simple framework to justify linear response theory, *Nonlinearity* **23** (2010), 909–922.
- [26] Jiang, M., Differentiating potential functions of SRB measures on hyperbolic attractors, *Ergodic Theory Dynam. Systems* **32** (2012), 1350–1369.
- [27] Jiang, M., de la Llave, R., Linear response function for coupled hyperbolic attractors, *Comm. Math. Phys.* **261** (2006), 379–404.
- [28] Jiang, M., de la Llave, R., Smooth dependence of thermodynamic limits of SRB-measures, *Comm. Math. Phys.* **211** (2000), 303–333.
- [29] Kato, T., *Perturbation Theory for Linear Operators*. Second corrected printing of the second edition, Springer-Verlag, Berlin, 1984.
- [30] Katok, A., Hasselblatt, B., *Introduction to the modern theory of dynamical systems*. Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.
- [31] Katok, A., Knieper, G., Pollicott, M., Weiss, H., Differentiability and analyticity of topological entropy for Anosov and geodesic flows, *Invent. Math.* **98** (1989), 581–597.
- [32] Katok, A., Spatzier, R.J., First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity, *Publ. Math., Inst. Hautes Étud. Sci.* **79** (1994), 131–156.
- [33] Katok, A., Spatzier, R.J., Invariant measures for higher-rank hyperbolic abelian actions. *Ergodic Theory Dynam. Systems* **16** (1996), 751–778.
- [34] Keller, G., Stochastic stability in some dynamical systems, *Monatshefte Math.* **94** (1982), 313–333.
- [35] Keller, G., Howard, P.J., Klages, R., Continuity properties of transport coefficients in simple maps, *Nonlinearity* **21** (2008), 1719–1743.
- [36] Keller, G., Liverani, C., Stability of the spectrum for transfer operators, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **28** (1999), 141–152.
- [37] Keller, G., Nowicki, T., Spectral theory, zeta functions and the distribution of periodic points for Collet–Eckmann maps, *Comm. Math. Phys.* **149** (1992), 633–680.
- [38] Krzyżewski, K., Szlenk, W., On invariant measures for expanding differentiable mappings, *Studia Math.* **33** (1969), 83–92.
- [39] Lasota, A., Yorke, J.A., On the existence of invariant measures for piecewise monotonic transformations, *Trans. Amer. Math. Soc.* **186** (1973), 481–488.
- [40] de la Llave, R., Marco, J.M., Moriyón, R., Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation, *Ann. of Math.* **123** (1986), 537–611.
- [41] Mañé, R., Sad, P., Sullivan, D., On the dynamics of rational maps, *Ann. Sci. École Norm. Sup.* **16** (1983), 193–217.

- [42] Mazzolena, M., Dinamiche espansive unidimensionali: dipendenza della misura invariante da un parametro, *Master's Thesis* Roma 2, Tor Vergata (2007).
- [43] Nowicki, T., Przytycki, F., Topological invariance of the Collet-Eckmann property for S-unimodal maps, *Fund. Math.* **155** (1998), 33–43.
- [44] Ruelle, D., Differentiation of SRB states, *Comm. Math. Phys.* **187** (1997), 227–241.
- [45] Ruelle, D., Differentiation of SRB states: Corrections and complements, *Comm. Math. Phys.* **234** (2003), 185–190.
- [46] Ruelle, D., General linear response formula in statistical mechanics, and the fluctuation-dissipation theorem far from equilibrium, *Phys. Lett. A* **245** (1998), 220–224.
- [47] Ruelle, D., Nonequilibrium statistical mechanics near equilibrium: computing higher-order terms, *Nonlinearity* **11** (1998), 5–18.
- [48] Ruelle, D., Application of hyperbolic dynamics to physics: some problems and conjectures, *Bull. Amer. Math. Soc.* **41** (2004), 275–278.
- [49] Ruelle, D., Differentiating the absolutely continuous invariant measure of an interval map f with respect to f , *Comm. Math. Phys.* **258** (2005), 445–453.
- [50] Ruelle, D., Differentiation of SRB states for hyperbolic flows, *Ergodic Theory Dynam. Systems* **28** (2008), 613–631.
- [51] Ruelle, D., Structure and f -dependence of the A.C.I.M. for a unimodal map f of Misiurewicz Type, *Comm. Math. Phys.* **287** (2009), 1039–1070.
- [52] Ruelle, D., A review of linear response theory for general differentiable dynamical systems, *Nonlinearity* **22** (2009), 855–870.
- [53] Ruelle, D., Singularities of the susceptibility of a Sinai-Ruelle-Bowen measure in the presence of stable-unstable tangencies, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **369** (2011), 482–493.
- [54] Rychlik, M., Sorets, E., Regularity and other properties of absolutely continuous invariant measures for the quadratic family, *Comm. Math. Phys.* **150** (1992), 217–236.
- [55] Sauzin, D., Tiozzo, G., Generalised continuation by means of right limits, Preprint, arXiv:1301.1175 (2013).
- [56] Schnellmann, D., Typical points for one-parameter families of piecewise expanding maps of the interval, *Discrete Contin. Dyn. Syst.* **31** (2011), 877–911.
- [57] Schnellmann, D., Law of iterated logarithm and invariance principle for one-parameter families of interval maps, Preprint, arXiv:1309.2116 (2013), to appear PTRF.
- [58] Shen, W., On stochastic stability of nonuniformly expanding interval maps, *Proc. Lond. Math. Soc.* **107** (2013), 1091–1134.
- [59] Thunberg, H., Unfolding of chaotic unimodal maps and parameter dependence of natural measures, *Nonlinearity* **14** (2001), 323–338.
- [60] Tsujii, M., Positive Lyapunov exponents in families of one-dimensional dynamical systems, *Invent. Math.* **111** (1993), 113–137.
- [61] Tsujii, M., On continuity of Bowen-Ruelle-Sinai measures in families of one-dimensional maps, *Comm. Math. Phys.* **177** (1996), 1–11.

- [62] Whitney, H., Analytic extensions of differentiable functions defined in closed sets, *Trans. Amer. Math. Soc.* **36** (1934), 63–89.
- [63] Young, L.-S., Decay of correlations for certain quadratic maps, *Comm. Math. Phys.* **146** (1992), 123–138.
- [64] Young, L.-S., Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math.* **147** (1998), 585–650.
- [65] Young, L.-S., What are SRB measures, and which dynamical systems have them? *J. Statist. Phys.* **108** (2002), 733–754.

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