Dynamical zeta functions and generalised Fredholm determinants

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Abstract. We recall how dynamical zeta functions can be viewed as generalised Fredholm determinants of transfer operators in expanding cases. We discuss recent joint results with David Ruelle: we define a *sharp trace* $\mathrm{Tr}^{\#}\mathcal{M}$ and a *sharp determinant* $\mathrm{Det}^{\#}(1 - z\mathcal{M}) = \exp \sum_{m\ge 1}(z^m/m)\mathrm{Tr}^{\#}\mathcal{M}^m$ for weighted transfer operators $\mathcal{M}$ associated with one dimensional maps without any expansion assumption. The sharp determinant can be viewed as the inverse of a *Lefschetz zeta function* $\zeta^{-1}(z)$ which counts weighted periodic points. We relate $\mathrm{Det}^{\#}(1 - z\mathcal{M})$ with the Fredholm determinant $\mathrm{Det}\,(1 + \mathcal{D}(z))$ of a (nuclear) *kneading operator* $\mathcal{D}(z)$, which is a continuous version of the Milnor-Thurston kneading determinant. The zeroes of the sharp determinant are the inverses of the discrete eigenvalues of $\mathcal{M}$.

Introduction

We start by recalling two basic and very classical results, one from dynamical systems and the other from functional analysis.

A. Zeta functions for subshifts of finite type.

Let A be a finite $0-1$ transition matrix, and $\sigma_A : \Sigma_A \to \Sigma_A$ the associated one-sided subshift of finite type. If $g : \Sigma_A \to \mathbb{C}$ is a function such that $g(x_1, x_2, x_3, \dots)$ only depends on x_1, x_2, one considers the weighted matrix $A^g_{ij} = A_{ij}g_{ij}$, the weighted dynamical zeta function

$$\zeta_g(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x\in\Sigma_A\,,\sigma_A^m(x)=x} \prod_{k=0}^{m-1} g(\sigma_A^k(x))\,, \tag{1}$$

and the weighted transfer operator defined for $\phi : \Sigma_A \to \mathbb{C}$ by:

$$\mathcal{L}_g(\phi)(x) = \sum_{y\in\Sigma_A\,,\sigma_A(y)=x} \phi(y)g(y)\,. \tag{2}$$

When g is real-valued and strictly positive, the Perron-Frobenius theorem applied to A^g yields a (translation invariant) Gibbs (or equilibrium) state for the nearest-neighbour interaction lattice system with hard cores (σ_A, g). The correlation spectrum of this Gibbs state is thus described by the eigenvalues of the weighted transition matrix. In particular if A^g is irreducible and aperiodic, the Gibbs state is exponentially mixing.

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We make two elementary observations:

(1) The matrix A^g can be viewed as the matrix of $\mathcal{L}_g$ acting on the finite dimensional vector space of functions depending on x_1 only.

(2) The trivial *trace formula*

$$\mathrm{Tr}\,(A^g)^m = \sum_{x\in\Sigma_A\,,\sigma_A^m(x)=x} \prod_{k=0}^{m-1} g(\sigma_A^k(x))\,, \tag{3}$$

yields

$$\frac{1}{\zeta_g(z)} = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{Tr}\,(A^g)^m = \mathrm{Det}\,(1 - zA^g)\,. \tag{4}$$

In particular $1/\zeta_g(z)$ is a polynomial whose zeroes are the inverses of the nonzero eigenvalues of A^g.

See Bowen [1975], Ruelle [1978] (and Section 1 below) for an account of much more general results valid for infinite range interactions, and applications to Axiom A dynamical systems.

B. Classical Fredholm theory.

Let $I = [0,1]$ be the unit interval and dx Lebesgue measure on I. If $K(x,y)$ is a continuous (complex) function on $I \times I$, the linear operator defined on continuous $\varphi : I \to \mathbb{C}$ by

$$\mathcal{K}\varphi(x) = \int K(x,y)\varphi(y)\,dy \tag{5}$$

is a compact operator. Its eigenvalues are the inverses of the zeroes of the *Fredholm determinant*, which is the entire function defined by:

$$\mathrm{Det}\,(1 - z\mathcal{K}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{Tr}\,\mathcal{K}^m\,, \tag{6}$$

where $\mathrm{Tr}\,\mathcal{K} = \int K(x,x)\,dx$ (we use that $\mathcal{K}^m$ can also be written as a nuclear operator with a continuous kernel). See Fredholm [1903] for the original reference, and Grothendieck [1956] for later generalisations.

Our (naive) point is simply that the inverse of the weighted dynamical zeta function (1) of the subshift of finite type can be viewed as a *generalised Fredholm determinant* for the operator $\mathcal{L}_g$. In this article, we present various cases where it is possible to associate a *generalised trace* $\mathrm{Tr}\,\mathcal{M}$ to bounded linear operators $\mathcal{M}$ in some family (of not necessarily compact operators), in such a way that the *generalised Fredholm determinant* defined by

$$\mathrm{Det}\,(1 - z\mathcal{M}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{Tr}\,\mathcal{M}^m\,,$$

can be viewed as (or related to) a weighted dynamical zeta function, and is holomorphic in some neighbourhood of zero in the complex plane where the inverses of its zeroes describe the discrete spectrum of $\mathcal{M}$ in a neighbourhood of infinity.

In Section 1, we review a few settings where the inverse of a dynamical zeta function can be viewed as a generalised Fredholm determinant. These cases involve two kind of assumptions: expansion and smoothness. (We shall not discuss the truly hyperbolic case, see Rugh [1992], Fried [1993b] for recent results in a hyperbolic setting, and references to previous work of Pollicott, Ruelle, Fried, Haydn, etc.)

In Section 2, we present recent joint results with D. Ruelle for one-dimensional systems, which do not require any expansion assumption. These developments were inspired by the kneading theory of Milnor and Thurston. In the Appendix, written jointly with D. Ruelle, we explain how the one-dimensional results in Baladi-Keller [1990] can be applied to non positively expansive cases.

More references can be found in the recent reviews Artuso-Aurell-Cvitanović [1990], Parry-Pollicott [1990], Baladi [1993], Ruelle [1994a].

1. Dynamical zeta functions and generalised Fredholm determinants

Consider an *expanding map* f on a compact metric space X, with an α-Hölder continuous weight g.

Assume first that f is α-Hölder continuous (Ruelle [1990], Fried [1993a]). Because of the Markov property of expanding maps, the transfer operator

$$\mathcal{L}_g \phi(x) = \mathcal{L}\phi(x) = \sum_{y\,, f(y)=x} \phi(y) g(y) \tag{1.1}$$

preserves the Banach space of α-Hölder continuous functions $\phi : X \to \mathbb{C}$. The difference with the situation discussed in the introduction is that in general there is no invariant subspace on which the restriction of $\mathcal{L}$ is compact. Nevertheless, one can obtain an upper bound $r_{\text{ess}}(\mathcal{L}) \le R_{\text{ess}}$ on its *essential spectral radius*:

$$\begin{aligned} r_{\text{ess}}(\mathcal{L}) = \inf\{\rho \ge 0 \mid \ & \text{spectrum}(\mathcal{L}) \cap \{|z| > \rho\} \\ & \text{consists of isolated eigenvalues of finite multiplicity } \}\,. \end{aligned} \tag{1.2}$$

(Note that $r_{\text{ess}}(\mathcal{L})$ vanishes if $\mathcal{L}$ is a compact operator. We call the eigenvalues appearing in the definition of r_{ess} *discrete eigenvalues* of $\mathcal{L}$.) If g is strictly positive, one proves that $r_{\text{ess}}(\mathcal{L})$ is strictly smaller than the spectral radius $r(\mathcal{L})$. Write $\text{Fix}\, f^m = \{x \in X \mid f^m(x) = x\}$. By analogy with (3), one defines a *counting trace*

$$\text{Tr}\, \mathcal{L} = \sum_{x \in \text{Fix}\, f} g(x)\,, \tag{1.3}$$

and a corresponding generalised Fredholm determinant

$$\text{Det}\,(1 - z\mathcal{L}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \text{Tr}\, \mathcal{L}^m = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \text{Fix}\, f^m} \prod_{k=0}^{m-1} g(f^k(x))\,. \tag{1.4}$$

The main theorem is then that $\mathrm{Det}\,(1 - z\mathcal{L})$ is holomorphic in the disc of radius R_{ess}^{-1}, and its zeroes in this disc are exactly the inverses of the eigenvalues of the operator $\mathcal{L}$ (the order of the zero coincides with the multiplicity of the eigenvalue).

Assume now additionally that X is a differentiable manifold, and that f and g are C^k for some $1 \le k < \infty$. Again (Ruelle [1990], Fried [1993a]), the Markov assumption implies that the Banach space of C^k functions on M is preserved by $\mathcal{L}$, and one obtains an upper bound R_{ess} on the essential spectral radius r_{ess} of $\mathcal{L}$. (The higher k, the lower this upper bound.) In this smooth setting, the natural Fredholm determinant is obtained by using a *flat trace*:

$$\mathrm{Tr}^{\flat}\mathcal{L} = \sum_{x \in \mathrm{Fix}\, f} \frac{g(x)}{|\mathrm{Det}\,(1 - D(f^{-1})_x)|}\,. \tag{1.5}$$

(Flat traces were considered by Atiyah and Bott [1967,1968]. They follow from regularisation of the classical Fredholm trace formula from Part B of the Introduction.) Again, one proves that the corresponding *flat determinant* $\mathrm{Det}^{\flat}(1 - z\mathcal{L})$ is holomorphic in the disc of radius $1/R_{\mathrm{ess}}$, and its zeroes in this disc are exactly the inverses of the discrete eigenvalues of the transfer operator $\mathcal{L}$ acting on $C^k(X)$. See Ruelle [1976, 1990] for the relationship between flat determinants and weighted zeta functions.

In the case (studied by Ruelle [1976], and by Fried, Mayer, etc.) where X is an analytic manifold and f and g are analytic, regularisation is replaced by the Cauchy formula, the transfer operators are nuclear in the sense of Grothendieck (in particular compact), and Grothendieck's [1956] generalisation of the Fredholm theory applies. The trace also has the form (1.5). The corresponding Fredholm determinant is of course an entire function in the whole complex plane.

The three smooth expanding cases just discussed involved more or less explicitly a Markov property. If suitable care is taken to preserve this Markov property, the setting of finitely many inverse contracting branches of a given expanding map f, may be replaced by a finite or countable system of (smooth) contractions ψ_ω and (smooth) weights g_ω. The corresponding *generalised transfer operator* preserves a space of smooth functions and takes the form

$$\mathcal{M}\phi(x) = \sum_{\omega} g_\omega(x)\phi(\psi_\omega(x))\,. \tag{1.6}$$

One of the advantages of considering such generalised transfer operators is that they form an algebra.

We now restrict to the *one-dimensional situation*, this will allow us to drop first the Markov property, and then, in the next section, also the expansion assumption. A useful expansion assumption on continuous interval maps $f : I \to I$ is that f has finitely many laps of monotonicity, and the corresponding partition of I is generating. (Countably many laps of monotonicity may in fact be allowed.) Even if f and $g : I \to \mathbb{C}$ are smooth, the transfer operator $\mathcal{L}_g$ defined by (1.1) in general does not even preserve the Banach space of continuous functions on I. However, the Banach space $\mathcal{B}$ of functions of bounded variation is invariant if g is in $\mathcal{B}$ (see the

Appendix). Again, one may prove an upper bound R_{ess} for the essential spectral radius of $\mathcal{L}_g$ acting on $\mathcal{B}$. It turns out that the counting trace (3) gives here also the appropriate generalised Fredholm determinant.i.e., the function

$$\zeta_g(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \operatorname{Fix} f^m} \prod_{k=0}^{m-1} g(f^k x) \tag{1.7}$$

is meromorphic in the disc of radius R_{ess}^{-1} and its poles in this disc are exactly the inverses of the discrete eigenvalues of $\mathcal{L}_g$ acting on $\mathcal{B}$. There exist for the moment two different proofs of this result: in Baladi-Keller [1990], countable Markov extensions are used, in Ruelle [1994a], the system (f, g, I) is embedded in a Markov system $(\tilde{f}, \tilde{g}, \tilde{I})$. The results presented in Section 2 actually give a third proof when g is continuous and vanishes at the endpoints of the monotonicity laps.

2. Lefschetz dynamical zeta functions and sharp determinants for one-dimensional maps

Consider now a piecewise monotone continuous map $f : I \to I$ where $I = [0, 1]$, without assuming that the partition into intervals of monotonicity is generating. The map f may thus have homtervals, and in particular for some integers $m \geq 1$ the set $\operatorname{Fix} f^m$ may be infinite. Milnor and Thurston [1977] solved this problem by counting only *negative fixed points* of f^m, i.e. elements of $\operatorname{Fix} f^m$ which lie in monotonicity laps where f^m is decreasing. This strategy works very well in the unweighted case, and yields partial results in the weighted case (as explained in the Appendix below). A second strategy involves choosing a *representative set of periodic points*, whenever there is more than one point of $\operatorname{Fix} f^m$ in a monotonicity lap of f^m. This route was explored by Ruelle [1994a].

We shall discuss a third, perhaps more natural, approach to this problem (Baladi-Ruelle [1993], Ruelle [1993], Baladi [1994], and Baladi-Ruelle [1994]). The starting point is inspired by the insight of Milnor and Thurston that the itineraries of the endpoints of the intervals of monotonicity (the *turning points*) are the essential invariants. Power series converging in the open unit disc may be constructed from these invariants. Milnor and Thurston constructed a (finite) kneading matrix with coefficients these power series. Their extremely striking result is that the determinant $\Delta(z)$ of this finite matrix coincides with the inverse $1/\zeta^-(z)$ of the unweighted squared negative zeta function which we define in the Appendix. (Note that Milnor and Thurston did not consider transfer operators and their spectrum.)

We sketch the successive steps in the generalisation of this result (we also mention related results by Mori [1990,1992]):

One may first (Baladi-Ruelle [1993]) consider locally constant weights g (where the intervals of constancy may be smaller than the monotonicity laps of f) and define weighted (finite) kneading matrices. In order to recover the formula $\Delta_g(z) = 1/\zeta_g(z)$ one is led to introduce a *weighted Lefschetz dynamical function* $\zeta^L(z)$: instead of counting only periodic points of negative type, one counts *all* periodic points, but with signs $\{-1, 0, 1\}$ (cancellations may occur, in particular when $\operatorname{Fix} f^m$ is infinite) and weighted with g. The proof is essentially the homotopy argument of Milnor and Thurston. Note that for $g \equiv 1$, we have $\zeta^L(z) = \zeta^-(z)$.

Considering again locally constant weights, Ruelle [1993] studied the generalised setup of (1.6), where the contractions are not necessarily the inverse branches of a single map f. This gives the freedom to introduce dual transfer operators, dual zeta functions, and dual kneading matrices. The Milnor-Thurston result on zeta functions and kneading matrices may be proved by a modification of the homotopy argument. Then, using the dual objects one can relate the discrete spectrum of the generalised transfer operator (1.6) with the poles of the Lefschetz weighted dynamical zeta function.

In the functional setting given by a single interval map f, if g is of bounded variation and constant on homtervals of f, it is possible to define (Baladi [1994]) a corresponding Lefschetz zeta function $\zeta_g^L(z)$. If g is continuous at the periodic points of f, the classical Fredholm argument applied to the sequence of finite kneading matrices associated with locally constant approximations of g shows that the determinant of these matrices converge to $1/\zeta_g^L(z)$ in a suitable domain of the complex plane.

Finally, we may consider the fully general situation (Baladi-Ruelle [1994]): the system of contractions ψ_ω does not necessarily come from a single map, and the weights g_ω are only assumed to be of bounded variation. With an additional (probably technical) assumption of continuity and vanishing at endpoints of the weights, a continuous version of the kneading matrix may be defined and related to a weighted Lefschetz dynamical function, whose definition follows from a new trace, which is called the sharp trace $\mathrm{Tr}^{\#}\mathcal{M}$. The remainder of this section is devoted to a precise statement of these results.

The setup is the following: consider $X = \mathbb{R}$ or an open interval, a finite or countable family (ψ_ω) of homeomorphisms of X (let $\epsilon_\omega = +1$ if ψ_ω is increasing and $\epsilon_\omega = -1$ if ψ_ω is decreasing), and a corresponding family $g_\omega : X \to \mathbb{C}$ of continuous functions of bounded variation vanishing at the endpoints of X, and such that $\sum_\omega \mathrm{Var}\, g_\omega < \infty$. Consider two bounded linear operators acting on the Banach space $\mathcal{B}$ of functions ϕ of bounded variation on X:

$$\mathcal{M}\phi(x) = \sum_\omega g_\omega(x)\phi(\psi_\omega(x)), \qquad \widehat{\mathcal{M}}\phi(x) = \sum_\omega \epsilon_\omega g_\omega(\psi_\omega^{-1}x)\phi(\psi_\omega^{-1}(x)). \tag{2.1}$$

We may also let $\mathcal{M}$ and $\widehat{\mathcal{M}}$ act on the space of bounded complex-valued functions on X endowed with the uniform norm $\|\cdot\|_0$, and define

$$R = \lim_{m\to\infty} (\|\mathcal{M}^m\|_0)^{1/m}, \qquad \widehat{R} = \lim_{m\to\infty} (\|\widehat{\mathcal{M}}^m\|_0)^{1/m}. \tag{2.2}$$

To fix ideas, assume that $\widehat{R} \le R$. (In the other case, exchange the roles of $\mathcal{M}$ and $\widehat{\mathcal{M}}$.) Without any continuity assumption on the weights g_ω, one can prove:

Theorem (Ruelle [1993]).

(1) *The spectral radius r of $\mathcal{M}$ acting on $\mathcal{B}$ satisfies $\widehat{R} \le r \le R$.*

(2) *The essential spectral radius r_{ess} of $\mathcal{M}$ acting on $\mathcal{B}$ satisfies $r_{\mathrm{ess}} \le \widehat{R}$.*

(3) *If $g_\omega \ge 0$ for all ω then $r = R$. If furthermore $\widehat{R} < R$, then R is an eigenvalue of $\mathcal{M}$ and there is a corresponding eigenfunction $\varphi_R \ge 0$.*

(Under the two assumptions of (3), one may also, as pointed out in Ruelle [1994b], construct a positive eigenfunctional of a dual operator $\mathcal{M}^*$ acting on bounded measures.)

Since g_ω is of bounded variation, $d(g_\omega(x))$ is a bounded measure, with which we now define the *sharp trace* introduced in Baladi-Ruelle [1994]:

$$\mathrm{Tr}^{\#}\mathcal{M} = \sum_\omega \int d(g_\omega(x))\frac{1}{2}\mathrm{sgn}\,(\psi_\omega(x) - x) \tag{2.3}$$

and the *sharp determinant*

$$\mathrm{Det}^{\#}(1 - z\mathcal{M}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m}\mathrm{Tr}^{\#}\mathcal{M}^m\,. \tag{2.4}$$

(Of course, we set $\mathrm{sgn}\,(x) = \pm 1$ if $\pm x > 0$ and $\mathrm{sgn}\,(x) = 0$ if $x = 0$.) One has $\mathrm{Tr}^{\#}\mathcal{M} = -\mathrm{Tr}^{\#}\widehat{\mathcal{M}}$. Therefore, $\mathrm{Det}^{\#}(1 - z\mathcal{M}) = 1/\mathrm{Det}^{\#}(1 - z\widehat{\mathcal{M}})$. We may define $\zeta^L(z) = 1/\mathrm{Det}^{\#}(1 - z\mathcal{M})$, $\widehat{\zeta}^L(z) = 1/\mathrm{Det}^{\#}(1 - z\widehat{\mathcal{M}})$. The main result can now be stated:

Theorem (Baladi-Ruelle [1994]). *The sharp determinant $\Delta(z) = Det^{\#}(1 - z\mathcal{M})$ is holomorphic in the open disc of radius $\widehat{R}^{-1}$, and vanishes in this disc exactly at the points λ^{-1} where λ is a discrete eigenvalue of $\mathcal{M}$. The order of λ^{-1} as a zero of $\Delta(z)$ coincides with the multiplicity of λ as an eigenvalue of $\mathcal{M}$.*

As mentioned in the abstract, the key to the proof of this theorem is to relate $\Delta(z)$ to the classical Fredholm-Grothendieck determinant of a trace-class operator (in fact one also uses the symmetry $\mathcal{M} \leftrightarrow \widehat{\mathcal{M}}$). This relationship is a continuous version of the Milnor and Thurston kneading theory. More precisely, we introduce kneading operators $\mathcal{D}(z)$ and $\widehat{\mathcal{D}}(z)$ acting on $L^1(d\mu)$, where

$$\mu(dx) = \sum_\omega |dg_\omega(x)| + \sum_\omega |dg_\omega \circ \psi_\omega^{-1}(x)|\,, \tag{2.5}$$

by setting for $|z| < R^{-1}$

$$(\mathcal{D}(z)\varphi)(y) = \sum_\omega \int \varphi(x) d(zg_\omega(x))\big[(1 - z\mathcal{M})^{-1}\frac{1}{2}\mathrm{sgn}\,(\cdot - y)\big](\psi_\omega x)\,, \tag{2.6}$$

and analogously for $\widehat{\mathcal{D}}(z)$ (replacing g_ω, ψ_ω and $\mathcal{M}$ by $\epsilon_\omega g_\omega \circ \psi_\omega^{-1}$, ψ_ω^{-1} and $\widehat{\mathcal{M}}$). One checks that $\mathcal{D}(z)$ and $\widehat{\mathcal{D}}(z)$ are nuclear in the sense of Grothendieck [1956], so that one may consider their Fredholm-Grothendieck determinants. The analogue of the Milnor-Thurston theorem is:

Proposition (Baladi-Ruelle [1994]). *The following identities between power series hold:*

$$Det^{\#}(1 - z\mathcal{M}) = Det\,(1 + \widehat{\mathcal{D}}(z))\,, \qquad Det^{\#}(1 - z\widehat{\mathcal{M}}) = Det\,(1 + \mathcal{D}(z))\,. \tag{2.7}$$

(The proof of the proposition is by resummation of power series.)

We would like to end by mentioning that the analogy with the Milnor-Thurston theory yields a method of approaching the discrete spectrum of transfer operators with zeroes of determinants of *finite* kneading matrices.

Appendix:
Some properties of zeta functions associated with maps in one dimension
(by V. Baladi and D. Ruelle)

In this appendix, we show how results of Baladi-Keller [1990] on weighted zeta functions of piecewise monotone intervals maps for which the partition into intervals of monotonicity is *generating* can be applied to Milnor-Thurston-type "squared negative" weighted zeta functions when the partition is not necessarily generating (the idea, of course, is to collapse homtervals). This gives a different proof of some of the results subsequently obtained by Ruelle by less elementary methods [1994a], yielding in particular a proof of part of Conjecture 1.4 in Ruelle [1991].

Let X be a compact subset of $\mathbb{R}$ (assume $X \subset [0,1]$ to fix ideas) and consider maps $f : X \to X$ for which there are $0 = a_0 < a_1 < \ldots < a_N = 1$ ($2 \le N < \infty$) such that f restricted to each $J_i = [a_i, a_{i+1}] \cap X$ is strictly monotone and satisfies the Darboux property. (Defining a *closed interval* of X to be any set of the form $X \cap [u,v]$, the Darboux property means here simply that $f(J_i)$ is a closed interval of X. This implies that f is continuous.) We call such f *piecewise monotone maps.* We consider weights $g : X \to \mathbb{C}$ of bounded variation, i.e.

$$\operatorname{Var} g := \sup\{\sum_{i=1}^{m} |g(t_i) - g(t_{i-1})| \mid m \ge 2\,, 0 \le t_0 < t_1 < \cdots < t_m \le 1\} < \infty\,.$$

Let $\mathcal{B}$ be the Banach space of functions $\phi : X \to \mathbb{C}$ of bounded variation, with the norm $\|\phi\| = \operatorname{Var}\phi + \sup_{x\in X} |\phi(x)|$, and the linear operator $\mathcal{L} = \mathcal{L}_g$ defined on $\mathcal{B}$ by

$$\mathcal{L}\phi(x) = \sum_{y\,, fy=x} g(y)\phi(y)\,.$$

If the set $\operatorname{Fix} f^m = \{x \in X \mid f^m(x) = x\}$ is finite for all $m \ge 1$, we may define the dynamical zeta function $\zeta(z) = \zeta_g(z)$ by the power series

$$\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x\in \operatorname{Fix} f^m} \prod_{k=0}^{m-1} g(f^k x)\,.$$

With these notations, we start with a slightly strengthened version of the main result in Baladi-Keller [1990]:

Theorem A.1.

(1) *The operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ is bounded, and its essential spectral radius is not larger than*

$$\theta = \theta_g := \lim_{m\to\infty} \sup_{x\in X} \Big(\prod_{k=0}^{m-1} |g(f^k x)|\Big)^{1/m}\,.$$

(2) *Assume that the intervals J_i are disjoint, and that the corresponding partition of X is generating (i.e., each intersection $\cap_{m=0}^{\infty} f^{-m} J_{\ell(m)}$ consists of at most one point). Then, the function $d(z) = 1/\zeta(z)$ is holomorphic for $|z| < \theta^{-1}$, and its zeroes in this disc are precisely the inverses of the eigenvalues λ of $\mathcal{L}$ with $|\lambda| > \theta$. The order of each such zero coincides with the multiplicity of the corresponding eigenvalue.*

Proof. If the weight g is continuous on each J_i the statement is proved in Baladi-Keller [1990] where it was pointed out that the continuity assumption was only used to prove Proposition 1.1 in that article. However, Proposition 1.1 in Baladi-Keller [1990] follows from a result in Ruelle [1991, Proposition 1.3], which did not require piecewise continuity of g:

Proposition. *Let $\mathcal{B}_c$ be the quotient Banach space of $\mathcal{B}$ obtained by identifying functions which differ on an at most countable set, and let $\pi : \mathcal{B} \to \mathcal{B}_c$ be the quotient map. If $\mathcal{L}_c$ is the operator on $\mathcal{B}_c$ such that $\mathcal{L}_c \pi = \pi \mathcal{L}$, then λ with $|\lambda| > \theta$ is an eigenvalue of $\mathcal{L}$ of multiplicity $m \geq 1$ if and only if it is an eigenvalue of $\mathcal{L}_c$ of multiplicity m.* □

We wish to study dynamical zeta functions for general piecewise monotone maps. In order to do this, we first consider the case where the intervals J_i are disjoint (we shall see below how to remove this assumption) and we let $\epsilon(x) = \pm 1$ depending on whether $x \in X$ is in an increasing or decreasing interval of monotonicity for f. Then, for $m \geq 1$ we introduce the set of *negative fixed points* of f^m

$$\mathrm{Fix}^- f^m = \{x \in \mathrm{Fix}\, f^m \mid \prod_{k=0}^{m-1} \epsilon(f^k(x)) = -1\} .$$

(Since intersections of decreasing branches of f^m with the diagonal are transverse, the set $\mathrm{Fix}^- f^m$ is finite.) We now define (following Milnor-Thurston [1977] who considered the unweighted case) a *weighted (squared) negative zeta function*:

$$\zeta^-(z) = \exp 2 \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}^- f^m} \prod_{k=0}^{m-1} g(f^k x) .$$

If some of the intervals J_i meet at their endpoints, the solution is simply to double these endpoints and all their preimages (see Hofbauer-Keller [1982]). More precisely, we fix some small $\delta > 0$ and we construct a new compact subset X_1 of $\mathbb{R}$ by replacing each $x \in X$ such that $f^k(x) \in \{a_1, \ldots, a_{N-1}\}$ for some $k \geq 0$ (chosen minimal with this property) by two points $x_- < x_+$, inserting a interval of length (e.g.) $\delta/k!$ between them. This defines a map $\pi_1 : X_1 \to X$, and there is a piecewise monotone map $f_1 : X_1 \to X_1$ such that $f\pi_1 = \pi_1 f_1$. We set $J_1^1 = [a_0, a_{1-}] \cap X_1$, ..., $J_i^1 = [a_{i-1+}, a_{i-}] \cap X_1$, ..., $J_N^1 = [a_{N-1+}, a_N] \cap X_1$. Setting $g_1 = g \circ \pi_1$, we have $\mathrm{Var}_{X_1} g_1 = \mathrm{Var}_X g$ and $\theta_{g_1} = \theta_g$. We may thus define the (squared) negative g_1-weighted zeta function $\zeta_1^-(z)$ for f_1, and we *define*

$$\zeta^-(z) := \zeta_1^-(z) .$$

(If the intervals J_i were disjoint to begin with, the above construction yields $f_1 = f$, $g_1 = g$, $\zeta_1^-(z) = \zeta^-(z)$.)

We can now state the main result in this Appendix (which partially solves Conjecture 1.4 in Ruelle [1991], see Ruelle [1994a] for further results):

Theorem A.2. *Let $f : X \to X$ be a piecewise monotone map and $g : X \to \mathbb{C}$ be of bounded variation. Then the weighted (squared) negative zeta function $\zeta^-(z)$ for f and g is meromorphic in the open disc defined by $|z| < \theta^{-1}$.*

Proof of Theorem A.2. Since $\zeta^-(z) = \zeta_1^-(z)$ and $\theta_g = \theta_{g_1}$, we may assume that the intervals of monotonicity are disjoint, i.e., work with the system $f_1 : X_1 \to X_1$ constructed above.

If the partition into intervals of monotonicity is generating, then the dynamical function $\zeta_1(z)$ for f_1 and g_1 is well defined, and we may also consider the zeta function $\zeta_1^\epsilon(z)$ for f_1 and ϵg_1. Note that

$$\zeta_1^-(z) = \frac{\zeta_1(z)}{\zeta_1^\epsilon(z)}\,.$$

Since $\theta_{g_1} = \theta_{\epsilon g_1} = \theta$, Theorem A.1 tells us that both zeta functions are meromorphic in the open disc of radius θ^{-1}, which concludes the proof in the case of a generating partition.

In the case where the partition is not necessarily generating, we may construct a piecewise monotone map $f_2 : X_2 \to X_2$ from $f_1 : X_1 \to X_1$ as follows. The compact set X_2 is obtained from X_1 by collapsing to a point every interval $\cap_{m=0}^\infty f_1^{-m} J^1_{\ell(m)}$ which consists of more than one point (there are at most countably many such intervals). This defines a map $\pi_2 : X_1 \to X_2$ and there is a piecewise monotone map $f_2 : X_2 \to X_2$ such that $f_2 \pi_2 = \pi_2 f_1$. We observe that for any $x \in X_2$, the set $\pi_2^{-1}(x) \cap \cup_{m\ge 1} \mathrm{Fix}^-(f_1^m)$ contains at most one element. (Indeed, assume that $\pi_2^{-1}(x) = \cap_{m=0}^\infty f_1^{-m} J^1_{\ell(m,x)}$ is not reduced to a point and contains an f_1-periodic point y of minimal period k and such that f_1^k is decreasing on $\pi_2^{-1}(x)$. It follows that the sequence of intervals $(J^1_{\ell(m,x)})_{m\ge 0}$ must be periodic, that its period p divides k, so that f_1^p is decreasing on $\pi_2^{-1}(x)$. Thus, f_1^p has a fixed point y_0 in $\pi_2^{-1}(x)$. Since y_0 and y are both negative fixed points for f_1^k we must have $y = y_0$.) We define g_2 on X_2 by letting $g_2(x) = g_1(\pi_2^{-1}(x) \cap \cup_{m\ge 1} \mathrm{Fix}^-(f_1^m))$ when $\pi_2^{-1}(x) \cap \cup_{m\ge 1} \mathrm{Fix}^-(f_1^m)$ is not empty, and $g_2(x) = g_1(y)$ for an arbitrarily chosen $y \in \pi_2^{-1}(x)$ otherwise. Clearly, $\mathrm{Var}\, g_2 \le \mathrm{Var}\, g_1$ and $\theta_{g_1} \ge \theta_{g_2}$. We may consider the dynamical zeta functions $\zeta_2(z)$ for f_2 and g_2, and $\zeta_2^\epsilon(z)$ for f_2 and ϵg_2. Obviously, we have

$$\zeta_2^-(z) = \zeta_1^-(z) = \zeta^-(z)\,, \tag{A.3}$$

and

$$\zeta_2^-(z) = \frac{\zeta_2(z)}{\zeta_2^\epsilon(z)}\,. \tag{A.4}$$

Since $\zeta_2(z)$ and $\zeta_2^\epsilon(z)$ are both meromorphic in the disc of radius $\theta_{g_2}^{-1} \ge \theta_{g_1}^{-1}$ by Theorem A.1, the result follows from (A.3-A.4). $\square$

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