

# ON THE MEASURE OF MAXIMAL ENTROPY FOR FINITE HORIZON SINAI BILLIARD MAPS

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ABSTRACT. The Sinai billiard map  $T$  on the two-torus, i.e., the periodic Lorentz gas, is a discontinuous map. Assuming finite horizon, we propose a definition  $h_*$  for the topological entropy of  $T$ . We prove that  $h_*$  is not smaller than the value given by the variational principle, and that it is equal to the definitions of Bowen using spanning or separating sets. Under a mild condition of sparse recurrence to the singularities, we get more: First, using a transfer operator acting on a space of anisotropic distributions, we construct an invariant probability measure  $\mu_*$  of maximal entropy for  $T$  (i.e.,  $h_{\mu_*}(T) = h_*$ ), we show that  $\mu_*$  has full support and is Bernoulli, and we prove that  $\mu_*$  is the unique measure of maximal entropy and it is different from the smooth invariant measure except if all non grazing periodic orbits have multiplier equal to  $h_*$ . Second,  $h_*$  is equal to the Bowen–Pesin–Pitskel topological entropy of the restriction of  $T$  to a non-compact domain of continuity. Last, applying results of Lima and Matheus, as upgraded by Buzzi, the map  $T$  has at least  $Ce^{nh_*}$  periodic points of period  $n$  for all  $n \in \mathbb{N}$ .

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## 1. INTRODUCTION

**1.1. Bowen–Margulis Measures and Measures of Maximal Entropy.** Half a century ago<sup>1</sup>, Margulis [Ma1] proved in his dissertation the following analogue of the prime number theorem for the closed geodesics  $\Gamma$  of a compact manifold of strictly negative (not necessarily constant) curvature: Let  $h > 0$  be the topological entropy of the geodesic flow; then,

$$(1.1) \quad \#\{\Gamma \text{ such that } |\Gamma| \leq L\} \sim_{L \rightarrow \infty} \frac{e^{hL}}{hL}.$$

(I.e.  $\lim_{L \rightarrow \infty} (hLe^{-hL} \#\{\Gamma \text{ such that } |\Gamma| \leq L\}) = 1$ .) The main ingredient in the proof is an invariant probability measure for the flow, the Margulis (or Bowen–Margulis [Bo3]) measure  $\mu_{\text{top}}$ . This measure — which coincides with volume in constant curvature, but not in general — is mixing (thus ergodic), and it can be written as a local product of its stable and unstable conditionals, where these conditional measures scale by  $e^{\pm ht}$  under the action of the flow. These properties were essential to establish (1.1). The measure  $\mu_{\text{top}}$  enjoys other remarkable properties, such as equidistribution of closed geodesics. Finally, the measure  $\mu_{\text{top}}$  is the unique measure of maximal entropy of the flow, that is, the unique invariant measure with Kolmogorov entropy equal to the topological entropy of the flow.

These results were extended to more general smooth uniformly hyperbolic flows and diffeomorphisms, using the thermodynamic formalism of Bowen, Ruelle, and Sinai. In particular Parry–Pollicott [PaP] obtained a different proof of (1.1) using a dynamical zeta function. Later, based on Dolgopyat’s [Do1] groundbreaking thesis (proving exponential mixing for the measure and giving a pole-free vertical strip for a zeta function), exponential error terms were obtained [PS1] for the counting asymptotics (1.1) in the case of surfaces or 1/4 pinched manifolds. Using [Do1, PS1], Stoyanov [St2] obtained exponential error terms for the closed orbits of a class of open planar convex billiards, which are smooth hyperbolic flows on their nonwandering set, a compact (fractal) invariant set. We refer to Sharp’s survey in [Ma2] for more counting results in uniformly hyperbolic dynamics. We just mention here that, for some Axiom A flows with slower (non-exponential)

<sup>1</sup>See [Ma2] for the full english text.

mixing rates, it is possible [PS2] to get (weaker) error terms, of the form  $\frac{e^{hL}}{hL}(1 + O(L^{-\delta}))$ , for the asymptotics (1.1), by exploiting relevant operator bounds from [Do2] (corresponding to a resonance free domain for the transfer operator). This may be relevant for the Sinai billiards considered in the present work, as we do not expect them to mix exponentially fast for the measure of maximal entropy without additional assumptions.

Entropy is a fundamental invariant in dynamics and the study of measures of maximal entropy is a topic in its own right [Ka2]. Let us just mention here the discrete-time analogue of the counting theorem (1.1) which has been established in several situations (see also [Ka1] for more general results): Let  $h > 0$  be the topological entropy of uniformly hyperbolic (Axiom A) diffeomorphism  $T$ , set  $\text{Fix } T^m = \{x : T^m(x) = x\}$ , then Bowen showed [Bo1] that  $\lim_{m \rightarrow \infty} \frac{1}{m} \log \#\text{Fix } T^m = h$ . In fact [Bo4], there is a constant  $C > 0$  so that

$$(1.2) \quad Ce^{hm} \leq \#\text{Fix } T^m \leq C^{-1}e^{hm}, \quad \forall m \geq 1.$$

Uniqueness of the measure of maximal entropy has been extended to some geodesic flows in non-positive curvature (i.e. weakening the hyperbolicity requirement). The breakthrough result of Knieper [Kn] for compact rank 1 manifolds has been recently given a new dynamical proof [B-T] (using Bowen's ideas as revisited by Climenhaga and Thompson). This is currently a very active topic, see e.g. [CKW].

The present paper is devoted to the study of the measure of maximal entropy in a situation where uniform hyperbolicity holds, but the dynamics is not smooth: The singular set  $\mathcal{S}_{\pm 1}$ , i.e. those points where the map  $T$  (or the flow  $\Phi$ ) or its inverse are not  $C^1$ , is not empty. In this setting, the following integrability condition is crucial:

$$(1.3) \quad \int |\log d(x, \mathcal{S}_{\pm 1})| d\mu_{\text{top}} < \infty.$$

Following Lima–Matheus [LM], we shall say that a measure  $\mu$  satisfying the above integrability condition for a map  $T$  is  $T$ -adapted.

Condition (1.3) is prevalent in the rich literature about measures of maximal entropy for meromorphic maps of a compact Kähler manifold (see the survey [Fr], and e.g. [DDG2] and references therein) such as birational mappings. In this work, we are concerned with a different class of dynamics with singularities: the dispersing billiards introduced by Sinai [S] on the two-torus. A Sinai billiard on the torus is the periodic case of the planar Lorentz gas (1905) model for the motion of a single dilute electron in a metal. The scatterers (corresponding to the atoms of the metal) are assumed to be strictly convex, but they are not necessarily perfect discs. Such billiards have become foundational models in mathematical physics.

The Sinai billiard flow is continuous, but<sup>2</sup> not differentiable: the “grazing” orbits (those which are tangent to a scatterer) lead to singularities. Nevertheless, existence of a measure of maximal entropy for the billiard flow is granted, thanks to hyperbolicity. The topological entropy has been studied for the billiard flow [BFK]. However, uniqueness of the measure of maximal entropy, as well as mixing and the adapted condition (1.3) are not known. Since the transfer operator techniques we use are simpler to implement in the discrete-time case, we study in this paper the Sinai billiard map, which is the return map of the single point particle to the scatterers.

Sinai billiard maps preserve a smooth invariant measure  $\mu_{\text{SRB}}$  which has been studied extensively: With respect to  $\mu_{\text{SRB}}$ , the billiard is uniformly hyperbolic, ergodic, K-mixing and Bernoulli [S, GO, SC, ChH]. The measure  $\mu_{\text{SRB}}$  is  $T$ -adapted [KS]. Moreover, this measure enjoys exponential decay of correlations [Y] and a host of other limit theorems (see e.g. [CM, Chapter 7] or [DZ1]). The billiard has many periodic orbits and thus many other ergodic invariant measures  $\mu$ , but there are

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<sup>2</sup>In contrast, open billiards in the plane which satisfy a non-eclipsing condition do not have any singularities on their nonwandering set, so that they fit in the Axiom A category [St2].

very few results regarding other invariant measures and they apply only to perturbations of  $\mu_{\text{SRB}}$  [CWZ, DRZ]. Since the billiard map is discontinuous, the standard results [W] guaranteeing that the supremum of Kolmogorov entropy is attained and coincides with the topological entropy, do not hold. It is natural to ask whether a measure of maximal entropy exists, and, in the affirmative, whether it is unique, ergodic, and mixing.

Another natural goal is to establish (1.2). Chernov asked (see [Gu, Problems 5 and 6]) whether a slightly weaker property than (1.2), namely  $\lim_{m \rightarrow \infty} \frac{1}{m} \log \#\text{Fix } T^m = h_{\text{top}}$ , holds. (Chernov [Ch1] showed that  $\liminf_{m \rightarrow \infty} \frac{1}{m} \log \#\text{Fix } T^m \geq h_{\mu_{\text{SRB}}}$ . For a related class of billiards, Stoyanov [St1] found finite constants  $C$  and  $H$  so that  $\#\text{Fix } T^m \leq Ce^{Hm}$  for all  $m \geq 1$ .)

A detailed knowledge of the measure of maximal entropy, and the techniques developed to obtain this information, could potentially allow us not only to establish (1.2) for the billiard map, but also eventually to prove a prime number asymptotic of the form (1.1) for the billiard flow. Although lifting a measure of maximal entropy for the map should not directly give a measure of maximal entropy for the flow, we believe that the techniques and results of the present paper will be instrumental in understanding the measure of maximal entropy of the billiard flow.

We list our results in Section 1.2. In a nutshell, for all finite horizon planar Sinai billiards  $T$  satisfying a (mild) condition of “sparse recurrence” to the singular set, we construct a measure of maximal entropy, we show that it is unique, mixing (even Bernoulli), that it has full support, and that it is  $T$ -adapted. Our results combined with those of Lima–Matheus [LM] and a very recent preprint of Buzzi [Bu] give  $C > 0$  such that the lower bound in (1.2) holds.

Finally, we mention that our technique for constructing and studying the invariant measure, which uses transfer operators but avoids coding, is reminiscent both of the construction of Margulis [Ma2] and the techniques of “laminar currents” introduced by Dujardin for birational mappings [Du] (see also [DDG2]).

**1.2. Summary of Main Results.** A Sinai billiard table  $Q$  on the two-torus  $\mathbb{T}^2$  is a set  $Q = \mathbb{T}^2 \setminus B$ , with  $B = \cup_{i=1}^D B_i$  for some finite number  $D \geq 1$  of pairwise disjoint closed domains  $B_i$  with  $C^3$  boundaries having strictly positive curvature (in particular, the domains are strictly convex). The sets  $B_i$  are called scatterers; see Figure 2 for some common examples. The billiard flow is the motion of a point particle traveling in  $Q$  at unit speed and undergoing elastic (i.e., specular) reflections at the boundary of the scatterers. (By definition, at a tangential — also called grazing — collision, the reflection does not change the direction of the particle.) This is also called a periodic Lorentz gas. As mentioned above, a key feature is that, although the billiard flow is continuous if one identifies outgoing and incoming angles, the tangential collisions give rise to singularities in the derivative [CM].

We shall be concerned with the associated billiard map  $T$ , defined to be the first collision map on the boundary of  $Q$ . Grazing collisions cause discontinuities in the billiard map  $T : M \rightarrow M$ . We assume, as in [Y], that the billiard table  $Q$  has *finite horizon* in the sense that the billiard flow on  $Q$  does not have any trajectories making only tangential collisions.

The first step is to find a suitable notion of topological entropy  $h_*$  for the discontinuous map  $T$ .

Let  $M' \subset M$  be the ( $T$ -invariant but not compact) set of points whose future and past orbits are never grazing. By definition,  $T$  is continuous on  $M'$ . The (Bowen–Pesin–Pitskel) topological entropy  $h_{\text{top}}(F|_Z)$  can be defined for a map  $F$  on a non-compact set of continuity  $Z$  (see e.g. [Bo2] and [Pes, §11 and App. II]). Chernov [Ch1] studied the topological entropy for a class of billiard maps including those of the present paper. In particular, he gave [Ch1, Thm 2.2] a countable symbolic dynamics description of two  $T$ -invariant subsets of  $M'$  of full Lebesgue measure in  $M'$ , expressing their topological entropy in terms of those of the associated Markov chains. The entropies found there are both bounded above by  $h_{\text{top}}(T|_{M'})$ , although Chernov does not prove their equality.

These existing results are not convenient for our purposes, however, since we have no control a priori on the measure of  $M \setminus M'$ . This is why we introduce (Definition 2.1) an ad hoc definition  $h_*$  of the topological entropy for the billiard map  $T$  on the compact set  $M$ .

Our first main result (Theorem 2.3) says that the topological entropies of  $T$  defined by spanning sets and separating sets coincide with the topological entropy  $h_*$ , that  $h_*$  can also be obtained by using the refinements of partitions of  $M$  into maximal connected components on which  $T$  and  $T^{-1}$  are continuous, and that  $h_* \geq \sup\{h_\mu(T) : \mu \text{ is a } T\text{-invariant Borel probability measure on } M\}$ .

To state our other main results, we need to quantify the recurrence to the singular set: Fix an angle  $\varphi_0$  close to  $\pi/2$  and  $n_0 \in \mathbb{N}$ . We say that a collision is  $\varphi_0$ -grazing if its angle with the normal is larger than  $\varphi_0$  in absolute value. Let  $s_0 \in (0, 1]$  be the smallest number such that

$$(1.4) \quad \text{any orbit of length } n_0 \text{ has at most } s_0 n_0 \text{ collisions which are } \varphi_0\text{-grazing.}$$

Our sparse recurrence condition is

$$(1.5) \quad \text{there exist } n_0 \text{ and } \varphi_0 \text{ such that } h_* > s_0 \log 2.$$

(Due to the finite horizon condition, we can choose  $\varphi_0$  and  $n_0$  such that  $s_0 < 1$ . We refer to §2.4 for further discussion of the condition.)

Assuming (1.5), our second main result (Theorem 2.4) is that  $T$  admits a unique invariant Borel probability measure  $\mu_*$  of maximal entropy  $h_* = h_{\mu_*}(T)$ . In addition,  $\mu_*(O) > 0$  for any open set and  $\mu_*$  is<sup>3</sup> Bernoulli. Finally, the absolutely continuous invariant measure  $\mu_{\text{SRB}}$  may coincide with  $\mu_*$  *only* if all non grazing periodic orbits have the same Lyapunov exponent, equal to  $h_*$ . (No dispersing billiards which satisfy this condition are known. See also Remark 1.2.)

Our third result is (Theorem 2.5) that  $h_*$  coincides with the Bowen–Pesin–Pitskel entropy  $h_{\text{top}}(T|_{M'})$  (still assuming (1.5)).

Next, Theorem 2.6 contains a key technical<sup>4</sup> estimate on the measures of neighbourhoods of singularity sets, (2.2), used to prove Theorems 2.4 and 2.5 under the assumption (1.5). Theorem 2.6 also states that  $\mu_*$  has no atoms, that it gives zero mass to any stable or unstable manifold and any singularity set, that  $\mu_*$  is  $T$ -adapted (in the sense of (1.3)), and that  $\mu_*$ -almost every  $x \in M$  has stable and unstable manifolds of positive lengths.

Finally, we obtain a lower bound  $\#\text{Fix } T^m \geq C e^{h_* m}$  on the cardinality of the set of periodic orbits (Corollary 2.7 and the comments thereafter) whenever (1.5) holds.

**1.3. The Transfer Operator — Organisation of the Paper.** Our tool to construct the measure of maximal entropy is a transfer operator  $\mathcal{L} = \mathcal{L}_{\text{top}}$  with  $\mathcal{L}f = \frac{f \circ T^{-1}}{J^s T \circ T^{-1}}$  analogous to the transfer operator  $\mathcal{L}_{\text{SRB}}f = (f/|\text{Det } DT|) \circ T^{-1}$  which has proved very successful [DZ1] to study the measure  $\mu_{\text{SRB}}$ . An important difference is that our transfer operator,  $\mathcal{L}f$ , is weighted by an unbounded<sup>5</sup> function ( $1/J^s T$ , where the stable Jacobian may tend to zero near grazing orbits). Using “exact” stable leaves instead of admissible approximate stable leaves will allow us to get rid of the Jacobian after a leafwise change of variables — the same change of variables in [DZ1] for the transfer operator  $\mathcal{L}_{\text{SRB}}$  associated with  $\mu_{\text{SRB}}$  left them with  $J^s T$ , allowing countable sums over homogeneity layers to control distortion, and thus working with a Banach space giving a spectral gap and exponential mixing. In the present work, we relinquish the homogeneity layers to avoid unbounded sums (see e.g. the logarithm needed to obtain the growth Lemma 5.1) and obtain a bounded operator, with spectral radius  $e^{h_*}$ . The price to pay is that we do not have the distortion control needed for Hölder type moduli of continuity in the Banach norms of our weak and strong spaces  $\mathcal{B} \subset \mathcal{B}_w$ . The

<sup>3</sup>Recall that Bernoulli implies K-mixing, which implies strong mixing, which implies ergodic. In practice, we first show K-mixing and then bootstrap to Bernoulli.

<sup>4</sup>This estimate implies that almost every point approaches the singularity sets more slowly than any exponential rate (7.9), see e.g. [LM] for an application of such rates of approach.

<sup>5</sup>The naive idea to introduce a bounded cutoff in the weight does not seem to work.

weaker modulus of continuity than in [DZ1] does not yield a spectral gap. We thus do not claim exponential mixing properties for the measure of maximal entropy  $\mu_*$  constructed (in the spirit of the work of Gouëzel–Liverani [GL] for Axiom A diffeomorphisms) by combining right and left maximal eigenvectors  $\mathcal{L}\nu = e^{h_*}\nu$  and  $\mathcal{L}^*\tilde{\nu} = e^{h_*}\tilde{\nu}$  of the transfer operator.

The paper is organised as follows: In Section 2, we give formal statements of our main results. Section 3 contains the proof of Theorem 2.3 about equivalent formulations of  $h_*$ . In Section 4, we define our Banach spaces  $\mathcal{B}$  and  $\mathcal{B}_w$  of anisotropic distributions, and we state the “Lasota–Yorke” type estimates on our transfer operator  $\mathcal{L}$ . Section 5 contains key combinatorial growth lemmas, controlling the growth in complexity of the iterates of a stable curve. It also contains the definition of Cantor rectangles (Section 5.3.) We next prove the “Lasota–Yorke” Proposition 4.7, the compact embedding of  $\mathcal{B}$  in  $\mathcal{B}_w$ , and show that the spectral radius of  $\mathcal{L}$  is equal to  $e^{h_*}$  in Section 6. The invariant probability measure  $\mu_*$  is constructed in Section 7.1 by combining a right and left eigenvector ( $\nu$  and  $\tilde{\nu}$ ) of  $\mathcal{L}$ . Section 7.1 contains the proof of Theorem 2.6 about the measure of singular sets. Section 7.3 contains a key result of absolute continuity of the unstable foliation with respect to  $\mu_*$  as well as the proof that  $\mu_*$  has full support, exploiting  $\nu$ -almost everywhere positive length of unstable manifolds from Section 7.2. We establish upper and lower bounds on the  $\mu_*$ -measure of dynamical Bowen balls in Section 7.4, deducing from them a necessary condition for  $\mu_{\text{SRB}}$  and  $\mu_*$  to coincide. Using the absolute continuity from Section 7.3, we show in Section 7.5 that  $\mu_*$  is K-mixing. In this section we also use the upper bounds on Bowen balls to see that  $\mu_*$  is a measure of maximal entropy and prove the Bowen–Pesin–Pitskel Theorem 2.5. We deduce the Bernoulli property from K-mixing and hyperbolicity in Section 7.6, adapting<sup>6</sup> [ChH]. Finally, we show uniqueness in Section 7.7.

Our Hopf-argument proof of K-mixing requires showing absolute continuity of the unstable foliation for  $\mu_*$ , a new result of independent interest, which is the content of Corollary 7.9. The “fragmentation” lemmas from Section 5, needed to get the lower bound on the spectral radius of the transfer operator, are also new. They imply, in particular, that the length  $|T^{-n}W|$  of every local stable manifold  $W$  grows at the same exponential rate  $e^{nh_*}$  (Corollary 5.10).

We conclude this introduction with two remarks on the finite horizon condition.

**Remark 1.1** (Finite Horizon and Collision Time  $\tau$ ). *For  $x \in M$ , let  $\tau(x)$  denote the distance from  $x$  to  $T(x)$ . If  $\tau$  is unbounded, i.e., if there is a collision-free trajectory for the flow, then there must be a flow trajectory making only tangential collisions. The reverse implication, however, is not true. Our<sup>7</sup> finite horizon assumption therefore implies that  $\tau$  is bounded on  $M$ . Assuming only that  $\tau$  is bounded is sometimes also called finite horizon [CM]. (If the scatterers  $B_i$  are viewed as open, then tangential collisions simply do not occur and the two definitions of finite horizon are reconciled.)*

**Remark 1.2** (Billiard with Infinite Horizon). *Chernov [Ch1, §3.4] proved that the topological entropy of the Sinai billiard map  $T$  restricted to the non compact set  $M'$  is infinite if the horizon is not finite, and together with Troubetskoy [CT] constructed invariant measures with infinite metric entropy for this map. Since the entropy of the smooth measure  $\mu_{\text{SRB}}$  is finite, the measure  $\mu_{\text{SRB}}$  does not maximise entropy for infinite horizon billiards. Chernov conjectured [Ch1, Remark 3.3] that this property holds for more general billiards, in particular for Sinai billiards with finite horizon.*

## 2. FULL STATEMENT OF MAIN RESULTS

In this section, we formulate definitions of topological entropy for the billiard map that we shall prove are equivalent before stating formally all main results of this paper.

<sup>6</sup>As pointed out to us by Y. Lima, we could instead apply [Sa1, Thm 3.1] to the lift of  $\mu_*$  to the symbolic space constructed in [LM].

<sup>7</sup>We shall need the slightly stronger version e.g. in Lemmas 3.4 and 3.5.

**2.1. Definitions of Topological Entropy  $h_*$  of  $T$  on  $M$ .** We first introduce notation: Adopting the standard coordinates  $x = (r, \varphi)$ , for  $T$ , where  $r$  denotes arclength along  $\partial B_i$  and  $\varphi$  is the angle the post-collision trajectory makes with the normal to  $\partial B_i$ , the phase space of the map is the compact metric space  $M$  given by the disjoint union of cylinders,

$$M := \partial Q \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = \bigcup_{i=1}^D \partial B_i \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

We denote each connected component of  $M$  by  $M_i = \partial B_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ . In the coordinates  $(r, \varphi)$ , the billiard map  $T : M \rightarrow M$  preserves [CM, §2.12] the smooth invariant measure<sup>8</sup> defined by  $\mu_{\text{SRB}} = (2|\partial Q|)^{-1} \cos \varphi dr d\varphi$ .

We discuss next the discontinuity set of  $T$ : Letting  $\mathcal{S}_0 = \{(r, \varphi) \in M : \varphi = \pm\pi/2\}$  denote the set of tangential collisions, then for each nonzero  $n \in \mathbb{N}$ , the set

$$\mathcal{S}_{\pm n} = \bigcup_{i=0}^n T^{\mp i} \mathcal{S}_0$$

is the singularity set for  $T^{\pm n}$ . In this notation, the  $T$ -invariant (non compact) set  $M'$  of continuity of  $T$  is  $M' = M \setminus \bigcup_{n \in \mathbb{Z}} \mathcal{S}_n$ .

For  $k, n \geq 0$ , let  $\mathcal{M}_{-k}^n$  denote the partition of  $M \setminus (\mathcal{S}_{-k} \cup \mathcal{S}_n)$  into its maximal connected components. Note that all elements of  $\mathcal{M}_{-k}^n$  are open sets. The cardinality of the sets  $\mathcal{M}_0^n$  will play a key role in the estimates on the transfer operator in Section 4. We formulate the following definition with the idea that the growth rate of elements in  $\mathcal{M}_{-k}^n$  should define the topological entropy of  $T$ , by analogy with the definition using a generating open cover (for continuous maps on compact spaces).

**Definition 2.1.**  $h_* = h_*(T) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \#\mathcal{M}_0^n$ .

The fact that the limsup defining  $h_*$  is a limit, as well as several equivalent characterizations involving the cardinality of related dynamical partitions or a variational principle, are proved in Theorem 2.3 (see Lemma 3.3).

**Remark 2.2** ( $h_*(T) = h_*(T^{-1})$ ). *If  $A \in \mathcal{M}_0^n$ , then  $T^n A \in \mathcal{M}_{-n}^0$  since  $T^n \mathcal{S}_n = \mathcal{S}_{-n}$ . Thus  $\#\mathcal{M}_0^n = \#\mathcal{M}_{-n}^0$ , and so  $h_*(T) = h_*(T^{-1})$ .*

It will be convenient to express  $h_*$  in terms of the rate of growth of the cardinality of the refinements of a fixed partition, i.e.,  $\bigvee_0^n T^{-i} \mathcal{P}$ , for some fixed  $\mathcal{P}$ . Although  $\mathcal{M}_0^n$  is not immediately of this form, we will show that in fact  $h_*$  can be expressed in this fashion, obtaining along the way subadditivity of  $\log \#\mathcal{M}_0^n$ . For this, we introduce two sequences of partitions. Let  $\mathcal{P}$  denote the partition of  $M$  into maximal connected sets on which  $T$  and  $T^{-1}$  are continuous. Define  $\mathcal{P}_{-k}^n = \bigvee_{i=-k}^n T^{-i} \mathcal{P}$ . Then,  $n \mapsto \log \#\mathcal{P}_{-k}^n$  is subadditive for any fixed  $k$ , in particular the limit  $\lim_{n \rightarrow \infty} \frac{1}{n} \log \#\mathcal{P}_0^n$  exists.

The interior of each element of  $\mathcal{P}$  corresponds to precisely one element of  $\mathcal{M}_{-1}^1$ ; however, its refinements  $\mathcal{P}_{-k}^n$  may also contain some isolated points if three or more scatterers have a common tangential trajectory. Figure 1 displays two such examples (the pictures are local: we have not represented all discs needed to ensure finite horizon).

Let now  $\mathring{\mathcal{P}}_{-k}^n$  denote the collection of interiors of elements of  $\mathcal{P}_{-k}^n$ . Then  $\mathring{\mathcal{P}}_{-k}^n$  forms a finite partition of  $M$ , while  $\mathring{\mathcal{P}}_{-k}^n$  forms a partition of  $M \setminus (\mathcal{S}_{-k-1} \cup \mathcal{S}_{n+1})$  into open, connected sets. (We will show in Lemma 3.3 that  $\mathring{\mathcal{P}}_{-k}^n = \mathcal{M}_{-k-1}^{n+1}$ .)

Finally, we recall the classical Bowen [W] definitions of topological entropy for continuous maps using  $\varepsilon$ -separated and  $\varepsilon$ -spanning sets. Define the dynamical distance

$$(2.1) \quad d_n(x, y) := \max_{0 \leq i \leq n} d(T^i x, T^i y),$$

<sup>8</sup>All measures in this work are finite Borel measures.

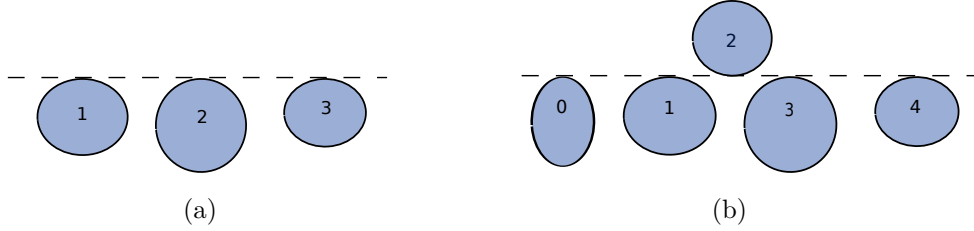


FIGURE 1. (a) The billiard trajectory corresponding to the dotted line has symbolic itinerary 123, but is an isolated point in  $\mathcal{P}_0^1$ . Any open set with symbolic itinerary 12 cannot land on scatterer 3 (unless it first wraps around the torus). (b) The billiard trajectory corresponding to the dotted line and having symbolic trajectory 1234 is not isolated since it belongs to the boundary of an open set with the same symbolic sequence; however, the addition of scatterer 0 on the common tangency forces the point with symbolic trajectory 01234 to be isolated.

where  $d(x, y)$  is the Euclidean metric on each  $M_i$ , and  $d(x, y) = 10D \cdot \max_i \text{diam}(M_i)$  if  $x$  and  $y$  belong to different  $M_i$  (this definition ensures we get a compact set), where  $D$  is the number of scatterers.

As usual, given  $\varepsilon > 0$ ,  $n \in \mathbb{N}$ , we call  $E$  an  $(n, \varepsilon)$ -separated set if for all  $x, y \in E$  such that  $x \neq y$ , we have  $d_n(x, y) > \varepsilon$ . We call  $F$  an  $(n, \varepsilon)$ -spanning set if for all  $x \in M$ , there exists  $y \in F$  such that  $d_n(x, y) \leq \varepsilon$ .

Let  $r_n(\varepsilon)$  denote the maximal cardinality of any  $(n, \varepsilon)$ -separated set, and let  $s_n(\varepsilon)$  denote the minimal cardinality of any  $(n, \varepsilon)$ -spanning set. We recall two related quantities:

$$h_{\text{sep}} = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(\varepsilon), \quad h_{\text{span}} = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\varepsilon).$$

Although  $\lim_{n \rightarrow \infty} \frac{1}{n} \log \#\mathcal{P}_0^n$ ,  $h_{\text{sep}}$ , and  $h_{\text{span}}$  are typically used for continuous maps, our first main result is that these naively defined quantities for the discontinuous billiard map  $T$  all agree with  $h_*$ , and they give an upper bound for the Kolmogorov entropy:

**Theorem 2.3** (Topological Entropy of the Billiard). *The limsup in Definition 2.1 is a limit, and in fact the sequence  $\log \#\mathcal{M}_0^n$  is subadditive. In addition, we have:*

- (1)  $h_* = \lim_{n \rightarrow \infty} \frac{1}{n} \log \#\mathcal{P}_0^n$ ;
- (2) the sequence  $\frac{1}{n} \log \#\hat{\mathcal{P}}_0^n$  also converges to  $h_*$  as  $n \rightarrow \infty$ ;
- (3)  $h_* = h_{\text{sep}}$  and  $h_* = h_{\text{span}}$ ;
- (4)  $h_* \geq \sup\{h_\mu(T) : \mu \text{ is a } T\text{-invariant Borel probability measure on } M\}$ .

The above theorem will follow from Lemmas 3.3, 3.4, 3.5, and 3.6.

(We shall obtain in Lemma 5.6 a superadditive property for  $\log \#\mathcal{M}_0^n$ .)

**2.2. The Measure  $\mu_*$  of Maximal Entropy.** Our next main result, existence and the Bernoulli property of a unique measure of maximal entropy, will be proved in Section 7, using the transfer operator  $\mathcal{L}$  studied in Section 4.

**Theorem 2.4** (Measure of Maximal Entropy for the Billiard). *If  $h_* > s_0 \log 2$  then*

$$h_* = \max\{h_\mu(T) : \mu \text{ is a } T\text{-invariant Borel probability measure on } M\}.$$

Moreover, there exists a unique  $T$ -invariant Borel probability measure  $\mu_*$  such that  $h_* = h_{\mu_*}(T)$ . In addition,  $\mu_*$  is Bernoulli and  $\mu_*(O) > 0$  for all open sets  $O$ . Finally, if there exists a non grazing periodic point  $x$  of period  $p$  such that  $\frac{1}{p} \log |\det(DT^{-p}|_{E^s(x)})| \neq h_*$  then  $\mu_* \neq \mu_{\text{SRB}}$ .

The above theorem follows from Propositions 7.11, 7.13, and 7.19, Corollary 7.17, and Proposition 7.21. (J. De Simoi has told us that [DKL, §4.4] the (possibly empty) set of planar billiard tables



satisfying a non-eclipsing condition (i.e., open billiards) for which  $\frac{1}{p} \log |\det(DT^{-p}|_{E^s(x)})| = h_*$  for all  $p$  and all non-grazing  $p$ -periodic points  $x$  has infinite codimension.)

The existence of  $\mu_*$  with  $h_{\mu_*}(T) = h_*$ , together with item (1) of Theorem 2.3 expressing  $h_*$  as a limit involving the refinements of a single partition, will allow us to interpret  $h_*$  as the Bowen–Pesin–Pitskel topological entropy of  $T|_{M'}$  in Section 7.5:

**Theorem 2.5** ( $h_*$  and Bowen–Pesin–Pitskel Entropy). *If  $h_* > s_0 \log 2$  then  $h_* = h_{\text{top}}(T|_{M'})$ .*

**2.3. A Key Estimate on Neighbourhood of Singularities.** We call a smooth curve in  $M$  a *stable curve* if its tangent vector at each point lies in the stable cone, and define an *unstable curve* similarly. As mentioned in Section 1, the sets  $\mathcal{S}_n$  are the singularity sets for  $T^n$ ,  $n \in \mathbb{Z} \setminus \{0\}$ . The set  $\mathcal{S}_n \setminus \mathcal{S}_0$  comprises [CM] a finite union of stable curves for  $n > 0$  and a finite union of unstable curves for  $n < 0$ . For any  $\epsilon > 0$  and any set  $A \subset M$ , we denote by  $\mathcal{N}_\epsilon(A) = \{x \in M \mid d(x, A) < \epsilon\}$  the  $\epsilon$ -neighbourhood of  $A$ .

The following key result gives information on the measure of neighbourhoods of the singularity sets (it is used in the proofs of Theorem 2.4 and, indirectly, Theorem 2.5).

**Theorem 2.6** (Measure of Neighbourhoods of Singularity Sets). *Assume that  $h_* > s_0 \log 2$  (where  $s_0$  is defined in (1.4)) and let  $\mu_*$  be the ergodic measure of maximal entropy constructed in (7.1). The measure  $\mu_*$  has no atoms, and for any local stable or unstable manifold  $W$  we have  $\mu_*(W) = 0$ . In addition  $\mu_*(\mathcal{S}_n) = 0$  for any  $n \in \mathbb{Z}$ .*

*More precisely, for any  $\gamma > 0$  so that  $2^{s_0\gamma} < e^{h_*}$  and  $n \in \mathbb{Z}$ , there exist  $C$  and  $\hat{C}_n < \infty$  such that for all  $\epsilon > 0$  and any smooth curve  $S$  uniformly transverse to the stable cone,*

$$(2.2) \quad \mu_*(\mathcal{N}_\epsilon(S)) < \frac{C}{|\log \epsilon|^\gamma}, \quad \mu_*(\mathcal{N}_\epsilon(\mathcal{S}_n)) < \frac{\hat{C}_n}{|\log \epsilon|^\gamma}.$$

*Since  $h_* > s_0 \log 2$  we may take  $\gamma > 1$ , and we have*

$$\int |\log d(x, \mathcal{S}_{\pm 1})| d\mu_* < \infty,$$

*(i.e.,  $\mu_*$  is  $T$ -adapted [LM]), and  $\mu_*$ -almost every  $x \in M$  has stable and unstable manifolds of positive length.*

Theorem 2.6 follows from Lemma 7.3 and Corollary 7.4.

This theorem is especially of interest for  $\gamma > 1$ , since in this case it implies that  $\mu_*$ -almost every point does not approach the singularity sets faster than some exponential, see (7.9). In addition, it allows us to give a lower bound on the number of periodic orbits: For  $m \geq 1$ , let  $\text{Fix } T^m$  denote the set  $\{x \in M \mid T^m(x) = x\}$ . By [BSC] and [Ch1, Cor 2.4], there exist  $h_C \geq h_{\mu_{\text{SRB}}}(T) > 0$  and  $C > 0$  with  $\#\text{Fix } T^m \geq Ce^{h_C m}$  for all  $m$ . Our result is that (possibly up to a period  $p$ ) we can take  $h_C = h_*$  if  $h_* > s_0 \log 2$ :

**Corollary 2.7** (Counting Periodic Orbits). *If  $h_* > s_0 \log 2$  then there exist  $C > 0$  and  $p \geq 1$  such that  $\#\text{Fix } T^{pm} \geq Ce^{h_* pm}$  for all  $m \geq 1$ .*

*Proof.* The corollary follows from the work of Lima–Matheus [LM], which in turn relies on work of Gurevič [G1, G2] (see the proof of [Sa2, Thm 1.1]). We recall briefly the setup of [LM, Theorem 1.3]: Under assumptions (A1)–(A6), the authors construct for any  $T$ -adapted measure  $\mu$  with positive Lyapunov exponent, a countable Markov partition that allows them to code a full  $\mu$ -measure set of points. Once this partition has been constructed, [LM, Corollary 1.2] implies the above lower bound on periodic orbits for  $T$  with rate given by  $h_\mu(T)$ .

[LM, Theorem 1.3] applies to our measure of maximal entropy  $\mu_*$  since it is  $T$ -adapted with positive Lyapunov exponent. In addition, conditions (A1)–(A4) of [LM] are requirements on the smoothness of the exponential map on the manifold, which are trivially satisfied in our setting since

$M$  is a finite union of cylinders and  $\mathcal{S}_{\pm 1}$  is a finite union of curves. Finally, conditions (A5) and (A6) are requirements on the rate at which  $\|DT\|$  and  $\|D^2T\|$  grow as one approaches  $\mathcal{S}_1$ . These are standard estimates for billiards and in the notation of [LM], if we choose  $a = 2$ , then conditions (A5) and (A6) hold, choosing there  $\beta = 1/4$  and any  $b > 1$ .  $\square$

After the first version of our paper was submitted, J. Buzzzi [Bu, v2] obtained results allowing one to bootstrap from Corollary 2.7 by exploiting the fact that  $T$  is topologically mixing, to show that if  $h_* > s_0 \log 2$  then there exists  $C > 0$  so that  $\#\text{Fix } T^m \geq Ce^{h_* m}$  for all  $m \geq 1$  [Bu, Theorem 1.5].

**2.4. On Condition (1.5) of Sparse Recurrence to Singularities.** We are not aware of any dispersing billiard on the torus for which the bound  $h_* > s_0 \log 2$  from (1.5) fails. Let us start by mentioning that if there are no triple tangencies on the table — a generic condition — then  $s_0 \leq 2/3$ . To discuss this condition further, our starting point is claim (4) of Theorem 2.3, which implies by the Pesin entropy formula [KS],

$$(2.3) \quad h_* \geq h_{\mu_{\text{SRB}}}(T) = \int \log J^u T \, d\mu_{\text{SRB}}.$$

Thus it suffices to check  $\chi_{\mu_{\text{SRB}}}^+ > s_0 \log 2$  in order to verify (1.5), where  $\chi_{\mu_{\text{SRB}}}^+ = \int \log J^u T \, d\mu_{\text{SRB}}$  is the Lyapunov exponent of  $\mu_{\text{SRB}}$ . (See footnote 11 for another sufficient condition involving the minimum free flight time and the minimum curvature.)

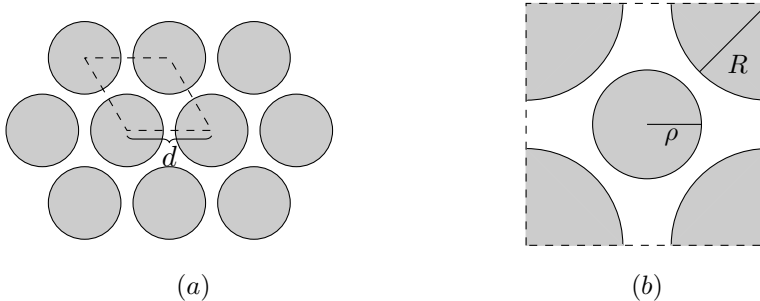


FIGURE 2. (a) The Sinai billiard on a triangular lattice studied in [BG] with angle  $\pi/3$ , scatterer of radius 1, and distance  $d$  between the centers of adjacent scatterers. (b) The Sinai billiard on a square lattice with scatterers of radii  $\rho < R$  studied in [Ga]. The boundary of a single cell is indicated by dashed lines in both tables.

Firstly, we mention two numerical case studies from the literature:

Baras and Gaspard [BG] studied the Sinai billiard corresponding to the periodic Lorentz gas with disks of radius 1 centered in a triangular lattice (Figure 2(a)). The distance  $d$  between points on the lattice is varied from  $d = 2$  (when the scatterers touch) to  $d = 4/\sqrt{3}$  (when the horizon becomes infinite). All computed values of the Lyapunov exponent<sup>9</sup> are greater than  $\frac{2}{3} \log 2$  [BG, Table 1]. (Notably  $\chi_{\mu_{\text{SRB}}}^+$  does not decay as the minimum free flight-time  $\tau_{\min}$  tends to zero.) For these billiard tables, since every segment with a double tangency is followed by *two* non-tangential collisions, one can choose  $\varphi_0$  and  $n_0$  so that (1.4) is satisfied with  $s_0 = 1/2$ . Thus (1.5) holds for all computed values in this family of tables.

Garrido [Ga] studied the Sinai billiard corresponding to the periodic Lorentz gas with two scatterers of radii  $\rho < R$  on the unit square lattice (Figure 2(b)). Setting  $R = 0.4$ , [Ga, Figure 6]

<sup>9</sup>The reported values in [BG] are for the billiard flow. These can be converted to Lyapunov exponents for the map via the well-known formula  $\chi_{\text{map}}^+ = \bar{\tau} \chi_{\text{flow}}^+$ , where  $\bar{\tau}$  is the average free flight time. For this billiard table,  $\bar{\tau} = \frac{d^2 \sqrt{3}}{4} - \frac{\pi}{2}$ , using [CM, eq. (2.32)].

computed  $\chi_{\mu_{\text{SRB}}}^+$  numerically for about 20 values of  $\rho$  ranging from  $\rho = 0.1$  (when the scatterers touch) to  $\rho = \frac{\sqrt{2}}{2} - 0.4$  (when the horizon becomes infinite). All computed values of  $\chi_{\mu_{\text{SRB}}}^+$  are greater than  $0.8 > \log 2$  so that (1.5) holds for all such tables. (For these tables as well, one can in fact choose  $s_0 = 1/2$ .)

Secondly, for the family of tables studied by Garrido, we obtain an open set of pairs of parameters  $(\rho, R)$  satisfying (1.5) as follows. To ensure finite horizon and disjoint scatterers, the constraints are

$$\frac{1}{2} < \rho + R < \frac{\sqrt{2}}{2}, \quad \rho < R < \frac{1}{2}, \quad \text{and} \quad R > \frac{\sqrt{2}}{4}.$$

Since  $\mu_{\text{SRB}}$  is a probability measure denoting by  $\mathcal{K}_{\min} > 0$  the minimum curvature and using a well-known [CM, eqs. (4.10) and (4.15)] bound for the unstable hyperbolicity exponent, we have,

$$\chi_{\mu_{\text{SRB}}}^+ \geq \frac{1}{2} \int_{-\pi/2}^{\pi/2} \log\left(1 + \frac{2\tau_{\min}\mathcal{K}_{\min}}{\cos\varphi}\right) \cos\varphi \, d\varphi.$$

We find that this is greater than  $(1/2)\log 2 \approx 0.34657$  whenever  $\tau_{\min}\mathcal{K}_{\min} > \frac{1}{7}$ . If  $\rho < \frac{\sqrt{2}-1}{2}$ , then  $\tau_{\min} = 1 - 2R$ , and  $\mathcal{K}_{\min} = R^{-1}$ , so that  $\tau_{\min}\mathcal{K}_{\min} = R^{-1} - 2$ . Thus if  $R < \frac{7}{15}$ , then (1.5) holds. The open set is defined by the inequalities  $\frac{\sqrt{2}}{4} < R < \frac{7}{15}$ ,  $\rho < \frac{\sqrt{2}-1}{2}$ , and  $\rho + R > \frac{1}{2}$ .

Thirdly, it seems true that if there are no periodic orbits making at least one grazing collision then, for any  $\epsilon > 0$ , the constants  $n_0$  and  $\varphi_0$  can be chosen to ensure  $s_0 < \epsilon$ . This has led P.-A. Guihéneuf to conjecture that there exists a natural topology<sup>10</sup> on the set of billiard tables so that, for any  $\epsilon > 0$ , the set of tables for which  $s_0 < \epsilon$  is generic (that is, open and dense). This would immediately imply that our condition (1.5) is generically satisfied.

Finally, we mention that Diller, Dujardin, and Guedj [DDG1, Example 4.6] construct a birational map  $F$  having a measure of maximal entropy which is mixing but not  $F$ -adapted, by showing that  $F$  violates the Bedford–Diller [BD] recurrence condition. The Bedford–Diller condition does not have a natural analogue in our setting since double tangencies always occur. One could interpret our sparse recurrence condition  $h_* > s_0 \log 2$  as its replacement. It would be interesting to find billiards for which  $h_* \leq s_0 \log 2$  and which admit a non  $T$ -adapted measure of maximal entropy.

### 3. PROOF OF THEOREM 2.3 (EQUIVALENT FORMULATIONS OF $h_*$ )

In this section, we shall prove Theorem 2.3 through Lemmas 3.3, 3.4, 3.5, and 3.6.

We first recall some facts about the uniform hyperbolicity of  $T$  to introduce notation which will be used throughout. It is well known [CM] that  $T$  is uniformly hyperbolic in the following sense: First, the cones  $C^u = \{(dr, d\varphi) \in \mathbb{R}^2 : \mathcal{K}_{\min} \leq d\varphi/dr \leq \mathcal{K}_{\max} + 1/\tau_{\min}\}$  and  $C^s = \{(dr, d\varphi) \in \mathbb{R}^2 : -\mathcal{K}_{\min} \geq d\varphi/dr \geq -\mathcal{K}_{\max} - 1/\tau_{\min}\}$ , are strictly invariant under  $DT$  and  $DT^{-1}$ , respectively, whenever these derivatives exist. Here,  $\mathcal{K}_{\max}$  represent the maximum curvature of the scatterer boundaries and  $\tau_{\max} < \infty$  is the largest free flight time between collisions. Second, denoting by  $\mathcal{K}_{\min} > 0$ ,  $\tau_{\min} > 0$  the minimum curvature and the minimum free flight time, and setting

$$\Lambda := 1 + 2\mathcal{K}_{\min}\tau_{\min},$$

there exists  $C_1 > 0$  such that for all  $n \geq 0$ ,

$$(3.1) \quad \|DT^n(x)v\| \geq C_1\Lambda^n\|v\|, \quad \forall v \in C^u, \quad \|DT^{-n}(x)v\| \geq C_1\Lambda^n\|v\|, \quad \forall v \in C^s,$$

for all  $x$  for which  $DT^n(x)$ , or respectively  $DT^{-n}(x)$ , is defined, so that  $\Lambda$  is a lower bound<sup>11</sup> on the hyperbolicity constant of the map  $T$ .

<sup>10</sup>For a fixed number of scatterers, a candidate is given by the distance defined in [DZ2, §2.2, §3.4, Remark 2.9(b)].

<sup>11</sup>Therefore,  $h_{\mu_{\text{SRB}}}(T) = \int \log J^u T \, d\mu_{\text{SRB}} > \log \Lambda$  and the bound  $\log(1 + 2\mathcal{K}_{\min}\tau_{\min}) > s_0 \log 2$  implies (1.5).

**3.1. Preliminaries.** The following lemma provides important information regarding the structure of the partitions  $\mathcal{P}_{-k}^n$ , which we will use to make an explicit connection between  $\mathcal{M}_{-k}^n$  and  $\mathring{\mathcal{P}}_{-k}^n$  in Lemma 3.3.

**Lemma 3.1.** *The elements of  $\mathcal{P}_{-k}^n$  are connected sets for all  $k \geq 0$  and  $n \geq 0$ .*

*Proof.* The statement is true by definition for  $\mathcal{P} = \mathcal{P}_0^0$ . We will prove the general statement by induction on  $k$  and  $n$  using the fact that  $\mathcal{P}_{-k}^{n+1} = \mathcal{P}_{-k}^n \vee T^{-1}\mathcal{P}_{-k}^n$ , and  $\mathcal{P}_{-k-1}^n = \mathcal{P}_{-k}^n \vee T\mathcal{P}_{-k}^n$ .

Fix  $k, n \geq 0$ , and assume the elements of  $\mathcal{P}_{-k}^n$  are connected sets. Let  $A_1, A_2 \in \mathcal{P}_{-k}^n$ . If  $T^{-1}A_1 \cap A_2$  is empty or is an isolated point, then it is connected. So suppose  $T^{-1}A_1 \cap A_2$  has nonempty interior.

Clearly,  $T^{-1}A_1$  is connected since  $T^{-1}$  is continuous on elements of  $\mathcal{P}_{-k}^n$  for all  $k, n \geq 0$ . Notice that the boundary of  $A_1$  is comprised of finitely many smooth stable and unstable curves in  $\mathcal{S}_{-k} \cup \mathcal{S}_n$ , as well as possibly a subset of  $\mathcal{S}_0$  ([CM, Prop 4.45 and Exercise 4.46], see also [CM, Fig 4.17]). We shall refer to these as the *stable* and *unstable parts of the boundary* of  $A_1$ . Similar facts apply to the boundaries of  $A_2$  and  $TA_1$ .

We consider whether a stable part of the boundary of  $T^{-1}A_1$  can cross a stable part of the boundary of  $A_2$ , and create two or more connected components of  $T^{-1}A_1 \cap A_2$ . Call these two boundary components  $\gamma_1$  and  $\gamma_2$  and notice that such an occurrence would force  $\gamma_1$  and  $\gamma_2$  to intersect in at least two points.

We claim the following fact: If a stable curve  $S_i \subset T^{-i}\mathcal{S}_0$  intersects  $S_j \subset T^{-j}\mathcal{S}_0$  for  $i < j$ , the  $S_j$  must terminate on  $S_i$ . This is because  $T^i S_i \subset \mathcal{S}_0$ , while  $T^i S_j \subset T^{i-j}\mathcal{S}_0$  is still a stable curve, terminating on  $\mathcal{S}_0$ . A similar property holds for unstable curves in  $\mathcal{S}_{-i}$  and  $\mathcal{S}_{-j}$ .

The claim implies that  $\gamma_1$  and  $\gamma_2$  both belong to  $T^{-j}\mathcal{S}_0$  for some  $1 \leq j \leq n$ . But when such curves intersect, again, one must terminate on the other (crossing would violate injectivity of  $T^{-1}$ ).

A similar argument precludes the possibility that unstable parts of the boundary cross one another multiple times. Thus the only intersections allowed are stable/unstable boundaries of  $T^{-1}A_1$  terminating on corresponding stable/unstable boundaries of  $A_2$ , or transverse intersections between stable components of  $\partial(T^{-1}A_1)$  and unstable components of  $\partial A_2$ , and vice versa. This last type of intersection cannot produce multiple connected components due to the *continuation of singularities*, which states that every stable curve in  $\mathcal{S}_{-n} \setminus \mathcal{S}_0$  is part of a monotonic and piecewise smooth decreasing curve which terminates on  $\mathcal{S}_0$  (see [CM, Prop 4.47]). A similar fact holds for unstable curves in  $\mathcal{S}_n \setminus \mathcal{S}_0$ . This implies that  $T^{-1}A_1 \cap A_2$  is a connected set, and since  $A_1$  and  $A_2$  were arbitrary, that  $\mathcal{P}_{-k}^{n+1}$  is comprised entirely of connected sets.

Similarly, considering  $TA_1 \cap A_2$  proves that all elements of  $\mathcal{P}_{-k-1}^n$  are connected.  $\square$

From the proof of Lemma 3.1, we can see that, aside from isolated points, elements of  $\mathcal{P}_{-k}^n$  consist of connected cells which are roughly ‘‘convex’’ and have boundaries comprised of stable and unstable curves.

**Lemma 3.2.** *There exists  $C > 0$ , depending on the table  $Q$ , such that for any  $k, n \in \mathbb{N}$ ,  $\#\mathring{\mathcal{P}}_{-k}^n \leq \#\mathcal{P}_{-k}^n \leq \#\mathring{\mathcal{P}}_{-k}^n + C(n+k+1)$ .*

*Proof.* It is clear from the definition of  $\mathring{\mathcal{P}}_{-k}^n$  and  $\mathcal{P}_{-k}^n$  that

$$\#\mathcal{P}_{-k}^n = \#\mathring{\mathcal{P}}_{-k}^n + \#\{\text{isolated points}\},$$

where the isolated points in  $\mathcal{P}_{-k}^n$  can be created by multiple tangencies aligning in a particular manner, as described above (see Figure 1). Thus the first inequality is trivial.

The set of isolated points created at each forward iterate is contained in  $\mathcal{S}_0 \cap T^{-1}\mathcal{S}_0$ , while the set of isolated points created at each backward iterate is contained in  $\mathcal{S}_0 \cap T\mathcal{S}_0$ . We proceed to estimate the cardinality of these sets.

Let  $r_0$  be sufficiently small such that for any segment  $S \subset \mathcal{S}_0$  of length  $r_0$ , the image  $TS$  comprises at most  $\tau_{\max}/\tau_{\min}$  connected curves on which  $T^{-1}$  is smooth [CM, Sect. 5.10]. For

each  $i$ , the number of points in  $\partial B_i \cap \mathcal{S}_0 \cap T^{-1}\mathcal{S}_0$  is thus bounded by  $2|\partial B_i|\tau_{\max}/(\tau_{\min}r_0)$ , where the factor 2 comes from the top and bottom boundary of the cylinder. Summing over  $i$ , we have  $\#(\mathcal{S}_0 \cap T^{-1}\mathcal{S}_0) \leq 2|\partial Q|\tau_{\max}/(\tau_{\min}r_0)$ . Due to reversibility, a similar estimate holds for  $\#(\mathcal{S}_0 \cap T\mathcal{S}_0)$ . Since this bound holds at each iterate, the second inequality holds with  $C = \frac{2|\partial Q|\tau_{\max}}{\tau_{\min}r_0}$ .  $\square$

**3.2. Formulations of  $h_*$  Involving  $\mathcal{P}$  and  $\mathring{\mathcal{P}}$ .** The following lemma gives claims (1) and (2) of Theorem 2.3:

**Lemma 3.3.** *The following holds for every  $k \geq 0$ . We have  $\mathring{\mathcal{P}}_{-k}^n = \mathcal{M}_{-k-1}^{n+1}$  for every  $n \geq 0$ . Moreover, the following limits exist and are equal to  $h_*$ :*

$$h_* = \lim_{n \rightarrow \infty} \frac{1}{n} \log \# \mathcal{M}_{-k}^n = \lim_{n \rightarrow \infty} \frac{1}{n} \log \# \mathring{\mathcal{P}}_{-k}^n = \lim_{n \rightarrow \infty} \frac{1}{n} \log \# \mathcal{P}_{-k}^n.$$

Finally, the sequence  $n \mapsto \log \# \mathcal{M}_{-k}^n$  is subadditive.

*Proof.* First notice that by Lemma 3.1, the elements of  $\mathring{\mathcal{P}}_{-k}^n$  are open, connected sets whose boundaries are curves in  $\mathcal{S}_{-k-1} \cup \mathcal{S}_{n+1}$ . Since the elements of  $\mathcal{M}_{-k-1}^{n+1}$  are the maximal open, connected sets with this property, it must be that  $\mathring{\mathcal{P}}_{-k}^n$  is a refinement of  $\mathcal{M}_{-k-1}^{n+1}$ . Now suppose that the union of  $O_1, O_2 \in \mathring{\mathcal{P}}_{-k}^n$  is contained in a single element  $A \in \mathcal{M}_{-k-1}^{n+1}$ . This is impossible since  $\partial O_1, \partial O_2 \subset \mathcal{S}_{-k-1} \cup \mathcal{S}_{n+1}$ , and at least part of these boundaries must lie inside  $A$ , contradicting the definition of  $A$ . So in fact,  $\mathring{\mathcal{P}}_{-k}^n = \mathcal{M}_{-k-1}^{n+1}$ .

We next show that the limit in terms of  $\# \mathcal{P}_{-k}^n$  exists and is independent of  $k$ . It will follow that the limits in terms of  $\# \mathcal{M}_{-k}^n$  and  $\# \mathring{\mathcal{P}}_{-k}^n$  exist and coincide using the relation  $\mathring{\mathcal{P}}_{-k}^n = \mathcal{M}_{-k-1}^{n+1}$  and Lemma 3.2.

Note that  $\# \mathcal{P}_{-j}^n \leq \# \mathcal{P}_{-k}^n$  whenever  $0 \leq j \leq k$ . For fixed  $k$ , we have  $\# \mathcal{P}_{-k}^{n+m} \leq \# \mathcal{P}_{-k}^n \cdot \# \left( \bigvee_{i=1}^m T^{-n-i}\mathcal{P} \right)$ , and since  $\# \left( \bigvee_{i=1}^m T^{-n-i}\mathcal{P} \right) = \# \left( \bigvee_{i=1}^m T^{-i}\mathcal{P} \right)$  because  $T$  is invertible, it follows that  $\# \mathcal{P}_{-k}^{n+m} \leq \# \mathcal{P}_{-k}^n \cdot \# \mathcal{P}_{-k}^m$ . Thus  $\log \# \mathcal{P}_{-k}^n$  is subadditive as a function of  $n$ , and the limit in  $n$  converges for each  $k$ . Applying this to  $k = 0$  implies that the limit defining  $h_*$  in Definition 2.1 exists.

Similar considerations show that  $\# \mathcal{P}_{-k}^n \leq \# \mathcal{P}_{-k}^0 \cdot \# \mathcal{P}_0^n$ , and so

$$h_* = \lim_{n \rightarrow \infty} \frac{1}{n} \log \# \mathcal{P}_0^n \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \# \mathcal{P}_{-k}^n \leq \lim_{n \rightarrow \infty} \frac{1}{n} (\log \# \mathcal{P}_{-k}^0 + \log \# \mathcal{P}_0^n) = h_*,$$

so that the limit exists and is independent of  $k$ .

For the final claim, we shall see that  $\log \# \mathring{\mathcal{P}}_{-k}^n$  is subadditive for essentially the same reason as  $\log \# \mathcal{P}_{-k}^n$ : Take an (nonempty) element  $P$  of  $\mathring{\mathcal{P}}_1^{n+m}$ . It is the interior of an intersection of elements of the form  $T^{-j}A_j$  for some  $A_j$  in  $\mathcal{P}$ , for  $j = 1$  to  $n+m$ . This is equal to the intersection of the interiors of  $T^{-j}A_j$ . But, since  $P$  is nonempty, none of the  $T^{-j}A_j$  can have empty interior and so none of the  $A_j$  can have empty interior. Thus the interiors of  $A_j$  are in  $\mathring{\mathcal{P}}$  as well. Now, splitting the intersection of the first  $n$  sets from the last  $m$ , we see that the intersection of the first  $n$  sets form an element of  $\mathring{\mathcal{P}}_1^n$ . For the last  $m$  sets, we can factor out  $T^{-n}$  at the price of making the set a bit bigger:

$$\text{int}(T^{-n-j}(A_{-n-j})) \subseteq T^{-n}(\text{int}(T^{-j}(A_{-n-j}))),$$

where  $\text{int}(\cdot)$  denotes the interior of a set. Doing this for  $j = 1$  to  $m$ , we see that this intersection is contained in  $T^{-n}$  of an element of  $\mathring{\mathcal{P}}_1^m$ . It follows that  $\# \mathring{\mathcal{P}}_1^{n+m} \leq \# \mathring{\mathcal{P}}_1^n \cdot \# \mathring{\mathcal{P}}_1^m$ , so taking logs, the sequence is subadditive. And then so is the sequence with  $\mathcal{M}_0^n$  in place of  $\mathring{\mathcal{P}}_1^{n-1}$ .  $\square$

**3.3. Comparing  $h_*$  with the Bowen Definitions.** We set  $\text{diam}^s(\mathcal{M}_{-k}^n)$  equal to the maximum length of a stable curve in any element of  $\mathcal{M}_{-k}^n$ . Similarly,  $\text{diam}^u(\mathcal{M}_{-k}^n)$  denotes the maximum length of an unstable curve in any element of  $\mathcal{M}_{-k}^n$  while  $\text{diam}(\mathcal{M}_{-k}^n)$  denotes the maximum diameter of any element of  $\mathcal{M}_{-k}^n$ .

The following lemma gives the first claim of (3) in Theorem 2.3:

**Lemma 3.4.**  $h_* = h_{\text{sep}}$ .

*Proof.* Fix  $\varepsilon > 0$ . Let  $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$  denote the lower bound on the hyperbolicity constant for  $T$  as in (3.1). Choose  $k_\varepsilon$  large enough that  $\text{diam}^s(\mathcal{M}_{-k_\varepsilon-1}^0) \leq C_1^{-1}\Lambda^{-k_\varepsilon} < c_1\varepsilon$ , for some  $c_1 > 0$  to be chosen below. It follows that  $\text{diam}^u(\mathcal{M}_{-k_\varepsilon-1}^{n+1}) \leq C_1^{-1}\Lambda^{-n} < c_1\varepsilon$  for each  $n \geq k_\varepsilon$ . Using the uniform transversality of stable and unstable cones, we may choose  $c_1 > 0$  such that  $\text{diam}(\mathcal{M}_{-k_\varepsilon-1}^{n+1}) < \varepsilon$  for all  $n \geq k_\varepsilon$ .

Now for  $n \geq k_\varepsilon$ , let  $E$  be an  $(n, \varepsilon)$ -separated set. Given  $x, y \in E$ , we will show that  $x$  and  $y$  cannot belong to the same set  $A \in \mathring{\mathcal{P}}_{-k_\varepsilon}^{k_\varepsilon+n}$ .

Since  $x, y \in E$ , there exists  $j \in [0, n]$  such that  $d(T^j(x), T^j(y)) > \varepsilon$ . If  $x \in A \in \mathring{\mathcal{P}}_{-k_\varepsilon}^{k_\varepsilon+n}$ , then  $x \in \cap_{i=-k_\varepsilon}^{k_\varepsilon+n} \text{int}(T^{-i}P_i)$  for some choice of  $P_i \in \mathcal{P}$ . Then

$$(3.2) \quad T^j x \in \cap_{i=-k_\varepsilon}^{k_\varepsilon+n-j} T^{-i}P_{i+j} \subset \cap_{-k_\varepsilon}^{k_\varepsilon} T^{-i}P_{i+j} \in \mathcal{P}_{-k_\varepsilon}^{k_\varepsilon}.$$

Note that the element of  $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon}$  to which  $T^j(x)$  belongs must have nonempty interior since  $T^{-i}P_i$  has non-empty interior for each  $i \in [-k_\varepsilon, k_\varepsilon + n]$ . If  $y \in A$ , then  $T^j y$  would belong to the same element of  $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon}$ , which is impossible since  $\text{diam}(\mathring{\mathcal{P}}_{-k_\varepsilon}^{k_\varepsilon}) < \varepsilon$  and taking the closure of such sets does not change the diameter.

Thus  $x, y \in E$  implies that  $x$  and  $y$  cannot belong to the same element of  $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$  with nonempty interior. On the other hand, if  $x$  belongs to an element of  $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$  with empty interior, then indeed the element containing  $x$  is an isolated point, and  $y$  cannot belong to the same element. Thus  $\#E \leq \#\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$ .

Since this bound holds for every  $(n, \varepsilon)$ -separated set, we have  $r_n(\varepsilon) \leq \#\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$ . Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log r_n(\varepsilon) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \#\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n} = h_*.$$

Since this bound holds for every  $\varepsilon > 0$ , we conclude  $h_{\text{sep}} \leq h_*$ .

To prove the reverse inequality, we claim that there exists  $\varepsilon_0 > 0$ , independent of  $n \geq 1$  and depending only on the table  $Q$ , such that

$$(3.3) \quad \text{if } x, y \text{ lie in different elements of } \mathcal{M}_0^n, \text{ then } d_n(x, y) \geq \varepsilon_0.$$

To each point  $x$  lying in an element of  $\mathcal{M}_0^n$ , we can associate an itinerary  $(i_0, i_1, \dots, i_n)$  such that  $T^{i_j}(x) \in M_{i_j}$ . If  $x, y$  have different itineraries, then for some  $0 \leq j \leq n$ , the points  $T^j(x)$  and  $T^j(y)$  lie in different components  $M_i$ , and so by definition (2.1) we have,  $d_n(x, y) = 10D \cdot \max_i \text{diam}(M_i)$ .

Now suppose  $x, y$  lie in different elements of  $\mathcal{M}_0^n$ , but have the same itinerary. By definition of  $\mathcal{M}_0^n$ , the elements containing  $x$  and  $y$  are separated by curves in  $\mathcal{S}_n$ . Let  $j$  be the minimum index of such a curve. Then  $T^{j-1}(x)$  and  $T^{j-1}(y)$  lie on different sides of a curve in  $\mathcal{S}_1 \setminus \mathcal{S}_0$ . Due to the finite horizon condition (our slightly stronger version is needed here), there exists  $\varepsilon_0 > 0$ , depending only on the structure of  $\mathcal{S}_1$ , such that the two one-sided  $\varepsilon_0$ -neighbourhoods of each curve in  $\mathcal{S}_1 \setminus \mathcal{S}_0$  are mapped at least  $\varepsilon_0$  apart. Thus either  $d(T^{j-1}(x), T^{j-1}(y)) \geq \varepsilon_0$  or  $d(T^j(x), T^j(y)) \geq \varepsilon_0$ .

With the claim proved, fix  $n \in \mathbb{N}$  and  $\varepsilon \leq \varepsilon_0$ , and define  $E$  to be a set comprising exactly one point from each element of  $\mathcal{M}_0^n$ . Then by the claim,  $E$  is  $(n, \varepsilon)$ -separated, so that  $\#\mathcal{M}_0^n \leq r_n(\varepsilon)$  for each  $\varepsilon \leq \varepsilon_0$ . Taking  $n \rightarrow \infty$  and  $\varepsilon \rightarrow 0$  yields  $h_* \leq h_{\text{sep}}$ .  $\square$

The following lemma gives the second claim of (3) in Theorem 2.3:

**Lemma 3.5.**  $h_* = h_{\text{span}}$ .

*Proof.* Fix  $\varepsilon > 0$  and choose  $k_\varepsilon$  as in the proof of Lemma 3.4 so that  $\text{diam}(\mathcal{M}_{-k_\varepsilon-1}^{n+1}) < \varepsilon$  for all  $n \geq k_\varepsilon$ . Choose one point  $x$  in each element of  $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$ , and let  $F$  denote the collection of these points. We will show that  $F$  is an  $(n, \varepsilon)$ -spanning set for  $T$ .

Let  $y \in M$  and let  $B_y$  be the element of  $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$  containing  $y$ . If  $B_y$  is an isolated point, then  $y \in F$  and there is nothing to prove. Otherwise, let  $x_y = F \cap B_y$ . For each  $j \in [0, n]$ , using the analogous calculation as in (3.2), we must have  $T^j(y), T^j(x_y) \in B_j \in \mathcal{P}_{-k_\varepsilon}^{k_\varepsilon}$ . Since  $\text{diam}(\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon}) < \varepsilon$ , this implies  $d(T^j(y), T^j(x_y)) < \varepsilon$  for all  $j \in [0, n]$ . Thus  $F$  is an  $(n, \varepsilon)$ -spanning set. We have,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log s_n(\varepsilon) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \#\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n} = h_*.$$

Since this is true for each  $\varepsilon > 0$ , it follows that  $h_{\text{span}} \leq h_*$ .

To prove the reverse inequality, recall  $\varepsilon_0$  from the proof of Lemma 3.4. For  $\varepsilon < \varepsilon_0$  and  $n \in \mathbb{N}$ , let  $F$  be an  $(n, \varepsilon)$ -spanning set. We claim  $\#F \geq \#\mathcal{M}_0^n$ . Suppose not. Then there exists  $A \in \mathcal{M}_0^n$  which contains no elements of  $F$ . Let  $y \in A$  and let  $x \in F$ . By the claim in the proof of Lemma 3.4,  $d_n(x, y) \geq \varepsilon_0$  since  $x$  and  $y$  lie in different elements of  $\mathcal{M}_0^n$ . Since this holds for all  $x \in F$ , it contradicts the fact that  $F$  is an  $(n, \varepsilon)$ -spanning set.

Since this is true for each  $(n, \varepsilon)$ -spanning set for  $\varepsilon < \varepsilon_0$ , we conclude that  $s_n(\varepsilon) \geq \#\mathcal{M}_0^n$ , and taking appropriate limits,  $h_{\text{span}} \geq h_*$ .  $\square$

**3.4. Easy Direction of the Variational Principle for  $h_*$ .** Recall that given a  $T$ -invariant probability measure  $\mu$  and a finite measurable partition  $\mathcal{A}$  of  $M$ , the entropy of  $\mathcal{A}$  with respect to  $\mu$  is defined by  $H_\mu(\mathcal{A}) = -\sum_{A \in \mathcal{A}} \mu(A) \log \mu(A)$ , and the entropy of  $T$  with respect to  $\mathcal{A}$  is  $h_\mu(T, \mathcal{A}) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i} \mathcal{A}\right)$ .

The following lemma gives the bound (4) in Theorem 2.3:

**Lemma 3.6.**  $h_* \geq \sup\{h_\mu(T) : \mu \text{ is a } T\text{-invariant Borel probability measure}\}$ .

*Proof.* Let  $\mu$  be a  $T$ -invariant probability measure on  $M$ . We note that  $\mathcal{P}$  is a generator for  $T$  since  $\bigvee_{i=-\infty}^{\infty} T^{-i} \mathcal{P}$  separates points in  $M$ . Thus  $h_\mu(T) = h_\mu(T, \mathcal{P})$  (see for example [W, Thm 4.17]). Then,

$$h_\mu(T, \mathcal{P}) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i} \mathcal{P}\right) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu(\mathcal{P}_0^{n-1}) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log(\#\mathcal{P}_0^{n-1}) = h_*.$$

Thus  $h_\mu(T) \leq h_*$  for every  $T$ -invariant probability measure  $\mu$ .  $\square$

#### 4. THE BANACH SPACES $\mathcal{B}$ AND $\mathcal{B}_w$ AND THE TRANSFER OPERATOR $\mathcal{L}$

The measure of maximal entropy for the billiard map  $T$  will be constructed out of left and right eigenvectors of a transfer operator  $\mathcal{L}$  associated with the billiard map and acting on suitable spaces  $\mathcal{B}$  and  $\mathcal{B}_w$  of anisotropic distributions. In this section we define these objects, state and prove the main bound, Proposition 4.7, on the transfer operator, and deduce from it Theorem 4.10, showing that the spectral radius of  $\mathcal{L}$  on  $\mathcal{B}$  is  $e^{h_*}$ .

Recalling that the stable Jacobian of  $T$  satisfies  $J^s T \approx \cos \varphi$  [CM, eq. (4.20)], the relevant transfer operator is defined on measurable functions  $f$  by

$$(4.1) \quad \mathcal{L}f = \frac{f \circ T^{-1}}{J^s T \circ T^{-1}}.$$

In order to define the Banach spaces of distributions on which the operator  $\mathcal{L}$  will act, we need preliminary notations: Let  $\mathcal{W}^s$  denote the set of all nontrivial connected subsets  $W$  of stable manifolds for  $T$  so that  $W$  has length at most  $\delta_0 > 0$ , where  $\delta_0 < 1$  will be chosen after (5.4),

using the growth Lemma 5.1. Such curves have curvature bounded above by a fixed constant [CM, Prop 4.29]. Thus,  $T^{-1}\mathcal{W}^s = \mathcal{W}^s$ , up to subdivision of curves.

For every  $W \in \mathcal{W}^s$ , let  $C^1(W)$  denote the space of  $C^1$  functions on  $W$  and for every  $\alpha \in (0, 1)$  we let  $C^\alpha(W)$  denote the closure<sup>12</sup> of  $C^1(W)$  for the  $\alpha$ -Hölder norm  $|\psi|_{C^\alpha(W)} = \sup_W |\psi| + H_W^\alpha(\psi)$ , where

$$(4.2) \quad H_W^\alpha(\psi) = \sup_{\substack{x, y \in W \\ x \neq y}} \frac{|\psi(x) - \psi(y)|}{d(x, y)^\alpha}.$$

We write  $\psi \in C^\alpha(\mathcal{W}^s)$  if  $\psi \in C^\alpha(W)$  for all  $W \in \mathcal{W}^s$ , with uniformly bounded Hölder norm.

**4.1. Definition of Norms and of the Spaces  $\mathcal{B}$  and  $\mathcal{B}_w$ .** Since the stable cone  $C^s$  is bounded away from the vertical, we may view each stable curve  $W \in \mathcal{W}^s$  as the graph of a function  $\varphi_W(r)$  of the arclength coordinate  $r$  ranging over some interval  $I_W$ , i.e.,

$$(4.3) \quad W = \{G_W(r) := (r, \varphi_W(r)) \in M : r \in I_W\}.$$

Given two curves  $W_1, W_2 \in \mathcal{W}^s$ , we may use this representation to define a distance<sup>13</sup> between them: Define

$$d_{\mathcal{W}^s}(W_1, W_2) = |I_{W_1} \Delta I_{W_2}| + |\varphi_{W_1} - \varphi_{W_2}|_{C^1(I_{W_1} \cap I_{W_2})}$$

if  $I_{W_1} \cap I_{W_2} \neq \emptyset$ . Otherwise, set  $d_{\mathcal{W}^s}(W_1, W_2) = \infty$ .

Similarly, given two test functions  $\psi_1$  and  $\psi_2$  on  $W_1$  and  $W_2$ , respectively, we define a distance between them by

$$d(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{C^0(I_{W_1} \cap I_{W_2})},$$

whenever  $d_{\mathcal{W}^s}(W_1, W_2) < \infty$ . Otherwise, set  $d(\psi_1, \psi_2) = \infty$ .

We are now ready to introduce the norms used to define the spaces  $\mathcal{B}$  and  $\mathcal{B}_w$ . Besides  $\delta_0 \in (0, 1)$ , and a constant  $\epsilon_0 > 0$  to appear below, they will depend on positive real numbers  $\alpha, \beta, \gamma$ , and  $\varsigma$  so that, recalling  $s_0 \in (0, 1)$  from<sup>14</sup> (1.4),

$$(4.4) \quad 0 < \beta < \alpha \leq 1/3, \quad 1 < 2^{s_0 \gamma} < e^{h_*}, \quad 0 < \varsigma < \gamma.$$

(The condition  $\alpha \leq 1/3$  is used in Lemma 4.4 which is used to prove embedding into distributions. The number  $1/3$  comes from the  $1/k^2$  decay in the width of homogeneity strips (4.5). The upper bound on  $\gamma$  arises from use of the growth lemma from Section 5.1. See (5.4).)

For  $f \in C^1(M)$ , define the weak norm of  $f$  by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha(W)} \leq 1}} \int_W f \psi \, dm_W.$$

Here,  $dm_W$  denotes unnormalized Lebesgue (arclength) measure on  $W$ .

Define the strong stable norm of  $f$  by<sup>15</sup>

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\beta(W) \\ |\psi|_{C^\beta(W)} \leq |\log |W||^\gamma}} \int_W f \psi \, dm_W,$$

<sup>12</sup>Working with the closure of  $C^1$  will give injectivity of the inclusion of the strong space in the weak.

<sup>13</sup> $d_{\mathcal{W}^s}$  is not a metric since it does not satisfy the triangle inequality; however, it is sufficient for our purposes to produce a usable notion of distance between stable manifolds. See [DRZ, Footnote 4] for a modification of  $d_{\mathcal{W}^s}$  which does satisfy the triangle inequality.

<sup>14</sup>If  $\gamma > 1$ , we can get good bounds in Theorem 2.6. This is only possible if  $h_* > s_0 \log 2$ .

<sup>15</sup>The logarithmic modulus of continuity in  $\|f\|_s$  is used to obtain a finite spectral radius.



(note that  $\|f\|_w \leq \max\{1, |\log \delta_0|^{-\gamma}\} \|f\|_s$ ). Finally, for  $\varsigma \in (0, \gamma)$ , define the strong unstable norm<sup>16</sup> of  $f$  by

$$\|f\|_u = \sup_{\varepsilon \leq \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d_{\mathcal{W}^s}(W_1, W_2) \leq \varepsilon}} \sup_{\substack{\psi_i \in C^\alpha(W_i) \\ |\psi_i|_{C^\alpha(W_i)} \leq 1 \\ d(\psi_1, \psi_2) = 0}} |\log \varepsilon|^\varsigma \left| \int_{W_1} f \psi_1 dm_{W_1} - \int_{W_2} f \psi_2 dm_{W_2} \right|.$$

**Definition 4.1** (The Banach spaces). *The space  $\mathcal{B}_w$  is the completion of  $C^1(M)$  with respect to the weak norm  $\|\cdot\|_w$ , while  $\mathcal{B}$  is the completion of  $C^1(M)$  with respect to the strong norm,  $\|\cdot\|_{\mathcal{B}} = \|\cdot\|_s + \|\cdot\|_u$ .*

In the next subsection, we shall prove the continuous embeddings  $\mathcal{B} \subset \mathcal{B}_w \subset (C^1(M))^*$ , i.e., elements of our Banach spaces are distributions of order at most one (see Proposition 4.2). Proposition 6.1 in Section 6.4 gives the compact embedding of the unit ball of  $\mathcal{B}$  in  $\mathcal{B}_w$ .

**4.2. Embeddings into Distributions on  $M$ .** In this section we describe elements of our Banach spaces  $\mathcal{B} \subset \mathcal{B}_w$  as distributions of order at most one on  $M$ . (This does not follow from the corresponding result in [DZ1], in particular since we use exact stable leaves to define our norms.) We will actually show that they belong to the dual of a space  $C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$  containing  $C^1(M)$  that we define next: We did not require elements of  $\mathcal{W}^s$  to be homogeneous. Now, defining the usual homogeneity strips

$$(4.5) \quad \mathbb{H}_k = \left\{ (r, \varphi) \in M_i : \frac{\pi}{2} - \frac{1}{k^2} \leq \varphi \leq \frac{\pi}{2} - \frac{1}{(k+1)^2} \right\}, \quad k \geq k_0,$$

and analogously for  $k \leq -k_0$ , we define  $\mathcal{W}_{\mathbb{H}}^s \subset \mathcal{W}^s$  to denote those stable manifolds  $W \in \mathcal{W}^s$  such that  $T^n W$  lies in a single homogeneity strip for all  $n \geq 0$ . We write  $\psi \in C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$  if  $\psi \in C^\alpha(W)$  for all  $W \in \mathcal{W}_{\mathbb{H}}^s$  with uniformly bounded Hölder norm. Similarly, we define  $C_{\cos}^\alpha(\mathcal{W}_{\mathbb{H}}^s)$  to comprise the set of functions  $\psi$  such that  $\psi \cos \varphi \in C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$ . Clearly  $C^\alpha(\mathcal{W}_{\mathbb{H}}^s) \subset C_{\cos}^\alpha(\mathcal{W}_{\mathbb{H}}^s)$ .

Due to the uniform hyperbolicity (3.1) of  $T$  and the invariance of  $\mathcal{W}^s$  and  $\mathcal{W}_{\mathbb{H}}^s$ , if  $\psi \in C^\alpha(\mathcal{W}^s)$  (resp.  $C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$ ), then  $\psi \circ T \in C^\alpha(\mathcal{W}^s)$  (resp.  $C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$ ). Also, since the stable Jacobian of  $T$  satisfies  $J^s T \approx \cos \varphi$  [CM, eq. (4.20)] and is  $1/3$  log-Hölder continuous on elements of  $\mathcal{W}_{\mathbb{H}}^s$  [CM, Lemma 5.27], then  $\frac{\psi \circ T}{J^s T} \in C_{\cos}^\alpha(\mathcal{W}_{\mathbb{H}}^s)$  for any  $\alpha \leq 1/3$ .

We can now state our first embedding result. An embedding  $\mathcal{B}_w \subset (\mathcal{F})^*$  (for  $\mathcal{F} = C^1(M)$  or  $\mathcal{F} = C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$ ) is understood in the following sense: for  $f \in \mathcal{B}_w$  there exists  $C_f < \infty$  such that, letting  $f_n \in C^1(M)$  be a sequence converging to  $f$  in the  $\mathcal{B}_w$  norm, for every  $\psi \in \mathcal{F}$  the following limit exists

$$(4.6) \quad f(\psi) = \lim_{n \rightarrow \infty} \int f_n \psi d\mu_{\text{SRB}}$$

and satisfies  $|f(\psi)| \leq C_f \|\psi\|_{\mathcal{F}}$ .

**Proposition 4.2** (Embedding into Distributions). *The continuous embeddings*

$$C^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^\alpha(\mathcal{W}_{\mathbb{H}}^s))^* \subset (C^1(M))^*$$

*hold, the first two embeddings<sup>17</sup> being injective. Therefore, since  $C^1(M) \subset \mathcal{B} \subset \mathcal{B}_w$  injectively and continuously, we have*

$$(\mathcal{B}_w)^* \subset \mathcal{B}^* \subset (C^1(M))^*.$$

**Remark 4.3** (Radon Measures). *Proposition 4.2 has the following important consequence: If  $f \in \mathcal{B}_w$  is such that  $f(\psi)$  defined by (4.6) is nonnegative for all nonnegative  $\psi \in \mathcal{F} = C^1(M)$ , then, by Schwartz's [Sch, §I.4] generalisation of the Riesz representation theorem, it defines an element*

<sup>16</sup>The logarithmic modulus of continuity appears in  $\|f\|_u$  because of the logarithmic modulus of continuity in  $\|f\|_s$ . Its presence in  $\|f\|_u$  causes the loss of the spectral gap.

<sup>17</sup>We do not expect the third embedding to be injective, due to the logarithmic weight in the norm.

of the dual of  $C^0(M)$ , i.e., a Radon measure on  $M$ . If, in addition,  $f(\psi) = 1$  for  $\psi$  the constant function 1, then this measure is a probability measure.

The following lemma is important for the third inclusion in Proposition 4.2. Recalling (4.2), we define  $H_{\mathcal{W}_{\mathbb{H}}^s}^\alpha(\psi) = \sup_{W \in \mathcal{W}_{\mathbb{H}}^s} H_W^\alpha(\psi)$ .

**Lemma 4.4.** *There exists  $C > 0$  such that for any  $f \in \mathcal{B}_w$  and  $\psi \in C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$ , recalling (4.6),*

$$|f(\psi)| \leq C|f|_w(|\psi|_\infty + H_{\mathcal{W}_{\mathbb{H}}^s}^\alpha(\psi)).$$

*Proof.* By density it suffices to prove the inequality for  $f \in C^1(M)$ . Let  $\psi \in C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$ . Since by our convention, we identify  $f$  with the measure  $f d\mu_{\text{SRB}}$ , we must estimate,

$$f(\psi) = \int f \psi d\mu_{\text{SRB}}.$$

In order to bound this integral, we disintegrate the measure  $\mu_{\text{SRB}}$  into conditional probability measures  $\mu_{\text{SRB}}^{W_\xi}$  on maximal homogeneous stable manifolds  $W_\xi \in \mathcal{W}_{\mathbb{H}}^s$  and a factor measure  $d\hat{\mu}_{\text{SRB}}(\xi)$  on the index set  $\Xi$  of homogeneous stable manifolds; thus  $\mathcal{W}_{\mathbb{H}}^s = \{W_\xi\}_{\xi \in \Xi}$ . According to the time reversal counterpart of [CM, Cor 5.30], the conditional measures  $\mu_{\text{SRB}}^{W_\xi}$  have smooth densities with respect to the arclength measure on  $W_\xi$ , i.e.,  $d\mu_{\text{SRB}}^{W_\xi} = |W_\xi|^{-1} \rho_\xi dm_{W_\xi}$ , where  $\rho_\xi$  is log-Hölder continuous with exponent  $1/3$ . Moreover,  $\sup_{\xi \in \Xi} |\rho_\xi|_{C^\alpha(W_\xi)} =: \bar{C} < \infty$  since  $\alpha \leq 1/3$ .

Using this disintegration, we estimate<sup>18</sup> the required integral:

$$\begin{aligned} (4.7) \quad |f(\psi)| &= \left| \int_{\xi \in \Xi} \int_{W_\xi} f \psi \rho_\xi |W_\xi|^{-1} dm_{W_\xi} d\hat{\mu}_{\text{SRB}}(\xi) \right| \\ &\leq \int_{\xi \in \Xi} |f|_w |\psi|_{C^\alpha(W_\xi)} |\rho_\xi|_{C^\alpha(W_\xi)} |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \\ &\leq \bar{C} |f|_w (|\psi|_\infty + H_{\mathcal{W}_{\mathbb{H}}^s}^\alpha(\psi)) \int_{\xi \in \Xi} |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi). \end{aligned}$$

This last integral is precisely that in [CM, Exercise 7.15] which measures the relative frequency of short curves in a standard family. Due to [CM, Exercise 7.22], the SRB measure decomposes into a proper family, and so this integral is finite.  $\square$

*Proof of Proposition 4.2.* The continuity and injectivity of the embedding  $C^1(M) \hookrightarrow \mathcal{B}$  is clear from the definition. The inequality  $|\cdot|_w \leq \|\cdot\|_s$  implies the continuity of  $\mathcal{B} \hookrightarrow \mathcal{B}_w$ , while the injectivity follows from the definition of  $C^\beta(W)$  as the closure of  $C^1(W)$  in the  $C^\beta$  norm, as described at the beginning of Section 4, so that  $C^\alpha(W)$  is dense in  $C^\beta(W)$ .

Finally, since  $C^1(M) \subset C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$ , the continuity of the third and fourth inclusions follow from Lemma 4.4.  $\square$

**4.3. The Transfer Operator.** We now move to the key bounds on the transfer operator. First, we revisit the definition (4.1) in order to let  $\mathcal{L}$  act on  $\mathcal{B}$  and  $\mathcal{B}_w$ : We may define the transfer operator  $\mathcal{L} : (C_{\cos}^\alpha(\mathcal{W}_{\mathbb{H}}^s))^* \rightarrow (C^\alpha(\mathcal{W}^s))^*$  by

$$\mathcal{L}f(\psi) = f\left(\frac{\psi \circ T}{J^s T}\right), \quad \psi \in C^\alpha(\mathcal{W}^s).$$

When  $f \in C^1(M)$ , we identify  $f$  with the measure<sup>19</sup>

$$(4.8) \quad f d\mu_{\text{SRB}} \in (C_{\cos}^\alpha(\mathcal{W}_{\mathbb{H}}^s))^*.$$

<sup>18</sup>This is where we use  $f\mu_{\text{SRB}}$ : Replacing  $\hat{\mu}_{\text{SRB}}$  by the factor measure with respect to Lebesgue, this integral would be infinite. Using  $\mathcal{W}^s$  rather than  $\mathcal{W}_{\mathbb{H}}^s$  may produce a finite integral with respect to Lebesgue, but the  $\rho_\xi$  may not be uniformly Hölder continuous on the longer curves.

<sup>19</sup>To show the claimed inclusion just use that  $d\mu_{\text{SRB}} = (2|\partial Q|)^{-1} \cos \varphi dr d\varphi$ .

The measure above is (abusively) still denoted by  $f$ . For  $f \in C^1(M)$  the transfer operator then indeed takes the form  $\mathcal{L}f = (f/J^s T) \circ T^{-1}$  announced in (4.1) since, due to our identification (4.8), we have  $\mathcal{L}f(\psi) = \int \mathcal{L}f \psi d\mu_{\text{SRB}} = \int f \frac{\psi \circ T}{J^s T} d\mu_{\text{SRB}}$ .

**Remark 4.5** (Viewing  $f \in C^1$  as a measure). *If we viewed instead  $f$  as the measure  $f dm$ , it is not clear whether the embedding Lemma 4.4 would still hold since the weight  $\cos W$  (crucial to [DZ1, Lemma 3.9]) is absent from the norms. Along these lines, we do not claim that Lebesgue measure belongs to our Banach spaces.*

*Slightly modifying [DZ1] due to the lack of homogeneity strips, we could replace  $|\psi|_{C^\alpha(W)} \leq 1$  by  $|\psi \cos \varphi|_{C^\alpha(W)} \leq 1$  in our norms. Then it would be natural to view  $f$  as  $f dm$ , and the embedding Lemma 4.4 would hold, but the transfer operator would have the form*

$$\mathcal{L}_{\cos} f = \frac{f \circ T^{-1}}{(J^s T \circ T^{-1})(JT \circ T^{-1})},$$

where  $JT$  is the full Jacobian of the map (the ratio of cosines). We do not make such a change since it would only complicate our estimates unnecessarily. Note that the potentials of the operators  $\mathcal{L}$  and  $\mathcal{L}_{\cos}$  differ by a coboundary, giving the same spectral radius.

It follows from submultiplicativity of  $\#\mathcal{M}_0^n$  that  $e^{nh_*} \leq \#\mathcal{M}_0^n$  for all  $n$ . In Section 5.3, we shall prove the supermultiplicativity statement Lemma 5.6 from which we deduce the following upper bound for  $\#\mathcal{M}_0^n$ :

**Proposition 4.6** (Exact Exponential Growth). *Let  $c_1 > 0$  be given by Lemma 5.6. Then for all  $n \in \mathbb{N}$ , we have  $e^{nh_*} \leq \#\mathcal{M}_0^n \leq \frac{2}{c_1} e^{nh_*}$ .*

The following proposition (proved in Section 6) gives the key norm estimates.

**Proposition 4.7.** *Let  $c_1$  be as in Proposition 4.6. There exist  $\delta_0, C > 0$ , and  $\varpi \in (0, 1)$  such that<sup>20</sup> for all  $f \in \mathcal{B}$ ,*

$$(4.9) \quad |\mathcal{L}^n f|_w \leq \frac{C}{c_1 \delta_0} e^{nh_*} |f|_w, \quad \forall n \geq 0;$$

$$(4.10) \quad \|\mathcal{L}^n f\|_s \leq \frac{C}{c_1 \delta_0} e^{nh_*} \|f\|_s, \quad \forall n \geq 0;$$

$$(4.11) \quad \|\mathcal{L}^n f\|_u \leq \frac{C}{c_1 \delta_0} (\|f\|_u + n^{\varpi} \|f\|_s) e^{nh_*}, \quad \forall n \geq 0.$$

*If  $h_* > s_0 \log 2$  (where  $s_0 < 1$  is defined by (1.4)) then in addition there exist  $\varsigma > 0$  and  $C > 0$  such that for all  $f \in \mathcal{B}$*

$$(4.12) \quad \|\mathcal{L}^n f\|_u \leq \frac{C}{c_1 \delta_0} (\|f\|_u + \|f\|_s) e^{nh_*}, \quad \forall n \geq 0.$$

**Remark 4.8.** *Replacing  $|\log \epsilon|$  by  $\log |\log \epsilon|$  in the definition of  $\|f\|_u$ , we can replace  $n^{\varpi}$  by a logarithm in (4.11).*

In spite of compactness of the embedding  $\mathcal{B} \subset \mathcal{B}_w$  (Proposition 6.1), the above bounds do *not* deserve to be called Lasota–Yorke estimates since (even replacing  $\|\cdot\|_s + \|\cdot\|_u$  by  $\|\cdot\|_s + c_u \|\cdot\|_u$  for small  $c_u$  and using footnote 20) they do not lead to bounds of the type  $\|(e^{-h_*} \mathcal{L})^n f\|_{\mathcal{B}} \leq \sigma^n \|f\|_{\mathcal{B}} + K_n |f|_w$  for some  $\sigma < 1$  and finite constants  $K_n$ . We will nevertheless sometimes refer to them as “Lasota–Yorke” estimates, in quotation marks.

Proposition 4.7 combined with the following lemma imply that  $\mathcal{L}$  is a bounded operator on both  $\mathcal{B}$  and  $\mathcal{B}_w$ :

<sup>20</sup>In fact the strong stable norm satisfies a stronger inequality:  $\|\mathcal{L}^n f\|_s \leq \frac{C}{c_1 \delta_0} (\sigma^n \|f\|_s + |f|_w) e^{nh_*}$ , for some  $\sigma < 1$ . We omit the proof since we do not use this.

**Lemma 4.9** (Image of a  $C^1$  Function). *For any  $f \in C^1(M)$  the image  $\mathcal{L}f \in (C^\alpha(\mathcal{W}^s))^*$  is the limit of a sequence of  $C^1$  functions in the strong norm  $\|\cdot\|_{\mathcal{B}}$ .*

*Proof.* Since our norms are weaker than the norms of [DZ1] (modulo the use of homogeneity layers there), the statement follows from replacing  $\mathcal{L}_{\text{SRB}}$  by  $\mathcal{L}$  in the proofs of Lemmas 3.7 and 3.8 in [DZ1], and checking that the absence of homogeneity layers does not affect the computations.  $\square$

Proposition 4.7 gives the upper bounds in the following result (the bounds (4.14) and (4.15) are needed to construct a nontrivial maximal eigenvector in Proposition 7.1):

**Theorem 4.10** (Spectral Radius of  $\mathcal{L}$  on  $\mathcal{B}$ ). *There exist  $\varpi \in (0, 1)$ ,  $C < \infty$  such that,*

$$(4.13) \quad \|\mathcal{L}^n\|_{\mathcal{B}} \leq Cn^\varpi e^{nh_*}, \quad \forall n \geq 0.$$

*There exists  $C > 0$  such that, letting  $1$  be the function  $f \equiv 1$ , we have,*

$$(4.14) \quad \|\mathcal{L}^n 1\|_s \geq |\mathcal{L}^n 1|_w \geq C e^{nh_*}, \quad \forall n \geq 0.$$

*Recalling (4.9), the spectral radius of  $\mathcal{L}$  on  $\mathcal{B}$  and  $\mathcal{B}_w$  is thus equal to  $\exp(h_*) > 1$ .*

*If  $h_* > s_0 \log 2$  (with  $s_0 < 1$  defined by (1.4)) then, if  $\varsigma > 0$  and  $\delta_0 > 0$  are small enough, there exists  $\tilde{C} < \infty$  such that,*

$$(4.15) \quad \|\mathcal{L}^n\|_{\mathcal{B}} \leq \tilde{C} e^{nh_*}, \quad \forall n \geq 0.$$

The above theorem is proved in Subection 6.3.

## 5. GROWTH LEMMA AND FRAGMENTATION LEMMAS

This section contains combinatorial growth lemmas, controlling the growth in complexity of the iterates of a stable curve. They will be used to prove the ‘‘Lasota–Yorke’’ Proposition 4.7, to show Lemma 5.2, used in Section 6.3 to get the lower bound (4.14) on the spectral radius, and to show absolute continuity in Section 7.3.

In view of the compact embedding Proposition 6.1, and also to get Lemma 5.4 from Lemma 5.2, we must work with a more general class of stable curves: We define a set of cone-stable curves  $\widehat{\mathcal{W}}^s$  whose tangent vectors all lie in the stable cone for the map, with length at most  $\delta_0$  and curvature bounded above so that  $T^{-1}\widehat{\mathcal{W}}^s \subset \widehat{\mathcal{W}}^s$ , up to subdivision of curves. Obviously,  $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$ . We define a set of cone-unstable curves  $\widehat{\mathcal{W}}^u$  similarly.

For  $W \in \widehat{\mathcal{W}}^s$ , let  $\mathcal{G}_0(W) = W$ . For  $n \geq 1$ , define  $\mathcal{G}_n(W) = \mathcal{G}_n^{\delta_0}(W)$  inductively as the smooth components of  $T^{-1}(W')$  for  $W' \in \mathcal{G}_{n-1}(W)$ , where elements longer than  $\delta_0$  are subdivided to have length between  $\delta_0/2$  and  $\delta_0$ . Thus  $\mathcal{G}_n(W) \subset \widehat{\mathcal{W}}^s$  for each  $n$  and  $\cup_{U \in \mathcal{G}_n(W)} U = T^{-n}W$ . Moreover, if  $W \in \mathcal{W}^s$ , then  $\mathcal{G}_n(W) \subset \mathcal{W}^s$ .

Denote by  $L_n(W)$  those elements of  $\mathcal{G}_n(W)$  having length at least  $\delta_0/3$ , and define  $\mathcal{I}_n(W)$  to comprise those elements  $U \in \mathcal{G}_n(W)$  for which  $T^i U$  is not contained in an element of  $L_{n-i}(W)$  for  $0 \leq i \leq n-1$ .

A fundamental fact [Ch2, Lemma 5.2] we will use is that the growth in complexity for the billiard is at most linear:

$$(5.1) \quad \exists K > 0 \text{ such that } \forall n \geq 0, \text{ the number of curves in } \mathcal{S}_{\pm n} \text{ that intersect} \\ \text{at a single point is at most } Kn.$$

**5.1. Growth Lemma.** Recall  $s_0 \in (0, 1)$  from (1.4). We shall prove:

**Lemma 5.1** (Growth Lemma). *For any  $m \in \mathbb{N}$ , there exists  $\delta_0 = \delta_0(m) \in (0, 1)$  such that for all  $n \geq 1$ , all  $\bar{\gamma} \in [0, \infty)$  and all  $W \in \widehat{\mathcal{W}}^s$ , we have*

$$\text{a) } \sum_{W_i \in \mathcal{I}_n(W)} \left( \frac{\log |W|}{\log |W_i|} \right)^{\bar{\gamma}} \leq 2^{(n+1)s_0\bar{\gamma}} (Km + 1)^{n/m};$$

$$\begin{aligned} \text{b)} \quad & \sum_{W_i \in \mathcal{G}_n(W)} \left( \frac{\log |W|}{\log |W_i|} \right)^{\bar{\gamma}} \\ & \leq \min \{ 2\delta_0^{-1} 2^{(n+1)s_0\bar{\gamma}} \# \mathcal{M}_0^n, 2^{2\bar{\gamma}+1} \delta_0^{-1} \sum_{j=1}^n 2^{js_0\bar{\gamma}} (Km+1)^{j/m} \# \mathcal{M}_0^{n-j} \}. \end{aligned}$$

Moreover, if  $|W| \geq \delta_0/2$ , then both factors  $2^{(n+1)s_0\bar{\gamma}}$  can be replaced by  $2^{\bar{\gamma}}$ .

*Proof.* First recall that if  $W \in \widehat{\mathcal{W}}^s$  is short, then

$$(5.2) \quad |T^{-1}W| \leq C|W|^{1/2} \quad \text{for some constant } C \geq 1, \text{ independent of } W \in \widehat{\mathcal{W}}^s,$$

[CM, Exercise 4.50]. The above bound can be iterated, giving  $|T^{-\ell}W| \leq C'|W|^{2^{-\ell}}$ , where  $C' \leq C^2$ , for any number of consecutive ‘‘nearly tangential’’ collisions (collisions with angle  $|\varphi| > \varphi_0$ ). Since in every  $n_0$  iterates, we have at most  $s_0n_0$  nearly tangential collisions and  $(1-s_0)n_0$  iterates that expand at most by a constant factor  $\Lambda_1 > 1$  depending only on  $\varphi_0$ , we see that

$$\begin{aligned} |T^{-n_0}W| & \leq C|W|^{2^{-s_0n_0}} \Lambda_1^{(1-s_0)n_0} \\ \implies |T^{-2n_0}W| & \leq C^{1+2^{-s_0n_0}} |W|^{2^{-2s_0n_0}} \Lambda_1^{(1-s_0)n_0} \Lambda_1^{(1-s_0)n_0}. \end{aligned}$$

Iterating this inductively, we conclude

$$(5.3) \quad |T^{-j}W| \leq C''|W|^{2^{-s_0j}} \quad \text{for all } j \geq 1,$$

where  $C'' > 0$  depends only on  $n_0$  and  $\Lambda_1$ . Therefore, if  $\delta_0$  is smaller than  $1/C''$ , we have

$$\left( \frac{\log |W|}{\log |W_i|} \right)^{\bar{\gamma}} \leq \left( 2^{s_0n} \left( 1 - \frac{\log C''}{\log |W_i|} \right) \right)^{\bar{\gamma}} \leq 2^{(n+1)s_0\bar{\gamma}}, \quad \forall W_i \in \mathcal{G}_n(W),$$

since  $|W_i| \leq \delta_0$ . Note that if  $|W_i| \leq |W|$ , then  $\frac{\log |W|}{\log |W_i|} \leq 1$ , so that such curves do not contribute large terms to the sums in parts (a) and (b) of the lemma.

(a) Using the above argument, for any  $W \in \widehat{\mathcal{W}}^s$ , we may bound the ratio of logs by  $2^{(n+1)s_0\bar{\gamma}}$ . Moreover, if  $|W| \geq \delta_0/2$ , then since  $|W_i| \leq \delta_0 < 2$ , we have

$$\frac{\log |W|}{\log |W_i|} \leq \frac{\log(\delta_0/2)}{\log \delta_0} = 1 + \frac{\log 2}{\log \delta_0} \leq 2.$$

Now, fixing  $m$  and using the linear bound on complexity, choose  $\delta_0 = \delta_0(m) > 0$  such that if  $|W| \leq \delta_0$ , then  $T^{-\ell}W$  comprises at most  $K\ell + 1$  connected components for  $0 \leq \ell \leq 2m$ . Such a choice is always possible by (5.2). Then for  $n = mj + \ell$ , we split up the orbit into  $j - 1$  increments of length  $m$  and the last increment of length  $m + \ell$ . Part (a) then follows by a simple induction, since elements of  $\mathcal{I}_{mj}(W)$  must be formed from elements of  $\mathcal{I}_{m(j-1)}(W)$  which have been cut by singularity curves in  $\mathcal{S}_{-m}$ . At the last step, this estimate also holds for elements of which have been cut by singularity curves in  $\mathcal{S}_{-m-\ell}$  by choice of  $\delta_0$ .

(b) The bound on the ratio of logs is the same as in part (a). The first bound on the cardinality of the sum follows by noting that each element of  $\mathcal{G}_n(W)$  is contained in one element of  $\mathcal{M}_0^n$ . Moreover, due to subdivision of long pieces, there can be no more than  $2\delta_0^{-1}$  elements of  $\mathcal{G}_n(W)$  in a single element of  $\mathcal{M}_0^n$ .

For the second bound in part (b), we may assume that  $|W| < \delta_0/2$ ; otherwise, we may bound the sum by  $2^{\bar{\gamma}+1}\delta_0^{-1}\#\mathcal{M}_0^n$ , which is optimal for what we need. For  $|W| < \delta_0/2$ , let  $F_1(W)$  denote those  $V \in \mathcal{G}_1(W)$  whose length is at least  $\delta_0/2$ . Inductively, define  $F_j(W)$ , for  $2 \leq j \leq n-1$ , to contain those  $V \in \mathcal{G}_j(W)$  whose length is at least  $\delta_0/2$ , and such that  $T^kV$  is not contained in an element of  $F_{j-k}(W)$  for any  $1 \leq k \leq j-1$ . Thus  $F_j(W)$  contains elements of  $\mathcal{G}_j(W)$  that are ‘‘long for the first time’’ at time  $j$ .

We group  $W_i \in \mathcal{G}_n(W)$  by its “first long ancestor” as follows. We say  $W_i$  has *first long ancestor*<sup>21</sup>  $V \in F_j(W)$  for  $1 \leq j \leq n-1$  if  $T^{n-j}W_i \subseteq V$ . Note that such a  $j$  and  $V$  are unique for each  $W_i$  if they exist. If no such  $j$  and  $V$  exist, then  $W_i$  has been forever short and so must belong to  $\mathcal{I}_n(W)$ . Denote by  $A_{n-j}(V)$  the set of  $W_i \in \mathcal{G}_n(W)$  corresponding to one  $V \in F_j(W)$ . Now

$$\begin{aligned} \sum_{W_i \in \mathcal{G}_n(W)} \left( \frac{\log |W|}{\log |W_i|} \right)^{\bar{\gamma}} &= \sum_{j=1}^{n-1} \sum_{V_\ell \in F_j(W)} \sum_{W_i \in A_{n-j}(V_\ell)} \left( \frac{\log |W|}{\log |W_i|} \right)^{\bar{\gamma}} + \sum_{W_i \in \mathcal{I}_n(W)} \left( \frac{\log |W|}{\log |W_i|} \right)^{\bar{\gamma}} \\ &\leq \sum_{j=1}^{n-1} \sum_{V_\ell \in F_j(W)} \left( \frac{\log |W|}{\log |V_\ell|} \right)^{\bar{\gamma}} \sum_{W_i \in A_{n-j}(V_\ell)} \left( \frac{\log |V_\ell|}{\log |W_i|} \right)^{\bar{\gamma}} + 2^{(n+1)s_0\bar{\gamma}} (Km+1)^{n/m} \\ &\leq \sum_{j=1}^{n-1} \sum_{V_\ell \in F_j(W)} \left( \frac{\log |W|}{\log |V_\ell|} \right)^{\bar{\gamma}} 2^{\bar{\gamma}+1} \delta_0^{-1} \# \mathcal{M}_0^{n-j} + 2^{(n+1)s_0\bar{\gamma}} (Km+1)^{n/m} \\ &\leq \sum_{j=1}^{n-1} 2^{(j+1)s_0\bar{\gamma}} (Km+1)^{j/m} 2^{\bar{\gamma}+1} \delta_0^{-1} \# \mathcal{M}_0^{n-j} + 2^{(n+1)s_0\bar{\gamma}} (Km+1)^{n/m} \\ &\leq 2^{2\bar{\gamma}+1} \delta_0^{-1} \sum_{j=1}^n 2^{js_0\bar{\gamma}} (Km+1)^{j/m} \# \mathcal{M}_0^{n-j}, \end{aligned}$$

where we have applied part (a) from time 1 to time  $j$  and the first estimate in part (b) from time  $j$  to time  $n$ , since each  $|V_\ell| \geq \delta_0/2$ .  $\square$

With the growth lemma proved, we can choose  $m$  and the length scale  $\delta_0$  of curves in  $\mathcal{W}^s$ . Recalling  $K$  from (5.1) and the condition on  $\gamma$  from (4.4), we fix  $m$  so large that

$$(5.4) \quad \frac{1}{m} \log(Km+1) < h_* - \gamma s_0 \log 2,$$

and we choose  $\delta_0 = \delta_0(m)$  to be the corresponding length scale from Lemma 5.1. If  $h_* > s_0 \log 2$ , then we take  $\gamma > 1$ , so that in fact  $\frac{1}{m} \log(Km+1) < h_* - s_0 \log 2$ .

**5.2. Fragmentation Lemmas.** The results in this subsection will be used in Sections 5.3 and 7.3. For  $\delta \in (0, \delta_0)$  and  $W \in \widehat{\mathcal{W}}^s$ , define  $\mathcal{G}_n^\delta(W)$  to be the smooth components of  $T^{-n}W$ , with long pieces subdivided to have length between  $\delta/2$  and  $\delta$ . (So  $\mathcal{G}_n^\delta(W)$  is defined exactly like  $\mathcal{G}_n(W)$ , but with  $\delta_0$  replaced by  $\delta$ .) Let  $L_n^\delta(W)$  denote the set of curves in  $\mathcal{G}_n^\delta(W)$  that have length at least  $\delta/3$  and let  $S_n^\delta(W) = \mathcal{G}_n^\delta(W) \setminus L_n^\delta(W)$ . Define  $\mathcal{I}_n^\delta(W)$  to be those curves in  $\mathcal{G}_n^\delta(W)$  that have no ancestors<sup>22</sup> of length at least  $\delta/3$ , as in the definition of  $\mathcal{I}_n(W)$  above. The following lemma and its corollary bootstrap from Lemma 5.1(a) and will be crucial to get the lower bound on the spectral radius:

**Lemma 5.2.** *For each  $\varepsilon > 0$ , there exist  $\delta \in (0, \delta_0]$  and  $n_1 \in \mathbb{N}$ , such that for  $n \geq n_1$ ,*

$$\frac{\#L_n^\delta(W)}{\#\mathcal{G}_n^\delta(W)} \geq \frac{1-2\varepsilon}{1-\varepsilon}, \quad \text{for all } W \in \widehat{\mathcal{W}}^s \text{ with } |W| \geq \delta/3.$$

*Proof.* Fix  $\varepsilon > 0$  and choose  $n_1$  so large that  $3C_1^{-1}(Kn_1+1)\Lambda^{-n_1} < \varepsilon$  and  $\Lambda^{n_1} > e$ . Next, choose  $\delta > 0$  sufficiently small that if  $W \in \widehat{\mathcal{W}}^s$  with  $|W| < \delta$ , then  $T^{-n}W$  comprises at most  $Kn+1$  smooth pieces of length at most  $\delta_0$  for all  $n \leq 2n_1$ .

Let  $W \in \widehat{\mathcal{W}}^s$  with  $|W| \geq \delta/3$ . We shall prove the following equivalent inequality for  $n \geq n_1$ :

$$\frac{\#S_n^\delta(W)}{\#\mathcal{G}_n^\delta(W)} \leq \frac{\varepsilon}{1-\varepsilon}.$$

<sup>21</sup>Note that “ancestor” refers to the backwards dynamics mapping  $W$  to  $W_i$ .

<sup>22</sup>For  $k < n$ , we say that  $U \in \mathcal{G}_k^\delta(W)$  is an *ancestor* of  $V \in \mathcal{G}_n^\delta(W)$  if  $T^{n-k}V \subseteq U$ .

For  $n \geq n_1$ , write  $n = kn_1 + \ell$  for some  $0 \leq \ell < n_1$ . If  $k = 1$ , the above inequality is clear since  $S_{n_1+\ell}^\delta(W)$  contains at most  $K(n_1 + \ell) + 1$  components by assumption on  $\delta$  and  $n_1$ , while  $|T^{-(n_1+\ell)}W| \geq C_1\Lambda^{n_1+\ell}|W| \geq C_1\Lambda^{n_1+\ell}\delta/3$ . Thus  $\mathcal{G}_n^\delta(W)$  must contain at least  $C_1\Lambda^{n_1+\ell}/3$  curves since each has length at most  $\delta$ . Thus,

$$\frac{\#S_{n_1+\ell}^\delta(W)}{\#\mathcal{G}_{n_1+\ell}^\delta(W)} \leq 3C_1^{-1} \frac{K(n_1 + \ell) + 1}{\Lambda^{n_1+\ell}} \leq 3C_1^{-1} \frac{Kn_1 + 1}{\Lambda^{n_1}} < \varepsilon,$$

where the second inequality holds for all  $\ell \geq 0$  as long as  $\frac{1}{n_1} \leq \log \Lambda$ , which is true by choice of  $n_1$ .

For  $k > 1$ , we split  $n$  into  $k - 1$  blocks of length  $n_1$  and the last block of length  $n_1 + \ell$ . We group elements  $W_i \in S_{kn_1+\ell}^\delta(W)$  by most recent<sup>23</sup> long ancestor  $V_j \in L_{qn_1}^\delta(W)$ :  $q$  is the greatest index in  $[0, k - 1]$  such that  $T^{(k-q)n_1+\ell}W_i \subseteq V_j$  and  $V_j \in L_{qn_1}^\delta(W)$ . Note that since  $|V_j| \geq \delta/3$ , then  $\mathcal{G}_{(k-q)n_1+\ell}^\delta(V_j)$  must contain at least  $C_1\Lambda^{(k-q)n_1}/3$  curves since each has length at most  $\delta$ . Thus using Lemma 5.1(a) with  $\bar{\gamma} = 0$ , we estimate

$$\begin{aligned} \frac{\#S_{kn_1+\ell}^\delta(W)}{\#\mathcal{G}_{kn_1+\ell}^\delta(W)} &= \frac{\sum_{W_i \in \mathcal{I}_{kn_1+\ell}^\delta(W)} 1}{\#\mathcal{G}_{kn_1+\ell}^\delta(W)} + \frac{\sum_{q=1}^{k-1} \sum_{V_j \in L_{qn_1}^\delta(W)} \sum_{W_i \in \mathcal{I}_{(k-q)n_1+\ell}^\delta(V_j)} 1}{\#\mathcal{G}_{kn_1+\ell}^\delta(W)} \\ (5.5) \quad &\leq \frac{(Kn_1 + 1)^k}{C_1\Lambda^{kn_1}/3} + \sum_{q=1}^{k-1} \frac{\sum_{V_j \in L_{qn_1}^\delta(W)} (Kn_1 + 1)^{k-q}}{\sum_{V_j \in L_{qn_1}^\delta(W)} C_1\Lambda^{(k-q)n_1}/3} \\ &\leq 3C_1^{-1} \sum_{q=1}^k (Kn_1 + 1)^q \Lambda^{-qn_1} \leq \sum_{q=1}^k \varepsilon^q \leq \frac{\varepsilon}{1 - \varepsilon}. \end{aligned}$$

□

The following corollary is used in Corollary 7.9 and in Lemma 7.7:

**Corollary 5.3.** *There exists  $C_2 > 0$  such that for any  $\varepsilon, \delta$  and  $n_1$  as in Lemma 5.2,*

$$\frac{\#L_n^\delta(W)}{\#\mathcal{G}_n^\delta(W)} \geq \frac{1 - 3\varepsilon}{1 - \varepsilon}, \quad \forall W \in \widehat{\mathcal{W}}^s, \quad \forall n \geq C_2 n_1 \frac{|\log(|W|/\delta)|}{|\log \varepsilon|}.$$

*Proof.* The proof is essentially the same as that for Lemma 5.2, except that for curves shorter than length  $\delta/3$  one must wait  $n \sim |\log(|W|/\delta)|$  for at least one component of  $\mathcal{G}_n^\delta(W)$  to belong to  $L_n^\delta(W)$ .

More precisely, fix  $\varepsilon > 0$  and the corresponding  $\delta$  and  $n_1$  from Lemma 5.2. Let  $W \in \widehat{\mathcal{W}}^s$  with  $|W| < \delta/3$  and take  $n > n_1$ . Decomposing  $\mathcal{G}_n^\delta(W)$  as in Lemma 5.2, we estimate the second term of (5.5) as before.

For the first term of (5.5),  $\#\mathcal{I}_n^\delta(W)/\#\mathcal{G}_n^\delta(W)$ , for  $\delta$  sufficiently small, notice that since the flow is continuous, either  $\#\mathcal{G}_\ell^\delta(W) \leq K\ell + 1$  by (5.1) or at least one element of  $\mathcal{G}_\ell^\delta(W)$  has length at least  $\delta/3$ . Let  $n_2$  denote the first iterate  $\ell$  at which  $\mathcal{G}_\ell^\delta(W)$  contains at least one element of length more than  $\delta/3$ . By the complexity estimate (5.1) and the fact that  $|T^{-n_2}W| \geq C_1\Lambda^{n_2}|W|$  by (3.1), there exists  $\bar{C}_2 > 0$ , independent of  $W \in \widehat{\mathcal{W}}^s$ , such that  $n_2 \leq \bar{C}_2 |\log(|W|/\delta)|$ .

Now for  $n \geq n_2$ , and some  $W' \in \mathcal{G}_{n_2}^\delta(W)$ ,

$$\#\mathcal{I}_n^\delta(W) \leq (Kn_2 + 1) \#\mathcal{I}_{n-n_2}^\delta(W') \leq (Kn_2 + 1)(Kn_1 + 1)^{\lfloor (n-n_2)/n_1 \rfloor},$$

while

$$\#\mathcal{G}_n^\delta(W) \geq C_1\Lambda^{n-n_2}/3.$$

<sup>23</sup>We only consider what happens at the beginning of a block of length  $n_1$ . It does not affect our argument if  $W_i$  belongs to a long piece at an intermediate time, since we only consider the cardinality of short pieces that can be created in each block of length  $n_1$  according to our choice of  $\delta$ .

Putting these together, we have,

$$\frac{\#\mathcal{I}_n^\delta(W)}{\#\mathcal{G}_n^\delta(W)} \leq \frac{(Kn_2 + 1)(Kn_1 + 1)^{\lfloor n/n_1 \rfloor}}{C_1 \Lambda^n / 3} \Lambda^{n_2} \leq \varepsilon^{\lfloor n/n_1 \rfloor} (Kn_2 + 1) \Lambda^{n_2}.$$

Since  $n_2 \leq \bar{C}_2 |\log(|W|/\delta)|$ , we may make this expression  $< \varepsilon$  by choosing  $n$  so large that  $n/n_1 \geq C_2 \frac{\log(|W|/\delta)}{\log \varepsilon}$ , for some  $C_2 > 0$ . For such  $n$ , the estimate (5.5) is bounded by  $\varepsilon + \frac{\varepsilon}{1-\varepsilon} \leq \frac{2\varepsilon}{1-\varepsilon}$ , which completes the proof of the corollary.  $\square$

Choose  $\varepsilon = 1/4$  and let  $\delta_1 \leq \delta_0$  and  $n_1$  be the corresponding  $\delta$  and  $n_1$  from Lemma 5.2. With this choice, we have

$$(5.6) \quad \#L_n^{\delta_1}(W) \geq \frac{2}{3} \#\mathcal{G}_n^{\delta_1}(W), \quad \text{for all } W \in \widehat{\mathcal{W}}^s \text{ with } |W| \geq \delta_1/3 \text{ and } n \geq n_1.$$

Notice that for  $W \in \mathcal{W}^s$ , each element  $V \in \mathcal{G}_n^{\delta_1}(W)$  is contained in one element of  $\mathcal{M}_0^n$  and its image  $T^n V \subset W$  is contained in one element of  $\mathcal{M}_{-n}^0$ . Indeed, there is a one-to-one correspondence between elements of  $\mathcal{M}_0^n$  and elements of  $\mathcal{M}_{-n}^0$ .

The boundary of the partition formed by  $\mathcal{M}_{-n}^0$  is comprised of unstable curves belonging to  $\mathcal{S}_{-n} = \cup_{j=0}^n T^j(\mathcal{S}_0)$ . Let  $L_u(\mathcal{M}_{-n}^0)$  denote the elements of  $\mathcal{M}_{-n}^0$  whose unstable diameter<sup>24</sup> is at least  $\delta_1/3$ . Similarly, let  $L_s(\mathcal{M}_0^n)$  denote the elements of  $\mathcal{M}_0^n$  whose stable diameter is at least  $\delta_1/3$ .

The following lemma will be used to get both lower and upper bounds on the spectral radius via Proposition 5.5:

**Lemma 5.4.** *Let  $\delta_1$  and  $n_1$  be associated with  $\varepsilon = 1/4$  by Lemma 5.2. There exist  $C_{n_1} > 0$  and  $n_2 \geq n_1$  such that for all  $n \geq n_2$ ,*

$$\#L_u(\mathcal{M}_{-n}^0) \geq C_{n_1} \delta_1 \#\mathcal{M}_{-n}^0 \quad \text{and} \quad \#L_s(\mathcal{M}_0^n) \geq C_{n_1} \delta_1 \#\mathcal{M}_0^n.$$

*Proof.* We prove the lower bound for  $L_u(\mathcal{M}_{-n}^0)$ . The lower bound for  $L_s(\mathcal{M}_0^n)$  then follows by time reversal.

Let  $I_u(\mathcal{M}_{-n}^0)$  denote the elements of  $\mathcal{M}_{-n}^0$  whose unstable diameter is less than  $\delta_1/3$ . Clearly,  $I_u(\mathcal{M}_{-n}^0) \cup L_u(\mathcal{M}_{-n}^0) = \mathcal{M}_{-n}^0$ . Similarly, let  $I_u(T^j \mathcal{S}_0)$  denote the set of unstable curves in  $T^j(\mathcal{S}_0)$  whose length is less than  $\delta_1/3$ .

We first prove the following claim:  $\#I_u(\mathcal{M}_{-n}^0) \leq 2 \sum_{j=1}^n \#I_u(T^j \mathcal{S}_0) + K_2 n$ . Recall that the boundaries of elements of  $\mathcal{M}_{-n}^0$  are comprised of elements of  $\mathcal{S}_{-n} = \cup_{i=0}^n T^i \mathcal{S}_0$ , which are unstable curves for  $i \geq 1$ . We use the following property established in Lemma 3.1: If a smooth unstable curve  $U_i \subset T^i \mathcal{S}_0$  intersects a smooth curve  $U_j \subset T^j \mathcal{S}_0$ , for  $i < j$ , then  $U_j$  must terminate on  $U_i$ . Thus if  $A \in I_u(\mathcal{M}_{-n}^0)$ , then either the boundary of  $A$  contains a short curve in  $T^j(\mathcal{S}_0)$  for some  $1 \leq j \leq n$ , or  $\partial A$  contains an intersection point of two curves in  $T^j(\mathcal{S}_0)$  for some  $1 \leq j \leq n$  (see Figure 3). But such intersections of curves within  $T^j(\mathcal{S}_0)$  are images of intersections of curves within  $T(\mathcal{S}_0)$ , and the cardinality of cells created by such intersections is bounded by some uniform constant  $K_2 > 0$  depending only on  $T(\mathcal{S}_0)$ . Then, since each short curve in  $T^j(\mathcal{S}_0)$  belongs to the boundary of at most two  $A \in I_u(\mathcal{M}_{-n}^0)$ , the claim follows.

Next, subdivide  $\mathcal{S}_0$  into  $\ell_0$  horizontal segments  $U_i$  such that  $TU_i$  is an unstable curve of length between  $\delta_1/3$  and  $\delta_1$  for each  $i$ . Analogous to stable curves, let  $\mathcal{G}_j^{\delta_1}(U)$  denote the decomposition of the union of unstable curves comprising  $T^j U$  at length scale  $\delta_1$ . Then for  $j \geq n_1$  using the time reversal of (5.6), we have

$$(5.7) \quad \#I_u(T^j \mathcal{S}_0) = \sum_{i=1}^{\ell_0} \#I_u(\mathcal{G}_{j-1}^{\delta_1}(TU_i)) \leq \frac{1}{2} \sum_{i=1}^{\ell_0} \#L_u(\mathcal{G}_{j-1}^{\delta_1}(TU_i)).$$

<sup>24</sup>Recall from Section 3 that the unstable diameter of a set is the length of the longest unstable curve contained in that set.



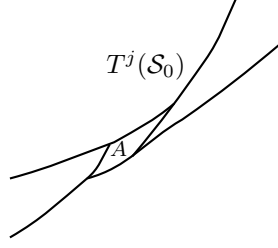


FIGURE 3. A short cell  $A \in I_u(\mathcal{M}_{-n}^0)$  created by long elements of  $T^j(\mathcal{S}_0)$ .

Using the claim and (5.7) we split the sum over  $j$  into 2 parts,

$$(5.8) \quad \#I_u(\mathcal{M}_{-n}^0) \leq K_2 n + 2 \sum_{j=1}^{n_1-1} \#I_u(T^j \mathcal{S}_0) + \sum_{j=n_1}^n \sum_{i=1}^{\ell_0} \#L_u(\mathcal{G}_{j-1}^{\delta_1}(TU_i)).$$

The cardinality of the sum over the first  $n_1$  terms is bounded by a fixed constant depending on  $n_1$ , but not on  $n$ ; let us call it  $\bar{C}_{n_1}$ . We want to relate the sum over the terms for  $j \geq n_1$  to  $L_u(\mathcal{M}_{-n}^0)$ . To this end, we follow the proof of Lemma 5.2 and split  $n - j$  into blocks of length  $n_1$ .

For each  $n_1 \leq j \leq n - n_1$ , write  $n - j = kn_1 + \ell$ , for some  $k \geq 1$ . If  $V \in L_u(\mathcal{G}_{j-1}^{\delta_1}(TU_i))$ , then  $|T^{n-j}V| \geq C_1 \Lambda^{n-j} \delta_1 / 3$ , while  $T^{n-j}V$  can be cut into at most  $(Kn_1 + 1)^k$  pieces. Since we have chosen  $\varepsilon = 1/4$  in the application of Lemma 5.2, by choice of  $n_1$ ,

$$\#L_u(\mathcal{G}_{j-1}^{\delta_1}(TU_i)) \geq 4^k \#L_u(\mathcal{G}_{j-1}^{\delta_1}(TU_i)) \text{ for each } n_1 \leq j \leq n - n_1 \text{ and } k = \lfloor (n - j)/n_1 \rfloor.$$

For  $n - n_1 < j \leq n$ , we perform the same estimate, but relating  $j$  with  $j + n_1$ ,

$$\#L_u(\mathcal{G}_{j+n_1-1}^{\delta_1}(TU_i)) \geq 4 \#L_u(\mathcal{G}_{j-1}^{\delta_1}(TU_i)) \text{ for each } n - n_1 + 1 \leq j \leq n.$$

Gathering these estimates together and using (5.8), we obtain,

$$(5.9) \quad \begin{aligned} \#I_u(\mathcal{M}_{-n}^0) &\leq K_2 n + \bar{C}_{n_1} + \sum_{j=n_1}^{n-n_1} 4^{-\lfloor (n-j)/n_1 \rfloor} \#L_u(T^n \mathcal{S}_0) + \sum_{j=n-n_1+1}^n \frac{1}{4} \#L_u(T^{j+n_1} \mathcal{S}_0) \\ &\leq 2K_2 n + \bar{C}_{n_1} + C\delta_1^{-1} n_1 \#L_u(\mathcal{M}_{-n}^0) + \sum_{j=n-n_1+1}^n C\delta_1^{-1} \#L_u(\mathcal{M}_{-j-n_1}^0), \end{aligned}$$

where the second inequality uses  $\#L_u(T^\ell \mathcal{S}_0) \leq C\delta_1^{-1} \#L_u(\mathcal{M}_{-\ell}^0) + K_2$  for  $\ell \geq n$ , which stems from the same non-crossing property used earlier: a curve in  $T^\ell(\mathcal{S}_0)$  must terminate on a curve in  $T^i(\mathcal{S}_0)$  if the two intersect for  $i < \ell$ .

To estimate the final sum in (5.9), note that if  $A \in L_u(\mathcal{M}_{-n-1}^0)$ , then  $A \subseteq A' \in L_u(\mathcal{M}_{-n}^0)$ . Moreover, there exists a constant  $B > 0$ , independent of  $n$ , such that each  $A' \in L_u(\mathcal{M}_{-n}^0)$  can contain at most  $B$  elements of  $L_u(\mathcal{M}_{-n-1}^0)$ . (Indeed by Lemma 3.3,  $B$  is at most  $|\hat{\mathcal{P}}|$ , and depends only on  $\mathcal{S}_1$ .) Inductively then,

$$\sum_{j=1}^{n_1} \#L_u(\mathcal{M}_{-n-j}^0) \leq \sum_{j=1}^{n_1} B^j \#L_u(\mathcal{M}_{-n}^0) \leq CB^{n_1} \#L_u(\mathcal{M}_{-n}^0).$$

Putting this estimate together with (5.9) yields,

$$\#I_u(\mathcal{M}_{-n}^0) \leq \#L_u(\mathcal{M}_{-n}^0) C\delta_1^{-1} (n_1 + B^{n_1}) + C_{n_1} + 2K_2 n.$$

Using  $\#\mathcal{M}_{-n}^0 = \#L_u(\mathcal{M}_{-n}^0) + \#I_u(\mathcal{M}_{-n}^0)$ , this implies,

$$\#L_u(\mathcal{M}_{-n}^0) \geq \frac{\#\mathcal{M}_{-n}^0 - C_{n_1} - 2K_2 n}{1 + C\delta_1^{-1} (n_1 + B^{n_1})}.$$

Since  $\#\mathcal{M}_{-n}^0$  increases at an exponential rate and  $n_1$  is fixed, there exists  $n_2 \in \mathbb{N}$  such that  $\#\mathcal{M}_{-n}^0 - \bar{C}_{n_1} - 2K_2n \geq \frac{1}{2}\#\mathcal{M}_{-n}^0$ , for  $n \geq n_2$ . Thus there exists  $C_{n_1} > 0$  such that for  $n \geq n_2$ ,  $\#L_u(\mathcal{M}_{-n}^0) \geq C_{n_1}\delta_1\#\mathcal{M}_{-n}^0$ , as required.  $\square$

**5.3. Exact Exponential Growth of  $\#\mathcal{M}_0^n$  — Cantor Rectangles.** It follows from submultiplicativity of  $\#\mathcal{M}_0^n$  that  $e^{nh_*} \leq \#\mathcal{M}_0^n$  for all  $n$ . In this subsection, we shall prove a supermultiplicativity statement (Lemma 5.6) from which we deduce the upper bound for  $\#\mathcal{M}_0^n$  in Proposition 4.6 giving the upper bound in Proposition 4.7, and ultimately the upper bound on the spectral radius of  $\mathcal{L}$  on  $\mathcal{B}$ .

The following key estimate is a lower bound on the rate of growth of stable curves having a certain length. The proof will crucially use the fact that the SRB measure is mixing in order to bootstrap from Lemma 5.4.

**Proposition 5.5.** *Let  $\delta_1$  be the value of  $\delta$  from Lemma 5.2 associated with  $\varepsilon = 1/4$  (see (5.6)). There exists  $c_0 > 0$  such that for all  $W \in \widehat{\mathcal{W}}^s$  with  $|W| \geq \delta_1/3$  and  $n \geq 1$ , we have  $\#\mathcal{G}_n(W) \geq c_0\#\mathcal{M}_0^n$ . The constant  $c_0$  depends on  $\delta_1$ .*

This will be used for the lower bound in Section 6.3. It also has the following important consequence.

**Lemma 5.6** (Supermultiplicativity). *There exists  $c_1 > 0$  such that  $\forall n, j \in \mathbb{N}$ , with  $j \leq n$ , we have*

$$\#\mathcal{M}_0^n \geq c_1\#\mathcal{M}_0^{n-j}\#\mathcal{M}_0^j.$$

We next introduce Cantor rectangles. Let  $W^s(x)$  and  $W^u(x)$  denote the maximal smooth components of the local stable and unstable manifolds of  $x \in M$ .

**Definition 5.7** ((Locally Maximal) Cantor Rectangles). *A solid rectangle  $D$  in  $M$  is a closed region whose boundary comprises precisely four nontrivial curves: two stable manifolds and two unstable manifolds. Given a solid rectangle  $D$ , the locally maximal Cantor rectangle  $R$  in  $D$  is formed by taking the union of all points in  $D$  whose local stable and unstable manifolds completely cross  $D$ . Locally maximal Cantor rectangles have a natural product structure: for any  $x, y \in R$ ,  $W^s(x) \cap W^u(y) \in R$ , where  $W^{s/u}(x)$  is the local stable/unstable manifold containing  $x$ . It is proved in [CM, Section 7.11] that such rectangles are closed and as such contain their outer boundaries, which coincide with the boundary of  $D$ . We shall refer to this pair of stable and unstable manifolds as the stable and unstable boundaries of  $R$ . In this case, we denote  $D$  by  $D(R)$  to emphasize that it is the smallest solid rectangle containing  $R$ . We shall sometimes drop the words “locally maximal” referring simply to Cantor rectangles  $R$ .*

**Definition 5.8** (Properly Crossing a (Locally Maximal) Cantor Rectangle). *For a (locally maximal) Cantor rectangle  $R$  such that*

$$(5.10) \quad \inf_{x \in R} \frac{m_{W^u}(W^u(x) \cap R)}{m_{W^u}(W^u(x) \cap D(R))} \geq 0.9,$$

*we<sup>25</sup> say a stable curve  $W \in \widehat{\mathcal{W}}^s$  properly crosses  $R$  if*

- a)  $W$  crosses both unstable sides of  $R$ ;*
- b) for every  $x \in R$ , the intersection  $W \cap W^s(x) \cap D(R) = \emptyset$ , i.e.,  $W$  does not cross any stable manifolds in  $R$ ;*
- c) for all  $x \in R$ , the point  $W \cap W^u(x)$  divides the curve  $W^u(x) \cap D(R)$  in a ratio between 0.1 and 0.9, i.e.,  $W$  does not come too close to either unstable boundary of  $R$ .*

<sup>25</sup>This is a version of Definition 7.85 of [CM] formulated with stable (instead of unstable) curves crossing  $R$ . We have also dropped any mention of homogeneous components, which are used in the construction in [CM].

**Remark 5.9.** *The (unstable analogue of) condition b) is not needed in its full strength, even in the proof of [CM, Lemma 7.90]. What is used there is that the fake unstable is trapped between two real unstable that it does not cross. Since the real unstable intersect and fully cross the target rectangle, this forces the fake unstable to do so as well. For us, we reverse time and consider stable manifolds. For real stable manifolds, condition (b) is not needed at all: If a real stable fully crosses the initial rectangle, then, when it intersects the target rectangle under iteration by  $T^{-n}$ , it must intersect a real stable manifold, and it must fully cross. (Otherwise, the preimage of a singularity would lie on a real stable manifold in the interior of the target rectangle. But this cannot be since real stable manifolds are never cut going forward and so do not intersect the preimages of singularity curves except at their end points.) When discussing proper crossing for real stable manifolds, we will drop condition (b) and allow  $W \in \mathcal{W}^s$  to be one of the stable manifolds defining  $R$ .*

*Proof of Proposition 5.5.* Using [CM, Lemma 7.87], we may cover  $M$  by Cantor rectangles  $R_1, \dots, R_k$  satisfying (5.10) whose stable and unstable boundaries have length at most  $\frac{1}{10}\delta_1$ , with the property that any stable curve of length at least  $\delta_1/3$  properly crosses at least one of them. The cardinality  $k$  is fixed, depending only on  $\delta_1$ .

Recall that  $L_u(\mathcal{M}_{-n}^0)$  denotes the elements of  $\mathcal{M}_{-n}^0$  whose unstable diameter is longer than  $\delta_1/3$ . We claim that for all  $n \in \mathbb{N}$ , at least one  $R_i$  is fully crossed in the unstable direction by at least  $\frac{1}{k}\#L_u(\mathcal{M}_{-n}^0)$  elements of  $\mathcal{M}_{-n}^0$ . Notice that if  $A \in \mathcal{M}_{-n}^0$ , then  $\partial A$  is comprised of unstable curves belonging to  $\cup_{i=1}^n T^i \mathcal{S}_0$ , and possibly  $\mathcal{S}_0$ . By definition of unstable manifolds,  $T^i \mathcal{S}_0$  cannot intersect the unstable boundaries of the  $R_i$ ; thus if  $A \cap R_i \neq \emptyset$ , then either  $\partial A$  terminates inside  $R_i$  or  $A$  fully crosses  $R_i$ . Thus elements of  $L_u(\mathcal{M}_{-n}^0)$  fully cross at least one  $R_i$  and so at least one  $R_i$  must be fully crossed by  $1/k$  of them, proving the claim.

For each  $n \in \mathbb{N}$ , denote by  $i_n$  the index of a rectangle  $R_{i_n}$  which is fully crossed by at least  $\frac{1}{k}\#L_u(\mathcal{M}_{-n}^0)$  elements of  $\mathcal{M}_{-n}^0$ . The main idea at this point will be to force every stable curve to properly cross  $R_{i_n}$  in a bounded number of iterates and so to intersect all elements of  $\mathcal{M}_{-n}^0$  that fully cross  $R_{i_n}$ .

To this end, fix  $\delta_* \in (0, \delta_1/10)$  and for  $i = 1, \dots, k$ , choose a ‘‘high density’’ subset  $R_i^* \subset R_i$  satisfying the following conditions:  $R_i^*$  has nonzero Lebesgue measure, and for any unstable manifold  $W^u$  such that  $W^u \cap R_i^* \neq \emptyset$  and  $|W^u| < \delta_*$ , we have  $\frac{m_{W^u}(W^u \cap R_i^*)}{|W^u|} \geq 0.9$ . (Such a  $\delta_*$  and  $R_i^*$  exist due to the fact that  $m_{W^u}$ -almost every  $y \in R_i$  is a Lebesgue density point of the set  $W^u(y) \cap R_i$  and the unstable foliation is absolutely continuous with respect to  $\mu_{\text{SRB}}$  or, equivalently, Lebesgue.)

Due to the mixing property of  $\mu_{\text{SRB}}$  and the finiteness of the number of rectangles  $R_i$ , there exist  $\varepsilon > 0$  and  $n_3 \in \mathbb{N}$  such that for all  $1 \leq i, j \leq k$  and all  $n \geq n_3$ ,  $\mu_{\text{SRB}}(R_i^* \cap T^{-n}R_j) \geq \varepsilon$ . If necessary, we increase  $n_3$  so that the unstable diameter of the set  $T^{-n}R_i$  is less than  $\delta_*$  for each  $i$ , and  $n \geq n_3$ .

Now let  $W \in \widehat{\mathcal{W}}^s$  with  $|W| \geq \delta_1/3$  be arbitrary. Let  $R_j$  be a Cantor rectangle that is properly crossed by  $W$ . Let  $n \in \mathbb{N}$  and let  $i_n$  be as above. By mixing,  $\mu_{\text{SRB}}(R_{i_n}^* \cap T^{-n_3}R_j) \geq \varepsilon$ . By [CM, Lemma 7.90], there is a component of  $T^{-n_3}W$  that fully crosses  $R_{i_n}^*$  in the stable direction. Call this component  $V \in \mathcal{G}_{n_3}^{\delta_1}(W)$ . By choice of  $R_{i_n}$ , this implies that  $\#\mathcal{G}_n(V) \geq \frac{1}{k}\#L_u(\mathcal{M}_{-n}^0)$ , and thus

$$\#\mathcal{G}_{n+n_3}(W) \geq \frac{1}{k}\#L_u(\mathcal{M}_{-n}^0) \implies \#\mathcal{G}_n(W) \geq \frac{C'}{k}\#L_u(\mathcal{M}_{-n}^0),$$

where  $C'$  is a constant depending only on  $n_3$  since at each refinement of  $\mathcal{M}_{-j}^0$  to  $\mathcal{M}_{-j-1}^0$ , the cardinality of the partition increases by a factor which is at most  $|\mathring{\mathcal{P}}|$ , as noted in the proof of Lemma 5.4. The final estimate needed is  $\#L_u(\mathcal{M}_{-n}^0) \geq C_{n_1}\delta_1\#\mathcal{M}_{-n}^0$ , for  $n \geq n_2$  from Lemma 5.4. Thus the proposition holds for  $n \geq \max\{n_2, n_3\}$ . It extends to all  $n \in \mathbb{N}$  since  $\#\mathcal{M}_0^n \leq (\#\mathcal{M}_0^1)^n$  and there are only finitely many values of  $n$  to correct for.  $\square$

*Proof of Lemma 5.6.* Recall the singularity sets  $\mathcal{S}_n = \cup_{i=0}^n T^{-i}\mathcal{S}_0$  and  $\mathcal{S}_{-k} = \cup_{i=0}^k T^i\mathcal{S}_0$  defined for  $n, k \in \mathbb{N}$ . Due to the relation,  $T^{-k}(\mathcal{S}_{-k} \cup \mathcal{S}_n) = \mathcal{S}_k \cup T^{-k}\mathcal{S}_n = \mathcal{S}_{n+k}$ , we have a one-to-one correspondence between elements of  $\mathcal{M}_{-k}^n$  and  $\mathcal{M}_0^{n+k}$ .

Now fix  $n, j \in \mathbb{N}$  with  $j < n$ . Using the above relation, we have,

$$\#\mathcal{M}_0^n = \#\mathcal{M}_{-j}^{n-j} = \#(\mathcal{M}_0^{n-j} \vee \mathcal{M}_{-j}^0).$$

In order to prove the lemma, it suffices to show that a positive fraction (independent of  $n$  and  $j$ ) of elements of  $\mathcal{M}_0^{n-j}$  intersect a positive fraction of elements of  $\mathcal{M}_{-j}^0$ . Note that  $\partial\mathcal{M}_0^{n-j}$  is comprised of stable curves, while  $\partial\mathcal{M}_{-j}^0$  is comprised of unstable curves.

Recall that  $L_u(\mathcal{M}_{-j}^0)$  denotes the elements of  $\mathcal{M}_{-j}^0$  whose unstable diameter is longer than  $\delta_1/3$ . Similarly,  $L_s(\mathcal{M}_0^{n-j})$  denotes those elements of  $\mathcal{M}_0^{n-j}$  whose stable diameter is longer than  $\delta_1/3$ . By Lemma 5.4,

$$\#L_s(\mathcal{M}_0^{n-j}) \geq C_{n_1} \delta_1 \#\mathcal{M}_0^{n-j}, \quad \text{for } n-j \geq n_2.$$

Let  $A \in L_s(\mathcal{M}_0^{n-j})$  and let  $V \in \widehat{\mathcal{W}}^s$  be a stable curve in  $A$  with length at least  $\delta_1/3$ . By Proposition 5.5,  $\#\mathcal{G}_j(V) \geq c_0 \#\mathcal{M}_0^j$ . Each component of  $\mathcal{G}_j(V)$  corresponds to one component of  $V \setminus \mathcal{S}_{-j}$  (up to subdivision of long pieces in  $\mathcal{G}_j(V)$ ). Thus  $V$  intersects at least  $c_0 \#\mathcal{M}_0^j = c_0 \#\mathcal{M}_{-j}^0$  elements of  $\mathcal{M}_{-j}^0$ . Since this holds for all  $A \in L_s(\mathcal{M}_0^{n-j})$ , we have

$$\#\mathcal{M}_0^n = \#(\mathcal{M}_0^{n-j} \vee \mathcal{M}_{-j}^0) \geq \#L_s(\mathcal{M}_0^{n-j}) \cdot c_0 \#\mathcal{M}_0^j \geq C_{n_1} \delta_1 c_0 \#\mathcal{M}_0^{n-j} \#\mathcal{M}_0^j,$$

proving the lemma with  $c_1 = c_0 C_{n_1} \delta_1$  when  $n-j \geq n_2$ . For  $n-j \leq n_2$ , since  $\#\mathcal{M}_0^{n-j} \leq (\#\mathcal{M}_0^1)^{n-j}$ , we obtain the lemma by decreasing  $c_1$  since there are only finitely many values to correct for.  $\square$

*Proof of Proposition 4.6.* Define  $\psi(n) = \#\mathcal{M}_0^n e^{-nh_*}$ , and note that  $\psi(n) \geq 1$  for all  $n$ . From Lemma 5.6 it follows that

$$(5.11) \quad \psi(n) \geq c_1 \psi(j) \psi(n-j), \quad \text{for all } n \in \mathbb{N}, \text{ and } 0 \leq j \leq n.$$

Suppose there exists  $n_1 \in \mathbb{N}$  such that  $\psi(n_1) \geq 2/c_1$ . Then using (5.11), we have

$$\psi(2n_1) \geq c_1 \psi(n_1) \psi(n_1) \geq \frac{4}{c_1}.$$

Iterating this bound, we have inductively for any  $k \geq 1$ ,

$$\psi(2kn_1) \geq c_1 \psi(2n_1) \psi(2(k-1)n_1) \geq c_1 \frac{4}{c_1} \frac{4^{k-1}}{c_1} = \frac{4^k}{c_1}.$$

This implies that  $\lim_{k \rightarrow \infty} \frac{1}{2kn_1} \log \psi(2kn_1) \geq \frac{\log 4}{2n_1}$ , which contradicts the definition of  $\psi(n)$  (since  $\lim_{n \rightarrow \infty} \frac{1}{n} \log \psi(n) = 0$ ). We conclude that  $\psi(n) \leq 2/c_1$  for all  $n \geq 1$ .  $\square$

Our final result of this section demonstrates the uniform exponential rate of growth enjoyed by all stable curves of length at least  $\delta_1/3$ .

**Corollary 5.10.** *For all stable curves  $W \in \widehat{\mathcal{W}}^s$  with  $|W| \geq \delta_1/3$  and all  $n \geq n_1$ , we have*

$$\frac{2\delta_1 c_0}{9} e^{nh_*} \leq |T^{-n}W| \leq \frac{4}{c_1} e^{nh_*}.$$

*Proof.* For  $W \in \widehat{\mathcal{W}}^s$  with  $|W| \leq \delta_1/3$ , Lemma 5.1(b) with  $\bar{\gamma} = 0$  together with Propositions 4.6 and 5.5 yield,

$$c_0 e^{nh_*} \leq c_0 \#\mathcal{M}_0^n \leq \#\mathcal{G}_n(W) \leq 2\delta_0^{-1} \#\mathcal{M}_0^n \leq \frac{4}{c_1 \delta_0} e^{nh_*}.$$

The upper bound of the corollary is completed by noting that

$$|T^{-n}W| = \sum_{W_i \in \mathcal{G}_n(W)} |W_i| \leq \delta_0 \#\mathcal{G}_n(W).$$

The lower bound follows using (5.6) since  $\#\mathcal{G}_n^{\delta_1}(W) \geq \#\mathcal{G}_n(W)$ ,

$$(5.12) \quad |T^{-n}W| = \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |W_i| \geq \frac{\delta_1}{3} \#L_n^{\delta_1}(W) \geq \frac{2\delta_1}{9} \#\mathcal{G}_n^{\delta_1}(W) \geq \frac{2\delta_1 c_0}{9} e^{nh_*}.$$

□

## 6. PROOF OF THE “LASOTA–YORKE” PROPOSITION 4.7 — SPECTRAL RADIUS

**6.1. Weak Norm and Strong Stable Norm Estimates.** We start with the weak norm estimate (4.9). Let  $f \in C^1(M)$ ,  $W \in \mathcal{W}^s$ , and  $\psi \in C^\alpha(W)$  be such that  $|\psi|_{C^\alpha(W)} \leq 1$ . For  $n \geq 0$  we use the definition of the weak norm on each  $W_i \in \mathcal{G}_n(W)$  to estimate

$$(6.1) \quad \int_W \mathcal{L}^n f \psi \, dm_W = \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f \psi \circ T^n \, dm_W \leq \sum_{W_i \in \mathcal{G}_n} |f|_w |\psi \circ T^n|_{C^\alpha(W_i)}.$$

Clearly,  $\sup |\psi \circ T^n|_{W_i} \leq \sup_W |\psi|$ . For  $x, y \in W_i$ , we have,

$$(6.2) \quad \frac{|\psi(T^n x) - \psi(T^n y)|}{d_W(T^n x, T^n y)^\alpha} \cdot \frac{d_W(T^n x, T^n y)^\alpha}{d_W(x, y)^\alpha} \leq C |\psi|_{C^\alpha(W)} |J^s T^n|_{C^0(W_i)}^\alpha \leq C \Lambda^{-\alpha n} |\psi|_{C^\alpha(W)},$$

so that  $H_{W_i}^\alpha(\psi \circ T^n) \leq C \Lambda^{-\alpha n} H_W^\alpha(\psi)$  and thus  $|\psi \circ T^n|_{C^\alpha(W_i)} \leq C |\psi|_{C^\alpha(W)}$ . Using this estimate and Lemma 5.1(b) with  $\bar{\gamma} = 0$  in equation (6.1), we obtain

$$\int_W \mathcal{L}^n f \psi \, dm_W \leq \sum_{W_i \in \mathcal{G}_n(W)} C |f|_w \leq C \delta_0^{-1} |f|_w (\#\mathcal{M}_0^n).$$

Taking the supremum over  $W \in \mathcal{W}^s$  and  $\psi \in C^\alpha(W)$  with  $|\psi|_{C^\alpha(W)} \leq 1$  yields (4.9), using the upper bound on  $\#\mathcal{M}_0^n$  in Proposition 4.6.

We now prove the strong stable norm estimate (4.10). Recall that our choice of  $m$  in (5.4) implies  $2^{s_0 \gamma} (Km + 1)^{1/m} < e^{h_*}$ , where  $K$  is from (5.1). Define

$$(6.3) \quad D_n = D_n(m, \gamma) := 2^{2\gamma+1} \delta_0^{-1} \sum_{j=1}^n 2^{js_0 \gamma} (Km + 1)^{j/m} \#\mathcal{M}_0^{n-j}.$$

We claim that it follows from Proposition 4.6 that

$$(6.4) \quad D_n \leq C e^{nh_*}.$$

Indeed, by choice of  $\gamma$  and  $m$ , setting  $\varepsilon_1 := h_* - \log(2^{s_0 \gamma} (Km + 1)^{1/m}) > 0$ , we have

$$\begin{aligned} D_n &= 2^{2\gamma+1} \delta_0^{-1} \sum_{j=1}^n 2^{js_0 \gamma} (Km + 1)^{j/m} \#\mathcal{M}_0^{n-j} \leq 2^{2\gamma+1} \delta_0^{-1} \sum_{j=1}^n e^{(h_* - \varepsilon_1)j} \frac{2}{c_1} e^{(n-j)h_*} \\ &\leq 2^{2\gamma+1} \delta_0^{-1} \frac{2}{c_1} e^{nh_*} \sum_{j=1}^n e^{-\varepsilon_1 j}. \end{aligned}$$

To prove the strong stable bound, let  $W \in \mathcal{W}^s$  and  $\psi \in C^\beta(W)$  with  $|\psi|_{C^\beta(W)} \leq |\log |W||^\gamma$ . Using equation (6.1), and applying the strong stable norm on each  $W_i \in \mathcal{G}_n(W)$ , we write

$$\int_W \mathcal{L}^n f \psi \, dm_W = \sum_i \int_{W_i} f \psi \circ T^n \, dm_W \leq \sum_i \|f\|_s |\log |W_i||^{-\gamma} |\psi \circ T^n|_{C^\beta(W_i)}.$$

From the estimate analogous to (6.2), we have  $|\psi \circ T^n|_{C^\beta(W_i)} \leq C |\psi|_{C^\beta(W)} \leq C |\log |W||^\gamma$ . (Note that the contraction coming from the negative power of  $\Lambda$  in (6.2) cannot be exploited, see footnote 20 and the comments after Remark 4.8.)

Thus,

$$\int_W \mathcal{L}^n f \psi \, dm_W \leq C \|f\|_s \sum_{W_i \in \mathcal{G}_n(W)} \left( \frac{\log |W|}{\log |W_i|} \right)^\gamma \leq C \|f\|_s D_n,$$

where we have used Lemma 5.1(b) with  $\bar{\gamma} = \gamma$ .

Taking the supremum over  $W$  and  $\psi$  and recalling (6.4) proves (4.10), since we have shown that  $\|\mathcal{L}^n f\|_s \leq CD_n \|f\|_s$ .

**6.2. Unstable Norm Estimate.** Fix  $\varepsilon \leq \varepsilon_0$  and consider two curves  $W^1, W^2 \in \mathcal{W}^s$  with  $d_{\mathcal{W}^s}(W^1, W^2) \leq \varepsilon$ . For  $n \geq 1$ , we describe how to partition  $T^{-n}W^\ell$  into ‘‘matched’’ pieces  $U_j^\ell$  and ‘‘unmatched’’ pieces  $V_i^\ell$ ,  $\ell = 1, 2$ .

Let  $\omega$  be a connected component of  $W^1 \setminus \mathcal{S}_{-n}$ . To each point  $x \in T^{-n}\omega$ , we associate a vertical line segment  $\gamma_x$  of length at most  $C\Lambda^{-n}\varepsilon$  such that its image  $T^n\gamma_x$ , if not cut by a singularity, will have length  $C\varepsilon$ . By [CM, §4.4], all the tangent vectors to  $T^i\gamma_x$  lie in the unstable cone  $C^u(T^i x)$  for each  $i \geq 1$  so that they remain uniformly transverse to the stable cone and enjoy the minimum expansion given by  $\Lambda$ .

Doing this for each connected component of  $W^1 \setminus \mathcal{S}_{-n}$ , we subdivide  $W^1 \setminus \mathcal{S}_{-n}$  into a countable collection of subintervals of points for which  $T^n\gamma_x$  intersects  $W^2 \setminus \mathcal{S}_{-n}$  and subintervals for which this is not the case. This in turn induces a corresponding partition on  $W^2 \setminus \mathcal{S}_{-n}$ .

We denote by  $V_i^\ell$  the pieces in  $T^{-n}W^\ell$  which are not matched up by this process and note that the images  $T^n V_i^\ell$  occur either at the endpoints of  $W^\ell$  or because the vertical segment  $\gamma_x$  has been cut by a singularity. In both cases, the length of the curves  $T^n V_i^\ell$  can be at most  $C\varepsilon$  due to the uniform transversality of  $\mathcal{S}_{-n}$  with the stable cone and of  $C^s(x)$  with  $C^u(x)$ .

In the remaining pieces the foliation  $\{T^n\gamma_x\}_{x \in T^{-n}W^1}$  provides a one-to-one correspondence between points in  $W^1$  and  $W^2$ . We further subdivide these pieces in such a way that the lengths of their images under  $T^{-i}$  are less than  $\delta_0$  for each  $0 \leq i \leq n$  and the pieces are pairwise matched by the foliation  $\{\gamma_x\}$ . We call these matched pieces  $U_j^\ell$ . Since the stable cone is bounded away from the vertical direction, we can adjust the elements of  $\mathcal{G}_n(W^\ell)$  created by artificial subdivisions due to length so that  $U_j^\ell \subset W_i^\ell$  and  $V_k^\ell \subset W_{i'}^\ell$  for some  $W_i^\ell, W_{i'}^\ell \in \mathcal{G}_n(W^\ell)$  for all  $j, k \geq 1$  and  $\ell = 1, 2$ , without changing the cardinality of the bound on  $\mathcal{G}_n(W^\ell)$ . There is at most one  $U_j^\ell$  and two  $V_j^\ell$  per  $W_i^\ell \in \mathcal{G}_n(W^\ell)$ .

In this way we write  $W^\ell = (\cup_j T^n U_j^\ell) \cup (\cup_i T^n V_i^\ell)$ . Note that the images  $T^n V_i^\ell$  of the unmatched pieces must be short while the images of the matched pieces  $U_j^\ell$  may be long or short.

We have arranged a pairing of the pieces  $U_j^\ell = G_{U_j^\ell}(I_j)$ ,  $\ell = 1, 2$ , with the property:

$$(6.5) \quad \text{If } U_j^1 = \{(r, \varphi_{U_j^1}(r)) : r \in I_j\} \text{ then } U_j^2 = \{(r, \varphi_{U_j^2}(r)) : r \in I_j\},$$

so that the point  $x = (r, \varphi_{U_j^1}(r))$  is associated with the point  $\bar{x} = (r, \varphi_{U_j^2}(r))$  by the vertical segment  $\gamma_x \subset \{(r, s)\}_{s \in [-\pi/2, \pi/2]}$ , for each  $r \in I_j$ .

Given  $\psi_\ell$  on  $W^\ell$  with  $|\psi_\ell|_{C^\alpha(W^\ell)} \leq 1$  and  $d(\psi_1, \psi_2) \leq \varepsilon$ , we must estimate

$$(6.6) \quad \left| \int_{W^1} \mathcal{L}^n f \psi_1 \, dm_W - \int_{W^2} \mathcal{L}^n f \psi_2 \, dm_W \right| \leq \sum_{\ell, i} \left| \int_{V_i^\ell} f \psi_\ell \circ T^n \, dm_W \right| \\ + \sum_j \left| \int_{U_j^1} f \psi_1 \circ T^n \, dm_W - \int_{U_j^2} f \psi_2 \circ T^n \, dm_W \right|.$$

We first estimate the differences of matched pieces  $U_j^\ell$ . The function  $\phi_j = \psi_1 \circ T^n \circ G_{U_j^1} \circ G_{U_j^2}^{-1}$  is well-defined on  $U_j^2$ , and we can estimate,

$$(6.7) \quad \left| \int_{U_j^1} f \psi_1 \circ T^n - \int_{U_j^2} f \psi_2 \circ T^n \right| \leq \left| \int_{U_j^1} f \psi_1 \circ T^n - \int_{U_j^2} f \phi_j \right| + \left| \int_{U_j^2} f(\phi_j - \psi_2 \circ T^n) \right|.$$

We bound the first term in equation (6.7) using the strong unstable norm. As before, (6.2) implies  $|\psi_1 \circ T^n|_{C^\alpha(U_j^1)} \leq C|\psi_1|_{C^\alpha(W^1)} \leq C$ . We have  $|G_{U_j^1} \circ G_{U_j^2}^{-1}|_{C^1} \leq C_g$ , for some  $C_g > 0$  due to the fact that each curve  $U_j^\ell$  has uniformly bounded curvature and slopes bounded away from infinity. Thus

$$(6.8) \quad |\phi_j|_{C^\alpha(U_j^2)} \leq CC_g |\psi_1|_{C^\alpha(W^1)}.$$

Moreover,  $d(\psi_1 \circ T^n, \phi_j) = \left| \psi_1 \circ T^n \circ G_{U_j^1} - \phi_j \circ G_{U_j^2} \right|_{C^0(I_j)} = 0$  by the definition of  $\phi_j$ .

To complete the bound on the first term of (6.7), we need the following estimate from [DZ1, Lemma 4.2]: There exists  $C > 0$ , independent of  $W^1$  and  $W^2$ , such that

$$(6.9) \quad d_{\mathcal{W}^s}(U_j^1, U_j^2) \leq C\Lambda^{-n}n\varepsilon =: \varepsilon_1, \quad \forall j.$$

In view of (6.8), we renormalize the test functions by  $CC_g$ . Then we apply the definition of the strong unstable norm with  $\varepsilon_1$  in place of  $\varepsilon$ . Thus,

$$(6.10) \quad \sum_j \left| \int_{U_j^1} f \psi_1 \circ T^n - \int_{U_j^2} f \phi_j \right| \leq (CC_g)C\delta_0^{-1} |\log \varepsilon_1|^{-\varsigma} \|f\|_u (\#\mathcal{M}_0^n),$$

where we used Lemma 5.1(b) with  $\bar{\gamma} = 0$  since there is at most one matched piece  $U_j^1$  corresponding to each component  $W_i^1 \in \mathcal{G}_n(W^1)$  of  $T^{-n}W^1$ .

It remains to estimate the second term in (6.7) using the strong stable norm.

$$(6.11) \quad \left| \int_{U_j^2} f(\phi_j - \psi_2 \circ T^n) \right| \leq \|f\|_s |\log |U_j^2||^{-\gamma} |\phi_j - \psi_2 \circ T^n|_{C^\beta(U_j^2)}.$$

In order to estimate the  $C^\beta$ -norm of the function in (6.11), we use that  $|G_{U_j^2}|_{C^1} \leq C_g$  and  $|G_{U_j^2}^{-1}|_{C^1} \leq C_g$  to write

$$(6.12) \quad |\phi_j - \psi_2 \circ T^n|_{C^\beta(U_j^2)} \leq C_g |\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{C^\beta(I_j)}.$$

The difference can now be bounded by the following estimate from [DZ1, Lemma 4.4]

$$(6.13) \quad |\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{C^\beta(I_j)} \leq C\varepsilon^{\alpha-\beta}.$$

Indeed, using (6.13) together with (6.12) yields by (6.11)

$$(6.14) \quad \begin{aligned} & \sum_j \left| \int_{U_j^2} f(\phi_j - \psi_2 \circ T^n) dm_W \right| \\ & \leq C\|f\|_s \sum_j |\log |U_j^2||^{-\gamma} \varepsilon^{\alpha-\beta} \leq C|\log \delta_0|^{-\gamma} \|f\|_s \varepsilon^{\alpha-\beta} 2\delta_0^{-1} (\#\mathcal{M}_0^n), \end{aligned}$$

where used (as in (6.10)) Lemma 5.1(b) with  $\bar{\gamma} = 0$  since there is at most one matched piece  $U_j^2$  corresponding to each component  $W_i^2 \in \mathcal{G}_n(W^2)$  of  $T^{-n}W^2$ . Since  $\delta_0 < 1$  is fixed, this completes the estimate on the second term of matched pieces in (6.7).

We next estimate over the unmatched pieces  $V_i^\ell$  in (6.6), using the strong stable norm. Note that by (6.2),  $|\psi_\ell \circ T^n|_{C^\beta(V_i^\ell)} \leq C|\psi_\ell|_{C^\alpha(W^\ell)} \leq C$ . The relevant sum for unmatched pieces in  $\mathcal{G}_n(W^1)$  is

$$(6.15) \quad \sum_i \int_{V_i^1} f \psi_1 \circ T^n dm_{V_i^1},$$

with a similar sum for unmatched pieces in  $\mathcal{G}_n(W^2)$ .

We say an unmatched curve  $V_i^1$  is created at time  $j$ ,  $1 \leq j \leq n$ , if  $j$  is the first time that  $T^{n-j}V_i^1$  is not part of a matched element of  $\mathcal{G}_j(W^1)$ . Indeed, there may be several curves  $V_i^1$  (in principle exponentially many in  $n-j$ ) such that  $T^{n-j}V_i^1$  belongs to the same unmatched element of  $\mathcal{G}_j(W^1)$ . Define

$$A_{j,k} = \{i : V_i^1 \text{ is created at time } j \\ \text{and } T^{n-j}V_i^1 \text{ belongs to the unmatched curve } W_k^1 \subset T^{-j}W^1\}.$$

Due to the uniform hyperbolicity of  $T$ , and, again, uniform transversality of  $\mathcal{S}_{-n}$  with the stable cone and of  $C^s(x)$  with  $C^u(x)$ , we have  $|W_k^1| \leq C\Lambda^{-j}\varepsilon$ .

Let  $\delta_1$  be the value of  $\delta \leq \delta_0$  from Lemma 5.2 associated with  $\varepsilon = 1/4$  (recall (5.6)). For a certain time, the iterate  $T^{-q}W_k^1$  remains shorter than length  $\delta_1$ . In this case, by Lemma 5.1(a) for  $\bar{\gamma} = 0$ , its complexity grows subexponentially,

$$(6.16) \quad \#\mathcal{G}_q(W_k^1) \leq (Km + 1)^{q/m}.$$

We would like to establish the maximal value of  $q$  as a function of  $j$ .

More precisely, we want to find  $q(j)$  so that any  $q \leq q(j)$  satisfies the conditions:

$$(a) \ T^{-q}W_k^1 \text{ remains shorter than length } \delta_1; \quad (b) \ \frac{|\log |T^{-q}W_k^1||^{-\gamma}}{|\log \varepsilon|^{-\varsigma}} \leq 1.$$

For (a), we use (5.3) together with the fact that  $|W_k^1| \leq C\Lambda^{-j}\varepsilon$  to estimate

$$|T^{-q}W_k^1| \leq \delta_1 \iff C''|W_k^1|^{2^{-s_0q}} \leq \delta_1 \iff C''\Lambda^{-j2^{-s_0q}}\varepsilon^{2^{-s_0q}} \leq \delta_1.$$

Omitting the  $\varepsilon^{2^{-s_0q}}$  factor and solving the last inequality for  $q$  yields,

$$(6.17) \quad q \leq \frac{\log j}{s_0 \log 2} + C_2, \text{ where } C_2 = \frac{\log\left(\frac{\log \Lambda}{|\log(\delta_1/C'')|}\right)}{s_0 \log 2}.$$

For (b), we again use (5.3) to bound  $|T^{-q}W_k^1| \leq C''(\Lambda^{-j}\varepsilon)^{2^{-s_0q}}$ , so that

$$(6.18) \quad \frac{|\log(\Lambda^{-j}\varepsilon)|^{2^{-s_0q}-\gamma}}{|\log \varepsilon|^{-\varsigma}} \leq 1 \implies 2^{\gamma s_0 q} |\log \varepsilon|^\varsigma \leq (|\log \varepsilon| + j \log \Lambda)^\gamma.$$

implies (b). In turn, (6.18) is implied by

$$(6.19) \quad q \leq \frac{(\gamma - \varsigma) \log j}{\gamma s_0 \log 2}.$$

Since the bound in (6.19) is smaller than that in (6.17) for  $j$  larger than some fixed constant depending only on  $\delta_1$ ,  $s_0$  and  $C''$ , we will use (6.19) to define  $q(j)$ .



Now we return to the estimate in (6.15). Grouping the unmatched pieces  $V_i^1$  by their creation times  $j$ , we estimate,<sup>26</sup>

$$\begin{aligned}
\sum_i \int_{V_i^1} f \psi_1 \circ T^n dm_{V_i^1} &= \sum_{j=1}^n \sum_{i \in A_{j,k}} \int_{T^{n-j} V_i^1} (\mathcal{L}^{n-j} f) \psi \circ T^j = \sum_{j=1}^n \sum_k \int_{W_k^1} (\mathcal{L}^{n-j} f) \psi \circ T^j \\
&\leq \sum_{j=1}^n \sum_k \sum_{V_\ell \in \mathcal{G}_{q(j)}(W_k^1)} \int_{V_\ell} (\mathcal{L}^{n-j-q(j)} f) \psi \circ T^{j+q(j)} \\
&\leq \sum_{j=1}^n \sum_k \sum_{V_\ell \in \mathcal{G}_{q(j)}(W_k^1)} \|\mathcal{L}^{n-j-q(j)} f\|_s C |\log |V_\ell||^{-\gamma} \\
&\leq C \|f\|_s \sum_{j=1}^n \#\mathcal{M}_0^j \#\mathcal{M}_0^{n-j-q(j)} (Km+1)^{q(j)/m} |\log(\Lambda^{-j}\varepsilon)|^{2-s_0q(j)-\gamma},
\end{aligned}$$

where we have used (6.16) to bound  $\#\mathcal{G}_{q(j)}(W_k^1)$ , the cardinality  $\#\mathcal{M}_0^j$  to bound the cardinality of the possible pieces  $W_k^1 \subset T^{-j}W^1$ , the estimate  $\|\mathcal{L}^{n-j-q(j)} f\|_s \leq C \#\mathcal{M}_0^{n-j-q(j)} \|f\|_s$ , and, again  $|T^{-q}W_k^1| \leq C''(\Lambda^{-j}\varepsilon)^{2-s_0q}$ . We also have, by the supermultiplicativity Lemma 5.6,

$$\#\mathcal{M}_0^j \#\mathcal{M}_0^{n-j-q(j)} \leq C e^{-q(j)h_*} \#\mathcal{M}_0^n.$$

Thus using (b) in the definition of  $q(j)$  (or, more precisely, (6.18)), we estimate

$$(6.20) \quad \sum_i \int_{V_i^1} f \psi_1 \circ T^n dm_{V_i^1} \leq C \|f\|_s |\log \varepsilon|^{-\varsigma} \#\mathcal{M}_0^n \sum_{j=1}^n (Km+1)^{q(j)/m} e^{-q(j)h_*}.$$

For the final sum over  $j$ , we let  $\varepsilon_2 = \frac{1}{m} \log(Km+1)$  and use (6.19),

$$\sum_{j=1}^n (Km+1)^{q(j)/m} e^{-q(j)h_*} = \sum_{j=1}^n e^{-q(j)(h_*-\varepsilon_2)} \leq \sum_{j=1}^n e^{-(h_*-\varepsilon_2) \frac{(\gamma-\varsigma) \log j}{\gamma s_0 \log 2}} = \sum_{j=1}^n j^{-(h_*-\varepsilon_2) \frac{\gamma-\varsigma}{\gamma s_0 \log 2}}.$$

Then by (6.20), since the exponent of  $j$  in the above sum is strictly negative by choice of  $m$  (see (5.4)), there exist  $C < \infty$  and  $\varpi \in [0, 1)$  such that the contribution to  $\|\mathcal{L}^n f\|_u$  of the unmatched pieces is bounded by

$$(6.21) \quad \sum_{\ell, i} \left| \int_{V_i^\ell} f \psi_\ell \circ T^n dm_W \right| \leq C |\log \varepsilon|^{-\varsigma} n^\varpi \#\mathcal{M}_0^n \|f\|_s.$$

Now we use (6.21) together with (6.10) and (6.14) to estimate (6.6)

$$\begin{aligned}
\left| \int_{W^1} \mathcal{L}^n f \psi_1 dm_W - \int_{W^2} \mathcal{L}^n f \psi_2 dm_W \right| &\leq C \delta_0^{-1} \|f\|_u |\log \varepsilon_1|^{-\varsigma} \#\mathcal{M}_0^n \\
&\quad + C \delta_0^{-1} (n^\varpi \|f\|_s |\log \varepsilon|^{-\varsigma} + \|f\|_s \varepsilon^{\alpha-\beta}) \#\mathcal{M}_0^n.
\end{aligned}$$

Dividing through by  $|\log \varepsilon|^{-\varsigma}$  and taking the appropriate suprema, we complete the proof of (4.11), recalling Proposition 4.6.

Finally, we study the consequences of the additional assumption  $h_* > s_0 \log 2$  on the estimate over unmatched pieces. In this case, again recalling (5.4) and following, we may choose  $\varsigma > 0$  small enough such that

$$\varepsilon_1 := h_* - \frac{1}{m} \log(Km+1) - \frac{\gamma}{\gamma-\varsigma} s_0 \log 2 > 0.$$

<sup>26</sup>When we sum the integrals in the first line over the different  $T^{n-j}V_i^1$ , we find the integral over  $W_k^1$  since the union of those pieces is precisely  $W_k^1$ .

Then

$$\sum_{j=1}^n j^{-(h_* - \varepsilon_2) \frac{\gamma - \varsigma}{\gamma s_0 \log 2}} = \sum_{j=1}^n j^{-1 - \varepsilon_1 \frac{\gamma - \varsigma}{\gamma s_0 \log 2}} < \infty.$$

Thus, by (6.20), the contribution to  $\|\mathcal{L}^n f\|_u$  of the unmatched pieces is bounded by

$$(6.22) \quad \sum_{\ell, i} \left| \int_{V_i^\ell} f \psi_\ell \circ T^n dm_W \right| \leq C |\log \varepsilon|^{-\varsigma} \#\mathcal{M}_0^n \|f\|_s$$

if  $h_* > s_0 \log 2$ . So we find (4.12) for  $h_* > s_0 \log 2$  by replacing (6.21) with (6.22).

**6.3. Upper and Lower Bounds on the Spectral Radius.** We now deduce the bounds of Theorem 4.10 from the inequalities of Proposition 4.7 and the rate of growth of stable curves proved in Proposition 5.5.

*Proof of Theorem 4.10.* The upper bounds (4.13) and (4.15) are immediate consequences of Proposition 4.7. To prove the lower bound on  $|\mathcal{L}^n 1|_w$ , recall the choice of  $\delta_1 = \delta > 0$  from Lemma 5.2 for  $\varepsilon = 1/4$ , giving (5.6). Let  $W \in \mathcal{W}^s$  with  $|W| \geq \delta_1/3$  and set the test function  $\psi \equiv 1$ . For  $n \geq n_1$ ,

$$(6.23) \quad \int_W \mathcal{L}^n 1 dm_W = \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} \int_{W_i} 1 dm_{W_i} = \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |W_i| \geq \frac{2\delta_1}{9} c_0 e^{nh_*},$$

by (5.12). Thus,

$$(6.24) \quad \|\mathcal{L}^n 1\|_s \geq |\mathcal{L}^n 1|_w \geq \frac{2\delta_1}{9} c_0 e^{nh_*}.$$

Letting  $n$  tend to infinity, one obtains  $\lim_{n \rightarrow \infty} \|\mathcal{L}^n\|_{\mathcal{B}}^{1/n} \geq h_*$ .  $\square$

**6.4. Compact Embedding.** The following compact embedding property is crucial to exploit Proposition 4.7 in order to construct  $\mu_*$  in Section 7.1.

**Proposition 6.1** (Compact Embedding). *The embedding of the unit ball of  $\mathcal{B}$  in  $\mathcal{B}_w$  is compact.*

*Proof.* Consider the set  $\widehat{\mathcal{W}}^s$  of (not necessarily homogeneous) cone-stable curves with uniformly bounded curvature and the distance  $d_{\mathcal{W}^s}(\cdot, \cdot)$  between them defined in Section 4.1. According to (4.3), each of these curves can be viewed as graphs of  $C^2$  functions of the position coordinate  $r$  with uniformly bounded second derivative,  $W = \{G_W(r)\}_{r \in I_W} = \{(r, \varphi_W(r))\}_{r \in I_W}$ . Thus they are compact in the  $C^1$  distance  $d_{\mathcal{W}^s}$ . Given  $\varepsilon > 0$ , we may choose finitely many  $V_i \in \widehat{\mathcal{W}}^s$ ,  $i = 1, \dots, N_\varepsilon$ , such that the balls of radius  $\varepsilon/2$  in the  $d_{\mathcal{W}^s}$  metric centered at the curves  $\{V_i\}_{i=1}^{N_\varepsilon}$  form a covering of  $\widehat{\mathcal{W}}^s$ .

Since  $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$ , we proceed as follows. In each ball  $B_{\varepsilon/2}(V_i)$  centered at  $V_i$  in the space of  $C^1$  graphs, if  $B_{\varepsilon/2}(V_i) \cap \mathcal{W}^s \neq \emptyset$ , then we choose one representative  $W_i \in B_{\varepsilon/2}(V_i) \cap \mathcal{W}^s$ . Otherwise, we discard  $B_{\varepsilon/2}(V_i)$ . The balls of radius  $\varepsilon$  in the  $d_{\mathcal{W}^s}$  metric centered at the curves  $\{W_i\}_{i=1}^{N_\varepsilon}$  constructed in this way form a covering of  $\mathcal{W}^s$ . (There may be fewer than  $N_\varepsilon$  such curves due to some balls having been discarded, but we will continue to use the symbol  $N_\varepsilon$  in any case.)

We now argue one component of the phase space,  $M_\ell = \partial B_\ell \times [-\pi/2, \pi/2]$ , at a time. Define  $\mathbb{S}_\ell^1$  to be the circle of length  $|\partial B_\ell|$  and let  $C_g$  be the graph constant from (6.8). Since the ball of radius  $C_g$  in the  $C^\alpha(\mathbb{S}_\ell^1)$  norm is compactly embedded in  $C^\beta(\mathbb{S}_\ell^1)$ , we may choose finitely many functions  $\bar{\psi}_j \in C^\alpha(\mathbb{S}_\ell^1)$  such that the balls of radius  $\varepsilon$  in the  $C^\beta(\mathbb{S}_\ell^1)$  metric centered at the functions  $\{\bar{\psi}_j\}_{j=1}^{L_\varepsilon}$  form a covering of the ball of radius  $C_g$  in  $C^\alpha(\mathbb{S}_\ell^1)$ .

Now let  $W = G_W(I_W) \in \mathcal{W}^s$ , and  $\psi \in C^\alpha(W)$  with  $|\psi|_{C^\alpha(W)} \leq 1$ . Viewing  $I_W$  as a subset of  $\mathbb{S}_\ell^1$ , we define the push down of  $\psi$  to  $I_W$  by  $\bar{\psi} = \psi \circ G_W$ . We extend  $\bar{\psi}$  to  $\mathbb{S}_\ell^1$  by linearly interpolating

between its two endpoint values on the complement of  $I_W$  in  $\mathbb{S}_\ell^1$ . Since  $I_W$  is much shorter than  $\mathbb{S}_\ell^1$ , this can be accomplished while maintaining  $|\bar{\psi}|_{C^\alpha(\mathbb{S}_\ell^1)} \leq C_g$ .

Choose  $W_i = G_{W_i}(I_{W_i})$  such that  $d_{\mathcal{W}^s}(W, W_i) < \varepsilon$  and  $\bar{\psi}_j$  such that  $|\bar{\psi} - \bar{\psi}_j|_{C^\beta(\mathbb{S}_\ell^1)} < \varepsilon$ . Define  $\psi_j = \bar{\psi}_j \circ G_{W_i}^{-1}$  and  $\tilde{\psi}_j = \bar{\psi}_j \circ G_W^{-1}$  to be the lifts of  $\bar{\psi}_j$  to  $W_i$  and  $W$ , respectively. Note that  $|\psi_j|_{C^\beta(W_i)} \leq C_g$ ,  $|\tilde{\psi}_j|_{C^\beta(W)} \leq C_g$ , while

$$d(\psi_j, \tilde{\psi}_j) = |\psi_j \circ G_{W_i} - \tilde{\psi}_j \circ G_W|_{C^0(I_{W_i} \cap I_W)} = 0, \quad \text{and} \quad |\psi - \tilde{\psi}_j|_{C^\beta(W)} \leq C_g \varepsilon.$$

Thus,

$$\begin{aligned} \left| \int_W f \psi \, dm_W - \int_{W_i} f \psi_j \, dm_{W_i} \right| &\leq \left| \int_W f(\psi - \tilde{\psi}_j) \, dm_W \right| + \left| \int_W f \tilde{\psi}_j \, dm_W - \int_{W_i} f \psi_j \, dm_{W_i} \right| \\ &\leq \|f\|_s |\log |W||^{-\gamma} |\psi - \tilde{\psi}_j|_{C^\beta(W)} + |\log \varepsilon|^{-\varsigma} \|f\|_u C_g \leq 2C_g \|f\|_B |\log \varepsilon|^{-\varsigma}. \end{aligned}$$

We have proved that for each  $\varepsilon > 0$ , there exist finitely many bounded linear functionals  $\ell_{i,j}(\cdot) = \int_{W_i} \cdot \psi_j \, dm_{W_i}$ , such that for all  $f \in \mathcal{B}$ ,

$$|f|_w \leq \max_{i \leq N_\varepsilon, j \leq L_\varepsilon} \ell_{i,j}(f) + 2C_g \|f\|_B |\log \varepsilon|^{-\varsigma},$$

which implies the relative compactness of  $\mathcal{B}$  in  $\mathcal{B}_w$ .  $\square$

## 7. THE MEASURE $\mu_*$

In this section, we assume throughout that  $h_* > s_0 \log 2$  (with  $s_0 < 1$  defined by (1.4)).

**7.1. Construction of the Measure  $\mu_*$  — Measure of Singular Sets (Theorem 2.6).** In this section, we construct a  $T$ -invariant probability measure  $\mu_*$  on  $M$  by combining in (7.1) a maximal eigenvector of  $\mathcal{L}$  on  $\mathcal{B}$  and a maximal eigenvector of its dual obtained in Proposition 7.1. In addition, the information on these left and right eigenvectors will give Lemma 7.3 and Corollary 7.4, which immediately imply Theorem 2.6.

We first show that such maximal eigenvectors exist and are in fact nonnegative Radon measures (i.e., elements of the dual of  $C^0(M)$ ).

**Proposition 7.1.** *If  $h_* > s_0 \log 2$  then there exist  $\nu \in \mathcal{B}_w$  and  $\tilde{\nu} \in \mathcal{B}_w^*$  such that  $\mathcal{L}\nu = e^{h_*}\nu$  and  $\mathcal{L}^*\tilde{\nu} = e^{h_*}\tilde{\nu}$ . In addition<sup>27</sup>  $\nu$  and  $\tilde{\nu}$  take nonnegative values on nonnegative  $C^1$  functions on  $M$  and are thus nonnegative Radon measures. Finally,  $\tilde{\nu}(\nu) \neq 0$  and  $\|\nu\|_u \leq \bar{C}$ .*

**Remark 7.2.** *The norm of the space  $\mathcal{B}$  depends on the parameter  $\gamma$  and is used in the proof of the proposition. However, this proof provides  $\nu$  and  $\tilde{\nu}$  which do not depend on  $\gamma$  (as soon as  $2^{s_0\gamma} < e^{h_*}$ ), and do not depend on the parameters  $\beta$  and  $\varsigma$  of  $\mathcal{B}$ .*

It is easy to see that  $|f\varphi|_w \leq |\varphi|_{C^1}|f|_w$  (use  $|\varphi\psi|_{C^\alpha(W)} \leq |\varphi|_{C^1}|\psi|_{C^\alpha(W)}$ ). Clearly, if  $f \in C^1$  and  $\varphi \in C^1$  then  $f\varphi \in C^1$ . Therefore, if  $h_* > s_0 \log 2$ , a bounded linear map  $\mu_*$  from  $C^1(M)$  to  $\mathbb{C}$  can be defined by taking  $\nu$  and  $\tilde{\nu}$  from Proposition 7.1 and setting

$$(7.1) \quad \mu_*(\varphi) = \frac{\tilde{\nu}(\nu\varphi)}{\tilde{\nu}(\nu)}.$$

This map is nonnegative for all nonnegative  $\varphi$  and thus defines a nonnegative measure  $\mu_* \in (C^0)^*$ , with  $\mu_*(1) = 1$ . Clearly,  $\mu_*$  is a  $T$  invariant probability measure since

$$\tilde{\nu}(\nu\varphi) = e^{-h_*}\tilde{\nu}(\varphi\mathcal{L}(\nu)) = e^{-h_*}\tilde{\nu}(\mathcal{L}(\nu(\varphi \circ T))) = \tilde{\nu}(\nu(\varphi \circ T)) = \tilde{\nu}(\nu)\mu_*(\varphi \circ T), \quad \forall \varphi \in C^1.$$

<sup>27</sup>Recall Proposition 4.2 and Remark 4.3.

*Proof of Proposition 7.1.* Let  $1$  denote the constant function<sup>28</sup> equal to one on  $M$ . We will take this as a seed in our construction of a maximal eigenvector. From (4.14) in Theorem 4.10 we see that  $\|\mathcal{L}^n 1\|_{\mathcal{B}} \geq \|\mathcal{L}^n 1\|_s \geq |\mathcal{L}^n 1|_w \geq C \# \mathcal{M}_0^n \geq C e^{nh_*}$ . Now, consider

$$(7.2) \quad \nu_n = \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} \mathcal{L}^k 1 \in \mathcal{B}, \quad n \geq 1.$$

By construction the  $\nu_n$  are nonnegative, and thus Radon measures. By our assumption on  $h_*$  and (4.15) in Theorem 4.10 they satisfy  $\|\nu_n\|_{\mathcal{B}} \leq \bar{C}$ , so using the relative compactness of  $\mathcal{B}$  in  $\mathcal{B}_w$  (Proposition 6.1), we extract a subsequence  $(n_j)$  such that  $\lim_j \nu_{n_j} = \nu$  is a nonnegative measure, and the convergence is in  $\mathcal{B}_w$ . (Changing the value of  $\gamma$  does not affect  $\nu$  since  $\mathcal{B}_w$  does not depend on  $\gamma$ .) Since  $\mathcal{L}$  is continuous on  $\mathcal{B}_w$ , we may write,

$$\begin{aligned} \mathcal{L}\nu &= \lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} e^{-kh_*} \mathcal{L}^{k+1} 1 \\ &= \lim_{j \rightarrow \infty} \left( \frac{e^{h_*}}{n_j} \sum_{k=0}^{n_j-1} e^{-kh_*} \mathcal{L}^k 1 - \frac{1}{n_j} e^{-h_*} \mathcal{L} 1 + \frac{1}{n_j} e^{-n_j h_*} \mathcal{L}^{n_j} 1 \right) = e^{h_*} \nu, \end{aligned}$$

where we used that the second and third terms go to 0 (in the  $\mathcal{B}$ -norm). We thus obtain a nonnegative measure  $\nu \in \mathcal{B}_w$  such that  $\mathcal{L}\nu = e^{h_*} \nu$ .

Although  $\nu$  is not a priori an element of  $\mathcal{B}$ , it does inherit bounds on the unstable norm from the sequence  $\nu_n$ . The convergence of  $(\nu_{n_j})$  to  $\nu$  in  $\mathcal{B}_w$  implies that

$$(7.3) \quad \lim_{j \rightarrow \infty} \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha(W)} \leq 1}} \left( \int_W \nu \psi \, dm_W - \int_W \nu_{n_j} \psi \, dm_W \right) = 0.$$

Since  $\|\nu_{n_j}\|_u \leq \bar{C}$ , it follows that  $\|\nu\|_u \leq \bar{C}$ , as claimed.

Next, recalling the bound  $|\int f \, d\mu_{\text{SRB}}| \leq \hat{C} |f|_w$  from Proposition 4.2, setting  $d\mu_{\text{SRB}} \in (\mathcal{B}_w)^*$  to be the functional defined on  $C^1(M) \subset \mathcal{B}_w$  by  $d\mu_{\text{SRB}}(f) = \int f \, d\mu_{\text{SRB}}$  and extended by density, we define<sup>29</sup>

$$(7.4) \quad \tilde{\nu}_n = \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} (\mathcal{L}^*)^k (d\mu_{\text{SRB}}).$$

Then, we have  $|\tilde{\nu}_n(f)| \leq C |f|_w$  for all  $n$  and all  $f \in \mathcal{B}_w$ . So  $\tilde{\nu}_n$  is bounded in  $(\mathcal{B}_w)^* \subset \mathcal{B}^*$ . By compactness of this embedding (Proposition 6.1), we can find a subsequence  $\tilde{\nu}_{\tilde{n}_j}$  converging to  $\tilde{\nu} \in \mathcal{B}^*$ . By the argument above, we have  $\mathcal{L}^* \tilde{\nu} = e^{h_*} \tilde{\nu}$ . The nonnegativity claim on  $\tilde{\nu}$  follows by construction.<sup>30</sup>

We next check that  $\tilde{\nu}$ , which in principle lies in the dual of  $\mathcal{B}$ , is in fact an element of  $(\mathcal{B}_w)^*$ . For this, it suffices to find  $\tilde{C} < \infty$  so that for any  $f \in \mathcal{B}$  we have

$$(7.5) \quad \tilde{\nu}(f) \leq \tilde{C} |f|_w.$$

Now, for  $f \in \mathcal{B}$  and any  $n \geq 1$ , we have

$$|\tilde{\nu}(f)| \leq |(\tilde{\nu} - \tilde{\nu}_n)(f)| + |\tilde{\nu}_n(f)| \leq |(\tilde{\nu} - \tilde{\nu}_n)(f)| + |f|_w.$$

<sup>28</sup>We could replace the seed function  $1$  by any  $C^1$  positive function  $f$  on  $M$ .

<sup>29</sup>We could again replace the seed  $\mu_{\text{SRB}}$  by  $f\mu_{\text{SRB}}$  for any  $C^1$  positive function  $f$  on  $M$ .

<sup>30</sup>To check  $\gamma$ -independence of  $\tilde{\nu}$ , note that if  $\tilde{\gamma} > \gamma$  then, since the dual norms satisfy  $\|\tilde{\nu}_{\tilde{n}_j} - \tilde{\nu}\|_{*,\tilde{\gamma}} \leq \|\tilde{\nu}_{\tilde{n}_j} - \tilde{\nu}\|_{*,\gamma}$ , the subsequence converges to  $\tilde{\nu}$  in the  $\|\cdot\|_{*,\tilde{\gamma}}$ -norm as well. If  $\tilde{\gamma} < \gamma$  then a further subsequence of  $\tilde{n}_j$  must converge to some  $\tilde{\nu}_{\tilde{\gamma}}$  in the  $\|\cdot\|_{*,\tilde{\gamma}}$  norm. The domination then implies  $\tilde{\nu} = \tilde{\nu}_{\tilde{\gamma}}$ .

Since  $\tilde{\nu}_n \rightarrow \tilde{\nu}$  in  $\mathcal{B}^*$ , we conclude  $|\tilde{\nu}(f)| \leq |f|_w$  for all  $f \in \mathcal{B}$ . Since  $\mathcal{B}$  is dense in  $\mathcal{B}_w$ , by [RS, Thm I.7]  $\tilde{\nu}$  extends uniquely to a bounded linear functional on  $\mathcal{B}_w$ , satisfying (7.5). It only remains to see that  $\tilde{\nu}(\nu) > 0$ .

Let  $(n_j)$  (resp.  $(\tilde{n}_j)$ ) denote the subsequence such that  $\nu = \lim_j \nu_{n_j}$  (resp.  $\tilde{\nu} = \lim_j \tilde{\nu}_{\tilde{n}_j}$ .) Since  $\tilde{\nu}$  is continuous on  $\mathcal{B}_w$ , we have on the one hand

$$(7.6) \quad \tilde{\nu}(\nu) = \lim_{j \rightarrow \infty} \tilde{\nu}(\nu_{n_j}) = \lim_j \frac{1}{n_j} \sum_{k=0}^{n_j-1} e^{-kh_*} \tilde{\nu}(\mathcal{L}^k 1) = \lim_j \frac{1}{n_j} \sum_{k=0}^{n_j-1} \tilde{\nu}(1) = \tilde{\nu}(1),$$

where we have used that  $\tilde{\nu}$  is an eigenvector for  $\mathcal{L}^*$ . On the other hand,

$$(7.7) \quad \tilde{\nu}(1) = \lim_{j \rightarrow \infty} \frac{1}{\tilde{n}_j} \sum_{k=0}^{\tilde{n}_j-1} e^{-kh_*} (\mathcal{L}^*)^k d\mu_{\text{SRB}}(1) = \lim_j \frac{1}{\tilde{n}_j} \sum_{k=0}^{\tilde{n}_j-1} e^{-kh_*} \int \mathcal{L}^k 1 d\mu_{\text{SRB}}.$$

Next, we disintegrate  $\mu_{\text{SRB}}$  as in the proof of Lemma 4.4 into conditional measures  $\mu_{\text{SRB}}^{W_\xi}$  on maximal homogeneous stable manifolds  $W_\xi \in \mathcal{W}_{\text{III}}^s$  and a factor measure  $d\hat{\mu}_{\text{SRB}}(\xi)$  on the index set  $\Xi$  of stable manifolds. Recall that  $\mu_{\text{SRB}}^{W_\xi} = |W_\xi|^{-1} \rho_\xi dm_W$ , where  $\rho_\xi$  is uniformly log-Hölder continuous so that

$$(7.8) \quad 0 < c_\rho \leq \inf_{\xi \in \Xi} \inf_{W_\xi} \rho_\xi \leq \sup_{\xi \in \Xi} |\rho_\xi|_{C^\alpha(W_\xi)} \leq C_\rho < \infty.$$

Let  $\Xi^{\delta_1}$  denote those  $\xi \in \Xi$  such that  $|W_\xi| \geq \delta_1/3$  and note that  $\hat{\mu}_{\text{SRB}}(\Xi^{\delta_1}) > 0$ . Then, disintegrating as usual, we get by (6.23) for  $k \geq n_1$ ,

$$\begin{aligned} \int \mathcal{L}^k 1 d\mu_{\text{SRB}} &= \int_{\Xi} \int_{W_\xi} \mathcal{L}^k 1 \rho_\xi |W_\xi|^{-1} dm_{W_\xi} d\hat{\mu}_{\text{SRB}}(\xi) \\ &\geq \int_{\Xi^{\delta_1}} \int_{W_\xi} \mathcal{L}^k 1 dm_{W_\xi} c_\rho 3\delta_1^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \geq c_\rho \frac{2c_0}{3} e^{kh_*} \hat{\mu}_{\text{SRB}}(\Xi^{\delta_1}). \end{aligned}$$

Combining this with (7.6) and (7.7) yields  $\tilde{\nu}(\nu) = \tilde{\nu}(1) \geq \frac{2c_\rho c_0}{3} \hat{\mu}_{\text{SRB}}(\Xi^{\delta_1}) > 0$  as required.  $\square$

We next study the measure of neighbourhoods of singularity sets and stable manifolds, in order to establish Theorems 2.6 and 2.2.

**Lemma 7.3.** *For any  $\gamma > 0$  such that  $2^{s_0\gamma} < e^{h_*}$  and any  $k \in \mathbb{Z}$ , there exists  $C_k > 0$  such that*

$$\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_k)) \leq C_k |\log \varepsilon|^{-\gamma}, \quad \forall \varepsilon > 0.$$

*In particular, for any  $p > 1/\gamma$  (one can choose  $p < 1$  if  $\gamma > 1$ ),  $\eta > 0$ , and  $k \in \mathbb{Z}$ , for  $\mu_*$ -almost every  $x \in M$ , there exists  $C > 0$  such that*

$$(7.9) \quad d(T^n x, \mathcal{S}_k) \geq C e^{-\eta n^p}, \quad \forall n \geq 0.$$

*Proof.* First, for each  $k \geq 0$ , we claim that there exists  $C_k > 0$  such that for all  $\varepsilon > 0$ ,

$$(7.10) \quad |\nu(\mathcal{N}_\varepsilon(\mathcal{S}_{-k}))| \leq C |1_{k,\varepsilon} \nu|_w \leq C_k |\log \varepsilon|^{-\gamma},$$

where  $1_{k,\varepsilon}$  is the indicator function of the set  $\mathcal{N}_\varepsilon(\mathcal{S}_{-k})$ . To prove the first inequality in (7.10), first note that since  $\mathcal{S}_{-k}$  comprises finitely many smooth curves, uniformly transverse to the stable cone, this also holds for the boundary curves of the set  $\mathcal{N}_\varepsilon(\mathcal{S}_{-k})$ . By [DZ3, Lemma 5.3], we have  $1_{k,\varepsilon} f \in \mathcal{B}$  for  $f \in \mathcal{B}$ ; similarly (and by a simpler approximation) if  $f \in \mathcal{B}_w$ , then  $1_{k,\varepsilon} f \in \mathcal{B}_w$ . So the first inequality in (7.10) follows from Lemma 4.4.

We next prove the second inequality in (7.10). Let  $W \in \mathcal{W}^s$  and  $\psi \in C^\alpha(W)$  with  $|\psi|_{C^\alpha(W)} \leq 1$ . Due to the uniform transversality of the curves in  $\mathcal{S}_{-k}$  with the stable cone, the intersection  $W \cap \mathcal{N}_\varepsilon(\mathcal{S}_{-k})$  can be expressed as a finite union of cardinality bounded by a constant  $A_k$  (depending only on  $\mathcal{S}_{-k}$ ) of stable manifolds  $W_i \in \mathcal{W}^s$ , of lengths at most  $C\varepsilon$ . Therefore, for any  $f \in C^1$ ,

$$(7.11) \quad \int_{W_\xi} f 1_{k,\varepsilon} \psi dm_W = \sum_i \int_{W_i} f \psi dm_{W_i} \leq \sum_i |f|_w |\psi|_{C^\alpha(W_i)} \leq C A_k |f|_w.$$

It follows that  $|1_{k,\varepsilon}f|_w \leq A_k|f|_w$  for all  $f \in \mathcal{B}_w$ . Similarly,  $|1_{k,\varepsilon}f|_w \leq A_k\|f\|_s|\log \varepsilon|^{-\gamma}$  for all  $f \in \mathcal{B}$ . Now recalling  $\nu_n$  from (7.2), we estimate,

$$|1_{k,\varepsilon}\nu|_w \leq |1_{k,\varepsilon}(\nu - \nu_n)|_w + |1_{k,\varepsilon}\nu_n|_w \leq A_k|\nu - \nu_n|_w + C'_k|\log \varepsilon|^{-\gamma}\|\nu_n\|_{\mathcal{B}}.$$

Since  $\|\nu_n\|_{\mathcal{B}} \leq \bar{C}$  for all  $n \geq 1$ , we take the limit as  $n \rightarrow \infty$  to conclude that  $|1_{k,\varepsilon}\nu|_w \leq C_k|\log \varepsilon|^{-\gamma}$ , concluding the proof of (7.10).

Next, applying (7.5), we have

$$\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_{-k})) = \tilde{\nu}(1_{k,\varepsilon}\nu) \leq \tilde{C}|1_{k,\varepsilon}\nu|_w \leq \tilde{C}C_k|\log \varepsilon|^{-\gamma}, \quad \forall k \geq 0.$$

To obtain the analogous bound for  $\mathcal{N}_\varepsilon(\mathcal{S}_k)$ , for  $k > 0$ , we use the invariance of  $\mu_*$ . It follows from the time reversal of (5.2) that  $T(\mathcal{N}_\varepsilon(\mathcal{S}_1)) \subset \mathcal{N}_{C\varepsilon^{1/2}}(\mathcal{S}_{-1})$ . Thus,

$$\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_1)) \leq \mu_*(\mathcal{N}_{C\varepsilon^{1/2}}(\mathcal{S}_{-1})) \leq C_1|\log(C\varepsilon^{1/2})|^{-\gamma} \leq C'_1|\log \varepsilon|^{-\gamma}.$$

The estimate for  $\mathcal{N}_\varepsilon(\mathcal{S}_k)$ , for  $k \geq 2$ , follows similarly since  $T^k\mathcal{S}_k = \mathcal{S}_{-k}$ .

Finally, fix  $\eta > 0$ ,  $k \in \mathbb{Z}$  and  $p > 1/\gamma$ . Since

$$(7.12) \quad \sum_{n \geq 0} \mu_*(\mathcal{N}_{e^{-\eta n p}}(\mathcal{S}_k)) \leq \tilde{C}C_k\eta^{-\gamma} \sum_{n \geq 1} n^{-p\gamma} < \infty,$$

by the Borel–Cantelli Lemma,  $\mu_*$ -almost every  $x \in M$  visits  $\mathcal{N}_{e^{-\eta n p}}(\mathcal{S}_k)$  only finitely many times, and the last statement of the lemma follows.  $\square$

Lemma 7.3 will imply the following:

**Corollary 7.4.** *a) For any  $\gamma > 0$  so that  $2^{s_0\gamma} < e^{h^*}$  and any  $C^1$  curve  $S$  uniformly transverse to the stable cone, there exists  $C > 0$  such that  $\nu(\mathcal{N}_\varepsilon(S)) \leq C|\log \varepsilon|^{-\gamma}$  and  $\mu_*(\mathcal{N}_\varepsilon(S)) \leq C|\log \varepsilon|^{-\gamma}$  for all  $\varepsilon > 0$ .*

*b) The measures  $\nu$  and  $\mu_*$  have no atoms, and  $\mu_*(W) = 0$  for all  $W \in \mathcal{W}^s$  and  $W \in \mathcal{W}^u$ .*

*c)  $\int |\log d(x, \mathcal{S}_{\pm 1})| d\mu_* < \infty$ .*

*d)  $\mu_*$ -almost every point in  $M$  has a stable and unstable manifold of positive length.*

*Proof.* a) This follows immediately from the bounds in the proof of Lemma 7.3 since the only property required of  $\mathcal{S}_{-k}$  is that it comprises finitely many smooth curves uniformly transverse to the stable cone.

b) That  $\nu$  and  $\mu_*$  have no atoms follows from part (a). If  $\mu_*(W) = a > 0$ , then by invariance,  $\mu_*(T^n W) = a$  for all  $n > 0$ . Since  $\mu_*$  is a probability measure and  $T^n$  is continuous on stable manifolds,  $\cup_{n \geq 0} T^n W$  must be the union of only finitely many smooth curves. Since  $|T^n W| \rightarrow 0$  there is a subsequence  $(n_j)$  such that  $\cap_{j \geq 0} T^{n_j} W = \{x\}$ . Thus  $\mu_*(\{x\}) = a$ , which is impossible. A similar argument applies to  $W \in \mathcal{W}^u$ , using the fact that  $T^{-n}$  is continuous on such manifolds.

c) Choose  $\gamma > 1$  and  $p > 1/(\gamma - 1)$ . Then by Lemma 7.3,

$$\begin{aligned} \int |\log d(x, \mathcal{S}_1)| d\mu_* &= \sum_{n \geq 0} \int_{\mathcal{N}_{e^{-np}}(\mathcal{S}_1) \setminus \mathcal{N}_{e^{-(n+1)p}}(\mathcal{S}_1)} |\log d(x, \mathcal{S}_1)| d\mu_* \\ &\leq \sum_{n \geq 0} (n+1)^p \mu_*(\mathcal{N}_{e^{-np}}(\mathcal{S}_1)) \leq 1 + \sum_{n \geq 1} C_1 n^{p(1-\gamma)} (1+1/n)^p < \infty. \end{aligned}$$

A similar estimate holds for  $\int \log d(x, \mathcal{S}_{-1}) d\mu_*$ .

d) The existence of stable and unstable manifolds for  $\mu_*$ -almost every  $x$  follows from the Borel–Cantelli estimate (7.12) by a standard argument if we choose  $\gamma > 1$ ,  $p = 1$  and  $e^\eta < \Lambda$  (see, for example, [CM, Sect. 4.12]).  $\square$

Lemma 7.3 and Corollary 7.4 prove all the items of Theorem 2.6.

**7.2.  $\nu$ -Almost Everywhere Positive Length of Unstable Manifolds.** We establish almost everywhere positive length of unstable manifolds in the sense of the measure  $\nu$  (the maximal eigenvector of  $\mathcal{L}$ ). The proof of this fact, as well as some arguments in subsequent sections, will require viewing elements of  $\mathcal{B}_w$  as *leafwise distributions*, see Definition 7.5 below. Indeed, to prove Lemma 7.6, we make in Lemma 7.7 an explicit connection<sup>31</sup> between the element  $\nu \in \mathcal{B}_w$  viewed as a measure on  $M$ , and the family of leafwise measures defined on the set of stable manifolds  $\mathcal{W}^s$ .

While  $\nu$  is not an invariant measure, the almost-everywhere existence of positive length unstable manifolds on *every* stable manifold  $W \in \mathcal{W}^s$  follows from the regularity inherited from the strong stable norm. This property may have some independent interest as it has not been proved in previous uses of this type of norm [DZ1, DZ3], and it will be important for proving the absolute continuity of the unstable foliation for  $\mu_*$  (Corollary 7.9), which relies on the analogous property for the measure  $\nu$  (Proposition 7.8). Lemmas 7.6 and 7.7 will also be useful to obtain that  $\mu_*$  has full support (Proposition 7.11).

**Definition 7.5** (Leafwise distributions and leafwise measures). *For  $f \in C^1(M)$  and  $W \in \mathcal{W}^s$ , the map defined on  $C^\alpha(W)$  by*

$$\psi \mapsto \int_W f \psi \, dm_W,$$

*can be viewed as a distribution of order  $\alpha$  on  $W$ . Since  $|\int_W f \psi \, dm_W| \leq |f|_w |\psi|_{C^\alpha(W)}$ , the map sending  $f \in C^1$  to this distribution of order  $\alpha$  on  $W$  can be extended to  $f \in \mathcal{B}_w$ . We denote this extension by  $\int_W \psi f$  or  $\int_W f \psi \, dm_W$ , and we call the corresponding family of distributions (indexed by  $W$ ) the leafwise distribution  $(f, W)_{W \in \mathcal{W}^s}$  associated with  $f \in \mathcal{B}_w$ . Note that if  $f \in \mathcal{B}_w$  is such that  $\int_W \psi f \geq 0$  for all  $\psi \geq 0$  then using again [Sch, §I.4], the leafwise distribution on  $W$  extends to a bounded linear functional on  $C^0(W)$ , i.e., it is a Radon measure. If this holds for all  $W \in \mathcal{W}^s$ , the leafwise distribution is called a leafwise measure.*

**Lemma 7.6** (Almost Everywhere Positive Length of Unstable Manifolds, for  $\nu$ ). *For  $\nu$ -almost every  $x \in M$  the stable and unstable manifolds have positive length. Moreover, viewing  $\nu$  as a leafwise measure, for every  $W \in \mathcal{W}^s$ ,  $\nu$ -almost every  $x \in W$  has an unstable manifold of positive length.*

Recall the disintegration of  $\mu_{\text{SRB}}$  into conditional measures  $\mu_{\text{SRB}}^{W_\xi}$  on maximal homogeneous stable manifolds  $W_\xi \in \mathcal{W}_{\mathbb{H}}^s$  and a factor measure  $d\hat{\mu}_{\text{SRB}}(\xi)$  on the index set  $\Xi$  of homogeneous stable manifolds, with  $d\mu_{\text{SRB}}^{W_\xi} = |W_\xi|^{-1} \rho_\xi dm_W$ , where  $\rho_\xi$  is uniformly log-Hölder continuous as in (7.8).

**Lemma 7.7.** *Let  $\nu^{W_\xi}$  and  $\hat{\nu}$  denote the conditional measures and factor measure obtained by disintegrating  $\nu$  on the set of homogeneous stable manifolds  $W_\xi \in \mathcal{W}_{\mathbb{H}}^s$ ,  $\xi \in \Xi$ . Then for any  $\psi \in C^\alpha(M)$ ,*

$$\int_{W_\xi} \psi \, d\nu^{W_\xi} = \frac{\int_{W_\xi} \psi \rho_\xi \nu}{\int_{W_\xi} \rho_\xi \nu} \quad \forall \xi \in \Xi, \quad \text{and} \quad d\hat{\nu}(\xi) = |W_\xi|^{-1} \left( \int_{W_\xi} \rho_\xi \nu \right) d\hat{\mu}_{\text{SRB}}(\xi).$$

*Moreover, viewed as a leafwise measure,  $\nu(W) > 0$  for all  $W \in \mathcal{W}^s$ .*

*Proof.* First, we establish the following claim: For  $W \in \mathcal{W}^s$ , let  $n_2 \leq \bar{C}_2 |\log(|W|/\delta_1)|$  be the constant from the proof of Corollary 5.3. (This is the first time  $\ell$  such that  $\mathcal{G}_\ell(W)$  has at least one element of length at least  $\delta_1/3$ .) Then there exists  $\bar{C} > 0$  such that for all  $W \in \mathcal{W}^s$ ,

$$(7.13) \quad \int_W \nu \geq \bar{C} |W|^{h_* \bar{C}_2}.$$

<sup>31</sup>This connection is used in Section 7.3.

Indeed, recalling (7.2) and using (6.23), we have for  $\bar{C} = \frac{2c_0}{9} \delta_1^{1-h_*} \bar{C}_2$ ,

$$\begin{aligned} \int_W \nu &= \lim_{n_j} \frac{1}{n_j} \sum_{k=0}^{n_j-1} e^{-kh_*} \int_W \mathcal{L}^k 1 dm_W \geq \lim_{n_j} \frac{1}{n_j} \sum_{k=n_2}^{n_j-1} e^{-kh_*} \sum_{W_i \in \mathcal{G}_{n_2}(W)} \int_{W_i} \mathcal{L}^{k-n_2} 1 dm_{W_i} \\ &\geq \lim_{n_j} \frac{1}{n_j} \sum_{k=n_2}^{n_j-1} e^{-kh_*} \frac{2\delta_1}{9} c_0 e^{h_*(k-n_2)} \geq \frac{2\delta_1}{9} c_0 e^{-h_* n_2} \geq \bar{C} |W|^{h_* \bar{C}_2}. \end{aligned}$$

This proves the last statement of the lemma.

Next, for any  $f \in C^1(M)$ , according to our convention, we view  $f$  as an element of  $\mathcal{B}_w$  by considering it as a measure integrated against  $\mu_{\text{SRB}}$ . Now suppose  $(\nu_n)_{n \in \mathbb{N}}$  is the sequence of functions from (7.2) such that  $|\nu_n - \nu|_w \rightarrow 0$ . For any  $\psi \in C^\alpha(M)$ , we have

$$\begin{aligned} (7.14) \quad \nu_n(\psi) &= \int_M \nu_n \psi d\mu_{\text{SRB}} = \int_{\Xi} \int_{W_\xi} \nu_n \psi \rho_\xi dm_{W_\xi} |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \\ &= \int_{\Xi} \frac{\int_{W_\xi} \nu_n \psi \rho_\xi dm_{W_\xi}}{\int_{W_\xi} \nu_n \rho_\xi dm_{W_\xi}} d(\hat{\mu}_{\text{SRB}})_n(\xi), \end{aligned}$$

where  $d(\hat{\mu}_{\text{SRB}})_n(\xi) = |W_\xi|^{-1} \int_{W_\xi} \nu_n \rho_\xi dm_{W_\xi} d\hat{\mu}_{\text{SRB}}(\xi)$ . By definition of convergence in  $\mathcal{B}_w$  (see for example (7.3)) since  $\psi, \rho_\xi \in C^\alpha(W_\xi)$ , the ratio of integrals converges (uniformly in  $\xi$ ) to  $\int_{W_\xi} \psi \rho_\xi \nu / \int_{W_\xi} \rho_\xi \nu$ , and the factor measure converges to  $|W_\xi|^{-1} \int_{W_\xi} \rho_\xi d\nu d\hat{\mu}_{\text{SRB}}(\xi)$ . Note that since  $\rho_\xi$  is uniformly log-Hölder, and due to (7.13), we have  $\int_{W_\xi} \nu \rho_\xi dm_{W_\xi} > 0$  with lower bound depending only on the length of  $W_\xi$ .

Finally, by Proposition 4.2 and Lemma 4.4, we have  $\nu_n(\psi)$  converging to  $\nu(\psi)$ . Disintegrating  $\nu$  according to the statement of the lemma yields the claimed identifications.  $\square$

*Proof of Lemma 7.6.* The statement about stable manifolds of positive length follows from the characterization of  $\hat{\nu}$  in Lemma 7.7, since the set of points with stable manifolds of zero length has zero  $\hat{\mu}_{\text{SRB}}$ -measure [CM].

We fix  $W \in \mathcal{W}^s$  and prove the statement about  $\nu$  as a leafwise measure. This will imply the statement regarding unstable manifolds for the measure  $\nu$  by Lemma 7.7.

Fix  $\varepsilon > 0$  and  $\hat{\Lambda} \in (\Lambda, 1)$ , and define  $O = \cup_{n \geq 1} O_n$ , where

$$O_n = \{x \in W : n = \min j \text{ such that } d_u(T^{-j}x, \mathcal{S}_1) < \varepsilon C_e \hat{\Lambda}^{-j}\},$$

and  $d_u$  denotes distance restricted to the unstable cone. By [CM, Lemma 4.67], any  $x \in W \setminus O$  has unstable manifold of length at least  $2\varepsilon$ . We proceed to estimate  $\nu(O) = \sum_{n \geq 1} \nu(O_n)$ , where equality holds since the  $O_n$  are disjoint. In addition, since  $O_n$  is a finite union of open subcurves of  $W$ , we have

$$(7.15) \quad \int_W 1_{O_n} \nu = \lim_{j \rightarrow \infty} \int_W 1_{O_n} \nu_{\ell_j} = \lim_{j \rightarrow \infty} \ell_j^{-1} \sum_{k=0}^{\ell_j-1} e^{-kh_*} \int_W 1_{O_n} \mathcal{L}^k 1 dm_W.$$

We estimate two cases.

*Case I:  $k < n$ .* Write  $\int_{W \cap O_n} \mathcal{L}^k 1 dm_W = \sum_{W_i \in \mathcal{G}_k(W)} \int_{W_i \cap T^{-k}O_n} 1 dm_{W_i}$ .

If  $x \in T^{-k}O_n$ , then  $y = T^{-n+k}x$  satisfies  $d_u(y, \mathcal{S}_1) < \varepsilon C_e \hat{\Lambda}^{-n}$  and thus  $d_u(Ty, \mathcal{S}_{-1}) \leq C\varepsilon^{1/2} \hat{\Lambda}^{-n/2}$ . Due to the uniform transversality of stable and unstable cones, as well as the fact that elements of  $\mathcal{S}_{-1}$  are uniformly transverse to the stable cone, we have  $d_s(Ty, \mathcal{S}_{-1}) \leq C\varepsilon^{1/2} \hat{\Lambda}^{-n/2}$  as well, with possibly a larger constant  $C$ .

Let  $r_{-j}^s(x)$  denote the distance from  $T^{-j}x$  to the nearest endpoint of  $W^s(T^{-j}x)$ , where  $W^s(T^{-j}x)$  is the maximal local stable manifold containing  $T^{-j}x$ . From the above analysis, we see that  $W_i \cap T^{-k}O_n \subseteq \{x \in W_i : r_{-n+k+1}^s(x) \leq C\varepsilon^{1/2} \hat{\Lambda}^{-n/2}\}$ . By the time reversal of the growth lemma



[CM, Thm 5.52], we have  $m_{W_i}(r_{-n+k+1}^s(x) \leq C\varepsilon^{1/2}\hat{\Lambda}^{-n/2}) \leq C'\varepsilon^{1/2}\hat{\Lambda}^{-n/2}$  for a constant  $C'$  that is uniform in  $n$  and  $k$ . Thus, using Proposition 4.6, we find

$$\int_{W \cap O_n} \mathcal{L}^k 1 \, dm_W \leq \#\mathcal{G}_k(W) C' \varepsilon^{1/2} \hat{\Lambda}^{-n/2} \leq C e^{kh_*} \varepsilon^{1/2} \hat{\Lambda}^{-n/2}.$$

*Case II:  $k \geq n$ .* Using the same observation as in Case I, if  $x \in T^{-n+1}O_n$ , then  $x$  satisfies  $d_s(x, \mathcal{S}_{-1}) \leq C\varepsilon^{1/2}\hat{\Lambda}^{-n/2}$ . We change variables to estimate the integral precisely at time  $-n+1$ , again using Proposition 4.6,

$$\begin{aligned} \int_{W \cap O_n} \mathcal{L}^k 1 \, dm_W &= \sum_{W_i \in \mathcal{G}_{n-1}(W)} \int_{W_i \cap T^{-n+1}O_n} \mathcal{L}^{k-n+1} 1 \, dm_{W_i} \\ &\leq \sum_{W_i \in \mathcal{G}_{n-1}(W)} |\log |W_i \cap T^{-n+1}O_n||^{-\gamma} \|\mathcal{L}^{k-n+1} 1\|_s \\ &\leq \sum_{W_i \in \mathcal{G}_{n-1}(W)} |\log(C\varepsilon^{1/2}\hat{\Lambda}^{-n/2})|^{-\gamma} C e^{(k-n+1)h_*} \leq |\log(C\varepsilon^{1/2}\hat{\Lambda}^{-n/2})|^{-\gamma} C e^{kh_*}. \end{aligned}$$

Using the estimates of Cases I and II in (7.15) and using the weaker bound, we see that,

$$\int_W 1_{O_n} \nu_{\ell_j} \leq C |\log(C\varepsilon^{1/2}\hat{\Lambda}^{-n/2})|^{-\gamma}.$$

Summing over  $n$ , we have,  $\int_W 1_O \nu_{\ell_j} \leq C' |\log \varepsilon|^{1-\gamma}$ , uniformly in  $j$ . Since  $\nu_{\ell_j}$  converges to  $\nu$  in the weak norm, this bound carries over to  $\nu$ . Since  $\gamma > 1$  and  $\varepsilon > 0$  was arbitrary, this implies  $\nu(O) = 0$ , completing the proof of the lemma.  $\square$

**7.3. Absolute Continuity of  $\mu_*$  — Full Support.** *In this subsection, we assume throughout that  $\gamma > 1$  (this is possible since we assumed  $h_* > s_0 \log 2$  to construct  $\mu_*$ ).*

Our proof of the Bernoulli property relies on showing first that  $\mu_*$  is K-mixing (Proposition 7.16). As a first step, we will prove that  $\mu_*$  is ergodic (see the Hopf-type Lemma 7.15). This will require establishing absolute continuity of the unstable foliation for  $\mu_*$  (Corollary 7.9), which will be deduced from the following absolute continuity result for  $\nu$ :

**Proposition 7.8.** *Let  $R$  be a Cantor rectangle. Fix  $W^0 \in \mathcal{W}^s(R)$  and for  $W \in \mathcal{W}^s(R)$ , let  $\Theta_W$  denote the holonomy map from  $W^0 \cap R$  to  $W \cap R$  along unstable manifolds in  $\mathcal{W}^u(R)$ . Then  $\Theta_W$  is absolutely continuous with respect to the leafwise measure  $\nu$ .*

*Proof.* Since by Lemma 7.6 unstable manifolds comprise a set of full  $\nu$ -measure, it suffices to fix a set  $E \subset W^0 \cap R$  with  $\nu$ -measure zero, and prove that the  $\nu$ -measure of  $\Theta_W(E) \subset W$  is also zero.

Since  $\nu$  is a regular measure on  $W^0$ , for  $\varepsilon > 0$ , there exists an open set  $O_\varepsilon \subset W^0$ ,  $O_\varepsilon \supset E$ , such that  $\nu(O_\varepsilon) \leq \varepsilon$ . Indeed, since  $W^0$  is compact, we may choose  $O_\varepsilon$  to be a finite union of intervals. Let  $\psi_\varepsilon$  be a smooth function which is 1 on  $O_\varepsilon$  and 0 outside of an  $\varepsilon$ -neighbourhood of  $O_\varepsilon$ . We may choose  $\psi_\varepsilon$  so that  $|\psi_\varepsilon|_{C^1(W^0)} \leq 2\varepsilon^{-1}$ .

Using (6.2), we choose  $n = n(\varepsilon)$  such that  $|\psi_\varepsilon \circ T^n|_{C^1(T^{-n}W^0)} \leq 1$ . Note this implies in particular that  $\Lambda^{-n} \leq \varepsilon$ . Following the procedure described at the beginning of Section 6.2, we subdivide  $T^{-n}W^0$  and  $T^{-n}W$  into matched pieces  $U_j^0, U_j$  and unmatched pieces  $V_i^0, V_i$ . With this construction, none of the unmatched pieces  $T^n V_i^0$  intersect an unstable manifold in  $\mathcal{W}^u(R)$  since unstable manifolds are not cut under  $T^{-n}$ .

Indeed, on matched pieces, we may choose a foliation  $\Gamma_j = \{\gamma_x\}_{x \in U_j^0}$  such that:

- i)  $T^n \Gamma_j$  contains all unstable manifolds in  $\mathcal{W}^u(R)$  that intersect  $T^n U_j^0$ ;
- ii) between unstable manifolds in  $\Gamma_j \cap T^{-n}(\mathcal{W}^u(R))$ , we interpolate via unstable curves;

iii) the resulting holonomy  $\Theta_j$  from  $T^n U_j^0$  to  $T^n U_j$  has uniformly bounded Jacobian<sup>32</sup> with respect to arc-length, with bound depending on the unstable diameter of  $D(R)$ , by [BDL, Lemmas 6.6, 6.8];

iv) pushing forward  $\Gamma_j$  to  $T^n \Gamma_j$  in  $D(R)$ , we interpolate in the gaps using unstable curves; call  $\bar{\Gamma}$  the resulting foliation of  $D(R)$ ;

v) the associated holonomy map  $\bar{\Theta}_W$  extends  $\Theta_W$  and has uniformly bounded Jacobian, again by [BDL, Lemmas 6.6 and 6.8].

Using the map  $\bar{\Theta}_W$ , we define  $\tilde{\psi}_\varepsilon = \psi_\varepsilon \circ \bar{\Theta}_W^{-1}$ , and note that  $|\tilde{\psi}_\varepsilon|_{C^1(W)} \leq C|\psi_\varepsilon|_{C^1(W^0)}$ , where we write  $C^1(W)$  for the set of Lipschitz functions on  $W$ , i.e.,  $C^\alpha$  with  $\alpha = 1$ .

Next, we modify  $\psi_\varepsilon$  and  $\tilde{\psi}_\varepsilon$  as follows: We set them equal to 0 on the images of unmatched pieces,  $T^n V_i^0$  and  $T^n V_i$ , respectively. Since these curves do not intersect unstable manifolds in  $\mathcal{W}^u(R)$ , we still have  $\psi_\varepsilon = 1$  on  $E$  and  $\tilde{\psi}_\varepsilon = 1$  on  $\Theta_W(E)$ . Moreover, the set of points on which  $\psi_\varepsilon > 0$  (resp.  $\tilde{\psi}_\varepsilon > 0$ ) is a finite union of open intervals that cover  $E$  (resp.  $\Theta_W(E)$ ).

Following Section 6.2, we estimate

$$(7.16) \quad \begin{aligned} \int_{W^0} \psi_\varepsilon \nu - \int_W \tilde{\psi}_\varepsilon \nu &= e^{-nh_*} \left( \int_{W^0} \psi_\varepsilon \mathcal{L}^n \nu - \int_W \tilde{\psi}_\varepsilon \mathcal{L}^n \nu \right) \\ &= e^{-nh_*} \sum_j \int_{U_j^0} \psi_\varepsilon \circ T^n \nu - \int_{U_j} \phi_j \nu + \int_{U_j} (\phi_j - \tilde{\psi}_\varepsilon \circ T^n) \nu, \end{aligned}$$

where  $\phi_j = \psi_\varepsilon \circ T^n \circ G_{U_j^0} \circ G_{U_j}^{-1}$ , and  $G_{U_j^0}$  and  $G_{U_j}$  represent the functions defining  $U_j^0$  and  $U_j$ , respectively, defined as in (6.5). Next, since  $d(\psi_\varepsilon \circ T^n, \phi_j) = 0$  by construction, and using (6.9) and the assumption that  $\Lambda^{-n} \leq \varepsilon$ , we have by (6.10),

$$(7.17) \quad e^{-nh_*} \left| \sum_j \int_{U_j^0} \psi_\varepsilon \circ T^n \nu - \int_{U_j} \phi_j \nu \right| \leq C |\log \varepsilon|^{-\varsigma} \|\nu\|_u.$$

It remains to estimate the last term in (7.16). This we do using the weak norm,

$$(7.18) \quad \int_{U_j} (\phi_j - \tilde{\psi}_\varepsilon \circ T^n) \nu \leq |\phi_j - \tilde{\psi}_\varepsilon \circ T^n|_{C^\alpha(U_j)} |\nu|_w.$$

By (6.12), we have

$$|\phi_j - \tilde{\psi}_\varepsilon \circ T^n|_{C^\alpha(U_j)} \leq C |\psi_\varepsilon \circ T^n \circ G_{U_j^0} - \tilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}|_{C^\alpha(I_j)},$$

where  $I_j$  is the common  $r$ -interval on which  $G_{U_j^0}$  and  $G_{U_j}$  are defined.

Fix  $r \in I_j$ , and let  $x = G_{U_j^0}(r) \in U_j$  and  $\bar{x} = G_{U_j}(r)$ . Since  $U_j^0$  and  $U_j$  are matched, there exists  $y \in U_j^0$  and an unstable curve  $\gamma_y \in \Gamma_j$  such that  $\gamma_y \cap U_j = \bar{x}$ . By definition of  $\tilde{\psi}_\varepsilon$ , we have  $\tilde{\psi}_\varepsilon \circ T^n(\bar{x}) = \psi_\varepsilon \circ T^n(y)$ . Thus,

$$\begin{aligned} |\psi_\varepsilon \circ T^n \circ G_{U_j^0}(r) - \tilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}(r)| & \\ &\leq |\psi_\varepsilon \circ T^n(x) - \psi_\varepsilon \circ T^n(y)| + |\psi_\varepsilon \circ T^n(y) - \tilde{\psi}_\varepsilon \circ T^n(\bar{x})| \\ &\leq |\psi_\varepsilon \circ T^n|_{C^1(U_j^0)} d(x, y) \leq C \Lambda^{-n} \leq C \varepsilon, \end{aligned}$$

where we have used the fact that  $d(x, y) \leq C \Lambda^{-n}$  due to the uniform transversality of stable and unstable curves.

Now given  $r, s \in I_j$ , we have on the one hand,

$$|\psi_\varepsilon \circ T^n \circ G_{U_j^0}(r) - \tilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}(r) - \psi_\varepsilon \circ T^n \circ G_{U_j^0}(s) + \tilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}(s)| \leq 2C \varepsilon,$$

<sup>32</sup>Indeed, [BDL] shows the Jacobian is Hölder continuous, but we shall not need this here.

while on the other hand,

$$\begin{aligned} & |\psi_\varepsilon \circ T^n \circ G_{U_j^0}(r) - \tilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}(r) - \psi_\varepsilon \circ T^n \circ G_{U_j^0}(s) + \tilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}(s)| \\ & \leq (|\psi_\varepsilon \circ T^n|_{C^1(U_j^0)} + |\tilde{\psi}_\varepsilon \circ T^n|_{C^1(U_j)})C|r - s|, \end{aligned}$$

where we have used the fact that  $G_{U_j^0}^{-1}$  and  $G_{U_j}^{-1}$  have bounded derivatives since the stable cone is bounded away from the vertical.

The difference is bounded by the minimum of these two expressions. This is greatest when the two are equal, i.e., when  $|r - s| = C\varepsilon$ . Thus  $H^\alpha(\psi_\varepsilon \circ T^n \circ G_{U_j^0} - \tilde{\psi}_\varepsilon \circ T^n \circ G_{U_j}) \leq C\varepsilon^{1-\alpha}$ , and so  $|\phi_j - \tilde{\psi}_\varepsilon \circ T^n|_{C^\alpha(U_j)} \leq C\varepsilon^{1-\alpha}$ . Putting this estimate together with (7.17) and (7.18) in (7.16), we conclude,

$$(7.19) \quad \left| \int_{W^0} \psi_\varepsilon \nu - \int_W \tilde{\psi}_\varepsilon \nu \right| \leq C|\log \varepsilon|^{-\varsigma} \|\nu\|_u + C\varepsilon^{1-\alpha} |\nu|_w.$$

Now since  $\int_{W^0} \psi_\varepsilon \nu \leq 2\varepsilon$ , we have

$$(7.20) \quad \int_W \tilde{\psi}_\varepsilon \nu \leq C' |\log \varepsilon|^{-\varsigma},$$

where  $C'$  depends on  $\nu$ . Since  $\tilde{\psi}_\varepsilon = 1$  on  $\Theta_W(E)$  and  $\tilde{\psi}_\varepsilon > 0$  on an open set containing  $\Theta_W(E)$  for every  $\varepsilon > 0$ , we have  $\nu(\Theta_W(E)) = 0$ , as required.  $\square$

We next state our main absolute continuity result:

**Corollary 7.9** (Absolute Continuity of  $\mu_*$  with Respect to Unstable Foliations). *Let  $R$  be a Cantor rectangle with  $\mu_*(R) > 0$ . Fix  $W^0 \in \mathcal{W}^s(R)$  and for  $W \in \mathcal{W}^s(R)$ , let  $\Theta_W$  denote the holonomy map from  $W^0 \cap R$  to  $W \cap R$  along unstable manifolds in  $\mathcal{W}^u(R)$ . Then  $\Theta_W$  is absolutely continuous with respect to the measure  $\mu_*$ .*

To deduce the corollary from Proposition 7.8, we shall introduce a set  $M^{reg}$  of regular points and a countable cover of this set by Cantor rectangles. The set  $M^{reg}$  is defined by

$$M^{reg} = \{x \in M : d(x, \partial W^s(x)) > 0, \quad d(x, \partial W^u(x)) > 0\}.$$

At each  $x \in M^{reg}$ , by [CM, Prop 7.81], we construct a (closed) locally maximal<sup>33</sup> Cantor rectangle  $R_x$ , containing  $x$ , which is the direct product of local stable and unstable manifolds (recall Section 5.3). By trimming the sides, we may arrange it so that  $\frac{1}{2} \text{diam}^s(R_x) \leq \text{diam}^u(R_x) \leq 2 \text{diam}^s(R_x)$ .

**Lemma 7.10** (Countable Cover of  $M^{reg}$  by Cantor Rectangles). *There exists a countable set  $\{x_j\}_{j \in \mathbb{N}} \subset M^{reg}$ , such that  $\cup_j R_{x_j} = M^{reg}$  and each  $R_j := R_{x_j}$  satisfies (5.10).*

*Proof.* Let  $n_\delta \in \mathbb{N}$  be such that  $1/n_\delta \leq \delta_0$ . As already mentioned, in the proof of Proposition 5.5, for each  $n \geq n_\delta$ , by [CM, Lemma 7.87], there exists a finite number of  $R_x$  such that any stable manifold of length at least  $1/n$  properly crosses one of the  $R_x$  (see Section 5.3 for the definition of proper crossing, recalling that each  $R_x$  must satisfy (5.10)). This fact follows from the compactness of the set of stable curves in the Hausdorff metric. Call this finite set of rectangles  $\{R_{n,i}\}_{i \in \tilde{I}_n}$ .

Fix  $y \in M^{reg}$  and define  $\epsilon = \min\{d(y, \partial W^s(y)), d(y, \partial W^u(y))\} > 0$ . Choose  $n \geq n_\delta$  such that  $2/n < \epsilon$ . By construction, there exists  $i \in \tilde{I}_n$  such that  $W^s(y)$  properly crosses  $R_{n,i}$ . Now  $\text{diam}^s(R_{n,i}) \leq 1/n$ , which implies  $\text{diam}^u(R_{n,i}) \leq 2/n < \epsilon$ . Thus  $W^u(y)$  crosses  $R_{n,i}$  as well. By maximality,  $y \in R_{n,i}$ .  $\square$

<sup>33</sup>Recall that, as in Section 5.3, by locally maximal we mean that  $y \in R_x$  if and only if  $y \in D(R_x)$  and  $y$  has stable and unstable manifolds that completely cross  $D(R_x)$ .

Let  $\{R_{n,i} : n \geq n_\delta, i \in \tilde{I}_n\}$  be the Cantor rectangles from the proof of Lemma 7.10. Since  $\mu_*(M^{reg}) = 1$ , by discarding any  $R_{n,i}$  of zero measure, we obtain a countable collection of Cantor rectangles

$$(7.21) \quad \{R_j\}_{j \in \mathbb{N}} := \{R_{n,i} : n \geq n_\delta, i \in \tilde{I}_n\}$$

such that  $\mu_*(R_j) > 0$  for all  $j$  and  $\mu_*(\cup_j R_j) = 1$ . In the rest of the paper we shall work with this countable collection of rectangles.

Given a Cantor rectangle  $R$ , define  $\mathcal{W}^s(R)$  to be the set of stable manifolds that completely cross  $D(R)$ , and similarly for  $\mathcal{W}^u(R)$ .

*Proof of Corollary 7.9.* In order to prove absolute continuity of the unstable foliation with respect to  $\mu_*$ , we will show that the conditional measures  $\mu_*^W$  of  $\mu_*$  are equivalent to  $\nu$  on  $\mu_*$ -almost every  $W \in \mathcal{W}^s(R)$ .

Fix a Cantor rectangle  $R$  satisfying (5.10) with  $\mu_*(R) > 0$ , and  $W^0$  as in the statement of the corollary. Let  $E \subset W^0 \cap R$  satisfy  $\nu(E) = 0$ , for the leafwise measure  $\nu$ .

For any  $W \in \mathcal{W}^s(R)$ , we have the holonomy map  $\Theta_W : W^0 \cap R \rightarrow W \cap R$  as in the proof of Proposition 7.8. For  $\varepsilon > 0$ , we approximate  $E$ , choose  $n$  and construct a foliation  $\bar{\Gamma}$  of the solid rectangle  $D(R)$  as before. Define  $\psi_\varepsilon$  and use the foliation  $\bar{\Gamma}$  to define  $\tilde{\psi}_\varepsilon$  on  $D(R)$ . We have  $\tilde{\psi}_\varepsilon = 1$  on  $\bar{E} = \cup_{x \in E} \tilde{\gamma}_x$ , where  $\tilde{\gamma}_x$  is the element of  $\bar{\Gamma}$  containing  $x$ . We extend  $\tilde{\psi}_\varepsilon$  to  $M$  by setting it equal to 0 on  $M \setminus D(R)$ .

It follows from the proof of Proposition 7.8, in particular (7.20), that  $\tilde{\psi}_\varepsilon \nu \in \mathcal{B}_w$ , and  $|\tilde{\psi}_\varepsilon \nu|_w \leq C' |\log \varepsilon|^{-\varsigma}$ . Now,

$$(7.22) \quad \begin{aligned} \mu_*(\tilde{\psi}_\varepsilon) &= \tilde{\nu}(\tilde{\psi}_\varepsilon \nu) = \lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} e^{-kh_*} (\mathcal{L}^*)^k d\mu_{\text{SRB}}(\tilde{\psi}_\varepsilon \nu) \\ &= \lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} e^{-kh_*} \mu_{\text{SRB}}(\mathcal{L}^k(\tilde{\psi}_\varepsilon \nu)). \end{aligned}$$

For each  $k$ , using the disintegration of  $\mu_{\text{SRB}}$  as in the proof of Lemma 7.7 with the same notation as there, we estimate,

$$\begin{aligned} \mu_{\text{SRB}}(\mathcal{L}^k(\tilde{\psi}_\varepsilon \nu)) &= \int_{\Xi} \int_{W_\xi} \mathcal{L}^k(\tilde{\psi}_\varepsilon \nu) \rho_\xi dm_{W_\xi} |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \\ &\leq C \int_{\Xi} |\mathcal{L}^k(\tilde{\psi}_\varepsilon \nu)|_w |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \leq C e^{kh_*} |\tilde{\psi}_\varepsilon \nu|_w \leq C e^{kh_*} |\log \varepsilon|^{-\varsigma}, \end{aligned}$$

where we have used (4.9) in the last line. Thus  $\mu_*(\tilde{\psi}_\varepsilon) \leq C |\log \varepsilon|^{-\varsigma}$ , for each  $\varepsilon > 0$ , so that  $\mu_*(\bar{E}) = 0$ .

Disintegrating  $\mu_*$  into conditional measures  $\mu_*^{W_\xi}$  on  $W_\xi \in \mathcal{W}^s$  and a factor measure  $d\hat{\mu}_*(\xi)$  on the index set  $\Xi_R$  of stable manifolds in  $\mathcal{W}^s(R)$ , it follows that  $\mu_*^{W_\xi}(\bar{E}) = 0$  for  $\hat{\mu}_*$ -almost every  $\xi \in \Xi_R$ . Since  $E$  was arbitrary, the conditional measures of  $\mu_*$  on  $\mathcal{W}^s(R)$  are absolutely continuous with respect to the leafwise measure  $\nu$ .

To show that in fact  $\mu_*^W$  is equivalent to  $\nu$ , suppose now that  $E \subset W^0$  has  $\nu(E) > 0$ . For any  $\varepsilon > 0$  such that  $C' |\log \varepsilon|^{-\varsigma} < \nu(E)/2$ , where  $C'$  is from (7.20), choose  $\psi_\varepsilon \in C^1(W^0)$  such that  $\nu(|\psi_\varepsilon - 1_E|) < \varepsilon$ , where  $1_E$  is the indicator function of the set  $E$ . As above, we extend  $\psi_\varepsilon$  to a function  $\tilde{\psi}_\varepsilon$  on  $D(R)$  via the foliation  $\bar{\Gamma}$ , and then to  $M$  by setting  $\tilde{\psi}_\varepsilon = 0$  on  $M \setminus D(R)$ .

We have  $\tilde{\psi}_\varepsilon \nu \in \mathcal{B}_w$  and by (7.19)

$$(7.23) \quad \nu(\tilde{\psi}_\varepsilon 1_W) \geq \nu(\psi_\varepsilon 1_{W^0}) - C' |\log \varepsilon|^{-\varsigma}, \quad \text{for all } W \in \mathcal{W}^s(R).$$

Now following (7.22) and disintegrating  $\mu_{\text{SRB}}$  as usual, we obtain,

$$(7.24) \quad \begin{aligned} \mu_*(\tilde{\psi}_\varepsilon) &= \lim_n \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} \int_{\Xi} \int_{W_\xi} \mathcal{L}^k(\tilde{\psi}_\varepsilon \nu) \rho_\xi dm_{W_\xi} d\hat{\mu}_{\text{SRB}}(\xi) \\ &= \lim_n \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} \int_{\Xi} \left( \sum_{W_{\xi,i} \in \mathcal{G}_k(W_\xi)} \int_{W_{\xi,i}} \tilde{\psi}_\varepsilon \rho_\xi \circ T^k \nu \right) d\hat{\mu}_{\text{SRB}}(\xi). \end{aligned}$$

To estimate this last expression, we estimate the cardinality of the curves  $W_{\xi,i}$  which properly cross the rectangle  $R$ .

By Corollary 5.3 and the choice of  $\delta_1$  in (5.6), there exists  $k_0$ , depending only on the minimum length of  $W \in \mathcal{W}^s(R)$ , such that  $\#L_k^{\delta_1}(W_\xi) \geq \frac{1}{3}\#\mathcal{G}_k(W_\xi)$  for all  $k \geq k_0$ .

By choice of our covering  $\{R_i\}$  from (7.21), all  $W_{\xi,j} \in L_k^{\delta_1}(W_\xi)$  properly cross one of finitely many  $R_i$ . By the topological mixing property of  $T$ , there exists  $n_0$ , depending only on the length scale  $\delta_1$ , such that some smooth component of  $T^{-n_0}W_{\xi,j}$  properly crosses  $R$ . Thus, letting  $\mathcal{C}_k(W_\xi)$  denote those  $W_{\xi,i} \in \mathcal{G}_k(W_\xi)$  which properly cross  $R$ , we have

$$\#\mathcal{C}_k(W_\xi) \geq \#L_{k-n_0}^{\delta_1}(W_\xi) \geq \frac{1}{3}\#\mathcal{G}_{k-n_0}(W_\xi) \geq \frac{1}{3}ce^{(k-n_0)h_*},$$

for all  $k \geq k_0 + n_0$ , where  $c > 0$  depends on  $c_0$  from Proposition 5.5 as well as the minimum length of  $W \in \mathcal{W}^s(R)$ .

Using this lower bound on the cardinality together with (7.23) yields,

$$\mu_*(\tilde{\psi}_\varepsilon) \geq \frac{1}{3}ce^{-n_0h_*} (\nu(\psi_\varepsilon) - C'|\log \varepsilon|^{-\varsigma}) \geq C''(\nu(E) - |\log \varepsilon|^{-\varsigma}).$$

Taking  $\varepsilon \rightarrow 0$ , we have  $\mu_*(\bar{E}) \geq C''\nu(E)$ , and so  $\mu_*^W(\bar{E}) > 0$  for almost every  $W \in \mathcal{W}^s(R)$ .  $\square$

A consequence of the proof of Corollary 7.9 is the positivity of  $\mu_*$  on open sets.

**Proposition 7.11** (Full Support). *We have  $\mu_*(O) > 0$  for any open set  $O$ .*

*Proof.* Suppose  $R$  is a Cantor rectangle with index set of stable leaves  $\Xi_R$ . We call  $I \subset \Xi_R$  an interval if  $a, b \in I$  implies that  $c \in I$  for all  $c \in \Xi_R$  such that  $W_c$  lies between  $W_a$  and  $W_b$ .<sup>34</sup> It follows from the proof of Corollary 7.9 that for any interval  $I \subset \Xi_R$  such that  $\hat{\mu}_{\text{SRB}}(I) > 0$ , then  $\mu_*(\cup_{\xi \in I} W_\xi) > 0$ . Indeed, by Lemma 7.7,  $\hat{\nu}$  is equivalent to  $\hat{\mu}_{\text{SRB}}$  (since  $\nu(W) > 0$  for all  $W \in \mathcal{W}^s$ , when  $\nu$  is viewed as a leafwise measure), so that  $\hat{\mu}_{\text{SRB}}(I) > 0$  implies  $\hat{\nu}(I) > 0$ . Then by Lemma 7.6 there exists a Cantor rectangle  $R'$  with  $D(R') \subset D(R)$  and  $\Xi_{R'} \subset I$  such that  $\nu(R') > 0$ . Then we simply apply (7.24) and the argument following it with  $\tilde{\psi}_\varepsilon$  replaced by the characteristic function of  $\cup_{\xi \in \Xi_{R'}} W_\xi$ .

Then if  $O$  is an open set in  $M$ , it contains a Cantor rectangle  $R$  such that  $D(R) \subset O$  and  $\mu_{\text{SRB}}(R) > 0$ . It follows that  $\hat{\mu}_{\text{SRB}}(\Xi_R) > 0$ , and so  $\mu_*(\cup_{\xi \in \Xi_R} W_\xi) > 0$ .  $\square$

**7.4. Bounds on Dynamical Bowen Balls — Comparing  $\mu_*$  and  $\mu_{\text{SRB}}$ .** In this section we show upper and lower bounds on the  $\mu_*$ -measure of dynamical Bowen balls, from which we establish a necessary condition for  $\mu_*$  and  $\mu_{\text{SRB}}$  to coincide. (The lower bound will use results from Section 7.3.)

For  $\epsilon > 0$  and  $x \in M$ , we denote by  $B_n(x, \epsilon)$  the dynamical (Bowen) ball at  $x$  of length  $n \geq 1$  for  $T^{-1}$ , i.e.,

$$B_n(x, \epsilon) = \{y \in M \mid d(T^{-j}(y), T^{-j}(x)) \leq \epsilon, \forall 0 \leq j \leq n\}.$$

For  $\eta, \delta > 0$  and  $p \in (1/\gamma, 1]$ , let  $M^{\text{reg}}(\eta, p, \delta)$  denote those  $x \in M^{\text{reg}}$  such that  $d(T^{-n}x, \mathcal{S}_{-1}) \geq \delta e^{-\eta n^p}$ . It follows from Lemma 7.3 that  $\mu_*(\cup_{\delta > 0} M^{\text{reg}}(\eta, p, \delta)) = 1$ .

<sup>34</sup>Notice that if  $I \subset \Xi_j$  is an interval such that  $\hat{\mu}_{\text{SRB}}(I) > 0$ , then  $\cup_{\xi \in I} W_\xi \cap R_j$  is a Cantor rectangle which contains a subset satisfying the high density condition (5.10), so we can talk about proper crossings.

**Proposition 7.12** (Topological Entropy and Measure of Dynamical Balls). *Assume that  $h_* > s_0 \log 2$ . There exists  $A < \infty$  such that for all  $\epsilon > 0$  sufficiently small,  $x \in M$ , and  $n \geq 1$ , the measure  $\mu_*$  constructed in (7.1) satisfies*

$$(7.25) \quad \mu_*(B_n(x, \epsilon)) \leq \mu_*(\overline{B_n(x, \epsilon)}) \leq Ae^{-nh_*}.$$

Moreover, for all  $\eta, \delta > 0$  and  $p \in (1/\gamma, 1]$ , for each  $x \in M^{reg}(\eta, p, \delta)$ , and all  $\epsilon > 0$  sufficiently small, there exists  $C(x, \epsilon, \eta, p, \delta) > 0$  such that for all  $n \geq 1$ ,

$$(7.26) \quad C(x, \epsilon, \eta, p, \delta) e^{-nh_* - \eta h_* \bar{C}_2 n^p} \leq \mu_*(B_n(x, \epsilon)),$$

where  $\bar{C}_2 > 0$  is the constant from the proof of Corollary 5.3.

*Proof.* Assume  $\gamma > 1$ . Fix  $\epsilon > 0$  such that  $\epsilon \leq \min\{\delta_0, \epsilon_0\}$ , where  $\epsilon_0$  is from the proof of Lemma 3.4. For  $x \in M$  and  $n \geq 0$ , define  $1_{n, \epsilon}^B$  to be the indicator function of the dynamical ball  $B_n(x, \epsilon)$ .

Since  $\nu$  is attained as the (averaged) limit of  $\mathcal{L}^n 1$  in the weak norm and  $\int_W (\mathcal{L}^n 1) \psi dm_W \geq 0$  whenever  $\psi \geq 0$ , it follows that, viewing  $\nu$  as a leafwise distribution,

$$(7.27) \quad \int_W \psi \nu \geq 0, \quad \text{for all } \psi \geq 0.$$

Then the inequality  $|\int_W \psi \nu| \leq \int_W |\psi| \nu$  implies that the supremum in the weak norm can be obtained by restricting to  $\psi \geq 0$ .

Let  $W \in \mathcal{W}^s$  be a curve intersecting  $B_n(x, \epsilon)$ , and let  $\psi \in C^\alpha(W)$  satisfy  $\psi \geq 0$  and  $|\psi|_{C^\alpha(W)} \leq 1$ . Then, since  $\mathcal{L}\nu = e^{h_*} \nu$ , we have

$$(7.28) \quad \int_W \psi 1_{n, \epsilon}^B \nu = \int_W \psi 1_{n, \epsilon}^B e^{-nh_*} \mathcal{L}^n \nu = e^{-nh_*} \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} \psi \circ T^n 1_{n, \epsilon}^B \circ T^n \nu.$$

We claim that  $1_{n, \epsilon}^B \nu \in \mathcal{B}_w$  (and indeed in  $\mathcal{B}$ ). To see this, note that

$$1_{n, \epsilon}^B = \prod_{j=0}^n 1_{\mathcal{N}_\epsilon(T^{-j}x)} \circ T^{-j} = \prod_{j=0}^n \mathcal{L}_{\text{SRB}}^j(1_{\mathcal{N}_\epsilon(T^{-j}x)}),$$

where, as in Section 1.3,  $\mathcal{L}_{\text{SRB}}$  denotes the transfer operator with respect to  $\mu_{\text{SRB}}$ . Since  $\mathcal{L}_{\text{SRB}}$  preserves  $\mathcal{B}$  and  $\mathcal{B}_w$  ([DZ3, Lemma 3.6]), it suffices to show that  $1_{\mathcal{N}_\epsilon(T^{-j}x)}$  satisfies the assumptions of [DZ3, Lemma 5.3]. This follows from the fact that  $\partial \mathcal{N}_\epsilon(T^{-j}x)$  comprises a single circular arc, possibly together with a segment of  $\mathcal{S}_0$ , which satisfies the weak transversality condition of that lemma with  $t_0 = 1/2$ . Then applying [DZ3, Lemma 5.3] successively for each  $j$  yields the claim.

In the proof of Lemma 3.4, it was shown that if  $x, y$  lie in different elements of  $\mathcal{M}_0^n$ , then  $d_n(x, y) \geq \epsilon_0$ , where  $d_n(\cdot, \cdot)$  is the dynamical distance defined in (2.1). Since  $B_n(x, \epsilon)$  is defined with respect to  $T^{-1}$ , we will use the time reversal counterpart of this property. Thus since  $\epsilon < \epsilon_0$ , we conclude that  $B_n(x, \epsilon)$  is contained in a single component of  $\mathcal{M}_{-n}^0$ , i.e.,  $B_n(x, \epsilon) \cap \mathcal{S}_{-n} = \emptyset$ , so that  $T^{-n}$  is a diffeomorphism of  $B_n(x, \epsilon)$  onto its image. Note that  $1_{n, \epsilon}^B \circ T^n = 1_{T^{-n}(B_n(x, \epsilon))}$  and that  $T^{-n}(B_n(x, \epsilon))$  is contained in a single component of  $\mathcal{M}_0^n$ , denoted  $A_{n, \epsilon}$ .

It follows that for each  $W_i \in \mathcal{G}_n(W)$  we have  $W_i \cap A_{n, \epsilon} = W_i$ . By (7.27), we have

$$\int_{W_i} (\psi \circ T^n) 1_{T^{-n}(B_n(x, \epsilon))} \nu \leq \int_{W_i} \psi \circ T^n \nu.$$

Moreover, there can be at most two  $W_i \in \mathcal{G}_n(W)$  having nonempty intersection with  $T^{-n}(B_n(x, \epsilon))$ . This follows from the facts that  $\epsilon \leq \delta_0$ , and that, in the absence of any cuts due to singularities, the only subdivisions occur when a curve has grown to length longer than  $\delta_0$  and is subdivided into two curves of length at least  $\delta_0/2$ .

Using these facts together with (6.2), we sum over  $W'_i \in \mathcal{G}_n(W)$  such that  $W'_i \cap T^{-n}(B_n(x, \epsilon)) \neq \emptyset$ , to obtain

$$\int_W \psi 1_{n,\epsilon}^B \nu \leq e^{-nh_*} \sum_i \int_{W'_i} \psi \circ T^n \nu \leq 2Ce^{-nh_*} |\nu|_w.$$

This implies that  $|1_{n,\epsilon}^B \nu|_w \leq 2Ce^{-nh_*} |\nu|_w$ . Applying (7.5), implies (7.25).

Next we prove (7.26). Fix  $\eta, \delta > 0$  with  $e^\eta < \Lambda$  and  $p \in (1/\gamma, 1]$ , and let  $x \in M^{reg}(\eta, p, \delta)$ . By [CM, Lemma 4.67] the length of the local stable manifold containing  $x$  is at least  $\delta C_1$ , where  $C_1$  is from (3.1). So by [CM, Lemma 7.87], there exists a Cantor rectangle  $R_x$  containing  $x$  such that  $\mu_{\text{SRB}}(R_x) > 0$  and whose diameter depends only on the length scale  $\delta C_1$ . By the proof of Proposition 7.11, we also have  $\mu_*(R_x) > 0$ . In particular,  $\hat{\mu}_*(\Xi_{R_x}) = c_x > 0$ , where  $\Xi_{R_x}$  is the index set of stable manifolds comprising  $R_x$ . Let  $\delta' > 0$  denote the minimum length of  $W_\xi \cap D(R_x)$  for  $\xi \in \Xi_{R_x}$ , where  $D(R_x)$  is the smallest solid rectangle containing  $R_x$ , as in Definition 5.7.

Choose  $\epsilon > 0$  such that  $\epsilon \leq \min\{\delta_0, \epsilon_0, \delta'\}$ . As above, we note that  $B_n(x, \epsilon)$  is contained in a single component of  $\mathcal{M}_{-n}^0$ , and thus  $T^{-n}(B_n(x, \epsilon))$  is contained in a single component of  $\mathcal{M}_0^n$ . Moreover,  $T^{-n}$  is smooth on  $W^u(x) \cap D(R_x)$ . Now suppose  $y \in W^u(x) \cap R_x$ . Then since  $x \in M^{reg}(\eta, p, \delta)$ ,

$$d(T^{-n}y, \mathcal{S}_{-1}) \geq d(T^{-n}x, \mathcal{S}_{-1}) - d(T^{-n}y, T^{-n}x) \geq \delta e^{-\eta n^p} - C_1 \Lambda^{-n} \geq \frac{\delta}{2} e^{-\eta n^p},$$

for  $n$  sufficiently large. It follows that for each  $\xi \in \Xi_{R_x}$ , there exists  $W_{\xi,i} \in \mathcal{G}_n(W_\xi)$  such that  $W'_{\xi,i} = W_{\xi,i} \cap T^{-n}(B_n(x, \epsilon))$  is a single curve and  $|W'_{\xi,i}| \geq \min\{\frac{\delta}{2} e^{-\eta n^p}, \epsilon\} \geq \frac{\epsilon}{2} e^{-\eta n^p}$ . Thus recalling (7.13) and following (7.28) with  $\psi \equiv 1$ ,

$$\int_{W_\xi} 1_{n,\epsilon}^B \nu \geq e^{-nh_*} \int_{W'_{\xi,i}} \nu \geq \bar{C} e^{-nh_*} |W'_{\xi,i}|^{h_* \bar{C}_2} \geq C' e^{-nh_* - \eta h_* \bar{C}_2 n^p},$$

where  $C'$  depends on  $\epsilon$ .

Finally, using the fact from the proof of Corollary 7.9 that  $\mu_*^W$  is equivalent to  $\nu$  on  $\mu_*$ -a.e.  $W \in \mathcal{W}^s$ , we estimate,

$$\begin{aligned} \mu_*(B_n(x, \epsilon)) &\geq \mu_*(B_n(x, \epsilon) \cap D(R_x)) = \int_{\Xi_{R_x}} \mu_*^{W_\xi}(B_n(x, \epsilon)) d\hat{\mu}_*(\xi) \\ &\geq C \int_{\Xi_{R_x}} \nu(B_n(x, \epsilon) \cap W_\xi) d\hat{\mu}_*(\xi) \geq C'' e^{-nh_* - \eta h_* \bar{C}_2 n^p} \hat{\mu}_*(\Xi_{R_x}). \end{aligned}$$

□

Periodic points whose orbit do not have grazing collisions belong to  $M^{reg}$ . We call them *regular*.

**Proposition 7.13** ( $\mu_*$  and  $\mu_{\text{SRB}}$ ). *Assume  $h_* > s_0 \log 2$ . If there exists a regular periodic point  $x$  of period  $p$  such that  $\lambda_x = \frac{1}{p} \log |\det(DT^{-p}|_{E^s(x)})| \neq h_*$ , then  $\mu_* \neq \mu_{\text{SRB}}$ .*

Although  $h_*$  may not be known a priori, using Proposition 7.13 it suffices to find two regular periodic points  $x, y$  such that  $\lambda_x \neq \lambda_y$ , to conclude that  $\mu_* \neq \mu_{\text{SRB}}$ . (All known examples of dispersing billiard tables satisfy this condition.)

Proposition 7.13 relies on the following lemma.

**Lemma 7.14.** *Let  $x \in M^{reg}$  be a regular periodic point. There exists  $A > 0$  such that for all  $\epsilon > 0$  sufficiently small, there exists  $C(x, \epsilon) > 0$  such that for all  $n \geq 1$ ,*

$$C(x, \epsilon) e^{-n\lambda_x} \leq \mu_{\text{SRB}}(B_n(x, \epsilon)) \leq A e^{-n\lambda_x}.$$

*Proof.* Let  $x$  be a regular periodic point for  $T$  of period  $p$ . For  $\epsilon$  sufficiently small,  $T^{-i}(\mathcal{N}_\epsilon(x))$  belongs to a single homogeneity strip for  $i = 0, 1, \dots, p$ . Thus if  $y \in B_n(x, \epsilon) \cap W^s(x)$ , then the stable Jacobians  $J^s T^n(x)$  and  $J^s T^n(y)$  satisfy the bounded distortion estimate,  $|\log \frac{J^s T^n(x)}{J^s T^n(y)}| \leq$

$C_d d(x, y)^{1/3}$ , for a uniform  $C_d > 0$  [CM, Lemma 5.27]. It follows that the conditional measure on  $W^s(x)$  satisfies

$$(7.29) \quad C_x^{-1} \epsilon e^{-n\lambda_x} \leq \mu_{\text{SRB}}^{W^s(x)}(B_n(x, \epsilon)) \leq C_x \epsilon e^{-n\lambda_x},$$

for some  $C_x \geq 1$ , depending on the homogeneity strips in which the orbit of  $x$  lies.

Next, using again [CM, Prop 7.81], we can find a Cantor rectangle  $R_x \subset \mathcal{N}_\epsilon(x)$  with diameter at most  $\epsilon/(2C_1)$  and  $\mu_{\text{SRB}}(R_x) \geq C \mu_{\text{SRB}}(\mathcal{N}_\epsilon(x))/(2C_1)^2$ , for a constant  $C > 0$  depending on the distortion of the measure. Note that  $W^u(x) \cap D(R_x)$  is never cut by  $\mathcal{S}_{-n}$  and lies in  $B_n(x, \epsilon)$  by (3.1). Thus each  $W \in \mathcal{W}^s(R_x)$  has a component in  $B_n(x, \epsilon)$  and this component has length satisfying the same bounds as (7.29). Integrating over  $B_n(x, \epsilon)$  as in the proof of Proposition 7.12 proves the lemma. An inspection of proof shows that the constant in the upper bound can be chosen independent of  $x$  when  $\epsilon$  is sufficiently small, while the constant in the lower bound cannot.  $\square$

*Proof of Proposition 7.13.* If  $x$  is a regular periodic point, then the upper and lower bounds on  $\mu_*(B_n(x, \epsilon))$  from Proposition 7.12 hold with<sup>35</sup>  $\eta = 0$  for  $\epsilon$  sufficiently small. If  $\lambda_x \neq h_*$ , these do not match the exponential rate in the bounds on  $\mu_{\text{SRB}}(B_n(x, \epsilon))$  from Lemma 7.14. Thus for  $n$  sufficiently large,  $\mu_*(B_n(x, \epsilon)) \neq \mu_{\text{SRB}}(B_n(x, \epsilon))$ .  $\square$

**7.5. K-mixing and Maximal Entropy of  $\mu_*$  — Bowen–Pesin–Pitskel Theorem 2.5.** In this section we use the absolute continuity results from Section 7.3 to establish  $K$ -mixing of  $\mu_*$ . We also show that  $\mu_*$  has maximal entropy, exploiting the upper bound from Section 7.4. Finally, we show that  $h_*$  coincides with the Bowen–Pesin–Pitskel entropy.

**Lemma 7.15** (Single Ergodic Component). *If  $R$  is a Cantor rectangle with  $\mu_*(R) > 0$ , then the set of stable manifolds  $\mathcal{W}^s(R)$  belongs to a single ergodic component of  $\mu_*$ .*

*Proof.* We follow the well-known Hopf strategy outlined in [CM, Section 6.4] of smooth ergodic theory to show that  $\mu_*$ -almost every stable and unstable manifold has a full measure set of points belonging to a single ergodic component: Given a continuous function  $\varphi$  on  $M$ , let  $\bar{\varphi}_+$ ,  $\bar{\varphi}_-$  denote the forward and backward ergodic averages of  $\varphi$ , respectively. Let  $M_\varphi = \{x \in M^{\text{reg}} : \bar{\varphi}_+(x) = \bar{\varphi}_-(x)\}$ . When the two functions agree, denote their common value by  $\bar{\varphi}$ .

Now fix a Cantor rectangle  $R$  with  $\mu_*(R) > 0$ . By Corollary 7.4, if  $\gamma > 1$  then  $\mu_*(M^{\text{reg}}) = 1$ . So, by the Birkhoff ergodic theorem,  $\mu_*(M_\varphi) = 1$ . Thus for  $\mu_*$  almost every  $W \in \mathcal{W}^s(R)$ , the conditional measure  $\mu_*^W$  satisfies  $\mu_*^W(M_\varphi) = 1$ . Due to the fact that forward ergodic averages are the same for any two points in  $W$ , it follows that  $\bar{\varphi}$  is constant on  $W \cap M_\varphi$ . The analogous fact holds for unstable manifolds in  $\mathcal{W}^u(R)$ .

Let  $G_\varphi = \{x \in M_\varphi : \bar{\varphi} \text{ is constant on a full measure subset of } W^u(x) \text{ and } W^s(x)\}$ . Clearly,  $\mu_*(G_\varphi) = 1$ , so the same facts apply to  $G_\varphi$  as  $M_\varphi$ .

Let  $W^0, W \in \mathcal{W}^s(R)$  be stable manifolds with  $\mu_*^{W^0}(G_\varphi) = \mu_*^W(G_\varphi) = 1$ . Let  $\Theta_W$  denote the holonomy map from  $W^0 \cap R$  to  $W \cap R$ . By absolute continuity, Corollary 7.9,  $\mu_*^W(\Theta_W(W^0 \cap G_\varphi)) > 0$ . Thus  $\bar{\varphi}$  is constant for almost every point in  $\Theta_W(W^0 \cap G_\varphi)$ . Let  $y$  be one such point and let  $x = \Theta_W^{-1}(y)$ . Then since  $x \in W^u(y) \cap G_\varphi$ ,

$$\bar{\varphi}(x) = \bar{\varphi}_-(x) = \bar{\varphi}_-(y) = \bar{\varphi}(y),$$

so that the values of  $\bar{\varphi}$  on a positive measure set of points in  $W^0$  and  $W$  agree. Since  $\bar{\varphi}$  is constant on  $G_\varphi$ , the values of  $\bar{\varphi}$  on a full measure set of points in  $W$  and  $W^0$  are equal. Since this applies to any  $W$  with  $\mu_*^W(G_\varphi) = 1$ , we conclude that  $\bar{\varphi}$  is constant almost everywhere on the set  $\cup_{W \in \mathcal{W}^s(R)} W$ . Finally, since  $\varphi$  was an arbitrary continuous function, the set  $\mathcal{W}^s(R)$  belongs (mod 0) to a single ergodic component of  $\mu_*$ .  $\square$

We are now ready to prove the  $K$ -mixing property of  $\mu_*$ .

<sup>35</sup>Here, it is convenient to have the role of  $\eta$  explicit in (7.26).



**Proposition 7.16.**  $(T, \mu_*)$  is  $K$ -mixing.

*Proof.* We begin by showing that  $(T^n, \mu_*)$  is ergodic for all  $n \geq 1$ . Recall the countable set of (locally maximal) Cantor rectangles  $\{R_i\}_{i \in \mathbb{N}}$  with  $\mu_*(R_i) > 0$ , such that  $\cup_i R_i = M^{reg}$  from (7.21).

We fix  $n$  and let  $R_1$  and  $R_2$  be two such Cantor rectangles. By Lemma 7.15,  $\mathcal{W}^s(R_i)$  belongs (mod 0) to a single ergodic component of  $\mu_*$ . Since  $T$  is topologically mixing, and using [CM, Lemma 7.90], there exists  $n_0 > 0$  such that for any  $k \geq n_0$ , a smooth component of  $T^{-k}(D(R_1))$  properly crosses  $D(R_2)$ . Let us call  $D_k$  the part of this smooth component lying in  $D(R_2)$ .

Since the set of stable manifolds is invariant under  $T^{-k}$ , by the maximality of the set  $\mathcal{W}^s(R_2)$ , we have that  $T^{-k}(\mathcal{W}^s(R_1)) \cap D_k \supseteq \mathcal{W}^s(R_2) \cap D_k$ . And since this set of stable manifolds in  $R_1$  has positive measure with respect to  $\hat{\mu}_*$ , it follows that  $\mu_*(T^{-k}(\mathcal{W}^s(R_1)) \cap \mathcal{W}^s(R_2)) > 0$ . Thus  $R_1$  and  $R_2$  belong to the same ergodic component of  $T$ . Indeed, since we may choose  $k = jn$  for some  $j \in \mathbb{N}$ ,  $R_1$  and  $R_2$  belong to the same ergodic component of  $T^n$ . Since this is true for each pair of Cantor rectangles  $R_i, R_j$  in our countable collection, and  $\mu_*(\cup_i R_i) = 1$ , we conclude that  $T^n$  is ergodic.

We shall use the Pinsker partition  $\pi(T) = \bigvee \{\xi : \xi \text{ finite partition of } M, h_{\mu_*}(T, \xi) = 0\}$ . Since  $T$  is an automorphism, the sigma-algebra generated by  $\pi(T)$  is  $T$ -invariant.

Given two measurable partitions  $\xi_1$  and  $\xi_2$ , the meet of the two partitions  $\xi_1 \wedge \xi_2$  is defined as the finest measurable partition with the property that  $\xi_1 \wedge \xi_2 \leq \xi_j$  for  $j = 1, 2$ . All definitions of measurable partitions and inequalities between them are taken to be mod 0, with respect to the measure  $\mu_*$ . It is a standard fact in ergodic theory (see e.g. [RoS]) that if  $\xi$  is a partition of  $M$  such that (i)  $T\xi \geq \xi$  and (ii)  $\bigvee_{n=0}^{\infty} T^n \xi = \epsilon$ , where  $\epsilon$  is the partition of  $M$  into points, then  $\bigwedge_{n=0}^{\infty} T^{-n} \xi \geq \pi(T)$  (mod 0).

Define  $\xi^s$  to be the partition of  $M$  into maximal local stable manifolds. If  $x \in M$  has no stable manifold or  $x$  is an endpoint of a stable manifold then define  $\xi^s(x) = \{x\}$ . Similarly, define  $\xi^u$  to be the partition of  $M$  into maximal local unstable manifolds. Note that  $\xi^s$  is a measurable partition of  $M$  since it is generated by the countable family of finite partitions given by the elements of  $\mathcal{M}_0^n$  and their closures. Similarly,  $\mathcal{M}_{-n}^0$  provides a countable generator for  $\xi^u$ .

It is a consequence of the uniform hyperbolicity of  $T$  that  $\xi^s$  satisfies (i) and (ii) above. Also,  $\xi^u$  satisfies these conditions with respect to  $T^{-1}$ , i.e.,  $T^{-1}\xi^u \geq \xi^u$  and  $\bigvee_{n=0}^{\infty} T^{-n}\xi^u = \epsilon$ . Thus  $\bigwedge_{n=0}^{\infty} T^n \xi^u \geq \pi(T)$ .

Define  $\eta_\infty = \bigwedge_{n=0}^{\infty} (T^n \xi^u \wedge T^{-n} \xi^s)$ , and notice that  $\eta_\infty \geq \pi(T)$  by the above. Then since  $\xi^s \wedge \xi^u \geq \eta_\infty$ , we have  $\xi^s \wedge \xi^u \geq \pi(T)$  as well.

We will show that each Cantor rectangle in our countable family belongs to one element of  $\xi^s \wedge \xi^u$  (mod 0). This will follow from the product structure of each  $R_i$  coupled with the absolute continuity of the holonomy map given by Corollary 7.9.

For brevity, let us fix  $i$  and set  $R = R_i$ . We index the curves  $W_\zeta^s \in \mathcal{W}^s(R)$  by  $\zeta \in Z$ . Define  $\mu_R = \frac{\mu_*|_R}{\mu_*(R)}$ . We disintegrate the measure  $\mu_R$  into a family of conditional probability measures  $\mu_R^{W_\zeta^s}$ ,  $W^s \in \mathcal{W}^s(R)$ , and a factor measure  $\hat{\mu}_R$  on the set  $Z$ . Then

$$\mu_R(A) = \int_{\zeta \in Z} \mu_R^{W_\zeta^s}(A) d\hat{\mu}_R(\zeta), \quad \text{for all measurable sets } A.$$

The set  $R$  belongs to a single element of  $\xi^s \wedge \xi^u$  if a full measure set of points can be connected by elements of  $\xi^s$  and  $\xi^u$  even after the removal of a set of  $\mu_*$ -measure 0. Let  $N \subset M$  be such that  $\mu_*(N) = 0$ . By the above disintegration, it follows that for  $\hat{\mu}_R$ -almost every  $\zeta \in Z$ , we have  $\mu_R^{W_\zeta^s}(N) = 0$ .

Let  $W_1^s$  and  $W_2^s$  be two elements of  $\mathcal{W}^s(R)$  such that  $\mu_R^{W_j^s}(N) = 0$ , for  $j = 1, 2$ . For all  $x \in W_1^s \cap R$ ,  $\xi^u(x)$  intersects  $W_2^s$ , and vice versa. Let  $\Theta$  denote the holonomy map from  $W_1^s$  to  $W_2^s$ . Then by Corollary 7.9, we have  $\mu_R^{W_2^s}(\Theta(W_1^s \cap N)) = 0$  and  $\mu_R^{W_1^s}(\Theta^{-1}(W_2^s \cap N)) = 0$ . Thus the set  $\Theta(W_1^s \setminus N)$  has full measure in  $W_2^s$  and vice versa. It follows that  $W_1^s$  and  $W_2^s$  belong to one element of  $\xi^s \wedge \xi^u$ . This proves that  $R$  belongs to a single element of  $\xi^s \wedge \xi^u$  (mod 0).

Since  $\xi^s \wedge \xi^u \geq \pi(T)$ , we have shown that each  $R_i$  belongs to a single element of  $\pi(T)$ , mod 0. Since  $\mu_*(R_i) > 0$  and  $\mu_*(\cup_i R_i) = 1$ , the ergodicity of  $T$  and the invariance of  $\pi(T)$  imply that  $\pi(T)$  contains finitely many elements, all having the same measure, whose union has full measure. The action of  $T$  is simply a permutation of these elements. Since  $(T^n, \mu_*)$  is ergodic for all  $n$ , it follows that  $\pi(T)$  is trivial. Thus  $(T, \mu_*)$  is K-mixing.  $\square$

Now that we know that  $\mu_*$  is ergodic, the upper bound in Proposition 7.12 will easily<sup>36</sup> imply that  $h_{\mu_*}(T) = h_*$ :

**Corollary 7.17** (Maximum Entropy). *For  $\mu_*$  defined as in (7.1), we have  $h_{\mu_*}(T) = h_*$ .*

*Proof.* Since  $\int |\log d(x, \mathcal{S}_{\pm 1})| d\mu_* < \infty$  by Theorem 2.6, and  $\mu_*$  is ergodic, we may apply [DWY, Prop 3.1]<sup>37</sup> to  $T^{-1}$ , which states that for  $\mu_*$ -almost every  $x \in M$ ,

$$\lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mu_*(B_n(x, \epsilon)) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu_*(B_n(x, \epsilon)) = h_{\mu_*}(T^{-1}).$$

Using (7.25) and (7.26) with  $p < 1$ , it follows that  $\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu_*(B_n(x, \epsilon)) = h_*$ , for any  $\epsilon > 0$  sufficiently small. Thus  $h_{\mu_*}(T) = h_{\mu_*}(T^{-1}) = h_*$ .  $\square$

Corollary 7.17 next allows us to prove Theorem 2.5 about the Bowen–Pesin–Pitskel entropy:

*Proof of Theorem 2.5.* To show  $h_* \leq h_{\text{top}}(T|_{M'})$ , we first use Corollary 7.17 and the fact that  $\mu_*(M') = 1$  (since  $\mu_*(\mathcal{S}_n) = 0$  for every  $n$  by Theorem 2.6) to see that

$$h_* = h_{\mu_*}(T) = \sup_{\mu: \mu(M')=1} h_{\mu}(T).$$

Then we apply the bound [Pes, (A.2.1)] or [PP, Thm 1] (by Remarks I and II there,  $T$  need not be continuous on  $M$ ) to get

$$\sup_{\mu: \mu(M')=1} h_{\mu}(T) \leq h_{\text{top}}(T|_{M'}).$$

To show  $h_{\text{top}}(T|_{M'}) \leq h_*$ , we use that [Pes, (11.12)] implies<sup>38</sup>  $h_{\text{top}}(T|_{M'}) \leq Ch_{M'}(T)$ , where  $Ch_{M'}(T)$  denotes the capacity topological entropy of the (invariant) set  $M'$ . Now, for any  $\delta > 0$ , the elements of  $\mathring{\mathcal{P}}_{-k}^k = \mathcal{M}_{-k-1}^{k+1}$  form an open cover of  $M'$  of diameter  $< \delta$ , if  $k$  is large enough (see the proof of Lemma 3.4). By adding finitely many open sets, we obtain an open cover  $\mathcal{U}_{\delta}$  of  $M$  of diameter  $< \delta$ . Next [Pes, (11.13)] gives that

$$Ch_{M'}(T) = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log \Lambda(M', \mathcal{U}_{\delta}, n),$$

where  $\Lambda(M', \mathcal{U}_{\delta}, n)$  is the smallest cardinality of a cover of  $M'$  by elements of  $\bigvee_{j=0}^n T^{-j} \mathcal{U}_{\delta}$ . Since for any  $n \geq 1$ , the sets of  $\bigvee_{j=0}^n T^{-j} \mathring{\mathcal{P}}_{-k}^k$  form a cover of  $M'$ , the second equality of Lemma 3.3 (i.e.,  $\lim_n \frac{1}{n} \log \#\mathring{\mathcal{P}}_{-k}^{k+n} = h_*$ ) implies that  $Ch_{M'}(T) \leq h_*$ .  $\square$

**7.6. Bernoulli Property of  $\mu_*$ .** In this section, we prove that  $\mu_*$  is Bernoulli by bootstrapping from K-mixing. The key ingredients of the proof, in addition to K-mixing, are Cantor rectangles with a product structure of stable and unstable manifolds, the absolute continuity of the unstable foliation with respect to  $\mu_*$ , and the bounds (2.2) on the neighbourhoods of the singularity sets. First, we recall some definitions, following Chernov–Haskell [ChH] and the notion of very weak Bernoulli partitions introduced by Ornstein [O].

<sup>36</sup>It is not much harder to deduce this fact in the absence of ergodicity, using only (7.26) with Theorem 2.3.

<sup>37</sup>This is a slight generalization of the Brin–Katok local theorem [BK], using [M, Lemma 2]. Continuity of the map is not used in the proof of the theorem, and so it applies to our setting.

<sup>38</sup>Just like in [PP, I and II], it is essential that  $M$  is compact, but the fact that  $T$  is not continuous on  $M$  is irrelevant. Note also that [Pes, (A.3'), p. 66] should be corrected, replacing “any  $\epsilon > \epsilon > 0$ ” by “any  $\epsilon > 1/m > 0$ ”.

Let  $(X, \mu_X)$  and  $(Y, \mu_Y)$  be two non-atomic Lebesgue probability spaces. A *joining*  $\lambda$  of the two spaces, is a measure on  $X \times Y$  whose marginals on  $X$  and  $Y$  are  $\mu_X$  and  $\mu_Y$ , respectively. Given finite partitions<sup>39</sup>  $\alpha = \{A_1, \dots, A_k\}$  of  $X$  and  $\beta = \{B_1, \dots, B_k\}$  of  $Y$ , let  $\alpha(x)$  denote the element of  $\alpha$  containing  $x \in X$  (and similarly for  $\beta$ ). Moreover, if  $x \in A_j$  and  $y \in B_j$  for the same value of  $j$  (which depends on the order in which the elements are listed), then we will write  $\alpha(x) = \beta(y)$ .

The distance  $\bar{d}$  defined below considers two partitions to be close if there is a joining  $\lambda$  such that most of the measure lies on the set of points  $(x, y)$  with  $\alpha(x) = \beta(y)$ : given two finite sequences of partitions  $\{\alpha_i\}_{i=1}^n$  of  $X$  and  $\{\beta_i\}_{i=1}^n$  of  $Y$ , define

$$\bar{d}(\{\alpha_i\}, \{\beta_i\}) = \inf_{\lambda} \int_{X \times Y} h(x, y) d\lambda,$$

where  $\lambda$  is a joining of  $X$  and  $Y$  and  $h$  is defined by

$$h(x, y) = \frac{1}{n} \#\{i \in [1, \dots, n] : \alpha_i(x) \neq \beta_i(y)\}.$$

We will adopt the following notation: If  $E \subset X$ , then  $\alpha|E$  denotes the partition  $\alpha$  conditioned on  $E$ , i.e., the partition of  $E$  given by elements of the form  $A \cap E$ , for  $A \in \alpha$ . Similarly,  $\mu_X(\cdot|E)$  is the measure  $\mu_X$  conditioned on  $E$ . If a property holds for all atoms of  $\alpha$  except for a collection whose union has measure less than  $\varepsilon$ , then we say the property holds for  $\varepsilon$ -almost every atom of  $\alpha$ .

If  $f : X \rightarrow X$  is an invertible, measure preserving transformation of  $(X, \mu_X)$ , and  $\alpha$  is a finite partition of  $X$ , then  $\alpha$  is said to be *very weak Bernoullian* (vwB) if for all  $\varepsilon > 0$ , there exists  $N > 0$  such that for every  $n > 0$  and  $N_0, N_1$  with  $N < N_0 < N_1$ , and for  $\varepsilon$ -almost every atom  $A$  of  $\bigvee_{N_0}^{N_1} f^i \alpha$ , we have

$$(7.30) \quad \bar{d}(\{f^{-i} \alpha\}_{i=1}^n, \{f^{-i} \alpha|A\}_{i=1}^n) < \varepsilon.$$

The following theorem from [OW] provides the essential connection between the Bernoulli property and vwB partitions. (See also Theorems 4.1 and 4.2 in [ChH].)

**Theorem 7.18.** *If a partition  $\alpha$  of  $X$  is vwB, then  $(X, \bigvee_{n=-\infty}^{\infty} f^{-n} \alpha, \mu_X, f)$  is a Bernoulli shift. Moreover, if  $\bigvee_{n=-\infty}^{\infty} f^{-n} \alpha$  generates the whole  $\sigma$ -algebra of  $X$ , then  $(X, \mu_X, f)$  is a Bernoulli shift.*

We are ready to state and prove the main result of this section.

**Proposition 7.19.** *The measure  $\mu_*$  is Bernoulli.*

*Proof.* First notice that since  $f$  is measure preserving in (7.30), then to prove that a partition  $\alpha$  is vwB, it suffices to show that for every  $\varepsilon > 0$ , there exist integers  $m$  and  $N > 0$  such that for every  $n, N_0, N_1$  with  $N < N_0 < N_1$ , and for  $\varepsilon$ -almost every atom  $A$  of  $\bigvee_{N_0-m}^{N_1-m} f^i \alpha$ ,

$$(7.31) \quad \bar{d}(\{f^{-i} \alpha\}_{i=1+m}^{n+m}, \{f^{-i} \alpha|A\}_{i=1+m}^{n+m}) < \varepsilon.$$

To prove Proposition 7.19, we will follow the arguments in Sections 5 and 6 of [ChH], only indicating where modifications should be made.

First, we remark that [ChH] decomposes the measure  $\mu_{\text{SRB}}$  into conditional measures on unstable manifolds and a factor measure on the set of unstable leaves. Due to Corollary 7.9, we prefer to decompose  $\mu_*$  into conditional measures on stable manifolds and the factor measure  $\hat{\mu}_*$ . For this reason, we exchange the roles of stable and unstable manifolds throughout the proofs of [ChH].

To this end, we take  $f = T^{-1}$  in the set-up presented above, and  $X = M$ . Moreover, we set  $\alpha = \mathcal{M}_{-1}^1$ , since this (mod 0) partition generates the full  $\sigma$ -algebra on  $M$ . We will follow the proof of [ChH] to show that  $\alpha$  is vwB, and so by Theorem 7.18,  $\mu_*$  will be Bernoulli with respect to  $T^{-1}$ , and therefore with respect to  $T$ . The proof in [ChH] proceeds in two steps.

<sup>39</sup>As we shall not need the norms of  $\mathcal{B}$  and  $\mathcal{B}_w$  in this section, we are free to use the letters  $\alpha$  and  $\beta$  to denote partitions instead of real parameters.

*Step 1. Construction of  $\delta$ -regular coverings.* Given  $\delta > 0$ , the idea is to cover  $M$ , up to a set of  $\mu_*$ -measure at most  $\delta$ , by Cantor rectangles of stable and unstable manifolds such that  $\mu_*$  restricted to each rectangle is arbitrarily close to a product measure. This is very similar to our covering  $\{R_i\}_{i \in \mathbb{N}}$  from (7.21); however, some adjustments must be made in order to guarantee uniform properties for the Jacobian of the relevant holonomy map.

On a Cantor rectangle  $R$  with  $\mu_*(R) > 0$ , we can define a product measure as follows.<sup>40</sup> Fix a point  $z \in R$ , and consider  $R$  as the product of  $R \cap W^s(z)$  with  $R \cap W^u(z)$ , where  $W^{s/u}(z)$  are the local stable and unstable manifolds of  $z$ , respectively. As usual, we disintegrate  $\mu_*$  on  $R$  into conditional measures  $\mu_{*,R}^W$ , on  $W \cap R$ , where  $W \in \mathcal{W}^s(R)$ , and a factor measure  $\hat{\mu}_*$  on the index set  $\Xi_R$  of the curves  $\mathcal{W}^s(R)$ .

Define  $\mu_{*,R}^p = \mu_{*,R}^{W^s(z)} \times \hat{\mu}_*$  and note that we can view  $\hat{\mu}_*$  as inducing a measure on  $W^u(z)$ . Corollary 7.9 implies that  $\mu_{*,R}^p$  is absolutely continuous with respect to  $\mu_*$ . The following definition is taken from [ChH] (as mentioned above, a  $\delta$ -regular covering of  $M$  is a collection of rectangles which covers  $M$  up to a set of measure  $\delta$ ).

**Definition 7.20.** For  $\delta > 0$ , a  $\delta$ -regular covering of  $M$  is a finite collection of disjoint Cantor rectangles  $\mathcal{R}$  for which,<sup>41</sup>

- a)  $\mu_*(\cup_{R \in \mathcal{R}} R) \geq 1 - \delta$ .
- b) Every  $R \in \mathcal{R}$  satisfies  $\left| \frac{\mu_{*,R}^p(R)}{\mu_*(R)} - 1 \right| < \delta$ . Moreover, there exists  $G \subset R$ , with  $\mu_*(G) > (1 - \delta)\mu_*(R)$ , such that  $\left| \frac{d\mu_{*,R}^p}{d\mu_*}(x) - 1 \right| < \delta$  for all  $x \in G$ .

By [ChH, Lemma 5.1], such coverings exist for any  $\delta > 0$ . The proof essentially uses the covering from (7.21), and then subdivides the rectangles into smaller ones on which the Jacobian of the holonomy between stable manifolds is nearly 1, in order to satisfy item (b) above. This argument relies on Lusin's theorem and goes through in our setting with no changes. Indeed, the proof in our case is simpler since the angles between stable and unstable subspaces are uniformly bounded away from zero, and the hyperbolicity constants in (3.1) are uniform for all  $x \in M$ .

*Step 2. Proof that  $\alpha = \mathcal{M}_{-1}^1$  is vwB.* Indeed, [ChH] prove that any  $\alpha$  with piecewise smooth boundary is vwB, but due to Theorem 7.18, it suffices to prove it for a single partition which generates the  $\sigma$ -algebra on  $M$ . Moreover, using  $\alpha = \mathcal{M}_{-1}^1$  allows us to apply the bounds (2.2) directly since  $\partial\alpha = \mathcal{S}_1 \cup \mathcal{S}_{-1}$ .

Fix  $\varepsilon > 0$ , and define

$$\delta = e^{-(\varepsilon/C')^{2/(1-\gamma)}},$$

where  $C' > 0$  is the constant from (7.33).

Let  $\mathcal{R} = \{R_1, R_2, \dots, R_k\}$  be a  $\delta$ -regular cover of  $M$  such that the diameters of the  $R_i$  are less than  $\delta$ . Define the partition  $\pi = \{R_0, R_1, \dots, R_k\}$ , where  $R_0 = M \setminus \cup_{i=1}^k R_i$ . For each  $i \geq 1$ , let  $G_i \subset R_i$  denote the set identified in Definition 7.20(b).

Since  $T^{-1}$  is  $K$ -mixing, there exists an even integer  $N = 2m$ , such that for any integers  $N_0, N_1$  such that  $N < N_0 < N_1$ ,  $\delta$ -almost every atom  $A$  of  $\vee_{N_0-m}^{N_1-m} T^{-i} \alpha$  satisfies,

$$(7.32) \quad \left| \frac{\mu_*(R|A)}{\mu_*(R)} - 1 \right| < \delta, \quad \text{for all } R \in \pi.$$

Now let  $n, N_0, N_1$  be given as above, and define  $\omega = \vee_{N_0-m}^{N_1-m} T^{-i} \alpha$ . [ChH] proceeds to show that  $c\varepsilon$ -almost every atom of  $\omega$  satisfies (7.31) with  $\varepsilon$  replaced by  $c\varepsilon$  for some uniform constant  $c > 0$ .

<sup>40</sup>We follow the definition in [ChH, Section 5.1], exchanging the roles of stable and unstable manifolds.

<sup>41</sup>The corresponding definition in [ChH] has a third condition, but this is trivially satisfied in our setting since our stable and unstable manifolds are one-dimensional and have uniformly bounded curvature.

The first set of estimates in the proof is to bound the measure of bad sets which must be thrown out, and to show that these add up to at most  $c\varepsilon$ .

The first set is  $\hat{F}_1$ , which is the union of all atoms in  $\omega$ , which do not satisfy (7.32). By choice of  $N$ , we have  $\mu_*(\hat{F}_1) < \delta$ .

The second set is  $\hat{F}_2$ . Let  $F_2 = \cup_{i=1}^k R_i \setminus G_i$ , and define  $\hat{F}_2$  to be the union of all atoms  $A \in \omega$ , for which either  $\mu_*(F_2|A) > \delta^{1/2}$ , or

$$\sum_{i=1}^k \frac{\mu_{*,R_i}^p(A \cap F_2)}{\mu_*(A)} > \delta^{1/2}.$$

It follows as in [ChH, Page 38], with no changes, that  $\mu_*(\hat{F}_2) < c\delta^{1/2}$ , for some  $c > 0$  independent of  $\delta$  and  $k$ .

Define  $F_3$  to be the set of all points  $x \in M \setminus R_0$  such that  $W^s(x)$  intersects the boundary of the element  $\omega(x)$  before it fully crosses the rectangle  $\pi(x)$ . Thus if  $x \in F_3$ , there exists a subcurve of  $W^s(x)$  connecting  $x$  to the boundary of  $(T^{-i}\alpha)(x)$  for some  $i \in [N_0 - m, N_1 - m]$ . Then since  $\pi(x)$  has diameter less than  $\delta$ ,  $T^i(x)$  lies within a distance  $C_1\Lambda^{-i}\delta$  of the boundary of  $\alpha$ , where  $C_1$  is from (3.1). Using (2.2), the total measure of such points must add up to at most

$$(7.33) \quad \sum_{i=N_0-m}^{N_1-m} \frac{C}{|\log(C_1\Lambda^{-i}\delta)|^\gamma} \leq C' |\log \delta|^{1-\gamma},$$

for some  $C' > 0$ . Letting  $\hat{F}_3$  denote the union of atoms  $A \in \omega$  such that  $\mu_*(F_3|A) > |\log \delta|^{\frac{1-\gamma}{2}}$ , it follows that  $\mu_*(\hat{F}_3) \leq C' |\log \delta|^{\frac{1-\gamma}{2}}$ . This is at most  $\varepsilon$  by choice of  $\delta$ .

Define  $F_4$  (following [ChH] Section 6.1, and not Section 6.2) to be the set of all  $x \in M \setminus R_0$  for which there exists  $y \in W^u(x) \cap \pi(x)$  such that  $h(x, y) > 0$ . This implies that  $W^u(x)$  intersects the boundary of the element  $(T^i\alpha)(x)$  for some  $i \in [1 + m, n + m]$ , remembering (7.31), and the definition of  $h$ . Using again the uniform hyperbolicity (3.1), this implies that  $T^{-i}(x)$  lies in a  $C_1\Lambda^{-i}\delta$ -neighbourhood of the boundary of  $\alpha$ . Thus the same estimate as in (7.33) implies  $\mu_*(F_4) \leq C' |\log \delta|^{1-\gamma}$ . Finally, letting  $\hat{F}_4$  denote the union of all atoms  $A \in \omega$  such that  $\mu_*(F_4|A) > |\log \delta|^{\frac{1-\gamma}{2}}$ , it follows as before that  $\mu_*(\hat{F}_4) \leq C' |\log \delta|^{\frac{1-\gamma}{2}}$ .

Finally, the bad set to be avoided in the construction of the joining  $\lambda$  is  $R_0 \cup (\cup_{i=1}^4 \hat{F}_i)$ . Its measure is less than  $c\varepsilon$  by choice of  $\delta$ . From this point, once the measure of the bad set is controlled, the rest of the proof in Section 6.2 of [ChH] can be repeated verbatim. This proves that (7.31) holds for  $c\varepsilon$ -almost every atom of  $\omega$ , and thus that  $\alpha$  is vwB.  $\square$

**7.7. Uniqueness of the measure of maximal entropy.** This subsection is devoted to the following proposition:

**Proposition 7.21.** *The measure  $\mu_*$  is the unique measure of maximal entropy.*

The proof of uniqueness relies on exploiting the fact that while the lower bound on Bowen balls (or elements of  $\mathcal{M}_{-n}^0$ ) cannot be improved for  $\mu_*$ -almost every  $x$ , yet if one fixes  $n$ , most elements of  $\mathcal{M}_{-n}^0$  should either have unstable diameter of a fixed length, or have previously been contained in an element of  $\mathcal{M}_{-j}^0$  with this property for some  $j < n$ . Such elements collectively satisfy stronger lower bounds on their measure. Since we have established good control of the elements of  $\mathcal{M}_{-n}^0$  and  $\mathcal{M}_0^n$  in the fragmentation lemmas of Section 5, we will work with these partitions instead of Bowen balls.

Recalling (5.1), choose  $m_1$  such that  $(Km_1 + 1)^{1/m_1} < e^{h^*/4}$ . Now choose  $\delta_2 > 0$  sufficiently small that for all  $n, k \in \mathbb{N}$ , if  $A \in \mathcal{M}_{-n}^k$  with  $\max\{\text{diam}^u(A), \text{diam}^s(A)\} \leq \delta_2$ , then  $A \setminus \mathcal{S}_{\pm m_1}$  consists of no more than  $Km_1 + 1$  connected components.

For  $n \geq 1$ , define

$$B_{-2n}^0 = \{A \in \mathcal{M}_{-2n}^0 : \forall j, 0 \leq j \leq n/2, T^{-j}A \subset E \in \mathcal{M}_{-n+j}^0 \text{ such that } \text{diam}^u(E) < \delta_2\},$$

with the analogous definition for  $B_0^{2n} \subset \mathcal{M}_0^{2n}$  with unstable diameter replaced by stable diameter. Next, set  $B_{2n} = \{A \in \mathcal{M}_{-2n}^0 : \text{either } A \in B_{-2n}^0 \text{ or } T^{-2n}A \in B_0^{2n}\}$ . Define  $G_{2n} = \mathcal{M}_{-2n}^0 \setminus B_{2n}$ .

Our first lemma shows that the set  $B_{2n}$  is small relative to  $\#\mathcal{M}_{-2n}^0$  for large  $n$ . Let  $n_1 > 2m_1$  be chosen so that for all  $A \in \mathcal{M}_{-n}^0$ ,  $\text{diam}^s(A) \leq C\Lambda^{-n} \leq \delta_2$  for all  $n \geq n_1$ .

**Lemma 7.22.** *There exists  $C > 0$  such that for all  $n \geq n_1$ ,*

$$\#B_{2n} \leq Ce^{3nh_*/2} (Km_1 + 1)^{\frac{n}{m_1} + 1} \leq Ce^{7nh_*/4}.$$

*Proof.* Fix  $n \geq n_1$  and suppose  $A \in B_{-2n}^0 \subset \mathcal{M}_{-2n}^0$ . For  $0 \leq j \leq \lfloor n/2 \rfloor$ , define  $A_j \in \mathcal{M}_{-\lfloor 3n/2 \rfloor - j}^0$  to be the element containing  $T^{-(\lfloor n/2 \rfloor - j)}A$  (note that  $T^{-k}A \in \mathcal{M}_{-2n+k}^k$  for each  $k \leq 2n$ ).

By definition of  $B_{-2n}^0$  and choice of  $n_1$ , we have  $\max\{\text{diam}^u(A_j), \text{diam}^s(A_j)\} \leq \delta_2$ . Thus the number of connected components of  $\mathcal{M}_{-\lfloor 3n/2 \rfloor}^{m_1}$  in  $A_0$  is at most  $Km_1 + 1$ . Thus the number of connected components of  $T^{m_1}A_0$  (one of which is  $A_{m_1}$ ) is at most  $Km_1 + 1$ . Since the stable and unstable diameters of  $A_{m_1}$  are again both shorter than  $\delta_2$  (since  $A \in B_{-2n}^0$ ) and  $n > 2m_1$ , we may apply this estimate inductively. Thus writing  $\lfloor n/2 \rfloor = \ell m_1 + i$  for some  $i < m_1$ , we have that  $\#\{A' \in B_{-2n}^0 : T^{-\lfloor n/2 \rfloor}A' \subset A_0\} \leq (Km_1 + 1)^{\ell+1}$ . Summing over all possible  $A_0 \in \mathcal{M}_{-\lfloor 3n/2 \rfloor}^0$  yields by Proposition 4.6 and choice of  $m_1$ ,

$$\#B_{-2n}^0 \leq \#\mathcal{M}_{-\lfloor 3n/2 \rfloor}^0 (Km_1 + 1)^{n/m_1 + 1} \leq Ce^{7nh_*/4}.$$

A similar estimate holds for  $\#B_0^{2n}$ . Given the one-to-one correspondence between elements of  $\mathcal{M}_{-2n}^0$  and  $\mathcal{M}_0^{2n}$ , it follows that  $\#B_{2n} \leq 2\#B_{-2n}^0$ , proving the required estimate.  $\square$

Next, the following lemma establishes the importance of long pieces in providing good lower bounds on the measure of partition elements.

**Lemma 7.23.** *There exists  $C_{\delta_2} > 0$ , such that for all  $n \geq 1$  and all  $A \in \mathcal{M}_{-j}^0$  such that  $\text{diam}^u(A) \geq \delta_2$  and  $\text{diam}^s(T^{-j}A) \geq \delta_2$ , we have<sup>42</sup>*

$$\mu_*(A) \geq C_{\delta_2} e^{-jh_*}.$$

*Proof.* As in the proof of Proposition 5.5, by [Ch1, Lemma 7.87], we may choose finitely many (maximal) Cantor rectangles,  $R_1, R_2, \dots, R_k$ , with  $\mu_*(R_i) > 0$ , and having the property that every unstable curve of length at least  $\delta_2$  properly crosses at least one of them in the unstable direction, and every stable curve of length at least  $\delta_2$  properly crosses at least one of them in the stable direction. Let  $\mathcal{R}_{\delta_2} = \{R_1, \dots, R_k\}$ .

Now let  $j \in \mathbb{N}$ , and  $A \in \mathcal{M}_{-j}^0$  with  $\text{diam}^u(A) \geq \delta_2$  and  $\text{diam}^s(T^{-j}A) \geq \delta_2$ . Notice that  $T^{-j}A \in \mathcal{M}_0^j$ . By construction,  $A$  properly crosses one rectangle  $R_i \in \mathcal{R}_{\delta_2}$ , and  $T^{-j}A$  properly crosses another rectangle  $R_{i'} \in \mathcal{R}_{\delta_2}$ . Let  $\Xi_i$  denote the index set for the family of stable manifolds comprising  $R_i$ . For  $\xi \in \Xi_i$ , let  $W_{\xi, A} = W_{\xi} \cap A$ . Since  $T^{-j}A$  properly crosses  $R_{i'}$  in the stable direction and  $T^{-j}$  is smooth on  $A$ , it follows that  $T^{-j}(W_{\xi, A})$  is a single curve that contains a stable manifold in the family comprising  $R_{i'}$ .

Let  $\ell_{\delta_2}$  denote the length of the shortest stable manifold in the finite set of rectangles comprising  $\mathcal{R}_{\delta_2}$ . Then using (7.28) and (7.13), we have for all  $\xi \in \Xi_i$ ,

$$\int_{W_{\xi, A}} \nu = e^{-jh_*} \int_{T^{-j}(W_{\xi, A})} \nu \geq e^{-jh_*} \bar{C} \ell_{\delta_2}^{h_*} \bar{C}_2,$$

<sup>42</sup>It also follows from the proof of Proposition 7.12 that the upper bound  $\mu_*(A) \leq Ce^{-jh_*}$  holds for all  $A \in \mathcal{M}_{-j}^0$  for some constant  $C > 0$  independent of  $j$  and  $\delta_2$ , but we shall not need this here.

where  $\bar{C}, \bar{C}_2 > 0$  are independent of  $\delta$  and  $j$ .

Lastly, denoting by  $D(R_i)$  the smallest solid rectangle containing  $R_i$  (as in Definition 5.7) and using the fact from the proof of Corollary 7.9 that  $\mu_*^W$  is equivalent to  $\nu$  on  $\mu_*$ -a.e.  $W \in \mathcal{W}^s$ , we estimate,

$$\begin{aligned} \mu_*(A) &\geq \mu_*(A \cap D(R_i)) \geq \int_{\Xi_i} \mu_*^{W_\xi}(A) d\hat{\mu}_*(\xi) \\ &\geq C \int_{\Xi_i} \nu(A \cap W_\xi) d\hat{\mu}_*(\xi) \geq C'_{\delta_2} e^{-jh_*} \hat{\mu}_*(\Xi_i), \end{aligned}$$

which proves the lemma since the family  $\mathcal{R}_{\delta_2}$  is finite.  $\square$

We may finally prove Proposition 7.21:

*Proof.* This follows from the previous two lemmas, adapting Bowen's proof of uniqueness of equilibrium states (see the use of [KH, Lemma 20.3.4] in [KH, Thm 20.3.7], as observed in the proof of [GL, Thm 6.4], noting that there is no need to check that boundaries have zero measure).

Since  $\mu_*$  is ergodic, it suffices by a standard argument (see e.g. the beginning of the proof of [KH, Thm 20.1.3]) to check that if  $\mu$  is a  $T$ -invariant probability measure so that there exists a Borel set  $F \subset M$  with  $T^{-1}(F) = F$  and  $\mu_*(F) = 0$  but  $\mu(F) = 1$  (that is,  $\mu$  is singular with respect to  $\mu_*$ ) then  $h_\mu(T) < h_{\mu_*}(T)$ .

Observe first that the billiard map  $T$  (as well as its inverse  $T^{-1}$ ) is expansive, that is, there exists  $\varepsilon_0 > 0$  so that if  $d(T^j(x), T^j(y)) < \varepsilon_0$  for some  $x, y \in M$  and all  $j \in \mathbb{Z}$ , then  $x = y$ . (Indeed, if  $x \neq y$  then there is  $n \geq 1$  and an element of either  $\mathcal{S}_n$  or  $\mathcal{S}_{-n}$  that separates them. So  $x$  and  $y$  get mapped to different sides of a singularity line and by (3.3) are separated by a minimum distance  $\varepsilon_0$ , depending on the table.)

For each  $n \in \mathbb{N}$ , we consider the partition  $\mathcal{Q}_n$  of maximal connected components of  $M$  on which  $T^{-n}$  is continuous. By Lemmas 3.2 and 3.3,  $\mathcal{Q}_n$  is  $\mathcal{M}_{-n}^0$  plus isolated points whose cardinality grows at most linearly with  $n$ . Thus  $G_n \subset \mathcal{Q}_n$  for each  $n$ . Define  $\tilde{B}_n = \mathcal{Q}_n \setminus G_n$ . The set  $\tilde{B}_n$  contains  $B_n$  plus isolated points, and so its cardinality is bounded by the expression in Lemma 7.22, by possibly adjusting the constant  $C$ .

By the uniform hyperbolicity of  $T$ , the diameters of the elements of  $T^{-[n/2]}(\mathcal{Q}_n)$  tend to zero as  $n \rightarrow \infty$ . This implies the following fact.

**Sublemma 7.24.** *For each  $n \geq n_1$  there exists a finite union  $\mathcal{C}_n$  of elements of  $\mathcal{Q}_n$  so that*

$$\lim_{n \rightarrow \infty} (\mu + \mu_*)((T^{-[n/2]}\mathcal{C}_n) \triangle F) = 0.$$

*Proof.* See [Bo3, Lemma 2]: Let  $\bar{\mu} = \mu + \mu_*$  and  $\tilde{\mathcal{Q}}_n = T^{-[n/2]}(\mathcal{Q}_n)$ . For  $\delta > 0$  pick compact sets  $K_1 \subset F$  and  $K_2 \subset M \setminus F$  so that  $\max\{\bar{\mu}(F \setminus K_1), \bar{\mu}((M \setminus F) \setminus K_2)\} < \delta$ . We have  $\eta = \eta_\delta := d(K_1, K_2) > 0$ . If  $\text{diam}(\tilde{Q}) < \eta/2$  then either  $\tilde{Q} \cap K_1 = \emptyset$  or  $\tilde{Q} \cap K_2 = \emptyset$ . Let  $n = n_\delta$  be so that the diameter of  $\tilde{\mathcal{Q}}_n$  is  $< \eta_\delta/2$ . Set  $\tilde{\mathcal{C}}_n = \cup\{\tilde{Q} \in \tilde{\mathcal{Q}}_n : \tilde{Q} \cap K_1 \neq \emptyset\}$ . Then  $K_1 \subset \tilde{\mathcal{C}}_n$  and  $\tilde{\mathcal{C}}_n \cap K_2 = \emptyset$ . Hence,  $\bar{\mu}(\tilde{\mathcal{C}}_n \triangle F) \leq \delta + \bar{\mu}(\tilde{\mathcal{C}}_n \triangle K_1) \leq \delta + \bar{\mu}(M \setminus (K_1 \cup K_2)) \leq 3\delta$ . Defining  $\mathcal{C}_n = T^{[n/2]}\tilde{\mathcal{C}}_n$  completes the proof.  $\square$

Remark that, since  $T^{-1}(F) = F$ , it follows that

$$(\mu + \mu_*)(\mathcal{C}_n \triangle F) = (\mu + \mu_*)((T^{[n/2]}\mathcal{C}_n) \triangle F)$$

also tends to zero as  $n \rightarrow \infty$ .

Since  $\mathcal{Q}_{2n}$  is generating for  $T^{2n}$ , we have

$$h_\mu(T^{2n}) = h_\mu(T^{2n}, \mathcal{Q}_{2n}) \leq H_\mu(\mathcal{Q}_{2n}) = - \sum_{Q \in \mathcal{Q}_{2n}} \mu(Q) \log \mu(Q).$$

By the proof of Sublemma 7.24, for each  $n$ , there exists a compact set  $K_1(n)$  that defines  $\tilde{\mathcal{C}}_n = T^{-\lfloor n/2 \rfloor} \mathcal{C}_n$ , and satisfying  $K_1(n) \nearrow F$  as  $n \rightarrow \infty$ . Next, we group elements  $Q \in \mathcal{Q}_{2n}$  according to whether  $T^{-n}Q \subset \tilde{\mathcal{C}}_n$  or  $T^{-n}Q \cap \tilde{\mathcal{C}}_n = \emptyset$ . Note that if  $Q$  is not an isolated point, and if  $T^{-n}Q \cap \tilde{\mathcal{C}}_n \neq \emptyset$ , then  $T^{-n}Q \in \mathcal{M}_{-n}^n$  is contained in an element of  $\mathcal{M}_{-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor}$  that intersects  $K_1(n)$ . Thus  $Q \subset T^n \tilde{\mathcal{C}}_n = T^{\lfloor n/2 \rfloor} \mathcal{C}_n$ . Therefore,

$$\begin{aligned} 2nh_\mu(T) &= h_\mu(T^{2n}) \leq - \sum_{Q \in \mathcal{Q}_{2n}} \mu(Q) \log \mu(Q) \\ &\leq - \sum_{Q \subset T^n \tilde{\mathcal{C}}_n} \mu(Q) \log \mu(Q) - \sum_{Q \in \mathcal{Q}_{2n} \setminus (T^n \tilde{\mathcal{C}}_n)} \mu(Q) \log \mu(Q) \\ &\leq \frac{2}{e} + \mu(T^n \tilde{\mathcal{C}}_n) \log \#(\mathcal{Q}_{2n} \cap T^n \tilde{\mathcal{C}}_n) + \mu(M \setminus (T^n \tilde{\mathcal{C}}_n)) \log \#(\mathcal{Q}_{2n} \setminus (T^n \tilde{\mathcal{C}}_n)), \end{aligned}$$

where we used in the last line that the convexity of  $x \log x$  implies that, for all  $p_j > 0$  with  $\sum_{j=1}^N p_j \leq 1$ , we have (see e.g. [KH, (20.3.5)])

$$- \sum_{j=1}^N p_j \log p_j \leq \frac{1}{e} + (\log N) \sum_{j=1}^N p_j.$$

Then, since  $-h_{\mu_*}(T) = (\mu(T^n \tilde{\mathcal{C}}_n) + \mu(M \setminus (T^n \tilde{\mathcal{C}}_n))) \log e^{-h_*}$ , for  $n \geq n_1$ , we write

$$\begin{aligned} (7.34) \quad &2n(h_\mu(T) - h_{\mu_*}(T)) - \frac{2}{e} \\ &\leq \mu(T^n \tilde{\mathcal{C}}_n) \log \sum_{Q \in \mathcal{Q}_{2n}: Q \subset T^n \tilde{\mathcal{C}}_n} e^{-2nh_*} + \mu(M \setminus (T^n \tilde{\mathcal{C}}_n)) \log \sum_{Q \in \mathcal{Q}_{2n} \setminus (T^n \tilde{\mathcal{C}}_n)} e^{-2nh_*} \\ &\leq \mu(\mathcal{C}_n) \log \left( \sum_{Q \in G_{2n}: Q \subset T^n \tilde{\mathcal{C}}_n} e^{-2nh_*} + \sum_{Q \in \tilde{B}_{2n}: Q \subset T^n \tilde{\mathcal{C}}_n} e^{-2nh_*} \right) \\ &\quad + \mu(M \setminus \mathcal{C}_n) \log \left( \sum_{Q \in G_{2n} \setminus (T^n \tilde{\mathcal{C}}_n)} e^{-2nh_*} + \sum_{Q \in \tilde{B}_{2n} \setminus (T^n \tilde{\mathcal{C}}_n)} e^{-2nh_*} \right), \end{aligned}$$

where we have used the invariance of  $\mu$  in the last inequality. By Lemma 7.22, both sums over elements in  $\tilde{B}_{2n}$  are bounded by  $Ce^{-nh_*/4}$ . It remains to estimate the sum over elements of  $G_{2n}$ .

First we provide the following characterization of elements of  $G_{2n}$ . Let  $Q \in G_{2n} \subset \mathcal{M}_{-2n}^0$ . Since  $Q \notin B_{-2n}^0$ , there exists  $0 \leq j \leq \lfloor n/2 \rfloor$  such that  $T^{-j}Q \subset E_j \in \mathcal{M}_{-2n+j}^0$  and  $\text{diam}^u(E_j) \geq \delta_2$ . We claim that there exists  $k \leq \lfloor n/2 \rfloor$  and  $\bar{E} \in \mathcal{M}_{-2n+j+k}^0$  such that  $E_j \subset \bar{E}$  and  $\text{diam}^s(T^{-2n+j+k}\bar{E}) \geq \delta_2$ .

The claim follows from the fact that  $T^{-2n}Q \notin B_0^{2n}$ . Thus there exists  $k \leq \lfloor n/2 \rfloor$  such that  $T^{-2n+k}Q \subset \tilde{E}_k \in \mathcal{M}_0^{2n-k}$  with  $\text{diam}^s(\tilde{E}_k) \geq \delta_2$ . But notice that  $T^{-2n+j+k}E_j \in \mathcal{M}_{-k}^{2n-j-k}$  contains  $T^{-2n+k}Q$ . Thus letting  $\tilde{E}$  denote the unique element of  $\mathcal{M}_0^{2n-j-k}$  containing both  $T^{-2n+j+k}E_j$  and  $\tilde{E}_k$ , we define  $\bar{E} = T^{2n-j-k}\tilde{E} \in \mathcal{M}_{-2n+j+k}^0$ , and  $\bar{E}$  has the required property since  $T^{-2n+j+k}\bar{E} \supset \tilde{E}_k$ .

By construction,  $\bar{E}$  satisfies the assumptions of Lemma 7.23 since  $\bar{E} \in \mathcal{M}_{-2n+j+k}^0$  with  $\text{diam}^u(\bar{E}) \geq \delta_2$ , and  $\text{diam}^s(T^{-2n+j+k}\bar{E}) \geq \delta_2$ . Thus,

$$(7.35) \quad \mu_*(\bar{E}) \geq C_{\delta_2} e^{(-2n+j+k)h_*}.$$

We call  $(\bar{E}, j, k)$  an *admissible triple* for  $Q \in G_{2n}$  if  $0 \leq j, k \leq \lfloor n/2 \rfloor$  and  $\bar{E} \in \mathcal{M}_{-2n+j+k}^0$ , with  $T^{-j}Q \subset \bar{E}$  and  $\min\{\text{diam}^u(\bar{E}), \text{diam}^s(T^{-2n+j+k}\bar{E})\} \geq \delta_2$ . Obviously, there may be many admissible triples associated to a given  $Q \in G_{2n}$ ; however, we define the unique *maximal triple* for  $Q$  by taking first the maximum  $j$ , and then the maximum  $k$  over all admissible triples for  $Q$ .



Let  $\mathcal{E}_{2n}$  be the set of maximal triples obtained in this way from elements of  $G_{2n}$ . For  $(\bar{E}, j, k) \in \mathcal{E}_{2n}$ , let  $\mathcal{A}_M(\bar{E}, j, k)$  denote the set of  $Q \in G_{2n}$  for which  $(\bar{E}, j, k)$  is the maximal triple. The importance of the set  $\mathcal{E}_{2n}$  lies in the following property.

**Sublemma 7.25.** *Suppose  $(\bar{E}_1, j_1, k_1), (\bar{E}_2, j_2, k_2) \in \mathcal{E}_{2n}$  with  $j_2 \geq j_1$  and  $(\bar{E}_1, j_1, k_1) \neq (\bar{E}_2, j_2, k_2)$ . Then  $T^{-(j_2-j_1)}\bar{E}_1 \cap \bar{E}_2 = \emptyset$ .*

*Proof.* Suppose, to the contrary, that there exist  $(\bar{E}_1, j_1, k_1), (\bar{E}_2, j_2, k_2) \in \mathcal{E}_{2n}$  with  $j_2 \geq j_1$  and  $T^{-(j_2-j_1)}\bar{E}_1 \cap \bar{E}_2 \neq \emptyset$ . Note that  $T^{-(j_2-j_1)}\bar{E}_1 \in \mathcal{M}_{-2n+j_2+k_1}^{j_2-j_1}$  while  $\bar{E}_2 \in \mathcal{M}_{-2n+j_2+k_2}^0$ .

Thus if  $k_1 \leq k_2$ , then  $T^{-(j_2-j_1)}\bar{E}_1 \subset \bar{E}_2$ , and so  $(\bar{E}_1, j_1, k_1)$  is not a maximal triple for all  $Q \in \mathcal{A}_M(\bar{E}_1, j_1, k_1)$ , a contradiction.

If, on the other hand,  $k_1 > k_2$ , then both  $T^{-(j_2-j_1)}\bar{E}_1$  and  $\bar{E}_2$  are contained in a larger element  $\bar{E}' \in \mathcal{M}_{-2n+j_2+k_1}^0$ . Since  $\bar{E}' \supset \bar{E}_2$ , we have  $\text{diam}^u(\bar{E}') \geq \delta_2$ , and since  $T^{-2n+j_2+k_1}\bar{E}' \supset T^{-2n+j_1+k_1}\bar{E}_1$ , we have  $\text{diam}^s(T^{-2n+j_2+k_1}\bar{E}') \geq \delta_2$ . Thus neither  $(\bar{E}_1, j_1, k_1)$  nor  $(\bar{E}_2, j_2, k_2)$  is a maximal triple, also a contradiction.  $\square$

Note that by definition, if  $Q \in T^n\tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k)$ , then  $T^{-n+j}\bar{E} \in \mathcal{M}_{-n+k}^{n-j}$  contains  $T^{-n}Q$ . Also, since  $j, k \leq \lfloor n/2 \rfloor$ ,  $T^{-n+j}\bar{E}$  is contained in the same element of  $\mathcal{M}_{-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor}$  that contains  $T^{-n}Q$  and intersects  $K_1(n)$ . Thus  $T^{-n+j}\bar{E} \subset \tilde{\mathcal{C}}_n$  whenever  $T^n\tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k) \neq \emptyset$ . This also implies that  $\mathcal{A}_M(\bar{E}, j, k) \subset T^n\tilde{\mathcal{C}}_n$  whenever  $T^n\tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k) \neq \emptyset$ .

Next, for a fixed  $(\bar{E}, j, k) \in \mathcal{E}_{2n}$ , by submultiplicativity, since  $\bar{E} \in \mathcal{M}_{-2n+j+k}^0$  and  $G_{2n} \subset \mathcal{M}_{-2n}^0$ , we have  $\#\mathcal{A}_M(\bar{E}, j, k) \leq \#\mathcal{M}_0^{j+k}$ . Now using Proposition 4.6 and (7.35), we estimate

$$\begin{aligned} \sum_{Q \in G_{2n}: Q \subset T^n\tilde{\mathcal{C}}_n} e^{-2nh_*} &\leq \sum_{(\bar{E}, j, k) \in \mathcal{E}_{2n}: \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} \sum_{Q \in \mathcal{A}_M(\bar{E}, j, k)} e^{-2nh_*} \\ &\leq \sum_{(\bar{E}, j, k) \in \mathcal{E}_{2n}: \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} C e^{(-2n+j+k)h_*} \leq \sum_{(\bar{E}, j, k) \in \mathcal{E}_{2n}: \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} C' \mu_*(\bar{E}) \\ &\leq \sum_{(\bar{E}, j, k) \in \mathcal{E}_{2n}: \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} C' \mu_*(T^{-n+j}\bar{E}) \leq C' \mu_*(\tilde{\mathcal{C}}_n) = C' \mu_*(\mathcal{C}_n), \end{aligned}$$

where the constant  $C'$  depends on  $\delta_2$ , but not on  $n$ . We have also used that  $T^{-n+j_1}\bar{E}_1 \cap T^{-n+j_2}\bar{E}_2 = \emptyset$  for all distinct triples  $(\bar{E}_1, j_1, k_1), (\bar{E}_2, j_2, k_2) \in \mathcal{E}_{2n}$ , by Sublemma 7.25, in order to sum over the elements of  $\mathcal{E}_{2n}$ . A similar bound holds for the sum over  $Q \in G_{2n} \setminus (T^n\tilde{\mathcal{C}}_n)$  since  $T^{-n+j}\bar{E} \subset M \setminus \tilde{\mathcal{C}}_n$  whenever  $T^n\tilde{\mathcal{C}}_n \cap \mathcal{A}(\bar{E}, j, k) = \emptyset$ . Putting these bounds together allows us to complete our estimate of (7.34),

$$\begin{aligned} 2n(h_\mu(T) - h_{\mu_*}(T)) - \frac{2}{e} &\leq \mu(\mathcal{C}_n) \log \left( C' \mu_*(\mathcal{C}_n) + C e^{-nh_*/4} \right) \\ &\quad + \mu(M \setminus \mathcal{C}_n) \log \left( C' \mu_*(M \setminus \mathcal{C}_n) + C e^{-nh_*/4} \right). \end{aligned}$$

Since  $\mu(\mathcal{C}_n)$  tends to 1 as  $n \rightarrow \infty$  while  $\mu_*(\mathcal{C}_n)$  tends to 0 as  $n \rightarrow \infty$  the limit of the right-hand side tends to  $-\infty$ . This yields a contradiction unless  $h_\mu(T) < h_{\mu_*}(T)$ .  $\square$

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