Dolgopyat's estimates and spectra of hyperbolic flows

Dmitri Dolgopyat's original techniques [3] were designed to investigate the rate of mixing of Anosov flows. Using suitable Markov sections of the flow and establishing fine estimates for a family of transfer operators related to the (uniformly expanding) Poincaré map, he has been able to show that, at least for negatively curved compact surfaces, the correlation functions of the flow enjoy an exponential decay as time goes to infinity, improving previous subexponential results of Chernov [2].

Seven years later, an increasing number of papers are using his seminal ideas with applications ranging from the geometry of negatively curved manifolds to number theory. Indeed, his work is based on a powerfull cancelation machinery which implies uniform contraction estimates for families of tranfer operators, and these estimates turn out to be usefull in many situations, especially in the analysis of various dynamically defined zeta functions.

In this course, after an expository lecture on the general aspects of Dolgopyat's method and its applications, I plan to demonstrate these ideas on a popular model, namely the modular surface $X = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2$ and its geodesic flow. Using Dolgopyat's method, I will give a purely dynamical [1] proof of the fact that the Selberg zeta function

$$Z_X(s) = \prod_{k\in\mathbb{N}} \prod_{\gamma\in\mathcal{P}} \left(1 - e^{-(s+k)l(\gamma)}\right),\ \Re(s) > 1,$$

admits a non-vanishing analytic continuation to a strip $\{1-\epsilon_0 \leq \Re(s) \leq 1\}$ with $\varepsilon_0 > 0$, except at $s = 1$ which is a simple zero. In the above Euler product formula for $Z_X(s)$, $\mathcal{P}$ is the set of primitive closed geodesics on X, and if $\gamma \in \mathcal{P}$, $l(\gamma)$ denotes its length.

A by product of this result (together with an estimate of the growth of $|Z_X(s)|$ in this strip) is an asymptotic of the counting function $N(T) = \#\{\gamma \in \mathcal{P}\ \ l(\gamma) \leq T\}$ of the type

$$N(T) = \mathrm{li}\left(e^T\right) + O(e^{\beta T}),$$

where $\mathrm{li}(x) = \int_2^x \frac{dt}{\log(t)}$ and β is a positive constant with $\beta < 1$. This kind of result was previously obtained by P. Sarnak [6] using Selberg's trace formula. Most of the material I will need for this alternative proof is contained in [1]. I will also use the thermodynamic formalism of the Gauss map [4, 5] and the classical coding of the geodesic flow on the modular surface via continued fractions, see for example [7].

References

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[1] i.e. without using Selberg's trace formula nor the harmonic analysis on X