Selberg's zeta function and Dolgopyat's estimates for the modular surface

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1 Introduction

The so-called Dolgopyat's estimates were introduced by D. Dolgopyat in his seminal paper [5] to answer a long standing problem of ergodic geometry. Let M be a strictly negatively curved compact surface (with possible variable curvature), let SM denote the unit tangent bundle to M on which the geodesic flow ϕ_t lives. For all ϕ_t-invariant equilibrium measure μ_φ on SM related to a Hölder continuous potential φ, for all Hölder continuous observables f, g on SM, we have exponential decay of correlations i.e. as $t \to \infty$,

$$\int_{SM} (f \circ \phi_t)\, g d\mu_\varphi = \int_{SM} f d\mu_\varphi \int_{SM} g d\mu_\varphi + O\left(e^{-\alpha t}\right),$$

with α depending on the Hölder regularity of the observables.

This result has been extended recently to higher dimensions by C. Liverani in [9]. The main obstacle in extending the work of Dolgopyat was the lack of regularity of the hyperbolic foliation and Liverani successfully applied a new functional analytic approach involving anisotropic function spaces to overcome this difficulty.

The original method of Dolgopyat turns out to be much more versatile than one might have thought at first sight. His estimates (when available) imply deep results on the zeros of dynamical zeta functions and this fact was used for the first time by Pollicott and Sharp in [17] where they obtained an exponentially small error term to the classical Margulis asymptotic for the counting function of closed geodesics. This work was followed by several other results (see for example [1, 24, 13, 14, 18]) on the asymptotic distribution of closed orbits for various flows of hyperbolic nature and counting problems on higher rank symmetric spaces. Some recent applications of Dolgopyat's techniques to number theory and algorithm theory were also obtained by Baladi-Vallée [3]. We also think that these dynamical ideas are very likely to be strong enough to reach interesting results in hyperbolic scattering theory that are beyond the scope of the standard tools of spectral theory, see [12] for a discussion around this topic.

In these notes, we will modestly try to give an overview of the ideas of Dmitri Dolgopyat on a popular example: the family of Ruelle-Mayer transfer operators related to the Gauss map. As explained later, the Gauss map induces a dynamical system which is in a sense a Poincaré first return map for a cross section of the geodesic flow on the modular surface $\mathcal{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2$. These transfer operator estimates imply therefore a zero-free strip for the Selberg zeta function $Z(s)$ defined for $\mathrm{Re}(s) > 1$ by

$$Z(s) = \prod_{k\in\mathbb{N}} \prod_{\gamma\in\mathcal{P}} \left(1 - e^{-(s+k)l(\gamma)}\right),$$

where the right product is over prime (i.e. primitive closed) geodesics γ on $\mathcal{M}$ whose lengths are denoted by $l(\gamma)$. Combined with a suitable growth estimate, it implies a precise asymptotic of the counting function for the closed geodesics, i.e. as $T \to +\infty$,

$$\#\{\gamma \in \mathcal{P} \ : \ l(\gamma) \leq T\} = \mathrm{Li}(e^T) + O\left(e^{\beta T}\right), \tag{1}$$

where $\mathrm{Li}(x) = \int_2^x (\log(t))^{-1} dt$ is the standard *integral logarithm*, and $0 < \beta < 1$ is a non-effective constant. This result is nothing new: Selberg's trace formula (see [7, 20]) implies a zero-free strip (and much more) for the Selberg's zeta function, together with a more effective asymptotic expansion for the closed geodesics. However the method sketched in these notes is purely dynamical and will seldom use the underlying arithmetic properties of $\mathcal{M}$. A first purely dynamical proof of the leading asymptotic term, namely

$$\#\{\gamma \in \mathcal{P} \ : \ l(\gamma) \leq T\} = \frac{e^T}{T} + o(1),$$

was first obtained by M. Pollicott in [16], see also [4] for a different approach.

If we think of $S\mathcal{M} = SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ as the moduli space of flat tori and the geodesic flow as the Teichmuller flow over this moduli space, then what we will discuss is just a "baby" version of the forthcoming work of Avila-Gouëzel-Yoccoz on the rate of decay of correlations for the Teichmuller flow over the moduli space of translation surfaces of higher genus.

These notes are far from being self-contained. Many elementary results from transfer operator theory will be stated without proof, but we hope that multiple references given throughout will be enough to fill that gap. Most of the statements hold in greater generality for "reasonable" families of expanding Markov maps, see the papers of Baladi-Vallée [3] for some abstract statements. We believe that by focusing on this particular non-trivial example the reader will have a better chance to grasp the very natural ideas hidden in the rather technical developments of Dolgopyat's method. Most of the proofs given here follow closely the original paper of Dolgopyat [5].

2 Closed geodesics and periodic points of the Gauss map

The discrete group $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{-I, +I\}$ acts by isometries (Moebius transforms)

$$z \mapsto \frac{az+b}{cz+d},$$

on the Poincaré upper Half plane $\mathbb{H}^2 = \{x + iy \ : \ y > 0\}$ endowed with its metric of constant negative curvature -1

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The quotient space $\mathcal{M} = PSL_2(\mathbb{Z})\backslash\mathbb{H}^2$ is a non-compact finite volume Riemann "surface" with a cuspidal end and two singularities (caused by elliptic elements in the group).

One of our goals is to investigate the asymptotic repartition of closed geodesics on $\mathcal{M}$ i.e. periodic orbits of the geodesic flow on the unit tangent bundle $S\mathcal{M}$. Recall that the geodesic flow on $S\mathcal{M}$ is nothing but motion at unit speed along circles in $\mathbb{H}^2$ orthogonal to the real line, modulo $PSL_2(\mathbb{Z})$ action.

The set of closed, primitive geodesics is denoted as usual by $\mathcal{P}$, and if $\gamma \in \mathcal{P}$, let $l(\gamma)$ denote its length. The length spectrum of $\mathcal{M}$ is by definition the set $\{l(\gamma) \ \ \gamma \in \mathcal{P}\}$, where lengths are repeated according to their multiplicities. The set of prime closed geodesics is, as it is the case for all hyperbolic manifolds, in bijection with the conjugacy classes of prime hyperbolic elements in $PSL_2(\mathbb{Z})$. Following a strong analogy with analytic number theory, if one wishes to study the asymptotic distribution of closed geodesics, it is a natural idea to define the *Ruelle zeta function* [1]

$$\zeta(s) = \prod_{\gamma \in \mathcal{P}} \left(1 - \frac{1}{N(\gamma)^s}\right)^{-1},$$

where $N(\gamma) = e^{l(\gamma)}$. The work of Atle Selberg has actually shown that the double product (called *Selberg zeta function*)

$$Z(s) = \prod_{k \in \mathbb{N}} \prod_{\gamma \in \mathcal{P}} \left(1 - e^{-(s+k)l(\gamma)}\right) = \prod_{k \in \mathbb{N}} \zeta(s+k)^{-1},$$

is in fact a more natural object to consider for this purpose. Some elementary volume estimates plus the fact that $PSL_2(\mathbb{Z})$ is a discrete group of the first kind (every point on the real line is an accumulation point of this group) imply that the Dirichlet series

$$\sum_{\gamma \in \mathcal{P}} e^{-sl(\gamma)},$$

are absolutly convergent for $\mathrm{Re}(s) > 1$, and thus $Z(s)$ is a non-vanishing analytic function on the half-plane $\{\mathrm{Re}(z) > 1\}$. Our goal in these notes is to give a dynamical proof of the following result.

Theorem 2.1 *The Selberg zeta function $Z(s)$ has an analytic continuation to $\{\mathrm{Re}(s) > \frac{1}{2}\}$ where it satisfies for all $\mathrm{Re}(s) \geq \sigma > \frac{1}{2}$, an upper bound of the type*

$$|Z(s)| \leq C_\sigma e^{C_\sigma |\mathrm{Im}(s)|^2}.$$

Moreover, there exists $\varepsilon > 0$ such that $Z(s)$ is non-vanishing in the vertical strip

$$\{1 - \varepsilon \leq \mathrm{Re}(s) \leq 1\},$$

except at $s = 1$ which is a simple zero.

The combination of these results with some standard methods of analytic number theory imply the asymtotics of formula (1). Roughly speaking, a classical Lemma of Tischmarch converts the upper bound on the growth of $Z(s)$ into an upper bound for the logarithmic derivative of $Z(s)$ in the non-vanishing strip. A "regularized enough" Perron formula and a contour deformation yields precise asymptotics for a regularized counting function over the lengths of closed geodesics. A standard finite difference method leads you to the conclusion.

In the remainder of this section we briefly recall how $Z(s)$ admits, for large $\mathrm{Re}(s)$, a representation involving the even periodic points of the Gauss map (see below). Let $x_0 \in (0,1)$ be an irrational number with periodic continued fraction expansion (i.e. x_0 is quadratic), say

$$x_0 = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ldots + \cfrac{1}{n_{2p} + \cfrac{1}{n_1 + \ldots}}}},$$

[1] whose definition mimics the Euler product of the Riemann zeta function

where $n_1, \ldots, n_{2p} \in \mathbb{N}^*$. Recall that the modular group $PSL_2(\mathbb{Z})$ is generated by $P_\pm(z) = z \pm 1$ and $Q(z) = \frac{-1}{z}$. It is easy to see that

$$R_0(x_0) := P_-^{n_{2p}} Q \ldots Q P_-^{n_2} Q P_+^{n_1} Q(x_0) = T^{2p}(x_0) = x_0,$$

where $Tx = \frac{1}{x} - \left[\frac{1}{x}\right]$ is the *Gauss map*. Since R_0 is obviously an element of $PSL_2(\mathbb{Z})$ and a hyperbolic isometry (x_0 is a repelling fixed point), we have by a standard result of hyperbolic geometry $(T^{2p})'(x_0) = R_0'(x_0) = e^{kl(\gamma)}$, where $k \in \mathbb{N}^*$ and $\gamma \in \mathcal{P}$. With a little more work (see Series [21, 22], but also [4]) one can show the following.

Proposition 2.2 *There is a one-to-one correspondence between the length spectrum (with multiplicities) of $\mathcal{M}$ and the set of values $\log |(T^{2p})'(x)|$ (with multiplicities), where*

$$x \in \overline{x} = \{x, T^2(x), \ldots, (T^2)^{p-1}(x)\},$$

$\overline{x}$ being a primitive periodic orbit of $T^2 : [0,1] \to [0,1]$.

Assuming that $\mathrm{Re}(s)$ is large enough to insure absolute convergence, we have therefore

$$Z(s) = \exp\left(-\sum_{m=1}^{+\infty} \frac{1}{m} \sum_{k,\gamma} e^{-m(s+k)l(\gamma)}\right) = \exp\left(-\sum_{m=1}^{+\infty} \frac{1}{m} \sum_{\gamma} \frac{e^{-msl(\gamma)}}{1 - e^{-ml(\gamma)}}\right)$$

$$= \exp\left(-\sum_{m=1}^{+\infty}\sum_{p=1}^{+\infty} \frac{1}{pm} \sum_{\substack{(T^2)^p(x)=x \\ p \text{ least period}}} \frac{|(T^{2p})'(x)|^{-sm}}{1 - |(T^{2p})'(x)|^{-m}}\right)$$

$$= \exp\left(-\sum_{n=1}^{+\infty} \frac{1}{n} \sum_{T^{2n}(x)=x} \frac{|(T^{2n})'(x)|^{-s}}{1 - |(T^{2n})'(x)|^{-1}}\right).$$

We will keep in mind this formula until the end of §3.1 where we will use it to prove analytic continuation of $Z(s)$ to the half plane $\{\mathrm{Re}(s) > \frac{1}{2}\}$.

3 The Ruelle-Mayer transfer operators

We recall that the Gauss map $T : I = [0,1] \to I$, is defined by

$$Tx = \begin{cases} 0 & \text{if } \ x = 0 \\ \frac{1}{x} - n & \text{if } \ x \in I_n = (\frac{1}{n+1}, \frac{1}{n}],\ n \geq 1 \end{cases}$$

The *Perron-frobenius* transfer operator $\mathcal{L} : L^1(I) \to L^1(I)$ is the unique (bounded) linear operator such that for all $(f, g) \in L^1(I) \times L^\infty(I)$,

$$\int_I \mathcal{L}(f) g dm = \int_I f(g \circ T) dm,$$

where m is the Lebesgue measure. It is an exercise to check that for almost all $x \in I$,

$$\mathcal{L}(f)(x) = \sum_{n=1}^{+\infty} |T'(\gamma_n x)|^{-1} f(\gamma_n x) = \sum_{n=1}^{+\infty} \frac{1}{(n+x)^2} f\left(\frac{1}{n+x}\right),$$

where γ_n is the inverse of $T|_{I_n}$. The Gauss density is the normalized eigenfunction

$$h(x) = \frac{1}{\log(2)} \frac{1}{1+x}$$

which satisfies $\mathcal{L}(h) = h$ so that hdm is a T-invariant measure. The Ruelle-Mayer transfer operator $\mathcal{L}_s$ is defined for all $s \in \mathbb{C}$, $\mathrm{Re}(s) > \frac{1}{2}$ by

$$\mathcal{L}_s(f) = \sum_{n=1}^{+\infty} \frac{1}{(n+x)^{2s}} f\left(\frac{1}{n+x}\right), \tag{2}$$

where f belongs to a suitable function space (see below). The spectral properties of the family $\mathcal{L}_s$ depend drastically on the function space used, and we shall describe below the spectrum of $\mathcal{L}_s$ when acting on a suitable Hilbert space of holomorphic functions. The family of transfer operator $\mathcal{L}_s$ will become trace class and we will obtain a representation of the Selberg zeta function $Z(s)$ as a product of Fredholm determinants,i.e. for $\mathrm{Re}(s) > \frac{1}{2}$,

$$Z(s) = \det(I - \mathcal{L}_s)\det(I + \mathcal{L}_s).$$

This is exactly what Mayer has done in [11], but we will slightly modify it using a Hilbert's space approach to obtain an a priori upper bound (required to take advantage of the zero-free strip). By using a nice trick of Mayer [10], it is actually possible to obtain a holomorphic continuation to the whole complex plane of the transfer operators (and thus of the Selberg zeta function), but we will not need it for our purpose.

3.1 Transfer operators on $H^2(D)$ and Selberg's zeta function

Let D be the open disc $D(1, \frac{3}{2}) \supset I$ in $\mathbb{C}$, and let $H^2(D)$ be the *Hardy space* of the disc D. This is the Hilbert space of holomorphic functions f on D such that

$$\|f\|_{H^2}^2 = \limsup_{r \to 3/2} \frac{1}{2\pi} \int_0^{2\pi} |f(1 + re^{i\theta})|^2 d\theta < +\infty.$$

Every element $f \in H^2(D)$ has a radial limit f^* defined almost everywhere by

$$f^*\left(1 + \frac{3}{2}e^{i\theta}\right) = \lim_{r \to 3/2} f(1 + re^{i\theta}),$$

and f^* is in $L^2(\partial D)$, with

$$\|f^*\|_{L^2}^2 = \int_{\partial D} |f^*|^2 d\mu = \|f\|_{H^2}^2,$$

where μ is the normalized Lebesgue measure on ∂D. Moreover, the Cauchy formula holds on the boundary: for all $f \in H^2(D)$, for all $z \in D$,

$$f(z) = \frac{1}{2i\pi} \int_{\partial D} \frac{f^*(\zeta)}{\zeta - z} d\zeta.$$

The Hardy space $H^2(D)$ can equivalently be defined as the Hilbert space of holomorphic functions on D having an expansion of the type

$$f(z) = \sum_{n \in \mathbb{N}} a_n e_n(z),$$

where $e_n(z) = \left(\frac{z-1}{3/2}\right)^n$, and $\sum_{n\in\mathbb{N}} |a_n|^2 = \|f\|_{H^2}^2 < +\infty$. For the proofs of these basic properties of Hardy spaces, see the standard reference [19].

Let us go back to the transfer operator $\mathcal{L}_s$. We remark that all the inverse branches $\gamma_n(z) = \frac{1}{n+z}$ are Moebius transforms mapping the disc D strictly into itself, i.e. for all $n \geq 1$,

(3) $$\gamma_n(D) \subset \overline{D(1,1)} \subset D.$$

Let $\mathrm{Log}(z)$ denote an holomorphic determination of the logarithm on $\mathbb{C} \setminus (-\infty, 0)$ such that

$$\left(\frac{1}{n+z}\right)^{2s} := e^{-2s\mathrm{Log}(n+z)}$$

is an holomorphic extension of $|(\gamma_n)'(z)|^s$ to the disc D. It is now easy to check that for all f holomorphic on D, $\mathcal{L}_s(f)$ defined by the series (2) makes sense for all $\mathrm{Re}(s) > \frac{1}{2}$ and is a bounded holomorphic function on D, thus a function in $H^2(D)$.

In fact using (3), one can prove that given f analytic on D, for all $\sigma > \frac{1}{2}$, there exist $C, C_\sigma > 0$ such that

(4) $$\sup_{z \in D} |\mathcal{L}_s(f)(z)| \leq C_\sigma e^{C|\mathrm{Im}(s)|} \sup_{\zeta \in D(1,1)} |f(\zeta)|.$$

A direct consequence of this observation is the following.

Proposition 3.1 *For all $\sigma > \frac{1}{2}$, $\mathcal{L}_s : H^2(D) \to H^2(D)$ is a compact operator.*

Proof. Pick a sequence $(f_n)_{n\geq 1} \in H^2(D)$ with $\|f_n\|_{H^2} \leq 1$. On every compact subset $K \subset D$, we have, using the Cauchy formula and the Schwartz inequality,

$$\sup_K |f_n| \leq \frac{3}{2} \|f_n\|_{H^2} \mathrm{dist}(K, \partial D)^{-1}.$$

This bound implies that $(f_n)_{n\geq 1}$ satisfies the Montel property and we can extract a subsequence $(f_{n_k})_{k\geq 1}$ converging uniformly on every compact subset of D to a holomorphic function $\widetilde{f}$. Because of (4), $L_s(\widetilde{f})$ is obviously in $H^2(D)$ and thus

$$\|\mathcal{L}_s(f_{n_k}) - \mathcal{L}_s(\widetilde{f})\|_{H^2} \leq \sup_D |\mathcal{L}_s(f_{n_k}) - \mathcal{L}_s(\widetilde{f})|,$$

which tends to zero using (4) and uniform convergence of $(f_{n_k})_{k\geq 1}$ on $D(1,1)$. □

In fact, when acting on $H^2(D)$, $\mathcal{L}_s$ is a trace class operator.

Proposition 3.2 *For all* $\mathrm{Re}(s) > \frac{1}{2}$, $\mathcal{L}_s : H^2(D) \to H^2(D)$ *is trace class, and the Fredholm determinants* $d_\pm(s)$ *defined by*

$$d_\pm(s) = \det(I \pm \mathcal{L}_s),$$

are analytic functions on the half plane $\{Re(s) > \frac{1}{2}\}$ *enjoying upper bounds*

$$|d_\pm(s)| \leq A_\sigma e^{A_\sigma |\mathrm{Im}(s)|^2},$$

for all $\sigma > \frac{1}{2}$ *and* $\mathrm{Re}(s) \geq \sigma$ *and a well chosen constant* $A_\sigma > 0$.

Proof. Our standard references for the theory of traces and determinants on Hilbert space are [6, 23]. We have to check that the sequence of singular values $(\mu_j(\mathcal{L}_s))_{j\geq 0}$ is summable. We recall that by definition $\mu_j(\mathcal{L}_s) = \lambda_j(\sqrt{\mathcal{L}_s^*\mathcal{L}_s})$, where

$$\lambda_0(\sqrt{\mathcal{L}_s^*\mathcal{L}_s}) \geq \lambda_1(\sqrt{\mathcal{L}_s^*\mathcal{L}_s}) \geq \lambda_2(\sqrt{\mathcal{L}_s^*\mathcal{L}_s}) \geq \ldots,$$

are the eigenvalues of the (positive) self-adjoint compact operator $\sqrt{\mathcal{L}_s^*\mathcal{L}_s}$. Courant's minimax principle (applied to $\sqrt{\mathcal{L}_s^*\mathcal{L}_s}$) shows that

$$\mu_j(\mathcal{L}_s) = \min_{\dim(F)=j} \max_{\substack{x\in F^\perp \\ \|x\|\leq 1}} \|\mathcal{L}_s(x)\|_{H^2},$$

which implies that for all Hilbert basis $(f_j)_{j\geq 0}$ of $H^2(D)$, we have for all $j \geq 0$,

$$\mu_j(\mathcal{L}_s) \leq \sum_{k=j}^{+\infty} \|\mathcal{L}_s(f_k)\|_{H^2} \leq +\infty.$$

If we choose the natural basis $(e_k)_{k\geq 0}$ defined above, then the estimate (4) yields

$$\|\mathcal{L}_s(e_k)\|_{H^2} \leq \sup_D |L_s(e_k)| \leq C_\sigma e^{C|\mathrm{Im}(s)|} \sup_{z\in D(1,1)} \left|\frac{z-1}{3/2}\right|^k \leq C_\sigma e^{C|\mathrm{Im}(s)|} \left(\frac{2}{3}\right)^k,$$

and we get for all $j \geq 0$,

$$\mu_j(\mathcal{L}_s) \leq 3C_\sigma e^{C|\mathrm{Im}(s)|} \left(\frac{2}{3}\right)^j. \tag{5}$$

The operator $\mathcal{L}_s$ is therefore trace class. Using Weyl's inequalities

$$\prod_{j=0}^{N} \Big(1 + |\lambda_j(\pm\mathcal{L}_s)|\Big) \leq \prod_{j=0}^{N} \Big(1 + \mu_j(\pm\mathcal{L}_s)\Big), \text{ for all } N \geq 0,$$

we deduce that the Fredholm determinants

$$d_\pm(s) := \prod_{j=0}^{+\infty} \Big(1 + \lambda_j(\pm\mathcal{L}_s)\Big)$$

are well defined for $\mathrm{Re}(s) > \frac{1}{2}$. A clever use of (5) combined with the Weyl inequalities shows that there indeed exists a constant $A > 0$, depending only on σ such that

$$|d_{\pm}(s)| \leq A_{\sigma} e^{A_{\sigma}|\mathrm{Im}(s)|^2},$$

for $\mathrm{Re}(s) \geq \sigma$. A point may remain unclear: we certainly cannot deduce from the definition of $d_{\pm}(s)$ as infinite products that they are holomorphic in the half plane $\{\mathrm{Re}(s) > \frac{1}{2}\}$. Analyticity of these determinants actually follows from the analyticity of $s \mapsto \mathcal{L}_s$ and the expansion of $\det(I \pm \mathcal{L}_s)$ as sums of traces of exterior powers of $\mathcal{L}_s$ or the so called Plemelj-Smithies formula, see [6]. □

It remains to finally relate the determinants $d_{\pm}(s)$ to the Selberg zeta function. To this end, we will need the following remark.

Lemma 3.3 *Let ϕ, ψ be holomorphic on D with continuous extensions to $\overline{D}$. Assume that $\phi(\overline{D}) \subset D$. Then*

$$L_{\phi} : \left\{ \begin{array}{ccc} H^2(D) & \to & H^2(D) \\ f & \mapsto & (f \circ \phi)\psi \end{array} \right.$$

is a trace class operator and

$$Tr(L_{\phi}) = \frac{\psi(z^*)}{1 - \phi'(z^*)},$$

where z^* is the unique fixed point of $\psi : D \to D$.

Proof. Compactness and nuclearity of the weighted composition operator L_{ϕ} follows from the same arguments as in the proof of Prop (3.1) and Prop (3.2). Note that because of the Cauchy formula, the composition operator L_{ϕ} can be thought as an integral operator with "smooth" kernel $K(z, \zeta) = \psi(z)(\zeta - \phi(z))^{-1}$, and the trace should be therefore given by the integral over the diagonal. This is precisely what we are going to show. Since $\phi(\overline{D}) \subset D$, we can certainly choose a smaller open disc $\widetilde{D}$ with $\phi(\overline{D}) \subset \widetilde{D}$ and $\overline{\widetilde{D}} \subset D$, so that we can write for all $z \in \overline{D}$,

$$L_{\phi}(f)(z) = \frac{1}{2i\pi} \int_{\partial \widetilde{D}} \frac{\psi(z) f(\zeta)}{\zeta - \phi(z)} d\zeta.$$

Assume that $\zeta \in \partial \widetilde{D}$. Since $z \mapsto \psi(z)(\zeta - \phi(z))^{-1}$ is holomorphic on D, the series (which is nothing but the taylor expansion at $z = 1$)

$$\sum_{k \geq 0} \left\langle \frac{\psi(z)}{\zeta - \phi(z)}, e_k(z) \right\rangle_{H^2} e_k(\zeta)$$

converges to $\psi(\zeta)(\zeta - \phi(\zeta))^{-1}$. An application of Fubini's theorem now shows that

$$\mathrm{Tr}(L_{\phi}) = \sum_{k=0}^{+\infty} \langle L_{\phi}(e_k), e_k \rangle_{H^2} = \sum_{k=0}^{+\infty} \frac{1}{2i\pi} \int_{\partial \widetilde{D}} \left\langle \frac{\psi(z)}{\zeta - \phi(z)}, e_k(z) \right\rangle_{H^2} e_k(\zeta) d\zeta.$$

We let the reader check that Lebesgue's theorem can comfortably be applied (this is where we use that $\widetilde{\widetilde{D}} \subset D$) to get

$$\operatorname{Tr}(L_\phi) = \frac{1}{2i\pi} \int_{\partial \widetilde{D}} \frac{\psi(\zeta)}{\zeta - \phi(\zeta)} d\zeta.$$

The holomorphic fixed point theorem (apply Montel's theorem) shows that $\phi : D \to D$ has a unique fixed point z^* where $|\phi'(z^*)| < 1$. The residue theorem concludes the proof. □

A straightforward computation shows that for all $f \in H^2(D)$,

$$\mathcal{L}_s^n(f)(z) = \sum_{\alpha \in \mathbb{N}_*^n} |\gamma'_\alpha|^s f(\gamma_\alpha z), \tag{6}$$

where we have set $\gamma_\alpha z = \gamma_{\alpha_1} \circ \ldots \circ \gamma_{\alpha_n} z$. Formula (6) (exercise) reveals that $\|\mathcal{L}_s^2\|_{H^2} \leq M_\sigma e^{\widetilde{C}|\mathrm{Im}(s)|}$, where $M_\sigma \to 0$ as $\sigma = \mathrm{Re}(s) \to +\infty$. This is enough (by the spectral radius formula) to conclude that the spectral radius $\rho(\mathcal{L}_s)$ tends to zero as $\sigma = \mathrm{Re}(s)$ tends to infinity and $|\mathrm{Im}(s)|$ stays bounded. Choose $R > 0$ large enough such that $\rho(\mathcal{L}_s) \leq \frac{1}{2}$ for all $s \in D(R, 1)$. We can write

$$d_\pm(s) = \exp\left(\sum_{i=0}^{+\infty} \log(1 \pm \lambda_i(\mathcal{L}_s))\right) = \exp\left(-\sum_{n=0}^{+\infty} \frac{(\mp 1)^n}{n} \sum_{i=0}^{+\infty} \lambda_i^n(\mathcal{L}_s)\right).$$

where $\log(1-z) = \sum_{n=1}^{+\infty} \frac{1}{n} z^n$, $|z| < 1$. Using Lidskii's Theorem, [2] we thus obtain

$$d_\pm(s) = \exp\left(-\sum_{n=0}^{+\infty} \frac{(\mp 1)^n}{n} \operatorname{Tr}(\mathcal{L}_s^n)\right), \ s \in D(R, 1).$$

Using (6) and estimates of singular values in the spirit of the proof of Proposition (3.2), one can check that

$$\|\mathcal{L}_s^n - \mathcal{L}_{s,N}^n\|_1 := \sum_{j=0}^{+\infty} \mu_j(\mathcal{L}_s^n - \mathcal{L}_{s,N}^n) \to 0$$

as $N \to +\infty$, where

$$\mathcal{L}_{s,N}^n(f)(z) = \sum_{\alpha \in \{1,\ldots,N\}^n} |\gamma'_\alpha(z)|^s f(\gamma_\alpha z).$$

By continuity of the trace (with respect to the Schatten norm $\|.\|_1$) and using Lemma (3.3), we get

$$\operatorname{Tr}(\mathcal{L}_s^n) = \lim_{N \to +\infty} \operatorname{Tr}(\mathcal{L}_{s,N}^n) = \sum_{T^n x = x} \frac{|(T^n)'(x)|^{-s}}{1 - |(T^n)'(x)|^{-1}}.$$

Therefore for all $s \in D(R, 1)$, with $R > 0$ large enough, we have shown that

$$d_+(s) d_-(s) = \exp\left(-\sum_{n=1}^{+\infty} \frac{1}{n} \left(\operatorname{Tr}(\mathcal{L}_s^n) + (-1)^n \operatorname{Tr}(\mathcal{L}_s^n)\right)\right)$$

[2]Lidskii's Theorem says that the trace of a compact trace class operator on a Hilbert space is indeed the sum of the eigenvalues, which is not a trivial fact and is false in the general Banach setting, see [6].

$$= \exp\left(-\sum_{m=1}^{+\infty} \frac{1}{m} \sum_{T^{2m}(x)=x} \frac{|(T^{2m})'(x)|^{-s}}{1-|(T^{2m})'(x)|^{-1}} \right) = Z(s).$$

By uniqueness of analytic continuation, we have reached the first part of Theorem 2.1. Moreover, for all $\frac{1}{2} < \mathrm{Re}(s) \leq 1$, we know that $Z(s) = 0$ iff 1 or -1 is an eigenvalue of $\mathcal{L}_s : H^2(D) \to H^2(D)$.

Rather than working on H^2, which would be a source of artificial problems caused by the complex values of $|\gamma'_\alpha|^\sigma$, we will study the behaviour of the iterates $\mathcal{L}_s^n$ when acting on the bigger space of C^1 functions defined on the interval I. We will pay a price to that by loosing compactness of transfer operators.

3.2 Transfer operators on $C^1(I)$

Let $C^1(I)$ be the Banach space of C^1 complex valued functions on I, endowed with the standard norm

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty.$$

We let the reader check that $\mathcal{L}_s : C^1(I) \to C^1(I)$ is well defined for all $\mathrm{Re}(s) > \frac{1}{2}$ and that for all $\sigma_0 > \frac{1}{2}$, there exists a constant $M_{\sigma_0} > 0$ such that for all $\mathrm{Re}(s) > \sigma_0$,

$$\|\mathcal{L}_s(f)\|_\infty \leq M_{\sigma_0} \|f\|_{C^1}, \text{and } \|(\mathcal{L}_s f)'\|_\infty \leq M_{\sigma_0}(1+|s|)\|f\|_{C^1}.$$

The key result of the spectral theory of $\mathcal{L}_s$ on $C^1(I)$ is the following.

Theorem 3.4 *For all real σ with $\sigma > \frac{1}{2}$, we have:*

- *The operator $\mathcal{L}_\sigma : C^1(I) \to C^1(I)$ has a maximal eigenvalue $\lambda_\sigma > 0$ i.e. the spectral radius $\rho(\mathcal{L}_\sigma) = \lambda_\sigma$. Moreover, λ_σ is algebraically simple and $\mathcal{L}_\sigma$ has no other eigenvalues on the circle or radius λ_σ.*
- *There exists a unique probability measure μ_σ on I such that $\mathcal{L}_\sigma^*(\mu_\sigma) = \lambda_\sigma \mu_\sigma$.*
- *The eigenspace $Ker(\mathcal{L}_\sigma - \lambda_\sigma)$ contains a unique positive eigenfunction h_σ such that $\int_I h_\sigma d\mu_\sigma = 1$, and $h_\sigma \mu_\sigma$ is T-invariant.*
- *The spectral radius of $\frac{1}{\lambda_\sigma}\mathcal{L}_\sigma - \mathbb{P}_\sigma$, where $\mathbb{P}_\sigma(f) = h_\sigma \int f d\mu_\sigma$, satisfies $\rho_\sigma = \rho(\frac{1}{\lambda_\sigma}\mathcal{L}_\sigma - \mathbb{P}_\sigma) < 1$.*
- *The essential spectral radius satisfies $\rho_{ess}(\mathcal{L}_\sigma) \leq \rho_\sigma \lambda_\sigma < \rho(\mathcal{L}_\sigma)$.*

This classical Ruelle-Perron-Frobenius Theorem can be proved in this setting by adapting directly the arguments in [2]. The key points are the topological mixing of T (implied by the strong Markov property $T(I_n) = I$ for all $n \geq 1$), the uniform hyperbolicity (see below), and the strong Renyi condition. An explicit estimale of the "spectral gap" (i.e the constant ρ_σ) can be derived by adapting Liverani's Birkhoff cones techniques [8].

Since $\sigma \mapsto \mathcal{L}_\sigma$ is continuous and λ_σ is simple, perturbation theory shows that $\lambda_\sigma, h_\sigma, \mathbb{P}_\sigma$ depend continuously on σ. For $\sigma = 1$, we have by uniqueness $\lambda_\sigma = 1$, $h_\sigma = h$ (the Gauss density), and μ_σ is the Lebesgue measure on I. All of our analysis will be in a small compact neighborhood of 1 (which will be schrinked several times according to our needs) so that we can always assume that h_σ and λ_σ are uniformly bounded from below and from above.

A first important step is to normalize the family $\mathcal{L}_s$ i.e. set

$$\widetilde{\mathcal{L}}_s(f) = \lambda_\sigma^{-1} h_\sigma^{-1} \mathcal{L}_s(h_\sigma f),$$

so that $\widetilde{\mathcal{L}}_\sigma(\mathbf{1}) = \mathbf{1}$. Another consequence of this normalization is that $\widetilde{\mathcal{L}}_\sigma^*(\nu_\sigma) = \nu_\sigma$, where $\nu_\sigma = h_\sigma \mu_\sigma$. A key fact in the following analysis is the so-called *Lasota-Yorke* inequality.

Lemma 3.5 *Let K be a small compact neighborhood of 1, there exists $C_K > 0$ such that for all $s = \sigma + it$ with $\sigma \in K$ and $t \in \mathbb{R}$, we have*

1. $\|\widetilde{\mathcal{L}}_s^n(f)\|_\infty \leq \|f\|_\infty$
2. $\|\widetilde{\mathcal{L}}_s^n(f)'\|_\infty \leq C_k \left\{|s|\|f\|_\infty + \left(\frac{1}{2}\right)^n \|f'\|_\infty\right\}.$

Proof. Inequality 1) is a direct consequence of the normalization. The proof of 2) is a straightforward estimate, but it is a good opportunity to recall some key facts about the Gauss map. According to the preceding §, we denote by $\gamma_\alpha = \gamma_{\alpha_1} \circ \ldots \circ \gamma_{\alpha_n}$ the inverse branches of T^n. They satisfy the following properties.

- *Uniform hyperbolicity.* For all $\alpha \in \mathbb{N}_*^n$, $\|(\gamma_\alpha)'\|_\infty \leq 2\left(\frac{1}{2}\right)^n$.
- *Bounded distortion* [3] *(or Renyi's condition).* There exists $M > 0$ such that for all $n \geq 1$, for $\alpha \in \mathbb{N}_*^n$,
$$\left\|\frac{(\gamma_\alpha)''}{(\gamma_\alpha)'}\right\|_\infty \leq M.$$
- *Bounded distortion for the third derivatives.* There exists $Q > 0$ such that for all $n \geq 1$, for $\alpha \in \mathbb{N}_*^n$,
$$\left\|\frac{(\gamma_\alpha)'''}{(\gamma_\alpha)'}\right\|_\infty \leq Q.$$

An important consequence of the bouded distortion property is that for all $n \geq 1$, for all $\alpha \in \mathbb{N}_*^n$, for all $x, y \in I$,

$$\frac{1}{L} \leq \frac{|\gamma_\alpha'(x)|}{|\gamma_\alpha'(y)|} \leq L,$$

where $L = e^M$. Let $f \in C^1(I)$, differentiating gives

$$\widetilde{\mathcal{L}}_s^n(f)' = -\frac{h_\sigma'}{h_\sigma}\widetilde{\mathcal{L}}_s^n(f) + \frac{1}{\lambda_\sigma^n} h_\sigma^{-1} \sum_{\alpha \in \mathbb{N}_*^n} r_\alpha',$$

[3] To prove these two distortion bounds, it is enough to check it for $n = 1$ and then to apply an induction argument together with the uniform hyperbolicity.

where $r_\alpha = |(\gamma_\alpha)'|^s(h_\sigma f)\circ\gamma_\alpha$. Since we have

$$|r'_\alpha| \leq |s||\gamma'_\alpha|^{\sigma-1}|\gamma''_\alpha|(|f|h_\sigma)\circ\gamma_\alpha + |\gamma'_\alpha|^{\sigma+1}(|h'_\sigma f| + |f'|h_\sigma)\circ\gamma_\alpha,$$

we get using hyperbolicity and Renyi's condition

$$|(\widetilde{\mathcal{L}}_s^n(f))'| \leq |s|M\widetilde{\mathcal{L}}_\sigma^n(|f|) + 2A_K\left(\tfrac{1}{2}\right)^n\widetilde{\mathcal{L}}_\sigma^n(|f|) + 2\left(\tfrac{1}{2}\right)^n\widetilde{\mathcal{L}}_\sigma^n(|f'|) + A_K\widetilde{\mathcal{L}}_\sigma^n(|f|),$$

where $A_K = \sup_{\sigma\in K}\|h'_\sigma\|_\infty\|h_\sigma^{-1}\|_\infty$ can be controlled on K by perturbation theory. The end follows by normalization. □

A first remark is the fact that if we set $\|f\|_{(t)} = \|f\|_\infty + \frac{1}{|t|}\|f'\|_\infty$ (for $t \neq 0$), we get that there exists $C_1 > 0$ such that for all σ in a compact neighborhood of 1 and for all $t \neq 0$,

$$\|\widetilde{\mathcal{L}}_s^n\|_{(t)} \leq C_1.$$

This observation suggests that we should be looking for contraction properties of the family with respect to the new norm $\|.\|_{(t)}$. What we want to prove is actually the following (called a Dolgopyat type estimate).

Theorem 3.6 *There exists $C > 0$ and $\beta > 0$ such that for all $|t|$ large and σ close to 1, we have*

$$\left\|\widetilde{\mathcal{L}}_s^{[C\log|t|]}\right\|_{(t)} \leq \frac{1}{|t|^\beta}. \tag{7}$$

In these notes, $[x]$ denote the smallest integer bigger than $x \in \mathbb{R}$.

Clearly for σ close to 1 and $|t|$ large, we have the spectral radius estimate (with respect to the C^1 norm)

$$\rho_{sp}(\mathcal{L}_s) \leq \lambda_\sigma \lim_{n\to+\infty}\left\|\widetilde{\mathcal{L}}_s^{n[C\log|t|]}\right\|_{(t)}^{\frac{1}{n[C\log|t|]}} \leq \lambda_\sigma e^{-\beta\frac{\log|t|}{[C_1\log|t|]}} \leq \rho_0 < 1,$$

as long as we take σ close enough to 1. Thus this Dolgopyat estimate clearly implies the zero free strip claimed in Theorem 2.1 for large imaginary parts 2.1, simply because a $H^2(D)$-eigenvalue of modulus one would produce the same C^1-eigenvalue and contradict our contraction estimate. The local analysis of $Z(s)$ close to $s = 1$ is postponed to the end of these notes.

3.3 Reduction to an L^2-estimate

The next step in Dolgopyat's train of ideas is to remark that the proof of the crucial estimate 7 can be reduced to an L^2-estimate. To this end we need the following Lemma.

Lemma 3.7 *For all $\sigma \in K$, where $1 \in K$ and K is small enough, there exists $A_\sigma > 0$ and $C'_K > 0$ with $A_\sigma \to 1$ as $\sigma \to 1$ such that*

$$\|\widetilde{\mathcal{L}}_\sigma^n(f)\|_\infty^2 \leq C_K A_\sigma^n \|\widetilde{\mathcal{L}}_1^n(|f|^2)\|_\infty.$$

Proof. Recall that

$$|\widetilde{\mathcal{L}}_\sigma^n(f)|^2 \le \frac{h_\sigma^{-2}}{\lambda_\sigma^{2n}}\left(\sum_{\alpha\in\mathbb{N}_*^n}|\gamma_\alpha'|^\sigma|h_\sigma f|\circ\gamma_\alpha\right)^2.$$

Assuming that K is small enough such that $\sigma>\frac{3}{4}$ and using Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}|\widetilde{\mathcal{L}}_\sigma^n(f)|^2 &\le \frac{h_\sigma^{-2}}{\lambda_\sigma^{2n}}\left(\sum_{\alpha\in\mathbb{N}_*^n}|\gamma_\alpha'|^{2\sigma-1}\right)\left(\sum_{\alpha\in\mathbb{N}_*^n}|\gamma_\alpha'||fh_\sigma|^2\circ\gamma_\alpha\right)\\ &\le \frac{B_K}{\lambda_\sigma^{2n}}\left(\sum_{\alpha\in\mathbb{N}_*^n}|\gamma_\alpha'|^{2\sigma-1}\right)\widetilde{\mathcal{L}}_1^n(|f|^2),\end{aligned}$$

where B_K is a suitable constant depending only on K. Using the normalization trick again, it is easy to see that

$$\sum_{\alpha\in\mathbb{N}_*^n}|\gamma_\alpha'|^{2\sigma-1}\le\sup_K\|h_{2\sigma-1}\|_\infty\|h_{2\sigma-1}^{-1}\|_\infty\lambda_{2\sigma-1}^n.$$

We have obtained for all $\sigma\in K$,

$$\|\mathcal{L}_\sigma^n(f)\|_\infty^2\le C_K\left(\frac{\lambda_{2\sigma-1}}{\lambda_\sigma^2}\right)^n\|\widetilde{\mathcal{L}}_1^n(|f|^2)\|_\infty.$$

The proof is done. □

Dolgopyat's L^2-type estimate is as follows.

Theorem 3.8 *There exist $\widetilde{\beta},\widetilde{C}>0$ such that for all $|t|$ large and σ close enough to 1, we have*

$$\int_I\left|\widetilde{\mathcal{L}}_s^{[\widetilde{C}\log|t|]}(f)\right|^2dx\le\frac{\|f\|_{(t)}^2}{|t|^{\widetilde{\beta}}}.$$

Let us show that Theorem 3.8 is enough to get the $\|.\|_{(t)}$-estimate. Take $C>\widetilde{C}$, where $\widetilde{C}$ is as in Theorem 3.8. Set $n_0(t)=[\widetilde{C}\log|t|]$, $n(t)=[C\log|t|]$. By positivity and using Lemma 3.7, we can write

$$\begin{aligned}\|\widetilde{\mathcal{L}}_s^{n(t)}(f)\|_\infty^2 &\le \left\|\widetilde{\mathcal{L}}_\sigma^{n(t)-n_0(t)}\left(\left|\widetilde{\mathcal{L}}_s^{n_0(t)}(f)\right|\right)\right\|_\infty^2\\ &\le C_KA_\sigma^{n(t)-n_0(t)}\left\|\widetilde{\mathcal{L}}_1^{n(t)-n_0(t)}\left(\left|\widetilde{\mathcal{L}}_s^{n_0(t)}(f)\right|^2\right)\right\|_\infty.\end{aligned}$$

Recall that because of the "gap" in the spectrum of $\widetilde{\mathcal{L}}_1$, we have for all $g\in C^1(I)$, as $n\to+\infty$,

$$\widetilde{\mathcal{L}}_1^n(g)=\int_I ghdx+O\left(\rho_0^n\|g\|_{C^1}\right).$$

Applying this asymptotic to $g = \left|\widetilde{\mathcal{L}}_s^{n_0(t)}(f)\right|^2$, and using the Lasota-Yorke inequality we get

$$\|\widetilde{\mathcal{L}}_s^{n(t)}(f)\|_\infty^2 \leq \widetilde{C_K} A_\sigma^{n(t)-n_0(t)} \left(\int_I \left|\widetilde{\mathcal{L}}_s^{n_0(t)}(f)\right|^2 dx + \rho_0^{n(t)-n_0(t)} |t| \|f\|_{(t)}^2 \right)$$

$$\leq \widetilde{C_K} A_\sigma^{n(t)-n_0(t)} \|f\|_{(t)}^2 \left(\frac{1}{|t|^{\widetilde{\beta}}} + \rho_0^{n(t)-n_0(t)} |t| \right).$$

To conclude, fix C large enough so that $\rho_0^{n(t)-n_0(t)}|t| = O\left(\frac{1}{|t|^{\widetilde{\beta}}}\right)$ and then make sure that σ is close enough to 1 so that for example $A_\sigma^{n(t)-n_0(t)} = O\left(|t|^{\widetilde{\beta}/2}\right)$ and the proof is done at least for the $\|.\|_\infty$ estimate. The case of the derivative proceeds along the same ideas and successive applications of the Lasota-Yorke inequality. □

4 Oscillatory integrals and L^2-contraction

In this section, we give the main ideas of the proof of Theorem 3.8. Perhaps it is better to give an overview of the ideas involved before we give a more detailed proof. For all $\alpha, \beta \in \mathbb{N}_*^n$, we set

$$\Psi_{\alpha,\beta}(x) = \log|\gamma'_\alpha(x)| - \log|\gamma'_\beta(x)|;$$

$$R_{\alpha,\beta}^\sigma(x) = h_\sigma^{-2}(x)|\gamma'_\alpha(x)|^\sigma |\gamma'_\beta(x)|^\sigma (h_\sigma f) \circ \gamma_\alpha(x) (h_\sigma \overline{f}) \circ \gamma_\beta(x);$$

$$\Delta(\alpha,\beta) = \inf_{x\in I} |\Psi'_{\alpha,\beta}(x)| = \inf_{x\in I} \left| \frac{\gamma''_\alpha(x)}{\gamma'_\alpha(x)} - \frac{\gamma''_\beta(x)}{\gamma'_\beta(x)} \right|.$$

We then have

$$\int_I \left|\widetilde{\mathcal{L}}_s^n(f)\right|^2 dx = \frac{1}{\lambda_\sigma^{2n}} \sum_{\alpha,\beta\in\mathbb{N}_*^n} \int_I e^{it\Psi_{\alpha,\beta}} R_{\alpha,\beta}^\sigma dx$$

$$= \sum_{\Delta(\alpha,\beta)\leq\epsilon} \frac{1}{\lambda_\sigma^{2n}} \int_I e^{it\Psi_{\alpha,\beta}} R_{\alpha,\beta}^\sigma dx + \sum_{\Delta(\alpha,\beta)>\epsilon} \frac{1}{\lambda_\sigma^{2n}} \int_I e^{it\Psi_{\alpha,\beta}} R_{\alpha,\beta}^\sigma dx = \mathcal{I}^-(\epsilon,n,t) + \mathcal{I}^+(\epsilon,n,t).$$

The first sum (called sum over closed pairs) $\mathcal{I}^-(\epsilon,n,t)$ will be treated using an ad hoc non-integrability argument (called UNI) to show that this sum is of size $\varepsilon \sim \frac{1}{|t|^\delta}$ if we take $n \sim C\log|t|$, for a good choice of δ and C. This will also involve distortion estimates for the measures ν_σ.
The second sum will be treated using a "non-stationnary phase" argument, since the "phases" $\Psi_{\alpha,\beta}$ of the above oscillatory integrals satisfy precisely $|\Psi'_{\alpha,\beta}(x)| \geq \epsilon$.

4.1 Dealing with the close terms and condition UNI

The so called ad hoc "uniform-non-integrability" condition defined by Baladi-Vallée in [3] is reminiscent of the non-triviality of the temporal distance function and its consequences for mixing Anosov flows (see [5]). Here is what we will precisely need for our purpose. Given $\alpha \in \mathbb{N}_*^n$, we denote by $J(\alpha, \epsilon)$ the union of intervals

$$J(\alpha, \epsilon) = \bigcup_{\Delta(\alpha,\beta)\leq\epsilon} \gamma_\beta(I).$$

Notice that all the $\gamma_\beta(I)$ have disjoint interiors. Given a subset J of I, $|J|$ will simply denote its Lebesgue measure.

Proposition 4.1 *The Gauss dynamical system satisfies a UNI condition i.e. there exists a uniform constant $\widetilde{M}$ such that for all $0 < \eta < 1$, for all $n \geq 1$, for all $\alpha \in \mathbb{N}_*^n$, we have*

$$\left|J\left(\alpha, \left(\tfrac{1}{2}\right)^{\eta n}\right)\right| \leq \widetilde{M}\left(\tfrac{1}{2}\right)^{\eta n}.$$

The proof is postponed to §5. The following Lemma is required to use the above estimate.

Lemma 4.2 *Let (once again) K be a small compact neighborhood of 1, and we assume that $\sigma \in K$. There exists a constant $\widetilde{C}_K > 0$ such that for all $\alpha \in \mathbb{N}_*^n$,*

$$\widetilde{C}_K^{-1}\frac{\|\gamma_\alpha'\|_\infty^\sigma}{\lambda_\sigma^n} \leq \nu_\sigma(\gamma_\alpha(I)) \leq \widetilde{C}_K\frac{\|\gamma_\alpha'\|_\infty^\sigma}{\lambda_\sigma^n}.$$

Moreover, for all subset $\mathcal{E} \subset \mathbb{N}_*^n$, then set $J = \cup_{\alpha\in\mathcal{E}}\gamma_\alpha(I)$, we have

$$\nu_\sigma(J) \leq B_K A_\sigma^{2n}|J|^{\frac{1}{2}},$$

where A_σ is the same constant as in Lemma 3.7.

Sketchy proof. If you think of $\gamma_\alpha(I)$ as a cylinder set, then the first estimate is nothing but the Gibbs distortion estimate for the measure ν_σ of cylinder sets. Remark that the identity

$$\int_I f d\nu_\sigma = \int_I \widetilde{\mathcal{L}}_\sigma^n(f) d\nu_\sigma$$

holds actually for all $f \in L^1_{\nu_\sigma}(I)$. Taking $f \equiv \chi_{\gamma_\alpha(I)}$, we get

$$\nu_\sigma(\gamma_\alpha(I)) = \frac{1}{\lambda_\sigma^n}\int_I h_\sigma^{-1}|\gamma_\alpha'|^\sigma h_\sigma \circ \gamma_\alpha d\nu_\sigma.$$

The bounded distortion property clearly ends the proof. The second estimate follows similar ideas and uses the Cauchy-Schwarz inequality as in the proof of Lemma 3.7. □

Let us go back to the estimate of the sum over the close pairs $\mathcal{I}^-(n, \epsilon, t)$. There clearly exists a constant $M_K > 0$ such that

$$|\mathcal{I}^-(n,\epsilon,t)| \leq M_K\frac{\|f\|_\infty^2}{\lambda_\sigma^{2n}}\sum_{\Delta(\alpha,\beta)\leq\epsilon}\int_I |\gamma_\alpha'|^\sigma|\gamma_\beta'|^\sigma dx$$

$$\leq \widetilde{M}_K \|f\|_\infty^2 \sum_{\Delta(\alpha,\beta)\leq\epsilon} \frac{\|\gamma'_\alpha\|_\infty^\sigma}{\lambda_\sigma^n} \frac{\|\gamma'_\beta\|_\infty^\sigma}{\lambda_\sigma^n}.$$

Using the Gibbs lower bound from Lemma 4.2, we have

$$|\mathcal{I}^-(n,\epsilon,t)| \leq M'_K \|f\|_\infty^2 \sum_{\Delta(\alpha,\beta)\leq\epsilon} \nu_\sigma(\gamma_\alpha(I))\nu_\sigma(\gamma_\beta(I)) = M'_K \|f\|_\infty^2 \sum_{\alpha\in\mathbb{N}_*^n} \nu_\sigma(\gamma_\alpha(I))\nu_\sigma(J(\alpha,\epsilon)).$$

Using the second estimate from Lemma 4.2 together with the UNI consequence, we get

$$|\mathcal{I}^-(n,\epsilon,t)| \leq M''_K \|f\|_\infty^2 A_\sigma^{2n} \sum_{\alpha\in\mathbb{N}_*^n} \nu_\sigma(\gamma_\alpha(I)) \left(\tfrac{1}{2}\right)^{\frac{\eta n}{2}} \left(\tfrac{1}{2}\right)^{\frac{\eta n}{2}} = M''_K \|f\|_\infty^2 A_\sigma^{2n} \left(\tfrac{1}{2}\right)^{\frac{\eta n}{2}},$$

with $\epsilon = \left(\frac{1}{2}\right)^{\eta n}$.

4.2 Decay of Oscillatory integrals

The main tool needed to deal with the oscillatory integrals appearing in $\mathcal{I}^+(n,\epsilon,t)$ is the following Lemma (called Van der Corput Lemma).

Lemma 4.3 *Let $\Phi \in C^2(I)$ with $\inf_I |\Phi'(x)| \geq \Delta > 0$ and $\|\Phi''\|_\infty \leq Q$. Let $r \in C^1(I)$, then for all $t \neq 0$, we have*

$$\left|\int_I e^{it\Phi(x)} r(x)dx\right| \leq \frac{\|r\|_{C^1}}{|t|}\left(\frac{3}{\Delta} + \frac{Q}{\Delta^2}\right).$$

Proof. Just integrate by parts. □

Let us check a few things before we apply Lemma 4.3. By the bounded distortion of third derivatives, it is straightforward to check that there exists $\widetilde{Q} > 0$ such that $\|\Psi''_{\alpha,\beta}\|_\infty \leq \widetilde{Q}$, uniformly in α,β and n. We will also use another Lasota-Yorke type estimate.

Lemma 4.4 *There exists $M_K > 0$ such that for all $\sigma \in K$,*

$$\|R^\sigma_{\alpha,\beta}\|_{C^1} \leq M_K \|\gamma'_\alpha\|_\infty^\sigma \|\gamma'_\beta\|_\infty^\sigma \|f\|_{(t)}^2 \left(1 + \left(\tfrac{1}{2}\right)^n |t|\right).$$

The proof follows similar ideas as in our proof of the Lasota-Yorke estimate.

Using all the above remarks and applying Lemma 4.3, we obtain

$$|\mathcal{I}^+(n,\epsilon,t)| \leq \|f\|_{(t)}^2 \frac{M_K}{\lambda_\sigma^{2n}} \sum_{\Delta(\alpha,\beta)>\varepsilon} \|\gamma'_\alpha\|^\sigma \|\gamma'_\beta\|_\infty^\sigma \mathcal{A}(t) \leq \widetilde{M}_K \|f\|_{(t)}^2 \mathcal{A}(t),$$

by using the bounded distortion property and the normalization, and where $\mathcal{A}(t)$ is equal to

$$\mathcal{A}(t) = \frac{\left(1 + \left(\frac{1}{2}\right)^n |t|\right)}{|t|}\left(\frac{3}{\epsilon} + \frac{\widetilde{Q}}{\epsilon^2}\right).$$

Now we choose $\epsilon = \left(\frac{1}{2}\right)^{\eta n}$ with $0 < \eta < 1$ and $n = [\widetilde{C} \log |t|]$. The "dangerous" terms in $\mathcal{A}(t)$ are precisely

$$\frac{3}{|t|\epsilon}, \ \frac{\widetilde{Q}}{|t|\epsilon^2}, \text{ and } \left(\tfrac{1}{2}\right)^n \frac{\widetilde{Q}}{\epsilon^2}.$$

For the last one, we just have to make sure that $0 < \eta < \frac{1}{2}$ and so we fix $\eta = \frac{1}{4}$. For the first two terms,

$$\frac{3}{|t|\epsilon} + \frac{\widetilde{Q}}{|t|\epsilon^2} = O\left(|t|^{\frac{\log 2}{2}\widetilde{C} - 1}\right),$$

and by taking $\widetilde{C}$ small enough we get a polynomial decay. To conclude the proof of Theorem 3.8, we just have to go back to the estimate of the close pairs $\mathcal{I}^-(n, \epsilon, t)$ with the above choice of $\epsilon(t)$, $n(t)$ and take σ close enough to 1.

5 Checking UNI

Let us prove Proposition 4.1. This the only place where we shall use some of the group properties of $GL_2(\mathbb{R})$. First recall that the inverse branches of T are the $\gamma_k(z) = \frac{1}{k+z}$, corresponding to $GL_2(\mathbb{R})$ matrices with determinant -1. As a consequence, all the higher order inverse branches can be written as

$$\gamma_\alpha(z) = \frac{a_\alpha z + b_\alpha}{c_\alpha z + d_\alpha},$$

with $a_\alpha d_\alpha - c_\alpha b_\alpha \in \{-1, +1\}$. The transposition will prove to be an important trick in the following. We can observe that given $\alpha \in \mathbb{N}_*^n$,

$$\gamma_\alpha^*(z) = \frac{a_\alpha z + c_\alpha}{b_\alpha z + d_\alpha}$$

is still an inverse branch corresponding to the reverse word [4] $\overline{\alpha} = \alpha_{i_n} \circ \ \ldots \circ \alpha_{i_1}$. We have to remark in addition that $\gamma'_\alpha(0) = \gamma'_{\overline{\alpha}}(0)$, for all α. This has the nice consequence that

$$\frac{1}{L^2} \leq \frac{|\gamma_\alpha(I)|}{|\gamma_{\overline{\alpha}}(I)|} = \frac{|\gamma'_{\overline{\alpha}}(0)||\gamma_\alpha(I)|}{|\gamma_{\overline{\alpha}}(I)||\gamma'_\alpha(0)|} \leq L^2$$

by bounded distortion property.

Let α be a fixed word of length n. Let $\beta \in \mathbb{N}_*^n$ be such that $\Delta(\alpha, \beta) \leq \epsilon$. We get

$$\epsilon \geq \inf_{x \in I} \left| \frac{\gamma''_\alpha(x)}{\gamma'_\alpha(x)} - \frac{\gamma''_\beta(x)}{\gamma'_\beta(x)} \right| = \inf_{x \in I} \frac{2|c_\alpha d_\beta - c_\beta d_\alpha|}{|(c_\alpha x + d_\alpha)(c_\beta x + d_\beta)|}.$$

Using the bounded distortion property (once again) we can write

$$|(c_\alpha x + d_\alpha)(c_\beta x + d_\beta)|^{-1} = |\gamma'_\alpha(x)|^{\frac{1}{2}} |\gamma'_\beta(x)|^{\frac{1}{2}} \geq L^{-1} \sqrt{|\gamma'_\alpha(0)||\gamma'_\beta(0)} = \frac{|d_\alpha d_\beta|}{L},$$

[4] This is of course due to the fact that $\gamma_k^* = \gamma_k$ for all k.

and thus

$$\epsilon \geq \frac{2}{L}\left|\frac{c_\alpha}{d_\alpha} - \frac{c_\beta}{d_\beta}\right| = \frac{2}{L}\left|\gamma_{\overline{\alpha}}(0) - \gamma_{\overline{\beta}}(0)\right|.$$

We now return to $J(\alpha, \epsilon)$ by remarking that

$$|J(\alpha,\epsilon)| = \sum_{\Delta(\alpha,\beta)\leq\epsilon} |\gamma_\beta(I)| \leq L^2 \left|\bigcup_{\Delta(\alpha,\beta)\leq\epsilon} \gamma_{\overline{\beta}}(I)\right| \leq 2L^2 \left(\frac{L}{2}\epsilon + \tfrac{1}{2}\left(\tfrac{1}{2}\right)^n\right),$$

and the proof is done if we take ϵ in the scale $\left(\frac{1}{2}\right)^{n\eta}$. □.

6 Zeros of $Z(s)$ on the line $\{\mathrm{Re}(s) = 1\}$

As remarked before, the conclusion of Theorem 7 does not give any information at all on what happens for small imaginary parts of s with real part of s close to one. To this end we will need the following important remark.

Lemma 6.1 *The following properties are equivalent.*

- *There exists $\beta \in \mathbb{R}$, $t_0 \neq 0$ such that $e^{i\beta}$ belongs to the C^1-spectrum of $\mathcal{L}_{1+it_0}$.*
- *There exists $f \in C^1(I)$, $|f| \equiv 1$ such that for all $n \geq 1$, for all $\alpha \in \mathbb{N}_*^n$,*

$$|\gamma_\alpha'(x)|^{it_0} f \circ \gamma_\alpha = e^{in\beta} f.$$

The proof of this fact follows word for word the classical arguments of the book [15], Proposition 6.2, and we skip it (or leave it as an exercise).

Assume now that $Z(s)$ has a zero on the line $\{\mathrm{Re}(s) = 1\}$ for $\mathrm{Im}(s) \neq 0$. There exists therefore $t_0 \neq 0$ such that $\varepsilon = \pm 1$ is an eigenvalue of $\mathcal{L}_{1+it_0} : C^1(I) \to C^1(I)$. By applying the preceding Lemma, we know that there exists $f_0 \in C^1(I)$ such that for all $j \in \mathbb{N}_*$,

$$(8) \qquad |\gamma_j'|^{it_0} f_0 \circ \gamma_j = \varepsilon f_0.$$

For all $p \geq 1$ set $t_p = pt_0$, $g_{(p)} = hf_0^p$. It is now straightforward to check using (8) that

$$\mathcal{L}_{1+it_p}(g_{(p)}) = \varepsilon^p g_{(p)},$$

which clearly contradicts (for p large enough) the fact that the family $\mathcal{L}_s$ is a strict contraction on the line $\{\mathrm{Re}(s) = 1\}$ for $|\mathrm{Im}(s)|$ large.

As a conclusion, $Z(s)$ can only vanish at $s = 1$ on the vertical line $\{\mathrm{Re}(s) = 1\}$, and this zero is simple by the Ruelle-Perron-Frobenius Theorem. The proof of Theorem 2.1 is now complete.

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