CHARACTERISTIC FUNCTIONS AS BOUNDED MULTIPLIERS ON ANISOTROPIC SPACES

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Abstract. We show that characteristic functions of domains with piecewise C^3 boundaries transversal to suitable cones are bounded multipliers on a recently introduced scale $\mathcal{U}_p^{\mathbf{C},t,s}$ of anisotropic Banach spaces, under the conditions $-1+1/p<s<-t<0$, with $p\in(1,\infty)$.

1. Introduction

A (not necessarily smooth) function $g : M \to \mathbb{C}$ is called a bounded multiplier on a Banach space $\mathcal{B}$ of distributions on a d-dimensional Riemann manifold M if there exists $C_g < \infty$ so that for all $\varphi \in \mathcal{B}$ the product $g\varphi$ is a well-defined element of $\mathcal{B}$ and, in addition, $\|g \cdot \varphi\| \le C_g \|\varphi\|$, where $\|\cdot\|$ is the norm of $\mathcal{B}$. One interesting special case is when g is the characteristic function 1_Λ of an open domain $\Lambda \subset M$: Half a century ago, Strichartz [16] proved that for any $d \ge 1$, if $M = \mathbb{R}^d$ and $\mathcal{B}$ is the Sobolev[1] space $H^t_p(\mathbb{R}^d)$ for $p \in (1,\infty)$ and $t \in \mathbb{R}$, then the characteristic function 1_Λ of a half-space is a bounded multiplier on $H^t_p(\mathbb{R}^d)$ if and only if $-1+1/p < t < 1/p$.

In the present work, we consider a newly introduced scale $\mathcal{U}_p^{\mathbf{C},t,s}$ of spaces of anisotropic distributions $\mathcal{B}$ on a manifold M, adapted to smooth hyperbolic dynamics, and we prove the bounded multiplier property for characteristic functions of suitable subsets $\Lambda \subset M$.

Fix $r > 1$, and suppose from now on that M is connected and compact. The simplest hyperbolic maps on M are transitive C^r Anosov diffeomorphisms T. The Ruelle transfer operator associated to such a map T and to a C^{r-1} function h on M (for example, $h = 1/|\det DT|$) is defined on C^{r-1} functions φ by

$$\mathcal{L}_h\varphi = (h\cdot\varphi)\circ T^{-1}\,. \tag{1}$$

Blank–Keller–Liverani [7] were the first to study the spectrum of such transfer operators on a suitable Banach space $\mathcal{B}$ of *anisotropic distributions* and to exploit this spectrum to get information on the Sinai–Ruelle–Bowen (physical) measure: The spectral radius of $\mathcal{L}_{1/|\det DT|}$ is equal to 1, there is a simple positive maximal eigenvalue, whose eigenvector is in fact a Radon measure μ, which is just the physical measure of T. Finally, the rest of the spectrum lies in a disc of radius strictly smaller

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[1] Recall that $\|\varphi\|_{H^t_p} = \|(\mathrm{id}+\Delta)^{t/2}\varphi\|_{L_p} = \|\mathbb{F}^{-1}(1+|\xi|^2)^{t/2}\mathbb{F}\varphi\|_{L_p}$, with Δ the Laplacian and $\mathbb{F}$ the Fourier transform.

than 1, which implies exponential decay of correlations $\int \varphi(\psi \circ T^n) \mathrm{d}\mu - \int \varphi \mathrm{d}\mu \int \psi \mathrm{d}\mu$ for Hölder observables ψ and φ as $n \to \infty$. (The first step in this analysis is to show the bound $\rho_{ess} < 1$ for the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$ on $\mathcal{B}$.)

Some natural dynamical systems originating from physics (such as Sinai billiards) enjoy uniform hyperbolicity, but are only *piecewise smooth.* Letting $M = \cup_i \Lambda_i$ be a (finite or countable) partition of M into domains where the dynamics is smooth, one can often reduce to the smooth hyperbolic case via the decomposition

$$\mathcal{L}_{1/|\det DT|}\varphi = \sum_i \frac{(1_{\Lambda_i} \cdot \varphi)}{|\det DT|} \circ T^{-1} \, . \tag{2}$$

This motivates studying bounded multiplier properties of characteristic functions.

In the 15 years since the publication of [7], dynamicists and semi-classical analysts have created a rich jungle of spaces of anisotropic distributions for hyperbolic dynamics (here, $d = d_s + d_u$ with $d_s \geq 1$ and $d_u \geq 1$). These spaces are usually scaled by two real numbers $v < 0$ and $t > 0$. Leaving aside the classical foliated anisotropic spaces of Triebel [17] (which are limited to "bunched" cases [4], and seem to fail for Sinai billiards), they come in two groups:

In the first, "geometric" group [7, 13], a class of d_s-dimensional "admissible" leaves Γ (having tangent vectors in stable cones for T) is introduced, and the norm of φ is obtained by fixing an integer $t \geq 1$ and taking a supremum, over all admissible leaves Γ, of the partial derivatives of φ of total order at most t, integrated against $C^{|v|}$ test functions on Γ. Modifications of this space, for suitable noninteger $0 < t < 1$ and $|v| < 1$, were introduced to work with piecewise smooth systems [8, 9] (only in dimension two). A version of these spaces for piecewise smooth hyperbolic flows in dimension three recently allowed to prove exponential mixing for Sinai billiard flows [3].

In the[2] second, "microlocal," group [5], a third parameter $p \in [1, \infty)$ is present, and the norm (in charts) of φ is the L_p average of $\Delta^{t,v}(\varphi)$, where the operator $\Delta^{t,v}$ interpolates smoothly between $(\mathrm{id} + \Delta)^{v/2}$ in *stable cones* in the cotangent space, and $(\mathrm{id} + \Delta)^{t/2}$ in *unstable cones* in the cotangent space. Powerful tools are available for this microlocal approach, allowing in particular to study the dynamical determinants and zeta functions[3] much more efficiently than for the geometric spaces. Variants of these microlocal spaces (usually in the Hilbert setting $p = 2$) have also been studied by the semi-classical community, starting from [10]. However, S. Gouëzel pointed out over ten years ago that *characteristic functions cannot be bounded multipliers* on spaces defined by conical wave front sets as in [5] or [10] (Gouëzel's counterexamples are presented in [2, App. 1]). The microlocal spaces of the type defined in [5, 6] or [10] thus appear *unsuitable* to study piecewise smooth dynamics.

In order to overcome this limitation of the microlocal approach, we recently introduced [2] a new scale $\mathcal{U}_p^{\mathbf{C},t,s}$ of microlocal anisotropic spaces, obtained by mimicking the construction of the geometric spaces of Gouëzel–Liverani [13] (with, morally, $s = v + t$). We showed in [2] the expected bound on the essential spectral radius of the transfer operator of a C^r Anosov diffeomorphism acting on $\mathcal{U}_p^{\mathbf{C},t,s}$ (if $t-(r-1) < s < -t < 0$), and we conjectured that characteristic functions of domains

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[2] This group could also be called pseudodifferential, or semi-classical, or Sobolev.

[3] The "kneading determinants" of by Milnor and Thurston from the 70's are revisited as "nuclear decompositions" in [1].

with piecewise smooth boundaries everywhere transversal to the stable cones should be bounded multipliers on $\mathcal{U}_p^{\mathbf{C},t,s}$, if s and t satisfy additional constraints depending on $p \in (0,1)$. *The main result*[4] *of the present paper, Theorem 3.1, implies this bounded multiplier property if* $\max\{t-(r-1), -1+1/p\} < s < -t < 0$.

This result opens the door to the spectral study, not only of hyperbolic maps with discontinuities in arbitrary dimensions, but also (using nuclear power decompositions [1, 2]) of the hitherto unexplored topic of the dynamical zeta functions of piecewise expanding and piecewise hyperbolic maps in any dimensions. This should include billiards maps [9] and their dynamical zeta functions in arbitrary dimensions. We also hope that the spaces $\mathcal{U}_p^{\mathbf{C},t,s}$ will allow to extend the scope of the renewal methods introduced in [14] to dynamical systems with infinite invariant measures. (The induction procedure used there introduces discontinuities in the dynamics.) Finally, it goes without saying that suitable version of the spaces $\mathcal{U}_p^{\mathbf{C},t,s}$ will be useful to study flows.

F. Faure and M. Tsujii [11] recently introduced new microlocal anisotropic spaces, for which the wave front set is more narrowly constrained than for previous microlocal spaces used for hyperbolic dynamics. It would be interesting to check whether characteristic functions are bounded multipliers on these new spaces. (Note however that, contrary to the spaces $\mathcal{U}_p^{\mathbf{C},t,s}$ or the spaces of [10, 5, 13, 9], spaces of [11] do not appear suitable for perturbations of hyperbolic maps or flows.)

2. $\mathcal{U}_p^{\mathbf{C},t,s}$: A Fourier version of the Demers–Gouëzel–Liverani spaces

We recall the "microlocal" spaces $\mathcal{U}_p^{\mathbf{C},t,s}$, for real numbers s and t (in the application, $s < -t < 0$) and $1 \le p \le \infty$, introduced in [2].

2.1. Basic notation.

Suppose that $d = d_s + d_u$ with $d_u \ge 1$ and $d_s \ge 1$. For $\ell \ge 1$ and $x \in \mathbb{R}^\ell$, $\xi \in \mathbb{R}^\ell$, we write $x\xi$ for the scalar product of x and ξ. The Fourier transform $\mathbb{F}$ and its inverse $\mathbb{F}^{-1}$ are defined on rapidly decreasing functions φ, ψ by

$$\mathbb{F}(\varphi)(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi}\varphi(x)\mathrm{d}x\,, \quad \xi \in \mathbb{R}^d\,, \tag{3}$$

$$\mathbb{F}^{-1}(\psi)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi}\psi(\xi)\mathrm{d}\xi\,, \quad x \in \mathbb{R}^d\,, \tag{4}$$

and extended to the space of temperate distributions φ, ψ as usual [15]. For suitable functions $a : \mathbb{R}^d \to \mathbb{R}$ (called "symbols", note that, in this paper, a depends only on ξ, while more general symbols may depend on x and ξ), we define an operator a^{Op} acting on suitable $\varphi : \mathbb{R}^d \to \mathbb{C}$, by

$$a^{Op}(\varphi) = \mathbb{F}^{-1}(a(\cdot)\cdot\mathbb{F}(\varphi)) = (\mathbb{F}^{-1}a) * \varphi\,. \tag{5}$$

Note that $\|a^{Op}\varphi\|_{L_p} \le \|\mathbb{F}^{-1}a\|_1 \|\varphi\|_{L_p}$ for each $1 \le p \le \infty$, by Young's inequality in L_p.

Fix a C^∞ function $\chi : \mathbb{R}_+ \to [0,1]$ with $\chi(x) = 1$ for $x \le 1$, and $\chi(x) = 0$ for $x \ge 2$. For $D \ge 1$, define $\psi_n^{(D)} : \mathbb{R}^D \to [0,1]$ for $n \in \mathbb{Z}_+$, by $\psi_0^{(D)}(\xi) = \chi(\|\xi\|)$, and

$$\psi_n^{(D)}(\xi) = \chi(2^{-n}\|\xi\|) - \chi(2^{-n+1}\|\xi\|)\,, \quad n \ge 1\,. \tag{6}$$

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[4] See Remark 2.5.

We set $\psi_n = \psi_n^{(d)}$. Note that

$$\mathbb{F}^{-1}\psi_n^{(D)} = 2^{D(n-1)}\mathbb{F}^{-1}\psi_1^{(D)}(2^{n-1}x) \text{ and } (\sum_{k\le n}\mathbb{F}^{-1}\psi_k^{(D)})(x) = 2^{Dn}\mathbb{F}^{-1}\chi(2^n x)\,,$$

so that, for any D,

$$\sup_n \|\mathbb{F}^{-1}\psi_n^{(D)}\|_{L_1(\mathbb{R}^D)} < \infty\,, \qquad \sup_n \|\sum_{k\le n}\mathbb{F}^{-1}\psi_k^{(D)}\|_{L_1(\mathbb{R}^D)} < \infty\,, \tag{7}$$

and for every multi-index β, there exists a constant C_β such that

$$\|\partial^\beta \psi_n^{(D)}\|_{L_\infty} \le C_\beta 2^{-n|\beta|}\,, \qquad \forall\, n\ge 0\,. \tag{8}$$

We shall work with the following operators $(\psi_n^{(D)})^{Op}$ (putting $\psi_n^{Op} = (\psi_n^{(d)})^{Op}$):

$$(\psi_n^{(D)})^{Op}(\varphi)(x) = \frac{1}{(2\pi)^d}\int_{y\in\mathbb{R}^d}\int_{\eta\in\mathbb{R}^d} e^{i(x-y)\eta}\psi_n^{(D)}(\eta)\varphi(y)\mathrm{d}\eta\mathrm{d}y\,.$$

Note finally the following almost orthogonality property

$$(\psi_n^{(D)})^{Op}\circ(\psi_m^{(D)})^{Op} \equiv 0 \qquad \text{if } |n-m|\ge 2\,. \tag{9}$$

2.2. The local anisotropic spaces $\mathcal{U}_p^{\mathbf{C}_+,t,s}(K)$ for compact $K\subset\mathbb{R}^d$.

Recall that a cone is a subset of $\mathbb{R}^d$ invariant under scalar multiplication. For two cones $\mathbf{C}$ and $\mathbf{C}'$ in $\mathbb{R}^d$, we write $\mathbf{C}\Subset\mathbf{C}'$ if $\overline{\mathbf{C}}\subset$ interior$(\mathbf{C}')\cup\{0\}$. We say that a cone $\mathbf{C}$ is d'-dimensional if $d'\ge 1$ is the maximal dimension of a linear subset of $\mathbf{C}$.

Definition 2.1. An *unstable cone* is a closed cone $\mathbf{C}_+$ with nonempty interior of dimension d_u in $\mathbb{R}^d$ so that $\mathbb{R}^{d_s}\times\{0\}$ is included in[5] $(\mathbb{R}^d\setminus\mathbf{C}_+)\cup\{0\}$.

Recall that $r>1$. The next key ingredient is adapted from [6]:

Definition 2.2 (Admissible (or fake) stable leaves)**.** Let $\mathbf{C}_+$ be an unstable cone, and let $C_{\mathcal{F}}>1$. Then $\mathcal{F}(\mathbf{C}_+,C_{\mathcal{F}},r)$ (or just $\mathcal{F}$) is the set of all C^r (embedded) submanifolds $\Gamma\subset\mathbb{R}^d$, of dimension d_s, with C^r norms of submanifold charts $\le C_{\mathcal{F}}$, and so that the straight line connecting any two distinct points in Γ is normal to a d_u-dimensional subspace contained in $\mathbf{C}_+$. Denote by π_- the orthogonal projection from $\mathbb{R}^d$ to the quotient $\mathbb{R}^{d_s}$ and by π_Γ its restriction to Γ. Our assumption implies that $\pi_\Gamma:\Gamma\to\mathbb{R}^{d_s}$ is a C^r diffeomorphism onto its image with a C^r inverse, whose C^r norm is bounded by a universal scalar multiple of $\mathcal{C}_F$. In the sequel, we replace $\mathcal{C}_F$ by this larger constant and *we restrict to those* Γ *so that* π_Γ *is surjective.*

Definition 2.3 (Isotropic norm on stable leaves)**.** Fix an unstable cone $\mathbf{C}_+$. Let $\Gamma\in\mathcal{F}(\mathbf{C}_+,C_{\mathcal{F}},r)$ and let $\varphi\in C^0(\Gamma)$. For $w\in\Gamma\subset\mathbb{R}^d$, we set

$$\psi_{\ell_s}^{Op(\Gamma)}(\varphi)(w) = \frac{1}{(2\pi)^{d_s}}\int_{z\in\mathbb{R}^{d_s}}\int_{\eta_s\in\mathbb{R}^{d_s}} e^{i(\pi_\Gamma(w)-z)\eta_s}\psi_{\ell_s}^{(d_s)}(\eta_s)\varphi(\pi_\Gamma^{-1}(z))\mathrm{d}\eta_s\mathrm{d}z\,, \tag{10}$$

where $\psi_k^{(d_s)}:\mathbb{R}^{d_s}\to[0,1]$ is defined in (6). For all real numbers $1\le p\le\infty$, and $-(r-1)<s<r-1$, define an auxiliary isotropic norm on $C^0(\Gamma)$ as

$$\|\varphi\|_{p,\Gamma}^s = \sup_{\ell_s\in\mathbb{Z}_+} 2^{\ell_s s}\|\psi_{\ell_s}^{Op(\Gamma)}(\varphi)\|_{L_p(\mu_\Gamma)}\,, \tag{11}$$

where μ_Γ is the Riemann volume on Γ induced by the standard metric on $\mathbb{R}^d$.

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[5] In Definitions 3.2 and 3.3, and 7 lines above Definition 3.2 of [2], the condition "$\mathbb{R}^{d_s}\times\{0\}$ is included in $\mathbf{C}_-$" can be replaced by this condition.

Note that (11) is equivalent, uniformly in $\Gamma \in \mathcal{F}$, to the ([15, §2.1, Def. 2]) classical d_s-dimensional Besov norm $B^s_{p,\infty}$ of φ in the chart given by π_Γ^{-1}:

$$\|\varphi\|^s_{p,\Gamma} \sim \|\varphi \circ \pi_\Gamma^{-1}\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} .$$

We next revisit the local space given in [2]:

Definition 2.4 (The local space $\mathcal{U}_p^{\mathbf{C}_+,t,s}(K)$)**.** Let $r > 1$, let $K \subset \mathbb{R}^d$ be a non-empty compact set. For an unstable cone $\mathbf{C}_+$, a constant $C_{\mathcal{F}} \geq 1$, and real numbers $1 \leq p \leq \infty$, and $t - (r-1) < s < -t < 0$, define for $\varphi \in L_\infty$ supported in K,

$$\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}} = \sup_{\Gamma \in \mathcal{F}(\mathbf{C}_+, C_{\mathcal{F}}, r)} \sup_{\ell \in \mathbb{Z}_+} 2^{\ell t} \|\psi_\ell^{Op}(\varphi)\|^s_{p,\Gamma} . \tag{12}$$

Set $\mathcal{U}_p^{t,s}(K) = \mathcal{U}_p^{\mathbf{C}_+,t,s}(K)$ to be the completion of $\{\varphi \in L_\infty(K) \mid \|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}} < \infty\}$ for the norm $\|\cdot\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}}$. (Note that $\mathcal{U}_p^{t,s}(K)$ also depends on r and $C_{\mathcal{F}}$.)

Remark 2.5. Beware that, in [2, Definition 3.3], the space $\mathcal{U}_p^{t,s}(K)$ was defined by completing $C^\infty(K)$ (or, equivalently, by [2, Lemma 3.4] and mollification, $C^{r-1}(K)$). We do not claim that $C^\infty(K)$ is dense in the space $\mathcal{U}_p^{t,s}(K)$ from Definition 2.4. (See however [9, Lemmas 3.7, 3.8].) But, since all results in [2] hold (except the heuristic remark after [2, Definition B.1]), with the same[6] proofs, for the completion used in Definition 2.4, we may (abusively) use here the same notation $\mathcal{U}_p^{t,s}(K)$. The new definition is useful to show that (13) implies that $1_\Lambda \mathcal{U}_p^{t,s}(K) \subset \mathcal{U}_p^{t,s}(K)$.

The following lemma was proved[7] in [2]:

Lemma 2.6 (Comparing $\mathcal{U}_p^{\mathbf{C}_+,t,s}(K)$ with classical spaces)**.** *Assume* $-(r-1) < s < -t < 0$. *For any* $u > t$, *there exists a constant* $C = C(u, K)$ *such that* $\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}} \leq C\|\varphi\|_{C^u}$ *for all* $\varphi \in C^u(K)$. *For any* $u > |t+s|$, *the space* $\mathcal{U}_p^{\mathbf{C}_+,t,s}(K)$ *is contained in the space of distributions of order* u *supported on* K.

2.3. The global spaces $\mathcal{U}_p^{\mathbf{C},t,s}$ of anisotropic distributions.

We finally introduce the global spaces $\mathcal{U}_p^{\mathbf{C},t,s}$ of distributions on a compact manifold M.

Definition 2.7. An *admissible chart system and partition of unity* is a finite system of local charts $\{(V_\omega, \kappa_\omega)\}_{\omega \in \Omega}$, with open subsets $V_\omega \subset M$, and C^∞ diffeomorphisms $\kappa_\omega : U_\omega \to V_\omega$ such that $M \subset \cup_\omega V_\omega$, and $U_\omega \subset \mathbb{R}^d$ is bounded and open, together with a C^∞ partition of unity $\{\theta_\omega\}_{\omega \in \Omega}$ for M, subordinate to the cover $\mathcal{V} = \{V_\omega\}$.

Definition 2.8 (Anisotropic spaces $\mathcal{U}_p^{\mathbf{C},t,s}$ on M)**.** Fix $r > 1$, an admissible chart system and partition of unity, $\mathcal{C}_F \geq 1$ and a system of cones $\mathbf{C} = \{\mathbf{C}_{\omega,+}\}_{\omega \in \Omega}$. Fix $1 \leq p \leq \infty$, and real numbers $-(r-1) < s < -t < 0$. The Banach space $\mathcal{U}_p^{\mathbf{C},t,s} = \mathcal{U}_p^{\mathbf{C},t,s,r,C_{\mathcal{F}}}$ is the completion (see Remark 2.5) of $\{\varphi \in L_\infty(M) \mid \|\varphi\|_{\mathcal{U}_p^{\mathbf{C},t,s}} < \infty\}$ for the norm $\|\varphi\|_{\mathcal{U}_p^{\mathbf{C},t,s}} := \max_{\omega \in \Omega} \|(\theta_\omega \cdot \varphi) \circ \kappa_\omega\|_{\mathcal{U}_p^{\mathbf{C}_{\omega,+},t,s}}$.

Remark 2.9 (Admissible systems $\{\mathbf{C}_{\omega,\pm}\}$). To get a spectral gap for the transfer operator $\mathcal{L}_{1/|\det DT|}$ associated to a $C^{\tilde r}$ Anosov diffeomorphism T for $\tilde r > 1$, one must take $r \leq \tilde r$ and consider an admissible chart system and partition of unity, with cones $\{\mathbf{C}_{\omega,+}\}$, satisfying the following conditions [2]:

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[6] In particular, [2, Lemma C.1] holds replacing $C^\infty(K)$ by compactly supported distributions.

[7] Injectivity of the embedding into distributions follows from injectivity of the embedding of the closure of the (larger) set of those tempered distributions φ so that $\|\varphi\|_{\mathcal{U}_p^{t,s}} < \infty$.

a) Let E^s and E^u be the stable, respectively unstable, bundles of T. Then if $x \in V_\omega$, the cone $(D\kappa_\omega^{-1})^*_x(\mathbf{C}_{\omega,+})$ contains the (d_u-dimensional) normal subspace of $E^s(x)$, and there exists a d_s-dimensional cone $\mathbf{C}_{\omega,-}$, with nonempty interior, so that $\mathbf{C}_{\omega,+} \cap \mathbf{C}_{\omega,-} = \{0\}$, and so that $(D\kappa_\omega^{-1})^*_x(\mathbf{C}_{\omega,-})$ contains the (d_s-dimensional) normal subspace of $E^u(x)$.

b) If $V_{\omega'\omega} = T(V_\omega) \cap V_{\omega'} \neq \emptyset$, the C^r map corresponding to T^{-1} in charts,

$$F = F_{\omega'\omega} = \kappa_\omega^{-1} \circ T^{-1} \circ \kappa_{\omega'} : \kappa_{\omega'}^{-1}(V_{\omega'\omega}) \to U_\omega \, ,$$

extends to a bilipschitz C^1 diffeomorphism of $\mathbb{R}^d$ so that (by definition, $\mathbf{C}_{\omega',-} \Subset (\mathbb{R}^{d_s} \setminus \mathbf{C}_{\omega',+})$)

$$DF_x^{tr}(\mathbb{R}^d \setminus \mathbf{C}_{\omega,+}) \Subset \mathbf{C}_{\omega',-} \, , \qquad \forall x \in \mathbb{R}^d \, .$$

c) Furthermore, there exists, for each x, y, a linear transformation $\mathbb{L}_{xy}$ so that

$$(\mathbb{L}_{xy})^{tr}(\mathbb{R}^d \setminus \mathbf{C}_{\omega,+}) \Subset \mathbf{C}_{\omega',-} \text{ and } \mathbb{L}_{xy}(x-y) = F(x) - F(y) \, .$$

A map F satisfying (b–c) is called *regular cone hyperbolic* from $\mathbf{C}_{\omega,\pm}$ to $\mathbf{C}_{\omega',\pm}$.

The anisotropic spaces $\mathcal{U}_1^{\mathbf{C},t,s}$ (with $p=1$) are analogues of the Blank–Keller–Gouëzel–Liverani [7, 13] spaces $\mathcal{B}^{t,|s+t|}$ associated to T, for integer t and $s < -t$. The spaces $\mathcal{U}_p^{\mathbf{C},t,s}$ are somewhat similar to the Demers–Liverani spaces [8] when $p > 1$ and $-1 + 1/p < s < -t < 0$. See [2].

3. CHARACTERISTIC FUNCTIONS AS BOUNDED MULTIPLIERS

3.1. **Statement of the main result.** Fix $r > 1$, $C_\mathcal{F} > 0$, $p \in (1, \infty)$, an admissible chart system and partition of unity on M (Definition 2.7), and an associated cone system $\mathbf{C} = \{\mathbf{C}_{\omega,+}\}$. Let $\tilde\Lambda \subset M$ be an open set so that $\partial\tilde\Lambda$ is a finite union of C^r hypersurfaces $\partial\tilde\Lambda_i$ so that the normal vector at any $x \in \partial\tilde\Lambda_i \cap V_\omega$ lies in $\mathbb{R}^d \setminus \mathbf{C}_{\omega,+}$ (a transversality condition). We claim that if $\max\{t-(r-1), -1+1/p\} < s < -t < 0$ then, for any[8] cone system $\tilde{\mathbf{C}}$ with[9] $\mathbf{C} \Subset \tilde{\mathbf{C}}$, there exists $C_{\tilde\Lambda,\tilde{\mathbf{C}}} < \infty$ so that

$$\|1_{\tilde\Lambda}\varphi\|_{\mathcal{U}_p^{\mathbf{C},t,s}} \le C_{\tilde\Lambda,\tilde{\mathbf{C}}} \|\varphi\|_{\mathcal{U}_p^{\tilde{\mathbf{C}},t,s}} \, , \quad \forall \varphi \, .$$

Since $t - (r-1) < s < -t$, by using suitable C^∞ partitions of unity h_j and C^r coordinates F_j (arbitrarily close to the identity, and thus regular cone hyperbolic from $\tilde{\mathbf{C}}$ to $\mathbf{C}$ if $\mathbf{C} \Subset \tilde{\mathbf{C}}$), and exploiting the Lasota–Yorke estimate [2, Lemma 4.2] for the corresponding transfer operators, we reduce to:

Theorem 3.1 (Characteristic functions of half-spaces)**.** *Fix $r > 1$, $C_\mathcal{F} > 0$, and an unstable cone $\mathbf{C}_+$. Let $K \subset \mathbb{R}^d$ be compact, and let $\tilde\Lambda \subset \mathbb{R}^d$ be a half-space whose unit normal vector $u_{\tilde\Lambda}$ lies in $\mathbb{R}^d \setminus \mathbf{C}_+$. Then for any*

$$1 < p < \infty \text{ and } \max\{t-(r-1), -1+\frac{1}{p}, \} < s < -t < 0 \, ,$$

there exists $C < \infty$ so that for any $\varphi \in \mathcal{U}_p^{\mathbf{C}_+,t,s}(K)$ we have,

$$\|1_{\tilde\Lambda}\varphi\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}} \le C \|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}} \, . \tag{13}$$

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[8] Given two cone systems with the same chart systems, $\mathbf{C} \Subset \tilde{\mathbf{C}}$ means $\mathbf{C}_{\omega,+} \Subset \tilde{\mathbf{C}}_{\omega,+}$ for all ω.

[9] Enlarging the cones is not a problem when studying $1_{\tilde\Lambda}((f\varphi) \circ F)$ for a C^{r-1} function f and a C^r regular cone-hyperbolic map F from $\mathbf{C}$ to $\tilde{\mathbf{C}}$ with $\mathbf{C} \Subset \tilde{\mathbf{C}}$, since the Lasota–Yorke estimate [2, Lemma 4.2] gives $\|(f\varphi) \circ F\|_{\mathcal{U}_p^{\tilde{\mathbf{C}},t,s}} \le C_{f,F} \|\varphi\|_{\mathcal{U}_p^{\mathbf{C},t,s}}$.

Since $1_{\tilde\Lambda}\varphi \in L_\infty$ if $\varphi \in L_\infty$ and since $\mathcal{U}_p^{\mathbf{C}_+,t,s}(K)$ is the completion of a set of bounded functions, the bound (13) implies that $1_{\tilde\Lambda}\varphi \in \mathcal{U}_p^{\mathbf{C}_+,t,s}(K)$ if $\varphi \in \mathcal{U}_p^{\mathbf{C}_+,t,s}(K)$ (use Cauchy sequences).

The conditions in the theorem imply $t < 1 - 1/p$. (This does not imply $t < 1/p$ if $p > 2$.)

Remark 3.2 (Heuristic proof via interpolation: $t < 1/p$ vs. $t < \min\{|s|, r-1-|s|\}$). A heuristic argument for the bounded multiplier property (13) under the conditions $-1 + 1/p < s < 0 < t < 1/p$ was sketched in [2, Remark 3.9], exploiting via interpolation the fact that ([15, Thm 4.6.3/1]) the characteristic function of a half-plane in $\mathbb{R}^n$ is a bounded multiplier on the Besov space $B^\tau_{p,\infty}(\mathbb{R}^n)$ if $\frac{1}{p} - 1 < \tau < \frac{1}{p}$. It does not seem easy to fill in details of this argument, and we shall prove Theorem 3.1 using paraproduct decompositions instead of interpolation. The restriction $t-(r-1) < s < -t$ is in any case necessary for applications to hyperbolic dynamics, and the bound for the essential spectral radius in [2] improves as $p \to 1$.

3.2. **Basic toolbox (Nikol'skij and Young bounds, paraproduct decomposition, and a crucial trivial observation on functions of a single variable).**

The proofs below use the *Nikol'skij inequality* (see e.g. [15, Remark 2.2.3.4, p. 32]) which says, in dimension $D \ge 1$, that for any $p > p_1 > 0$ there exists C so that for any $M > 1$, and any f with $\operatorname{supp}\mathbb{F}(f) \subset \{|\xi| \le M\}$,

$$\|f\|_{L_p(\mathbb{R}^D)} \le C M^{D(1/p_1 - 1/p)} \|f\|_{L_{p_1}(\mathbb{R}^D)} \,. \tag{14}$$

We shall also use the following *leafwise version of Young's inequality* (which can be proved like [6, Lemma 4.2], see [2], by using that any translation $\Gamma + x$ of $\Gamma \in \mathcal{F}$ also belongs to $\mathcal{F}$):

$$\|\tilde\psi * \varphi\|^s_{p,\Gamma} \le \|\tilde\psi\|_{L_1(\mathbb{R}^d)} \sup_{x\in\mathbb{R}^d} \|\varphi\|^s_{p,\Gamma+x} \le \|\tilde\psi\|_{L_1} \sup_{\tilde\Gamma\in\mathcal{F}} \|\varphi\|^s_{p,\tilde\Gamma} \,. \tag{15}$$

Write $S_k\varphi = \psi_k^{Op}(\varphi)$ for $k \ge 0$, set $S_{-1}\varphi \equiv 0$, and put $S^j\varphi = \sum_{k=0}^j S_k\varphi$ for integer $j \ge 0$. The (a priori formal) *paraproduct decomposition* (see [15, §4.4]) is

$$\begin{aligned}\varphi \cdot \upsilon &= \lim_{j\to\infty} (S^j\varphi)\cdot(S^j\upsilon)\\ &= \sum_{k=2}^\infty \sum_{j=0}^{k-2} S_j\varphi \cdot S_k\upsilon + \sum_{k=0}^\infty \sum_{j=k-1}^{k+1} S_j\varphi\cdot S_k\upsilon + \sum_{j=2}^\infty\sum_{k=0}^{j-2} S_j\varphi\cdot S_k\upsilon\\ &= \Pi_1(\varphi,\upsilon) + \Pi_2(\varphi,\upsilon) + \Pi_3(\varphi,\upsilon)\,,\end{aligned} \tag{16}$$

where we put

$$\Pi_1(\varphi,\upsilon) = \sum_{k=2}^\infty S^{k-2}\varphi\cdot S_k\upsilon\,, \qquad \Pi_2(\varphi,\upsilon) = \sum_{k=0}^\infty (S_{k-1}\varphi + S_k\varphi + S_{k+1}\varphi)\cdot S_k\upsilon\,,$$

$$\text{and} \qquad \Pi_3(\varphi,\upsilon) = \sum_{j=2}^\infty S_j\varphi\cdot S^{j-2}\upsilon = \Pi_1(\upsilon,\varphi)\,.$$

The two key facts motivating the decomposition (16) are

$$\operatorname{supp}\mathbb{F}(S^{k-2}\varphi\cdot S_k\upsilon) \subset \{2^{k-3} \le \|\xi\| \le 2^{k+1}\}\,, \quad \forall k \ge 2\,, \tag{17}$$

and

$$\operatorname{supp} \mathbb{F}\,(\sum_{j=k-1}^{k+1} S_j\varphi \cdot S_k\upsilon) \subset \{\|\xi\| \le 5 \cdot 2^k\}\,, \quad \forall k \ge 0\,. \tag{18}$$

Finally, the proof of Theorem 3.1 hinges on the fact that the singular set of a characteristic function is co-dimension one: We shall reduce there to the case $\partial\tilde\Lambda = \{x_1 = 0\}$ so that $1_{\tilde\Lambda}$ only depends on the first coordinate x_1 of $x \in \mathbb{R}^d$. We shall use below the fact that for such $\tilde\Lambda$ (see [15, Lemma 4.6.3.2 (ii), p. 209, Lemma 2.3.1/3, p. 48]) for all $p \in (1,\infty)$

$$\|1_{\tilde\Lambda}\|_{B^t_{p,q}(\mathbb{R}^d)} < \infty\,, \text{ if } 0 < t < 1/p \text{ and } 0 < q < \infty \text{ or } t = 1/p \text{ and } q = \infty\,. \tag{19}$$

We also note for further use the *trivial but absolutely essential fact* that if a function $\upsilon(x)$ only depends on x_1 then $S_k\upsilon = (\mathbb{F}^{-1}\psi_k) * \upsilon$ also only depends on x_1 for all k, and, more precisely,

$$S_k\upsilon(x) := (\mathbb{F}^{-1}\psi_k) * \upsilon(x) = (\mathbb{F}^{-1}\psi_k^{(1)}) * \upsilon(x_1)\,. \tag{20}$$

Indeed

$$(\mathbb{F}^{-1}\psi_k) * \upsilon(x) = \int (\mathbb{F}^{-1}\psi_k)(y)\mathrm{d}y_2 \dots \mathrm{d}y_d\, \upsilon(x_1 - y_1)\mathrm{d}y_1\,,$$

and, since $(2\pi)^{-(d-1)}\int_{\mathbb{R}^{d-1}} e^{i(y_2,\dots,y_d)(\xi_2,\dots,\xi_d)}dy_2\dots dy_d$ (the inverse Fourier transform of the constant function) is the Dirac mass at $(\xi_2,\dots,\xi_d) = 0$, we get,

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} (\mathbb{F}^{-1}\psi_k)(y_1, y_2, \dots, y_d)\mathrm{d}y_2 \dots \mathrm{d}y_d \\
&\quad = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}} e^{iy_1\xi_1}\psi_k(\xi)\mathrm{d}\xi_1\mathrm{d}\xi_2\dots\mathrm{d}\xi_d e^{i(y_2,\dots,y_d)(\xi_2,\dots,\xi_d)}\mathrm{d}y_2\dots\mathrm{d}y_d \\
&\quad = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iy_1\xi_1}\psi_k(\xi_1, 0)\mathrm{d}\xi_1 = (\mathbb{F}^{-1}\psi_k^{(1)})(y_1)\,,
\end{aligned}$$

where we used that $\psi_k^{(d)}(\xi_1, 0) = \psi_k^{(1)}(\xi_1)$.

3.3. **Multipliers depending on a single coordinate.** This subsection is devoted to a classical property of multipliers depending on a single coordinate, which is instrumental in the proof of Theorem 3.1. If $1 \le p \le \infty$, let $1 \le p' \le \infty$ be so that

$$\frac{1}{p} + \frac{1}{p'} = 1\,, \text{ i.e., } p' = \frac{p}{p-1}\,. \tag{21}$$

Lemma 3.3. *Let $d_s \ge 1$. Let $1 < p < \infty$ and let $-1 + \frac{1}{p} < s < 0$. Then there exists $C < \infty$ so that for all $f, g : \mathbb{R}^{d_s} \to \mathbb{C}$ with $g(x) = g(x_1)$,*

$$\|fg\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \le C\|f\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}(\|g\|_{B^{1/p'}_{p',\infty}(\mathbb{R})} + \|g\|_{L_\infty(\mathbb{R})})\,. \tag{22}$$

Remark 3.4. The bound (22) is a special case of a much more general result (see e.g. [15, Cor 4.6.2.1 (40)]) which also implies that if $g(x) = g(x_1)$ then

$$\|fg\|_{B^t_{p,\infty}(\mathbb{R}^{d_s})} \le C\|f\|_{B^t_{p,\infty}(\mathbb{R}^{d_s})}(\limsup_{q\to p}\|g\|_{B^{1/q}_{q,\infty}(\mathbb{R})} + \|g\|_{L_\infty(\mathbb{R})}) \text{ if } 0 < t < \frac{1}{p}\,, \tag{23}$$

for a constant C, which may depend on p and t, but not on f or g.

For the convenience of the reader, and as a warmup in the use of paraproducts, we include a proof of Lemma 3.3.

Proof of Lemma 3.3. The proof uses the decomposition $\tilde\Pi_1(f,g)+\tilde\Pi_2(f,g)+\tilde\Pi_3(f,g)$ obtained from (16) by replacing S_k and S^k by the d_s-dimensional operators

$$\tilde S_k := (\psi_k^{(d_s)})^{Op} f\,, \qquad \tilde S^k := \sum_{j=0}^{k} (\psi_j^{(d_s)})^{Op} f = \sum_{j=0}^{k} \tilde S_j f\,. \tag{24}$$

The bound for the contribution of $\tilde\Pi_3(f,g)$ is easy and does not require conditions on s or g: Indeed, (17) and the Young inequality with the first claim of (7) imply

$$\|\sum_{j=2}^{\infty} \tilde S_j f \tilde S^{j-2} g\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \le C \sup_{k\ge 2} 2^{ks} \sum_{\ell=-1}^{+3} \|\tilde S_{k+\ell} f \tilde S^{k+\ell-2} g\|_{L_p(\mathbb{R}^{d_s})}\,.$$

We focus on the term for $\ell = 0$ (the others are similar) and get

$$\begin{aligned}\sup_{k\ge 2} 2^{ks} \|\tilde S_k f \tilde S^{k-2} g\|_{L_p(\mathbb{R}^{d_s})} &\le C \sup_k 2^{ks} \|\tilde S_k f\|_{L_p(\mathbb{R}^{d_s})} \sup_k \|\tilde S^k g\|_{L_\infty} \\ &\le C \|f\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \|g\|_{L_\infty}\,,\end{aligned} \tag{25}$$

where we used the Hölder inequality and then the Young inequality, together with the second claim of (7).

For $\tilde\Pi_1(f,g)$, we do not require any condition on g, and the condition on s is limited to $s<0$: Indeed, exploiting again (17), we get

$$\|\sum_{j=2}^{\infty} \tilde S^{j-2} f \tilde S_j g\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \le C \sup_{k\ge 2} 2^{ks} \sum_{\ell=-1}^{+1} \|\tilde S^{k+\ell-2} f \tilde S_{k+\ell} g\|_{L_p(\mathbb{R}^{d_s})}\,.$$

Focusing again on the terms for $\ell = 0$, we find

$$\begin{aligned}\sup_{k\ge 2} 2^{ks} \|\tilde S^{k-2} f \tilde S_k g\|_{L_p(\mathbb{R}^{d_s})} &\le C \sup_k 2^{ks} \|\sum_{j=0}^{k-2} \tilde S_j f\|_{L_p(\mathbb{R}^{d_s})} \sup_k \|\tilde S_k g\|_{L_\infty} \\ &\le C \sup_k \big(\sum_{j=0}^{k-2} 2^{(k-j)s}\big) \sup_j 2^{js} \|\tilde S_j f\|_{L_p(\mathbb{R}^{d_s})} \|g\|_{L_\infty} \\ &\le C \|f\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \|g\|_{L_\infty}\,,\end{aligned} \tag{26}$$

where we used the Hölder inequality and then the Young inequality, together with the first claim of (7).

The computation for $\tilde\Pi_2(f,g)$ is trickier and will use the assumption $s > -1+1/p$ together with the Nikol'skij inequality (14). For $\ell \in \{0, \pm 1\}$, by (18), we get

$$\|\sum_{j=0}^{\infty} \tilde S_{j+\ell} f \tilde S_j g\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \le C \sum_{j=0}^{\infty} \sup_{k\ge 0} 2^{ks} \|\tilde S_k (\tilde S_{k+j+\ell} f \tilde S_{k+j} g)\|_{L_p(\mathbb{R}^{d_s})}\,. \tag{27}$$

In the sequel, we consider the terms with $\ell = 0$ (the other terms are almost identical). Setting $y = (x_2, \ldots, x_{d_s})$ and applying the one-dimensional Nikol'skij inequality (14) for $1 < p_1 < p$, we have, for any function υ,

$$
\begin{aligned}
(28) \qquad 2^{ks}\|\tilde{S}_k\upsilon\|_{L_p(\mathbb{R}^{d_s})} &= \Big(\int \Big[\big(\int 2^{ksp}|\tilde{S}_k\upsilon(x_1,y)|^p \mathrm{d}x_1\big)^{1/p}\Big]^p \mathrm{d}y\Big)^{1/p} \\
&\le \Big(\int \Big[\big(\int 2^{k(s+\frac{1}{p_1}-\frac{1}{p})p_1}|\tilde{S}_k\upsilon(x_1,y)|^{p_1} \mathrm{d}x_1\big)^{1/p_1}\Big]^p \mathrm{d}y\Big)^{1/p} \\
&= 2^{k(s+\frac{1}{p_1}-\frac{1}{p})} A(p,p_1,\tilde{S}_k\upsilon)\,,
\end{aligned}
$$

where

$$
(29) \qquad A(p,p_1,\tilde{S}_k\upsilon) = \Big(\int \Big[\big(\int |\tilde{S}_k\upsilon(x_1,y)|^{p_1} \mathrm{d}x_1\big)^{1/p_1}\Big]^p \mathrm{d}y\Big)^{1/p}\,.
$$

Since $s > -1 + 1/p$, we may choose $p_1 \in (1,p)$ close enough to 1 so that

$$
(30) \qquad s_1 = s + \frac{1}{p_1} - \frac{1}{p} > 0\,.
$$

Then, the right-hand side of (27) can be bounded as follows, using (28),

$$
\begin{aligned}
(31) \qquad \sum_{j=0}^{\infty} \sup_{k\ge 0} 2^{ks}\|\tilde{S}_k(\tilde{S}_{k+j}f\tilde{S}_{k+j}g)\|_{L_p} &\le \sum_{j=0}^{\infty} \sup_k 2^{ks_1} A(p,p_1,\tilde{S}_k(\tilde{S}_{k+j}f\tilde{S}_{k+j}g)) \\
&\le \big(\sum_{j=0}^{\infty} 2^{-js_1}\big) \sup_{k,j} 2^{(k+j)s_1} A(p,p_1,\tilde{S}_k(\tilde{S}_{k+j}f\tilde{S}_{k+j}g)) \\
&\le C \sup_{m\ge 0} 2^{ms_1} A(p,p_1,\tilde{S}_m f\tilde{S}_m g)\,.
\end{aligned}
$$

In the last line we used (18) to exploit that there exists $C < \infty$, depending on $p > 1$ and $p_1 > 1$, so that, for any $\{\upsilon_k\}_{k\ge 0}$ so that $\mathrm{supp}\,(\mathbb{F}(\upsilon_k)) \subset \{|\xi| \le 5\cdot 2^k\}$,

$$
A(p,p_1,\tilde{S}_k(\upsilon_{k+j})) \le C A(p,p_1,\upsilon_{k+j})\,, \quad \forall k \ge 0\,,\ j \ge 0\,.
$$

(The above basically follows from Young's inequality, see [15, Thm 2.6.3, (5), p. 96], noting that $p > 1$ and $p_1 > 1$, so that $\max\{0, 1/p - 1, 1/p_1 - 1\} = 0$, and noting that f_j in the right-hand side of [15, (5), p. 96] should be replaced by $f_{j+\ell}$, see [12, Thm 2.4.1.(II) and (III)].)

Next, recalling that g only depends on x_1, using (20), and applying the Hölder inequality in dx_1 for $1/p_1 = 1/p + 1/q$, we find C so that for all k

$$
\begin{aligned}
A(p,p_1,\tilde{S}_k f\tilde{S}_k g) &= \Big(\int \Big[\big(\int |\tilde{S}_k g(x_1)\tilde{S}_k f(x_1,y)|^{p_1} \mathrm{d}x_1\big)^{1/p_1}\Big]^p \mathrm{d}y\Big)^{1/p} \\
&\le C\Big(\int \Big[\big(\int |\tilde{S}_k g(x_1)|^q \mathrm{d}x_1\big)^{1/q}\big(\int |\tilde{S}_k f(x_1,y)|^p \mathrm{d}x_1\big)^{1/p}\Big]^p \mathrm{d}y\Big)^{1/p} \\
&\le C\big(\int |\tilde{S}_k g(x_1)|^q \mathrm{d}x_1\big)^{1/q}\Big(\int \Big[\big(\int |\tilde{S}_k f(x_1,y)|^p \mathrm{d}x_1\big)^{1/p}\Big]^p \mathrm{d}y\Big)^{1/p} \\
&= C\|\tilde{S}_k g\|_{L_q(\mathbb{R})}\|\tilde{S}_k f\|_{L_p(\mathbb{R}^{d_s})}\,.
\end{aligned}
$$

Note that (20) implies $\tilde S_k g = (\psi_k^{(1)})^{Op} g$. Finally, putting together (27) and (31), we find, recalling (30) and (21),

$$\begin{aligned}
\Big\|\sum_{j=0}^{\infty}\tilde S_j f \tilde S_j g\Big\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} &\le C\sup_{k\ge 0}\big(2^{ks_1}\|\tilde S_k g\|_{L_q(\mathbb{R})}\|\tilde S_k f\|_{L_p(\mathbb{R}^{d_s})}\big) \notag\\
&\le C\sup_{k\ge 0}\big(2^{k\frac{1}{q}}\|\tilde S_k g\|_{L_q(\mathbb{R})}\big)\sup_{k\ge 0}\big(2^{ks}\|\tilde S_k f\|_{L_p(\mathbb{R}^{d_s})}\big) \notag\\
&\le C\sup_{k\ge 0}\big(2^{k\frac{1}{q}}2^{k(\frac{1}{p'}-\frac{1}{q})}\|\tilde S_k g\|_{L_{p'}(\mathbb{R})}\big)\|f\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \tag{32}\\
&\le C\|g\|_{B^{1/p'}_{p'}(\mathbb{R})}\|f\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\,, \tag{33}
\end{aligned}$$

where we used the one-dimensional Nikol'skij inequality for $q > p' > 1$ in (32) (recalling (18)). Together, (25), (26), and (33) give (22). □

3.4. **Proof of Theorem 3.1.** To prove the theorem, we need one last lemma. The point is that if Γ is horizontal, i.e., $\Gamma = \mathbb{R}^{d_s}\times\{0\}$, then (9) implies

$$\tilde S_{k_s}((S^k\varphi)\circ\pi_\Gamma^{-1}|_{\mathbb{R}^{d_s}}) \equiv 0\,, \quad \forall k_s > k+2 \ge 2\,. \tag{34}$$

If Γ is an arbitrary admissible stable leaf, then we must work harder. To state the bound replacing the trivial decoupling property (34), we need notation: Defining $b:\mathbb{R}^d\to\mathbb{R}_+$ by $b(x)=1$ if $\|x\|\le 1$ and $b(x)=\|x\|^{-d-1}$ if $\|x\|>1$, we set $b_k(x) = 2^{dk}\cdot b(2^k x)$ for $k\ge 0$. (Note that $\|b_k\|_{L_1(\mathbb{R}^d)} = \|b\|_{L_1(\mathbb{R}^d)} < \infty$.)

Lemma 3.5 (Decoupled wave packets in $\mathbb{R}^d$ and the cotangent space of Γ)**.** *Fix a compact set $K\subset\mathbb{R}^d$. There exists $C_0\in[2,\infty)$ (depending on $\mathcal{C}_{\mathcal{F}}$, K) so that for any $k_s > k + C_0 \ge C_0$ and any $\Gamma\in\mathcal{F}$, the kernel $V(x,y)$ defined by $\tilde S_{k_s}((S^k\varphi)\circ\pi_\Gamma^{-1})(x) = \int_{y\in\mathbb{R}^d} V(x,y)\varphi(y)\mathrm{d}y$ for $x\in\mathbb{R}^{d_s}$ and φ supported in K satisfies*[10]

$$|V(x,y)| \le C_0 2^{-k_s r} b_k(\pi_\Gamma^{-1}(x) - y)\,, \quad \forall x\in\mathbb{R}^{d_s}\,, \forall y\in K\,. \tag{35}$$

The lemma implies that, if φ is supported in K, then $\int_{y\in\mathbb{R}^d} V(x,y)\varphi(y)\mathrm{d}y$ is bounded by a convolution with a function in $L_1(\mathbb{R}^d)$, for which (15) holds.

Proof. The kernel $V(x,y)$ is given by the formula[11]

$$\frac{1}{(2\pi)^{d_s+d}}\int_{z\in\mathbb{R}^{d_s}}\int_{\eta\in\mathbb{R}^d}\int_{\eta_s\in\mathbb{R}^{d_s}} e^{i(\pi_\Gamma^{-1}(z)-y)\eta}e^{i(x-z)\eta_s}\sum_{j=0}^{k}\psi_j(\eta)\psi_{k_s}^{(d_s)}(\eta_s)\mathrm{d}\eta_s\mathrm{d}\eta\mathrm{d}z\,.$$

As a warmup, let us prove (34) if Γ is horizontal or, more generally, affine: Letting $\eta = (\eta_-,\eta_+)$ with $\eta_- = \pi_-(\eta)\in\mathbb{R}^{d_s}$, we have $\pi_\Gamma^{-1}(z) = (z, A(z)+A_0)$ with $A_0\in\mathbb{R}^{d_u}$ and $A:\mathbb{R}^{d_s}\to\mathbb{R}^{d_u}$ linear ($A\equiv 0$ if Γ is horizontal), so that (using like

---

[10] The proof shows that the same bound holds for the kernel associated to $\tilde S_{k_s}((S_k\varphi)\circ\pi_\Gamma^{-1})(x)$.

[11] Strictly speaking, we must first integrate by parts d_s+1 times in the kernel $\int e^{i(\pi_\Gamma^{-1}(z)-y)\eta}\sum_{j=0}^{k}\psi_j(\eta)\mathrm{d}\eta$ of $(S^k\varphi)\circ\Pi_\Gamma^{-1}(z)$ for $d(z,K)>\epsilon$, to get an element of $L_1(dz)$.

in (20) that $\mathbb{F}^{-1}(1)$ is the Dirac at 0), $V(x,y)$ can be rewritten as

$$\frac{1}{(2\pi)^{d+d_s}}\int_{\mathbb{R}^{2d_s+d}} e^{-iy\eta}e^{ix\eta_s}e^{iA_0\eta_+}e^{iz(-\eta_s+\eta_-+A^{tr}\eta_+)}\sum_{j=0}^{k}\psi_j(\eta)\psi^{(d_s)}_{k_s}(\eta_s)\mathrm{d}\eta_s\mathrm{d}\eta\mathrm{d}z$$

$$=\frac{1}{(2\pi)^{d}}\int_{\mathbb{R}^{d}} e^{-iy\eta}e^{ix(\eta_-+A^{tr}\eta_+)}e^{iA_0\eta_+}\sum_{j=0}^{k}\psi_j(\eta)\psi^{(d_s)}_{k_s}(\eta_-+A^{tr}\eta_+)\mathrm{d}\eta\equiv 0\,,$$

since $\psi_j(\eta)$ and $\psi^{(d_s)}_{k_s}(\eta_-+A^{tr}\eta_+)$ have disjoint supports if $k_s > k + C_0$, where $C_0 \ge 2$ depends on $\|A\| \le \mathcal{C}_{\mathcal{F}}$.

More generally, $\Gamma \in \mathcal{F}$ is the graph of a C^r map γ (with $\|\gamma\|_{C^r} \le \mathcal{C}_{\mathcal{F}}$), i.e., $\pi_\Gamma^{-1}(z) = (z, \gamma(z))$ for $z \in \mathbb{R}^{d_s}$. The lemma is thus obtained integrating by parts r times (in the sense of [2, App. C] if r is not an integer) with respect to z in the kernel $V(x,y)$, using (8), and proceeding as in the end of the proof of [1, Lemma 2.34], mutatis mutandis (using that $\|y-\pi_\Gamma^{-1}(x)\| > 2^{-k}$ implies that either $\|y-\pi_\Gamma^{-1}(z)\| > 2^{-k+1}$ or $\|\pi_\Gamma^{-1}(z)-\pi_\Gamma^{-1}(x)\| > 2^{-k+1}$, choosing C_0 depending on $\mathcal{C}_{\mathcal{F}}$, so that $\|\pi_\Gamma^{-1}(z)-\pi_\Gamma^{-1}(x)\| > 2^{-k+1}$ implies $\|z-x\| \ge 2^{-k+1}/C_0$). □

Proof of Theorem 3.1. If G is a rotation about $0 \in \mathbb{R}^d$ then, since $\psi_n \circ G^{-1} = \psi_n$, we have $\psi_n^{Op}(\tilde\varphi \circ G) = ((\psi_n \circ G^{tr})^{Op}\tilde\varphi)\circ G = (\psi_n^{Op}\tilde\varphi)\circ G$ (use $G^{tr} = G^{-1}$), and thus $\|\tilde\varphi \circ G\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}} = \|\tilde\varphi\|_{\mathcal{U}_p^{G(\mathbf{C}_+),t,s}}$ for all φ (use $G\circ\pi_\Gamma^{-1} = \pi_{G(\Gamma)}^{-1}$). It thus suffices to show (13) for $\Lambda = \{x \in \mathbb{R}^d \mid x_1 > 0\}$. Indeed, the assumption on $u_{\tilde\Lambda}$ implies that the rotation G satisfying $1_{\tilde\Lambda}\varphi = (1_\Lambda(\varphi\circ G^{-1}))\circ G$ is such that $G(\mathbf{C}_+)$ is still an unstable cone, i.e., $\mathbb{R}^{d_s}\times\{0\}$ is included in $(\mathbb{R}^d \setminus G(\mathbf{C}_+))\cup\{0\}$ (note that $G(u_{\tilde\Lambda}) = (1,0,\dots,0)$, and consider the limiting case $u_{\tilde\Lambda}\to\partial\mathbf{C}_+$).

Next, since φ is supported in K, we can replace the half-space Λ by a strip $0 < x_1 < B$, still denoted Λ, and whose characteristic function $1_\Lambda(x)$ still only depends on $x_1 \in \mathbb{R}$. Without loss of generality, we may assume that $B = 1$.

Our starting point is then the decomposition (16) applied to $\upsilon = 1_\Lambda$. We consider first the term $\Pi_3(\varphi, 1_\Lambda)$. We will bootstrap from Lemma 3.3. Set

$$1_\Lambda^{k-2,\Gamma}(x_-) = (S^{k-2}1_\Lambda)(x_-,\gamma(x_-)) = \sum_{j=0}^{k-2}(\mathbb{F}^{-1}\psi_j * 1_\Lambda)(x_-,\gamma(x_-))\,. \tag{36}$$

Then $1_\Lambda^{k-2,\Gamma}(x_-)$ is a function of x_1 alone (recalling (20)), and the leafwise Young inequality (15), together with the second claim of (7) and the fact that $\|1_\Lambda\|_{B^{1/t}_{t,\infty}(\mathbb{R})} < \infty$ (for any $1 < t < \infty$, see e.g. [15, Lemma 2.3.1/3(ii), Lemma 2.3.5]), give that both $\|1_\Lambda^{k-2,\Gamma}\|_{B^{1/p'}_{p',\infty}(\mathbb{R})}$ and $\|1_\Lambda^{k-2,\Gamma}\|_{L_\infty(\mathbb{R})}$ are finite, uniformly in Γ and k. Next, by (17), (15), and (22), there exists a constant C so that for any $\ell \ge 0$, since $-1+1/p < s < 0$,

$$\begin{aligned}
2^{\ell t}\|S_\ell(\Pi_3(\varphi,1_\Lambda))\|^s_{p,\Gamma} &\le 2^{\ell t}\sum_{k=\ell-1}^{\ell+3}\|S_k\varphi\cdot S^{k-2}1_\Lambda\|^s_{p,\Gamma}\\
&\le 2^{\ell t}\sum_{k=\ell-1}^{\ell+3}\|S_k\varphi\|^s_{p,\Gamma}(\|1_\Lambda^{k-2,\Gamma}\|_{B^{1/p'}_{p',\infty}(\mathbb{R})}+\|1_\Lambda^{k-2,\Gamma}\|_{L_\infty(\mathbb{R})})\\
&\le C\sup_n 2^{nt}\|S_n\varphi\|^s_{p,\Gamma} \le C\|\varphi\|_{\mathcal{U}_p^{\mathbf{C},t,s}}\,,
\end{aligned}$$

where we used (22) from Lemma 3.3 for $f(x_-) = S_k\varphi(x_-, \gamma(x_-))$ with $\gamma = \gamma(\Gamma)$ from the proof of Sublemma 3.5, and $g(x_-) = 1_\Lambda^{k-2,\Gamma}(x_-)$. This concludes the bound for $\Pi_3(\varphi, 1_\Lambda)$, and we move to $\Pi_2(\varphi, 1_\Lambda)$. Setting

$$1^\Gamma_{\Lambda,k}(x_-) = (S_k 1_\Lambda)(x_-, \gamma(x_-)) = (\mathbb{F}^{-1}\psi_k * 1_\Lambda)(x_-, \gamma(x_-))\,, \tag{37}$$

we have that $1^\Gamma_{\Lambda,k}(x_-) = 1^\Gamma_{\Lambda,k}(x_1)$, and also, recalling (19), the leafwise Young inequality (15), together with the first claim of (7), we find

$$\sup_{k,\Gamma} \|1^\Gamma_{\Lambda,k}\|_{B^{1/p'}_{p',\infty}(\mathbb{R})} < \infty\,, \qquad \sup_{k,\Gamma} \|1^\Gamma_{\Lambda,k}\|_{L_\infty(\mathbb{R})} < \infty\,. \tag{38}$$

Thus, using (18), and applying (22) from Lemma 3.3 again, we find, since $t > 0$,

$$\begin{aligned} 2^{\ell t}\|S_\ell(\Pi_2(\varphi, 1_\Lambda))\|^s_{p,\Gamma} &\le 2^{\ell t} 3 \sum_{k \ge \ell-1} \|S_k\varphi \cdot S_k 1_\Lambda\|^s_{p,\Gamma} \\ &\le 3 \sup_k 2^{kt}\|S_k\varphi\|^s_{p,\Gamma}(\|1^\Gamma_{\Lambda,k}\|_{B^{1/p'}_{p',\infty}(\mathbb{R})} + \|1^\Gamma_{\Lambda,k}\|_{L_\infty(\mathbb{R})}) \sum_{k\ge \ell-1} 2^{(\ell-k)t} \\ &\le C \sup_k 2^{kt}\|S_k\varphi\|^s_{p,\Gamma} \le C\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}}\,, \qquad \forall \ell \ge 0\,. \end{aligned}$$

It remains to bound the contribution of $\Pi_1(\varphi, 1_\Lambda)$. This is the trickiest estimate. It will use Lemma 3.5 and our assumption $t - (r-1) < s < -t < 0$. For any $\ell \ge 0$, we have, using again (15), (17), and (7),

$$2^{\ell t}\|\psi_\ell^{Op}(\Pi_1(\varphi, 1_\Lambda))\|^s_{p,\Gamma} \le \sum_{k=\ell-1}^{\ell+3} 2^{\ell t}\|S^{k-2}\varphi \cdot S_k 1_\Lambda\|^s_{p,\Gamma}\,. \tag{39}$$

We may focus on the term $k = \ell$, as the others are almost identical. We will use the paraproduct decomposition $\tilde\Pi_1 + \tilde\Pi_2 + \tilde\Pi_3$ and the operators $\tilde S_j$ and $\tilde S^j$ (see (24)). Put $(S^{k-2}\varphi)^\Gamma = (S^{k-2}\varphi) \circ \pi_\Gamma^{-1}$. By (20) and (17), we have

(40)

$$2^{kt}\|S^{k-2}\varphi \cdot S_k 1_\Lambda\|^s_{p,\Gamma} \le \sum_{i=1}^{2} 2^{kt}\|\tilde\Pi_i((S^{k-2}\varphi)^\Gamma, 1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}} + 2^{kt}\mathcal{R}^\Gamma_{k,s,p,\Lambda}(\varphi)$$

$$+\, 2^{kt} \sum_{m=k-1}^{k+1} \sum_{j=m+2}^{m+2+C_0} \|\tilde S_j((S^{k-2}\varphi)^\Gamma)(\tilde S_m 1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}}\,, \tag{41}$$

taking $C_0 \ge 2$ from Lemma 3.5, using (9), and setting

$$\mathcal{R}^\Gamma_{k,s,p,\Lambda}(\varphi) = \sum_{m=k-1}^{k+1} \sum_{j=m+2+C_0+1}^{\infty} \|\tilde S_j((S^{k-2}\varphi)^\Gamma)(\tilde S_m 1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}}\,.$$

Lemma 3.3 and the Young inequality (thrice) give C so that for all j, k, m, and Γ

$$\begin{aligned} &\|\tilde S_j((S^{k-2}\varphi)^\Gamma)(\tilde S_m 1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \\ &\qquad \le C\|\tilde S_j((S^{k-2}\varphi)^\Gamma)\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}(\|1^\Gamma_{\Lambda,k}\|_{B^{1/p'}_{p',\infty}(\mathbb{R})} + \|1^\Gamma_{\Lambda,k}\|_{L_\infty(\mathbb{R})}) \\ &\qquad \le C\|\tilde S_j((S^{k-2}\varphi)^\Gamma)\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \le C\|(S^{k-2}\varphi)^\Gamma\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\,, \end{aligned} \tag{42}$$

where we applied (38) in the second inequality. Thus, Lemma 3.5 and the leafwise[12] Young inequality (15) applied to $k_s = j \ge k + 2 + C_0$ gives $k_0 \ge C_0$ so that for any

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[12] See §4 of Corrections and complements to [2] for the factor $2^{k(-s+\delta)}$.

$\delta \in (0,1)$ (recalling $0 < t-s < r-1 < r-\delta$)

$$\begin{aligned}\sup_{k\ge k_0,\Gamma} 2^{kt}\mathcal{R}^{\Gamma}_{k,s,p,\Lambda}(\varphi) &\le 3C_0 C \sup_{k,\Gamma} 2^{k(t-r-s+\delta)}\Big(\sum_{j=k+2+C_0}^{\infty} 2^{-(j-k)r}\Big)\,\|S^{k-2}\varphi\|^{s}_{p,\Gamma}\\ &\le 3C_0 C\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}}\,.\end{aligned}\tag{43}$$

Using again (42), the finite double sum in (41) is bounded by $(C_0+4)C\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}}$.

For the contribution of $\tilde\Pi_1$ in (40), using again (20) and (17), we find

$$2^{kt}\|\tilde\Pi_1((S^{k-2}\varphi)^\Gamma, 1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \le 2^{kt}\sum_{n=k-1}^{k+1}\|(\tilde S^{n-2}(S^{k-2}\varphi)^\Gamma)\cdot \tilde S_n(1^\Gamma_{\Lambda,k}))\|_{B^s_{p,\infty}}\,.$$

Setting $(S_j\varphi)^\Gamma = (S_j\varphi)\circ\pi_\Gamma^{-1}$, we bound the term for $n=k$ above[13] by the sum of

$$2^{kt}\sum_{\ell=-1}^{1} 2^{(k+\ell)s}\|[\sum_{j=0}^{k-2}\sum_{m=j+C_0}^{k-2}\tilde S_{k+\ell}(\tilde S_m(S_j\varphi)^\Gamma)]\cdot\tilde S_k(1^\Gamma_{\Lambda,k})\|_{L_p(\mathbb{R}^{d_s})}\,,$$

(which can be handled as in (43), by Lemma 3.5), and,

$$\begin{aligned}&2^{kt}\sum_{\ell=-1}^{1} 2^{(k+\ell)s}\|[\sum_{j=0}^{k-2}\sum_{m=0}^{j+C_0-1}\tilde S_{k+\ell}(\tilde S_m(S_j\varphi)^\Gamma)]\cdot\tilde S_k(1^\Gamma_{\Lambda,k})\|_{L_p(\mathbb{R}^{d_s})}\\ &\le \Big(\sup_{0\le j\le k-2}\sum_{m=0}^{j+C_0-1} 2^{(j-m)s}\Big)\\ &\qquad\cdot 2^{kt}\sum_{\ell=-1}^{1}\sum_{j=0}^{k-2}\sup_{0\le m<j+C_0} 2^{(k+\ell-j+m)s}\|[\tilde S_{k+\ell}(\tilde S_m(S_j\varphi)^\Gamma)]\cdot\tilde S_k(1^\Gamma_{\Lambda,k})\|_{L_p}\\ &\le C2^{C_0|s|}\sum_{j=0}^{k-2}\sup_{\substack{0\le m<j+C_0\\ -1\le\ell\le 1}} 2^{(k+\ell-j)(t+s)}2^{ms}2^{jt}\|\tilde S_{k+\ell}([\tilde S_m(S_j\varphi)^\Gamma]\cdot\tilde S_k(1^\Gamma_{\Lambda,k}))\|_{L_p(\mathbb{R}^{d_s})}\,,\end{aligned}$$

using that $s<0$. Now, since $s+t<0$, we get, using the Young inequality,

$$\begin{aligned}&\sum_{j=0}^{k-2}\sup_{\substack{0\le m<j+C_0\\ -1\le\ell\le 1}} 2^{(k+\ell-j)(t+s)}2^{ms}2^{jt}\|\tilde S_{k+\ell}([\tilde S_m(S_j\varphi)^\Gamma]\cdot\tilde S_k(1^\Gamma_{\Lambda,k}))\|_{L_p(\mathbb{R}^{d_s})}\\ &\qquad\le C\sup_m\sup_j 2^{ms}2^{jt}\|\tilde S_m(S_j\varphi)^\Gamma\|_{L_p(\mathbb{R}^{d_s})}\|\tilde S_k(1^\Gamma_{\Lambda,k})\|_{L_\infty(\mathbb{R})}\\ &\qquad\le C\sup_j 2^{jt}\|S_j\varphi\|^s_{p,\Gamma}\le C\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}}\,.\end{aligned}$$

Finally, using (20) once more, we bound the contribution of $\tilde\Pi_2$ in (40):

$$2^{kt}\|\tilde\Pi_2((S^{k-2}\varphi)^\Gamma, 1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}} \le 2^{kt}\sum_{\ell=-1}^{1}\|(\tilde S_{k+\ell}(S^{k-2}\varphi)^\Gamma)\cdot\tilde S_k(1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}}$$

$$\le 2^{kt}\tilde{\mathcal{R}}^\Gamma_{k,p,s,\Lambda}(\varphi) + 2^{kt}\sum_{\ell=-1}^{1}\sum_{\tilde\ell=2}^{C_0}\|(\tilde S_{k+\ell}(S_{k-\tilde\ell}\varphi)^\Gamma)\cdot\tilde S_k(1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\,,\tag{44}$$

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[13] The other terms are similar.

where

$$2^{kt}\tilde{\mathcal{R}}^{\Gamma}_{k,p,s,\Lambda}(\varphi) = 2^{kt} \sum_{\ell=-1}^{1} \sum_{\tilde\ell=C_0+1}^{k} \|(\tilde S_{k+\ell}(S_{k-\tilde\ell}\varphi)^{\Gamma}) \cdot \tilde S_k(1^{\Gamma}_{\Lambda,k})\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}$$

can be bounded similarly as (43), using Lemma 3.5. For the remaining finite double sum in (44), we focus on the contributions with $\ell = 0$ and $\tilde\ell = 2$, the others being similar. Then, applying Lemma 3.3, we find

$$\begin{aligned}
&\sup_{k,\Gamma} 2^{kt}\|(\tilde S_k(S_{k-2}\varphi)^{\Gamma}) \cdot \tilde S_k(1^{\Gamma}_{\Lambda,k})\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \\
&\quad \le \sup_{k,\Gamma} 2^{kt}\|(\tilde S_k(S_{k-2}\varphi)^{\Gamma})\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}(\|1^{\Gamma}_{\Lambda,k}\|_{B^{1/p'}_{p',\infty}(\mathbb{R})} + \|1^{\Gamma}_{\Lambda,k}\|_{L_\infty(\mathbb{R})}) \le C\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_+,t,s}}\,,
\end{aligned}$$

using (38) once more. This ends the proof of Theorem 3.1. □

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