BANACH SPACES FOR PIECEWISE CONE HYPERBOLIC MAPS

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Abstract. We consider piecewise cone hyperbolic systems satisfying a bunching condition and we obtain a bound on the essential spectral radius of the associated weighted transfer operators acting on anisotropic Sobolev spaces. The bunching condition is always satisfied in dimension two, and our results give a unifying treatment of the work of Demers-Liverani [DL08] and our previous work [BG09]. When the complexity is subexponential, our bound implies a spectral gap for the transfer operator corresponding to the physical measures in many cases (for example if T preserves volume, or if the stable dimension is equal to 1 and the unstable dimension is not zero).

1. Introduction

The "spectral" or "functional" approach to study statistical properties of dynamical systems with enough hyperbolicity, originally limited to one-dimensional dynamics, has greatly expanded its range of applicability in recent years. The following spectral gap result of[1] Blank–Keller–Liverani [BKL02] appeared in 2002:

Theorem 1.1. *Let $T : X \to X$ be a C^3 Anosov diffeomorphism on a compact Riemannian manifold, with a dense orbit. Define a bounded linear operator by*

$$\mathcal{L}\omega = \frac{\omega \circ T^{-1}}{|\det DT \circ T^{-1}|}, \quad \omega \in L^\infty(X). \tag{1.1}$$

Then there exist a Banach space $\mathcal{B}$ of distributions on X, containing $C^\infty(X)$, and a bounded operator on $\mathcal{B}$, coinciding with $\mathcal{L}$ on $\mathcal{B} \cap L^\infty(X)$ and denoted also by $\mathcal{L}$, with the following properties: The spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is equal to one, the essential spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is strictly smaller than one, $\mathcal{L}$ has a fixed point in $\mathcal{B}$. Finally, 1 is the only eigenvalue on the unit circle, and it is simple.

It is a remarkable fact that "Perron-Frobenius-type" spectral information as in the above theorem (possibly with a nonsimple real maximal eigenvalue of finite multiplicity and other eigenvalues on the unit circle) gives simpler proofs of many known theorems, but also new information. Among these consequences, let us just mention: Existence of finitely many physical measures whose basins have full measure (working with slightly more general transfer operators, one can treat other equilibrium states), exponential decay of correlations for physical measures and Hölder observables, statistical and stochastic stability, linear response and the

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[1] Various improvements of this result have been obtained since then, [GL06, GL08, BT07, BT08], in particular in the Axiom A setting.

linear response formula, central and local limit theorems, location of the poles of dynamical zeta functions and zeroes of dynamical determinants, smooth Anosov systems with holes, etc. (We just recall that the dual of $\mathcal{L}$ preserves Lebesgue measure, so that the fixed point of $\mathcal{L}$ corresponds to the physical measure. See [BT07] and [GL08].)

One of the advantages of this "functional approach" is that it bypasses the construction of Markov partitions and the need to introduce artificial "one-sided" expanding endomorphisms (such endomorphisms only retain a small part of the smoothness of the original hyperbolic diffeomorphism).

Billiards with convex scatterers, also called Sinai billiards, are among the most natural and interesting dynamical systems. They are uniformly hyperbolic, preserve Liouville measure, but they are only piecewise smooth. Analyzing the difficulties posed by the singularities has been an important challenge for mathematicians, and it is only in 1998 that L.-S. Young [You98] proved that the Liouville measure enjoys exponential decay of correlations for two-dimensional Sinai billiards (under a finite horizon condition, which was shortly thereafter removed by Chernov [Che99]). It should be noted that these results were in fact obtained for a discrete-time version of the billiard flow. Indeed the question of whether the original two-dimensional continuous-time Sinai billiard enjoys decay of correlations is to this day still open. (Chernov [Che07] recently obtained stretched exponential upper bounds.) It is well known that the continuous-time case is much more difficult, and it seems that the ideas of Dolgopyat [Dol98] which were exploited in several smooth hyperbolic situations are not compatible with the tools used in [You98] for example. We believe that a new, "functional," proof (via a spectral gap result for the transfer operator (1.1) on a suitable anisotropic Banach space of distributions) of exponential decay of correlations for *discrete-time* surface Sinai billiards will be a key stepping stone towards the expected proof of exponential decay of correlations for the *continuous-time* Sinai billiards.

The recent paper of Demers-Liverani [DL08] was a first breakthrough in this direction, as we explain next. Since none of the spaces of [GL06, GL08, BT07, BT08] behave well with respect to multiplication by characteristic functions of sets, they cannot be used for systems with singularities. Demers–Liverani [DL08] therefore introduced some new Banach spaces, on which transfer operators associated to two-dimensional piecewise hyperbolic systems admit a spectral gap. However, the construction and the argument of [DL08] are quite intricate, in particular, pieces of stable or unstable manifolds are iterated by the dynamics, and the way they are cut by the discontinuities has to be studied in a very careful way, in the spirit of [You98] and [Che99]. As a consequence, adapting the approach in [DL08] to billiards (which are not piecewise hyperbolic, stricto sensu, because their derivatives blow up along the singularity lines) is daunting.

Another progress in the direction of a modern proof of exponential decay of correlations for discrete-time billiards is our previous paper [BG09]. There, we showed that ideas of Strichartz [Str67] imply that classical anisotropic Sobolev spaces $H_p^{t,s}$ in the Triebel-Lizorkin class [Tri77] (Definition 2.6, these spaces had been introduced in dynamics in [Bal05]) are suitable for piecewise hyperbolic systems, under the condition that the system admits a smooth (at least C^1) stable foliation. Unfortunately, although it holds for several nontrivial examples, this condition is pretty restrictive: In general, the foliations are only measurable!

In the present paper, we consider piecewise smooth piecewise hyperbolic dynamics. We are able to remove the assumption of smoothness of the stable foliation, whenever the hyperbolicity exponents of the system satisfy a *bunching* condition (see (2.3) and (2.4) below). This condition is rather standard in smooth hyperbolic dynamics, where it ensures that the dynamical foliations are C^1 instead of the weaker Hölder condition which holds in full generality (see [HPS77], or, e.g., [HK95]). The bunching condition is always satisfied in codimension one (in particular, it holds in dimension two, so that our results apply to physical measures of all surface piecewise hyperbolic systems previously covered in [You98] or [DL08], in particular to hyperbolic Lozi maps possessing a compact invariant domain, see Appendix D). The present paper requires the dynamics to be $C^{1+\alpha}$ on each (closed) domain of smoothness, and therefore does not apply directly to discrete-time Sinai billiard. However, we expect that it will be possible to adapt the methods here to obtain the desired functional proof of exponential decay of correlations for two-dimensional Sinai billiards. We shall use the terminology "cone-hyperbolic" to stress that hyperbolicity is defined in terms of cones and that there is *a priori* no invariant stable distribution, contrary to our previous paper [BG09].

We use the Triebel spaces $H_p^{t,s}$ as building blocks in the construction of our new Banach spaces $\mathbf{H}_p^{t,s}(R)$ (Definition 2.12) and $\mathbf{H}$ (see (2.20)). As a consequence, we may exploit, as we did in [BG09], the rich existing theory (in particular regarding interpolation), and use again the results of Strichartz [Str67].

The new ingredient with respect to [BG09] is that we define our norm by considering the Triebel norm in $\mathbb{R}^d$ through suitable C^1 charts, taking now the *supremum* over *all* cone-admissible charts $\mathcal{F}$ (Definition 2.7). We use the bunching assumption to show that the family is invariant under iteration (Lemma 3.3). *Indeed, this is how we avoid the necessity for a smooth stable foliation.* As in [BG09], we do not iterate single stable or unstable manifolds (contrary to [You98, Che99, DL08]), and we do not need to match nearby stable or unstable manifolds: Everything follows from an appropriate functional analytic framework.

Our main result, Theorem 2.5, is an upper bound on the essential spectral radius of weighted transfer operators associated to cone hyperbolic systems satisfying the bunching condition and acting on a Banach space $\mathbf{H}$ of anisotropic distributions. If the complexity growth (as measured by (2.2)) is subexponential, and if either $\det DT \equiv 1$, or $d_s = 1$ and $d_u > 0$, then one can always choose the Banach space so that the transfer operator (1.1) has essential spectral radius strictly smaller than 1, and thus a spectral gap. This spectral gap property gives finiteness and exponential mixing (up to a finite period) of the physical measures (see e.g. Theorem 33 in [BG09], or its generalization below, Theorem D.5).

Let us mention here that all existing results on piecewise hyperbolic systems, including the present one, require some kind of transversality condition between the discontinuity hypersurfaces and the stable or unstable dynamical directions or cones (see Definition 2.3). (This condition is satisfied for billiards, modulo Remark 2.4.)

The paper is organized as follows. In Section 2 we define formally the dynamical systems for which our results hold and the anisotropic spaces $\mathbf{H}$ on which the transfer operator will act: Subsection 2.1 contains the assumptions on the dynamics and the statement of our main result, Theorem 2.5. In Subsection 2.2, we recall the definition of the Triebel spaces $H_p^{t,s}$ and we define the cone-admissible foliations $\mathcal{F}(C_0, C_1)$, depending on two parameters C_0 and C_1 that should be suitably chosen. In Subsection 2.3, we combine these two ingredients, together with a "zoom" by a large factor $R > 1$, to construct the Banach spaces of distributions $\mathbf{H}_p^{t,s}(R, C_0, C_1)$. Subsection 2.4 contains a technical step which reduces our main result to a more convenient form, Theorem 2.14, constructing along the way the final Banach spaces $\mathbf{H}$ from the $\mathbf{H}_p^{t,s}(R, C_0, C_1)$.

Section 3 is devoted to the proof of invariance of the class $\mathcal{F}$ of admissible foliations. This is the heart of our argument, and the main new technical ingredient is Lemma 3.3. Its proof is based on the usual Hadamard-Perron graph transform ideas (see (3.11)–(3.13)), but requires to be spelt out in full detail in order to discover the appropriate conditions in Definition 2.7.

Section 4 contains various results on the local spaces $H_p^{t,s}$, in particular the corresponding "Leibniz" (Lemma 4.1) and "chain-rule" (Lemmas 4.6 and 4.7) estimates, and the fact that characteristic functions of appropriate sets are bounded multipliers (Lemma 4.2). These results are mostly adapted from [BG09]. Subsection 4.1 also contains a compactness embedding statement for spaces $\mathbf{H}_p^{t,s}(R, C_0, C_1)$ (Lemma 4.4) which is crucial for our Lasota-Yorke-type estimate in the proof of our main result.

Finally, Section 5 contains the proof of Theorem 2.14.

Four appendices contain some complements: Appendices A and B contain useful technical results, Appendix C describes some extensions of our main result (which allow us in particular to sometimes weaken our transversality assumption), and Appendix D gives consequences concerning physical measures of our main result and its extension.

Note that the methods in this paper do not allow to exploit the additional smoothness available if T is Anosov or Axiom A and C^r for $r > 2$ (even if they satisfy the bunching condition), contrarily to [GL06, GL08, BT07, BT08]. The present work is thus complementary to the approach of [GL06, GL08, BT07, BT08] which gives more information in the smooth case (but fails when there are singularities).

2. Definitions and statement of the spectral theorem

2.1. **The main result.** Let X be a Riemannian manifold of dimension $d \geq 2$ without boundary, and let X_0 be a compact subset of X. We view $1 \leq d_s \leq d-1$ and $d_u = d - d_s \geq 1$ as being fixed integers, so constants may depend on these numbers[2]. We call C^1 hypersurface a codimension-one C^1 submanifold of X, possibly with boundary. We say that a function g is C^r for $r > 0$ if g is $C^{[r]}$ and all partial derivatives of order $[r]$ are $r - [r]$-Hölder. The norm of a vector (in the tangent space of X, or in $\mathbb{R}^d$) will be denoted by $|v|$.

Hyperbolicity will be defined in terms of cones, and we shall need the cones to satisfy some form of convexity (in (3.8)). Even the simplest linear cone $|x|^2 \leq |y_1|^2 + |y_2|^2$ in $\mathbb{R}^3$ is not convex in the usual sense (it contains $(1,1,0)$ and $(1,0,1)$ but not $(1, 1/2, 1/2)$). Therefore, we introduce the following definition:

Definition 2.1. *A cone of dimension $d' \in [1, d-1]$ in $\mathbb{R}^d$ is a closed subset C of $\mathbb{R}^d$ with nonempty interior, invariant under scalar multiplication, such that d' is the maximal dimension of a vector subspace included in C.*

A cone C of dimension d' is transverse to a vector subspace E of $\mathbb{R}^d$ if E contains a subspace of dimension $d - d'$ which intersects C only at 0.

A cone C is convexly transverse to a vector subspace E if C is transverse to E and, additionally, for all $z \in \mathbb{R}^d$, $C \cap (E + z)$ is convex.

Two cones C_u and C_s, of respective dimensions d_u and d_s, with $d_u + d_s = d$, are convexly transverse if $C_u \cap C_s = \{0\}$, for any vector subspace $E_s \subset C_s$ the cone C_u is convexly transverse to E_s, and for any vector subspace $E_u \subset C_u$ the cone C_s is convexly transverse to E_u.

We claim that if $A : \mathbb{R}^{d_s} \to \mathbb{R}^k$ is a nonzero linear map then the set $C_A = \{(x, y) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_s} \mid |x| \leq |Ay|\}$ (which obviously contains the d_s-dimensional vector subspace $\{(0, y)\}$) is a d_s-dimensional cone which is convexly transverse to $\{(x, 0)\}$. See Appendix B for the easy proof of this claim. It follows that, if C_A and $C_{A'}$ are cones in $\mathbb{R}^d$ associated (not necessarily for the same coordinates) to

[2]Our methods also work when $d_s = 0$ or $d_u = 0$, but they do not improve on the results of [BG09] since the stable and unstable manifolds are automatically smooth in this case.

nonzero linear maps $A : \mathbb{R}^{d_s} \to \mathbb{R}^{k_1}$ and $A' : \mathbb{R}^{d_u} \to \mathbb{R}^{k_2}$, then C_A and $C_{A'}$ are convexly transverse if and only if $C_A \cap C_{A'} = \{0\}$. Definition 2.1 is slightly more flexible than such linear cones. More importantly, it sheds light on the essence of the convexity assumption.

Definition 2.2 (Piecewise $C^{1+\alpha}$ cone hyperbolic maps)**.** *Let $\alpha \in (0,1]$. A piecewise $C^{1+\alpha}$ (cone) hyperbolic map is a map $T : X_0 \to X_0$ such that there exist finitely many pairwise disjoint open subsets $(O_i)_{i\in I}$, covering Lebesgue almost all X_0, so that each ∂O_i is a finite union of C^1 hypersurfaces, and so that for each $i \in I$:*

(1) There exists a $C^{1+\alpha}$ map T_i defined on a neighborhood $\widetilde{O}_i$ of $\overline{O_i}$ in X, which is a diffeomorphism onto its image and such that $T|_{O_i} = T_i|_{O_i}$.

(2) There exist two families of convexly transverse cones $\mathcal{C}_i^{(u)}(q)$ and $\mathcal{C}_i^{(s)}(q)$ in the tangent space $\mathcal{T}_qX$, depending continuously on $q \in \overline{O_i}$, so that $\mathcal{C}_i^{(u)}(q)$ is d_u-dimensional and $\mathcal{C}_i^{(s)}(q)$ is d_s-dimensional, and such that:

(2.a) For each $q \in \overline{O_i} \cap T_i^{-1}(\overline{O_j})$, then $DT_i(q)\mathcal{C}_i^{(u)}(q) \subset \mathcal{C}_j^{(u)}(T_i(q))$, and there exists $\lambda_{i,u}(q) > 1$ such that

$$|DT_i(q)v| \ge \lambda_{i,u}(q)|v| \,, \forall v \in \mathcal{C}_i^{(u)}(q) \,.$$

(2.b) For each $q \in \overline{O_i} \cap T_i^{-1}(\overline{O_j})$, then $DT_i^{-1}(T_i(q))\mathcal{C}_j^{(s)}(T_i(q)) \subset \mathcal{C}_i^{(s)}(q)$, and there exists $\lambda_{i,s}(q) \in (0,1)$ such that

$$|DT_i^{-1}(T_i(q))v| \ge \lambda_{i,s}^{-1}(q)|v| \,, \forall v \in \mathcal{C}_j^{(s)}(T_i(q)) \,.$$

Note that we do not assume that T is continuous or injective on X_0.

We introduce some notation. For $n \ge 1$, and $\mathbf{i} = (i_0, \dots, i_{n-1}) \in I^n$ we let $T_{\mathbf{i}}^n = T_{i_{n-1}} \circ \cdots \circ T_{i_0}$, which is defined on a neighborhood of $\overline{O_{\mathbf{i}}}$, where $O_{(i_0)} = O_{i_0}$, and

$$O_{(i_0,\dots,i_{n-1})} = \{q \in O_{i_0} \mid T_{i_0}(q) \in O_{(i_1,\dots,i_{n-1})}\} \,. \tag{2.1}$$

Denote by $\lambda_{\mathbf{i},s}^{(n)}(q) < 1$ and $\lambda_{\mathbf{i},u}^{(n)}(q) > 1$ the weakest contraction and expansion coefficients of $T_{\mathbf{i}}^n$ at q, and by $\Lambda_{\mathbf{i},s}^{(n)}(q) \le \lambda_{\mathbf{i},s}^{(n)}(q)$ and $\Lambda_{\mathbf{i},u}^{(n)}(q) \ge \lambda_{\mathbf{i},u}^{(n)}(q)$ its strongest contraction and expansion coefficients. We put

$$\lambda_{s,n}(q) = \sup_{\mathbf{i}} \lambda_{\mathbf{i},s}^{(n)}(q) < 1 \,, \quad \lambda_{u,n}(q) = \inf_{\mathbf{i}} \lambda_{\mathbf{i},u}^{(n)}(q) > 1 \,,$$

where the infimum and the supremum are restricted to those $\mathbf{i}$ such that $q \in \overline{O_{\mathbf{i}}}$.

As is usual in piecewise hyperbolic settings, we shall require a transversality assumption on the discontinuity hypersurfaces[3]:

Definition 2.3 (Transversality condition)**.** *Let T be a piecewise $C^{1+\alpha}$ hyperbolic map. We say that T satisfies the transversality condition if each ∂O_i is a finite union of C^1 hypersurfaces $K_{i,k}$ which are everywhere transverse to the stable cones, i.e., for all $q \in K_{i,k}$, $\mathcal{T}_qK_{i,q}$ contains a d_u-dimensional subspace that intersects $\mathcal{C}_i^{(s)}(q)$ only at 0.*

Remark 2.4 (Transversality in the image). If the cone field is continuous (i.e., it does not really depend on i, as is the case with Sinai billiards), then one can weaken this requirement, by demanding only that the images $T(K_{i,q})$ are transverse to the stable cone (see Appendix C for details). When the cone fields are not globally continuous, the stronger requirement in Definition 2.3 is necessary to ensure that C^1 functions belong to the Banach space $\mathbf{H}$ we shall construct below (see the argument after Definition 2.12).

[3]This condition is unrelated to the "convex transversality" assumption on the cones!

To estimate dynamical complexity, we define the n-complexities at the beginning and at the end:

$$D_n^b = \max_{q \in X_0} \operatorname{Card}\{\mathbf{i} \in I^n \mid q \in \overline{O_{\mathbf{i}}}\}, \quad D_n^e = \max_{q \in X_0} \operatorname{Card}\{\mathbf{i} \in I^n \mid q \in \overline{T^n(O_{\mathbf{i}})}\}. \tag{2.2}$$

(If T is a globally invertible piecewise cone hyperbolic map for the covering (O_i, i), with n-complexity at the end D_n^e denoted by $D_n^e(T, (O_i, i))$, then T^{-1} is piecewise cone hyperbolic for the covering $(T(O_i))$, and its n-complexity at the beginning satisfies $D_n^b(T^{-1}, (T(O_i), i)) = D_n^e(T, (O_i, i))$. For $T(x) = 2x \bmod 1$ on $[0,1]$ we have $D_n^e = 2^n$, but fortunately this quantity plays no role for the transfer operator associated to $g = |\det DT|^{-1}$ when $d_s = 0$, up to taking p close enough to 1 in Theorem 2.5.)

Our main result can now be stated (all Jacobians in this paper are relative to Lebesgue measure, and $|\det DT|$ denotes the Jacobian of T):

Theorem 2.5 (Spectral theorem)**.** *Let $\alpha \in (0,1]$, and let T be a piecewise $C^{1+\alpha}$ cone hyperbolic map satisfying the transversality condition. Assume in addition the following bunching[4] condition: For some $n > 0$,*

$$\sup_{\mathbf{i} \in I^n,\, q \in \overline{O}_{\mathbf{i}}} \frac{\lambda_{\mathbf{i},s}^{(n)}(q)^{\alpha} \Lambda_{\mathbf{i},u}^{(n)}(q)}{\lambda_{\mathbf{i},u}^{(n)}(q)} < 1. \tag{2.3}$$

Let $\beta \in (0, \alpha)$ be small enough so that

$$\sup_{\mathbf{i} \in I^n,\, q \in \overline{O}_{\mathbf{i}}} \frac{\lambda_{\mathbf{i},s}^{(n)}(q)^{\alpha-\beta} \Lambda_{\mathbf{i},u}^{(n)}(q)^{1+\beta}}{\lambda_{\mathbf{i},u}^{(n)}(q)} < 1. \tag{2.4}$$

Let $1 < p < \infty$ and let $t, s \in \mathbb{R}$ be so that

$$1/p - 1 < s < 0 < t < 1/p, \quad -\beta < t - |s| < 0, \quad \alpha t + |s| < \alpha. \tag{2.5}$$

Then there exists a space $\mathbf{H} = \mathbf{H}(p,t,s)$ of distributions on X, containing C^1 and in which $L^\infty \cap \mathbf{H}$ is dense, and such that for any function $g : X_0 \to \mathbb{C}$ so that the restriction of g to each O_i admits a C^γ extension to $\overline{O_i}$ for some $\gamma > t + |s|$, the operator $\mathcal{L}_g$ defined on L^∞ by

$$(\mathcal{L}_g \omega)(q) = \sum_{T(q')=q} g(q')\omega(q')$$

extends continuously to $\mathbf{H}$. Moreover, its essential spectral radius on $\mathbf{H}$ is at most

$$\lim_{n\to\infty} (D_n^b)^{1/(pn)} \cdot (D_n^e)^{(1/n)(1-1/p)} \cdot \left\| g^{(n)} |\det DT^n|^{1/p} \max(\lambda_{u,n}^{-t}, \lambda_{s,n}^{-(t-|s|)}) \right\|_{L^\infty}^{1/n}, \tag{2.6}$$

where we set $g^{(n)}(q) = \prod_{k=0}^{n-1} g(T^k(q))$, for $n \geq 1$.

Our proof does not give good bounds on the spectral radius of $\mathcal{L}_g$ on $\mathbf{H}$. However, if $g = |\det DT|^{-1}$ and the bound in (2.6) is < 1, then Theorem 33 in [BG09] implies that the spectral radius is equal to 1, and that T has finitely many physical measures, attracting Lebesgue almost every point of the manifold. For details, we refer the reader to Appendix D, where we also explain how to iterate the map in the other direction of time to get different conditions under which this conclusion holds. (These conditions are satisfied whenever $d_s = 1$, they apply for instance to any Lozi map with a compact invariant domain X_0, see Corollary D.4.)

The limit in (2.6) exists by submultiplicativity. We can bound $\lambda_{s,n}$ and $\lambda_{u,n}^{-1}$ by λ^{-n}, where $\lambda > 1$ is the weakest rate of contraction/expansion of T. Therefore, if

[4] Condition (2.3) always holds if $d_u = 1$.

$g = |\det DT|^{-1}$ then the essential spectral radius is strictly smaller than 1 if there exist s, t and p as in Theorem 2.5 with

$$\lim_{n\to\infty} (D_n^b)^{1/(pn)} \cdot (D_n^e)^{(1/n)(1-1/p)} \cdot \Big\| |\det DT^n|^{1/p-1} \Big\|_{L^\infty}^{1/n} < \lambda^{\min(t,-(t-|s|))} .$$

In particular, if $g = |\det DT|^{-1} \equiv 1$, then the essential spectral radius is strictly smaller than 1 if $\lim_{n\to\infty}(D_n^b)^{1/(pn)}(D_n^e)^{(1/n)(1-1/p)} < \lambda^{\min(t,-(t-|s|))}$, that is, if hyperbolicity dominates complexity.

Subsections 2.2 and 2.3 are devoted to the definition of spaces $\mathbf{H}_p^{t,s}(R, C_0, C_1)$ which will give the space $\mathbf{H}$ of Theorem 2.5 via Proposition 2.15 (see (2.20)). Let us now describe briefly this space $\mathbf{H}$, which generalizes the spaces of [Bal05, BG09]. Intuitively, an element of $\mathbf{H}$ is a distribution which has t derivatives in L^p in all directions together with s derivatives in L^p in the stable direction. This amounts to $s+t$ derivatives in L^p in the stable direction, and t derivatives in the transverse "unstable" direction. Since $t > 0$ and $t+s = t-|s| < 0$, the transfer operator increases regularity in this space. The restriction $1/p - 1 < s < 0 < t < 1/p$ is designed so that this space is stable under multiplication by characteristic functions of nice sets (see [BG09, Lemma 23]) — this makes it possible to deal with discontinuous maps. If one assumes that there exists a C^1 stable direction, the above rough description can be made precise, using anisotropic Sobolev spaces: This was done in [BG09][5]. In our setting, there is in general not even a continuous stable direction, so we shall instead use a *class* of local foliations (with uniformly bounded $C^{1+\beta}$ norms) compatible with the stable cones, and define our norm as the supremum of the anisotropic Sobolev norms over all local foliations in this class (Definition 2.12). To ensure that the space so defined is invariant under the action of the transfer operator, one should make sure that the preimage under iterates of T of a foliation in our class remains in our class: This is the content of our key Lemma 3.3. Since we want those foliations to have bounded C^1 norm (otherwise, the argument for anisotropic Sobolev norms fails), we need the bunching condition (2.3) to prove this invariance. (In the smooth, i.e., Axiom A case, (2.3) would ensure that the stable foliation is C^1 — see e.g. [HK95, §19.1] in the case $\alpha = 1$ — and the strengthening (2.4) would even ensure that the stable foliation is $C^{1+\beta}$. In the general piecewise smooth case, the foliation is only measurable, even if (2.3) holds.)

2.2. **Anisotropic spaces $H_p^{t,s}$ in $\mathbb{R}^d$ and the class $\mathcal{F}(z_0, \mathcal{C}^s, C_0, C_1)$ of local foliations.** In this subsection, we recall the anisotropic spaces $H_p^{t,s}$ in $\mathbb{R}^d$ (which were used in [BG09]), and we define a class $\mathcal{F}$ of cone-admissible local foliations in $\mathbb{R}^d$ with uniformly bounded $C^{1+\beta}$ norms (in Lemma 3.3 we shall show that this class is invariant under iterations of the dynamics). These are the two building blocks that we shall use in Section 2.3 to define our spaces of distributions.

We write $z \in \mathbb{R}^d$ as $z = (x, y)$ where $x = (z_1, \dots, z_{d_u})$ and $y = (z_{d_u+1}, \dots, z_d)$. The subspaces $\{x\} \times \mathbb{R}^{d_s}$ of $\mathbb{R}^d$ will be referred to as the *stable leaves* in $\mathbb{R}^d$. We say that a diffeomorphism of $\mathbb{R}^d$ *preserves stable leaves* if its derivative has this property. For $r > 0$ and $z = (x, y) \in \mathbb{R}^d$, let us write $B(x, r) = \{x' \in \mathbb{R}^{d_u} \mid |x' - x| \le r\}$, $B(y, r) = \{y' \in \mathbb{R}^{d_s} \mid |y' - y| \le r\}$ and $B(z, r) = B(x, r) \times B(y, r)$. We denote the Fourier transform in $\mathbb{R}^d$ by $\mathbf{F}$. An element of the dual space of $\mathbb{R}^d$ will be written as (ξ, η) with $\xi \in \mathbb{R}^{d_u}$ and $\eta \in \mathbb{R}^{d_s}$.

The local anisotropic Sobolev spaces $H_p^{t,s}$ belong to a class of spaces first studied by Triebel [Tri77]:

[5] The local spaces in Definition 2.6 are the same as those in [BG09].

Definition 2.6 (Sobolev spaces $H_p^{t,s}$ and H_p^t in $\mathbb{R}^d$). *For $1 < p < \infty$, and t, $s \in \mathbb{R}$, let $H_p^{t,s}$ be the set of (tempered) distributions w in $\mathbb{R}^d$ such that*

$$\|w\|_{H_p^{t,s}} := \left\|\mathbf{F}^{-1}(a_{t,s}\mathbf{F}w)\right\|_{L^p} < \infty\,, \tag{2.7}$$

where

$$a_{t,s}(\xi,\eta) = (1+|\xi|^2+|\eta|^2)^{t/2}(1+|\eta|^2)^{s/2}. \tag{2.8}$$

For $1 < p < \infty$, $t \in \mathbb{R}$, the set $H_p^t = H_p^{t,0}$ is the standard (generalized) Sobolev space.

Triebel proved that rapidly decaying C^∞ functions are dense in each $H_p^{t,s}$ (see e.g. [BG09, Lemma 18]). In particular, we could equivalently define $H_p^{t,s}$ to be the closure of rapidly decaying C^∞ functions for the norm (2.7).

We shall work with local foliations indexed by points m in appropriate finite subsets of $\mathbb{R}^d$ (defined in (2.15) below). The following definition of the class of foliations is the key new ingredient of the present work. We view $\alpha \in (0,1]$ and $\beta \in (0,\alpha]$ as fixed (like in the statement of Theorem 2.5) while the constants $C_0 > 1$ and $C_1 > 2C_0$ will be chosen later. These constants play the following role: if C_0 is large, then the admissible foliation covers a large domain; if C_1 is large, then the leaves of the foliation are almost parallel.

Definition 2.7 (Sets $\mathcal{F}(m,\mathcal{C}^s,C_0,C_1)$ of cone-admissible foliations at $m\in\mathbb{R}^d$). *Let $\mathcal{C}^s$ be a d_s-dimensional cone in $\mathbb{R}^d$, transverse to $\mathbb{R}^{d_u}\times\{0\}$, let $m=(x_m,y_m)\in\mathbb{R}^d$, and let $1 < C_0 < C_1/2$. The set $\mathcal{F}(m,\mathcal{C}^s,C_0,C_1)$ of $\mathcal{C}^s$-admissible local foliations at m is the set of maps*

$$\phi = \phi_F : B(m,C_0)\to\mathbb{R}^d\,,\quad \phi_F(x,y) = (F(x,y),y)\,,$$

where $F : B(m,C_0)\to\mathbb{R}^{d_u}$ is C^1 and satisfies

$$(\partial_y F(z)w,w)\in\mathcal{C}^s\,,\forall w\in\mathbb{R}^{d_s}\,,\forall z\in B(m,C_0)\,;\quad F(x,y_m)=x\,,\,\forall x\in B(x_m,C_0)\,,$$

and, for all (x,y) and (x',y') in $B(m,C_0)$,

$$|DF(x,y)-DF(x,y')| \le |y-y'|^\alpha/C_1\,, \tag{2.9}$$

$$|DF(x,y)-DF(x',y)| \le |x-x'|^\beta/C_1\,, \tag{2.10}$$

and

$$|DF(x,y)-DF(x,y')-DF(x',y)+DF(x',y')| \le |x-x'|^\beta|y-y'|^{\alpha-\beta}/C_1\,. \tag{2.11}$$

The set $\mathcal{F}(m,\mathcal{C}^s,C_0,C_1)$ is large, as we explain next: If the cone $\mathcal{C}^s$ is d_s-dimensional and transverse to $\mathbb{R}^{d_u}$, then it contains a d_s-dimensional vector subspace E which is transverse to $\mathbb{R}^{d_u}$. Therefore, there exists a (possibly zero) linear map $\mathbb{E}:\mathbb{R}^{d_s}\to\mathbb{R}^{d_u}$ so that $E=\{(\mathbb{E}w,w)\,,w\in\mathbb{R}^{d_s}\}$. It follows that the affine map $F_\mathbb{E}(x,y)=x+\mathbb{E}(y-y_m)$ is such that $\phi_{F_\mathbb{E}}\in\mathcal{F}(m,\mathcal{C}^s,C_0,C_1)$. Then, it is easy to see that if F is $C^{1+\alpha}$, with $F(x,y_m)=F_\mathbb{E}(x,y_m)=x$, and F is close enough to $F_\mathbb{E}$, then $\phi_F\in\mathcal{F}(m,\mathcal{C}^s,C_0,C_1)$. (To check (2.11), consider separately the cases $|x-x'|\le|y-y'|$ and $|x-x'|>|y-y'|$.)

We now collect easy but important consequences of the above definition. (See also the remarks at the end of this subsection about the technical conditions (2.9)–(2.11).) We shall see in Lemma 2.8 that the graphs $\{(F(x,y),y)\mid |y-y_m|<C_0\}$ for $|x-x_m|<C_0$ form a partition of a neighborhood of m of size proportional to C_0 (through the R-zoomed charts to be introduced in Section 2.3, this will correspond to a neighborhood of size of the order of C_0/R in the manifold), and their tangent space is everywhere contained in $\mathcal{C}^s$. The map F thus defines a local foliation (justifying the terminology), and the map ϕ_F is a diffeomorphism straightening this foliation, i.e., the leaves of the foliation are the images of the stable leaves of

$\mathbb{R}^d$ under the map ϕ_F. (The maps $y \mapsto (F(x,y), y)$ for fixed x are sometimes called *plaques,* while $x \mapsto F(x,y)$ for y fixed is the *holonomy* between the transversals of respective heights y_m and y.) Moreover, if C_1 is very large, then DF is close to constant, i.e., ϕ_F is very close to an affine map. The conditions in the definition up to (2.9) imply that the local foliation defined by F is $C^{1+\alpha}$ along the leaves. Moreover, the next lemma shows that these conditions imply uniform bounds on F (independent of C_0).

Lemma 2.8 (Admissible foliations are $C^{1+\beta}$ foliations)**.** *For any d_s-dimensional cone $\mathcal{C}^s$ transverse to $\mathbb{R}^{d_u} \times \{0\}$, there exists a constant $C_\#$ depending only on $\mathcal{C}^s$ such that, for any $1 < C_0 < C_1/2$, and any $\phi_F \in \mathcal{F}(m, \mathcal{C}^s, C_0, C_1)$, the map ϕ_F is a diffeomorphism onto its image with $\|D\phi_F\|_{C^\beta} \le C_\#$ and $\left\|D\phi_F^{-1}\right\|_{C^\beta} \le C_\#$. Moreover, $\phi_F(B(m, C_0))$ contains $B(m, C_\#^{-1}C_0)$.*

The proof of these claims does not require (2.11).

Proof. Let $\phi = \phi_F \in \mathcal{F}(m, \mathcal{C}^s, C_0, C_1)$. We first check that $\|DF\|_{C^0} \le C_\#$. Observe first that $\partial_y F$ is bounded since the cone $\mathcal{C}^s$ is transverse to $\mathbb{R}^{d_u} \times \{0\}$. Since $F(x, y_m) = x$, we have $\partial_x F(x, y_m) = \mathrm{id}$, hence (2.9) gives

$$|\partial_x F(x,y) - \mathrm{id}| = |\partial_x F(x,y) - \partial_x F(x,y_m)| \le |y - y_m|^\alpha / C_1 \le C_0^\alpha / C_1 < 1/2\,. \tag{2.12}$$

In particular, $|\partial_x F|$ is uniformly bounded. This shows that $\|DF\|_{C^0} \le C_\#$. We next observe that condition (2.10) together with (2.9) imply that DF is β-Hölder: There exists a constant $C_\#$ (independent of C_0) such that, for all pairs (x,y) and (x', y') in $B(m, C_0)$,

$$|DF(x,y) - DF(x',y')| \le C_\# d((x,y),(x',y'))^\beta\,. \tag{2.13}$$

Indeed, (2.9) gives $|DF(x,y) - DF(x,y')| \le |y - y'|^\alpha$, and (2.10) gives $|DF(x,y') - DF(x',y')| \le |x - x'|^\beta$. Since $\beta \le \alpha$, (2.13) follows. We have shown that $\|D\phi\|_{C^\beta} \le C_\#$.

For any vector v, (2.12) shows that $\langle \partial_x F v, v\rangle \ge |v|^2/2$. Integrating this inequality on the segment between x and x', for $v = x' - x$, we get $\langle F(x,y) - F(x',y), x - x'\rangle \ge |x - x'|^2/2$. In particular,

$$|F(x,y) - F(x',y)| \ge |x - x'|/2\,. \tag{2.14}$$

By Lemma A.1, this implies that the map ϕ belongs to the class $\mathcal{D}(C_\#)$ defined in Subsection A.1, for some $C_\# > 0$ independent of C_0, C_1. In particular, ϕ is a diffeomorphism onto its image, and $|D\phi^{-1}| \le C_\#$. Since $D\phi$ is β-Hölder, it follows that $D\phi^{-1}$ is also β-Hölder, and $\left\|D\phi^{-1}\right\|_{C^\beta} \le C_\#$.

Finally, Lemma A.2 shows that $\phi(B(m, C_0))$ contains $B(\phi(m), C_\#^{-1}C_0)$. □

We end this subsection with the promised remarks on the conditions in Definition 2.7 involving α and β.

Remark 2.9 (Condition (2.9)). Condition (2.9) is used in the proof of Lemma 2.8 to ensure that $|DF|$ is uniformly bounded. It would seem more natural to replace (2.9) by the weaker condition $|DF| \le C$. However, it turns out that this weaker condition is never invariant under the graph transform, while (2.9) is invariant if (2.3) is satisfied (see (3.11)). If T is piecewise C^2 one can take $\alpha = 1$, and this is what is usually done in the literature ([HK95, §19], [Liv04, App. A]). In addition, because of the extra $C^{1+\alpha}$ smoothness in the y-direction given by (2.9), Lemma 3.3 produces diffeomorphisms Ψ and Ψ_m which belong to the space $D^1_{1+\alpha}$ from Definition 3.1. This is useful in view of the composition Lemma 4.7.

Remark 2.10 (Conditions (2.10) and (2.11): Hölder Jacobian). Lemma 4.4 about compact embeddings requires the foliations ϕ_F and their inverses ϕ_F^{-1} to have C^β Jacobians for some $\beta > 0$. (Beware that, even if T is volume-preserving, the class of foliations satisfying $|\det D\phi| \equiv 1$ is not invariant under the dynamics, because of the necessary reparametrizations in the proof of Lemma 3.3.) Lemma 2.8 shows that the conditions (2.9) and (2.10) imply that the Jacobians $J(x,y) = |\det D\phi_F|(x,y) = |\det \partial_x F|(x,y)$ and $\widetilde{J}(x,y) = |\det D\phi_F^{-1}|(x,y)$ are β-Hölder (with a C^β norm bounded independently of C_0).

Condition (2.10) will only be used to ensure that J and $\widetilde{J}$ are C^β. It turns out that the Hölder condition on the Jacobians, by itself, is not preserved when the foliation is iterated under hyperbolic maps, and neither is the condition (2.10) alone. However, the pair (2.10)–(2.11) is invariant if (2.4) is satisfied (see in particular Step 3 in the proof of Lemma 3.3).

2.3. **Extended cones, suitable charts and spaces of distributions.** In this subsection, we introduce appropriate cones $\mathcal{C}^s_{i,j}$ and coordinate patches $\kappa_{i,j}$ on the manifold in order to glue together (via a partition of unity) the local spaces $H_p^{t,s}$ and define a space $\mathbf{H}_p^{t,s}(R)$ of distributions[6] by using the charts in $\mathcal{F}(m, \mathcal{C}^s_{i,j}, C_0, C_1)$.

Definition 2.11. *An extended cone $\mathcal{C}$ is a set of four cones $(\mathcal{C}^s, \mathcal{C}^s_0, \mathcal{C}^u, \mathcal{C}^u_0)$ such that $\mathcal{C}^s$ and $\mathcal{C}^u$ are convexly transverse, $\mathcal{C}^s_0$ contains $\{0\}\times\mathbb{R}^{d_s}$, $\mathcal{C}^u_0$ contains $\mathbb{R}^{d_u}\times\{0\}$ and $\mathcal{C}^s_0 - \{0\}$ is contained in the interior of $\mathcal{C}^s$, $\mathcal{C}^u_0 - \{0\}$ is contained in the interior of $\mathcal{C}^u$. Given two extended cones $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, we say that an invertible matrix $M : \mathbb{R}^d \to \mathbb{R}^d$ sends $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ compactly if $M\mathcal{C}^u$ is contained in $\widetilde{\mathcal{C}}^u_0$, and $M^{-1}\widetilde{\mathcal{C}}^s$ is contained in $\mathcal{C}^s_0$.*

For all $i \in I$, we fix once and for all a finite number of open sets $U_{i,j,0}$ of X_0, for $1 \le j \le N_i$, covering $\overline{O_i}$, and included in the fixed neighborhood $\widetilde{O}_i$ of $\overline{O_i}$ where the extension T_i of $T_{|O_i}$ is defined. Let also $\kappa_{i,j} : U_{i,j,0} \to \mathbb{R}^d$, for $i \in I$ and $1 \le j \le N_i$, be a finite family of C^∞ charts, and let $\mathcal{C}_{i,j}$ be extended cones in $\mathbb{R}^d$ such that, wherever $\kappa_{i',j'} \circ T_i \circ \kappa_{i,j}^{-1}$ is defined, its differential sends $\mathcal{C}_{i,j}$ to $\mathcal{C}_{i',j'}$ compactly. Such charts and cones exist, as we explain now. Since the map is hyperbolic and the image of the unstable cone is included in the unstable cone, small enlargements of the unstable cones are sent strictly into themselves by the map. Therefore, if one considers charts with small enough supports, and locally constant cones $\mathcal{C}^s_{i,j}$, $\mathcal{C}^u_{i,j}$ slightly larger than the cones $D\kappa_{i,j}(q)\mathcal{C}_i^{(s)}(q)$, $D\kappa_{i,j}(q)\mathcal{C}_i^{(u)}(q)$, and slightly smaller cones $\mathcal{C}^s_{i,j,0}, \mathcal{C}^u_{i,j,0}$, they satisfy the previous requirements. (Convex transversality in the extended cone follows from our convex transversality assumption on $\mathcal{C}_i^{(s)}$ and $\mathcal{C}_i^{(u)}$.) We also fix open sets $U_{i,j,1}$ covering X_0 such that $\overline{U_{i,j,1}} \subset U_{i,j,0}$, and we let $V_{i,j,k} = \kappa_{i,j}(U_{i,j,k})$, $k = 0, 1$.

The spaces of distributions will depend on a large parameter $R \ge 1$ which will play the part of a "zoom:" If $R \ge 1$ and W is a subset of $\mathbb{R}^d$, denote by W^R the set $\{R \cdot z \mid z \in W\}$. Let also $\kappa^R_{i,j}(q) = R\kappa_{i,j}(q)$, so that $\kappa^R_{i,j}(U_{i,j,k}) = V^R_{i,j,k}$. Let

$$\mathcal{Z}_{i,j}(R) = \{m \in V^R_{i,j,0} \cap \mathbb{Z}^d \mid B(m, C_0) \cap V^R_{i,j,1} \ne \emptyset\}\,, \tag{2.15}$$

and

$$\mathcal{Z}(R) = \{(i,j,m) \mid i \in I, 1 \le j \le N_i, m \in \mathcal{Z}_{i,j}(R)\}\,. \tag{2.16}$$

To $\zeta = (i,j,m) \in \mathcal{Z}(R)$ is associated the point $q_\zeta := (\kappa^R_{i,j})^{-1}(m)$ of X. These are the points around which we shall construct local foliations, as follows. Let us first introduce useful notations: We write

$$O_\zeta = O_i\,, \quad \kappa^R_\zeta = \kappa^R_{i,j} \text{ and } \mathcal{C}_\zeta = \mathcal{C}_{i,j} \quad \text{for } \zeta = (i,j,m) \in \mathcal{Z}(R)\,.$$

[6]This is a modification of the space denoted $\widetilde{\mathcal{H}}_p^{t,s}$ in [BG09].

These are respectively the partition set, the chart and the extended cone that we use around q_ζ. Let us fix some constants $C_0 > 1$ and $C_1 > 2C_0$. If R is large enough, say $R \geq R_0(C_0, C_1)$, then, for any $\zeta = (i,j,m) \in \mathcal{Z}(R)$ and any chart $\phi_\zeta \in \mathcal{F}(m, \mathcal{C}^s_\zeta, C_0, C_1)$, we have $\phi_\zeta(B(m, C_0)) \subset V^R_{i,j,0}$. For $\zeta = (i,j,m) \in \mathcal{Z}(R)$, we can therefore consider the set of charts (R, C_0 and C_1 do not appear in the notation for the sake of brevity)

$$\mathcal{F}(\zeta) := \{\Phi_\zeta = (\kappa^R_\zeta)^{-1} \circ \phi_\zeta : B(m, C_0) \to X \, , \, \phi_\zeta \in \mathcal{F}(m, \mathcal{C}^s_\zeta, C_0, C_1)\} \, . \tag{2.17}$$

The image under a chart $\Phi_\zeta \in \mathcal{F}(\zeta)$ of the stable foliation in $\mathbb{R}^d$ is a local foliation around the point q_ζ, whose tangent space is everywhere contained in $(D\kappa^R_\zeta)^{-1}(\mathcal{C}^s_\zeta)$. This set is almost contained in the stable cone $\mathcal{C}^{(s)}_i(q_\zeta)$, by our choice of charts $\kappa_{i,j}$ and extended cones $\mathcal{C}_{i,j}$.

Let us fix once and for all a C^∞ function[7] $\rho : \mathbb{R}^d \to [0,1]$ such that

$$\rho(z) = 0 \text{ if } |z| \geq d \qquad \text{and} \qquad \sum_{m \in \mathbb{Z}^d} \rho(z - m) = 1 \, .$$

For $\zeta = (i,j,m) \in \mathcal{Z}(R)$, let $\rho_m(z) = \rho(z - m)$, and

$$\rho_\zeta := \rho_\zeta(R) = \rho_m \circ \kappa^R_\zeta : X \to [0,1] \, .$$

Since ρ_m is compactly supported in $\kappa^R_{i,j}(U_{i,j,0})$ if $m \in \mathcal{Z}_{i,j}(R)$ (and R is large enough, depending on d), the above expression is well-defined. This gives a partition of unity in the following sense:

$$\sum_{m \in \mathcal{Z}_{i,j}(R)} \rho_{i,j,m}(q) = 1 \, , \forall q \in U_{i,j,1} \, , \quad \rho_{i,j,m}(q) = 0 \, , \forall q \notin U_{i,j,0} \, .$$

Our choices ensure that the intersection multiplicity of this partition of unity is bounded, uniformly in R, i.e., for any point q, the number of functions such that $\rho_\zeta(q) \neq 0$ is bounded independently of R.

The space we shall consider depends in an essential way on the parameters p, t, and s. It will also depend, in an inessential way, on the choices we have made (i.e., the reference charts $\kappa_{i,j}$, the extended cones $\mathcal{C}_{i,j}$, the constants C_0 and C_1, the function ρ, and $R \geq R_0(C_0, C_1)$): Different choices would lead to different spaces, but all such spaces share the same features. We emphasize the dependence on R, C_0 and C_1 in the notations, since all the other choices will be fixed once and for all.

Definition 2.12 (Spaces $\mathbf{H}^{t,s}_p(R, C_0, C_1)$ of distributions on X)**.** *Let $1 < p < \infty$, $s, t \in \mathbb{R}$, let $1 < C_0 < C_1/2$ and let $R \geq R_0(C_0, C_1)$. For any system of charts $\Phi = \{\Phi_\zeta \in \mathcal{F}(\zeta) \mid \zeta \in \mathcal{Z}(R)\}$, let for $\omega \in L^\infty(X_0)$*

$$\|\omega\|_\Phi = \left(\sum_{\zeta \in \mathcal{Z}(R)} \left\| (\rho_\zeta(R) \cdot 1_{O_\zeta} \omega) \circ \Phi_\zeta \right\|^p_{H^{t,s}_p} \right)^{1/p} \, , \tag{2.18}$$

and put $\|\omega\|_{\mathbf{H}^{t,s}_p(R,C_0,C_1)} = \sup_\Phi \|\omega\|_\Phi$, the supremum ranging over all such systems of charts Φ.

The space $\mathbf{H}^{t,s}_p(R, C_0, C_1)$ is the closure of $\{\omega \in L^\infty(X_0) \mid \|\omega\|_{\mathbf{H}^{t,s}_p(R,C_0,C_1)} < \infty\}$ for the norm $\|\omega\|_{\mathbf{H}^{t,s}_p(R,C_0,C_1)}$.

For fixed R, the sum in (2.18) involves a uniformly bounded number of terms. Since the charts Φ_ζ have a uniformly bounded C^1 norm, the functions $(\rho_\zeta(R) \cdot \omega) \circ \Phi_\zeta$ are uniformly bounded in C^1 if ω is C^1. Moreover, $H^{t,s}_p$ contains the space of compactly supported C^1 functions on $\mathbb{R}^d$ when $|t| + |s| \leq 1$. Therefore, if there

[7] Such a function exists since the balls of radius d centered at points in $\mathbb{Z}^d$ cover $\mathbb{R}^d$.

were no multiplication by 1_{O_ς} in (2.18), then $\|\omega\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)}$ would be finite for any C^1 function ω. When $s,t \in (1/p-1, 1/p)$, multiplication by $1_{O_\varsigma} \circ \Phi_\varsigma$ leaves the space $H_p^{t,s}$ invariant (see Lemma 4.2 below). Therefore, all C^1 functions belong to $\mathbf{H}_p^{t,s}(R,C_0,C_1)$ in this case.

Remark 2.13. A priori, the space $\mathbf{H}_p^{t,s}(R,C_0,C_1)$ is not isomorphic to a Triebel space $H_p^{t,s}(X_0)$. However, our assumptions ensure that $\mathbf{H}_p^{t,0}(R,C_0,C_1)$ is isomorphic to the Sobolev-Triebel space $H_p^{t,0}(X_0)$ (whatever the value of R, C_0, C_1) when $-\beta < t < 1+\beta$. See Lemma 4.4 for various embedding claims on the spaces $\mathbf{H}_p^{t,s}(R,C_0,C_1)$.

2.4. **Reduction of the main result.** In this subsection, we shall deduce Theorem 2.5 from the following result about the spaces introduced in Subsection 2.3.

To simplify the statements, we will use the following convention throughout this article: the sentence "for all large enough $x, y, z, \dots$" means that, if x is large enough, then, if y is large enough (possibly depending on x), then if z is large enough (possibly depending on x and y), ...

Theorem 2.14. *Let T, g, and p, t, s satisfy the assumptions of Theorem 2.5. There exist $C_0 > 1$ and $C_\# > 0$ such that, for any $N > 0$, any large enough $C_1 > 2C_0$, any large enough integer n which is a multiple of N, and any large enough R, the operator $\mathcal{L}_g^n$ is bounded on $\mathbf{H}_p^{t,s}(R,C_0,C_1)$, and its essential spectral radius is at most*

$$(C_\# N)^{n/N}(D_n^b)^{1/p}\cdot(D_n^e)^{1-1/p}\cdot\left\| g^{(n)}|\det DT^n|^{1/p}\max(\lambda_{u,n}^{-t}, \lambda_{s,n}^{-(t-|s|)})\right\|_{L^\infty}. \tag{2.19}$$

The above theorem will be proved in Section 5. Below, we deduce Theorem 2.5 from Theorem 2.14, using the following proposition (which will be proved at the end of Section 5).

Proposition 2.15. *Let T, g, and p, t, s satisfy the assumptions of Theorem 2.5, and let C_0 be given by Theorem 2.14. For any large enough $C_1 > 0$ and $R > 0$, and any large enough $C_1' > 0$ and $R' > 0$, then for any large enough N, $\mathcal{L}_g^N$ is continuous from $\mathbf{H}_p^{t,s}(R,C_0,C_1)$ to $\mathbf{H}_p^{t,s}(R',C_0,C_1')$.*

Proof that Theorem 2.14 implies Theorem 2.5. Theorem 2.14 does not claim that the space $\mathbf{H}_p^{t,s}(R,C_0,C_1)$ is invariant under $\mathcal{L}_g$. This issue is easy to deal with: Consider C_1, n and R such that Theorem 2.14 applies to $\mathcal{L}_g^n$ acting on $\mathbf{H}_p^{t,s}(R,C_0,C_1)$, and let $H(n,R,C_0,C_1) = H(p,t,s,n,R,C_0,C_1)$ be the closure of $L^\infty(X_0)$ for the norm

$$\|\omega\|_{H(p,t,s,n,R,C_0,C_1)} = \sum_{j=0}^{n-1} \left\|\mathcal{L}_g^j \omega\right\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)}. \tag{2.20}$$

Since $\left\|\mathcal{L}_g^n\omega\right\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)} \le C\|\omega\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)}$ by Theorem 2.14, it follows that the operator $\mathcal{L}_g$ is continuous on $H(n,R,C_0,C_1)$.

Moreover, for any C^1 function ω and any j, the function $\mathcal{L}_g^j\omega = \sum_{\mathbf{i}} 1_{T_\mathbf{i}O_\mathbf{i}}(g^{(j)}\omega)\circ T_\mathbf{i}^{-j}$ is a sum of C^γ functions multiplied by characteristic functions of nice sets. The discussion following Definition 2.12 (with C^1 replaced by C^γ) implies that $\mathcal{L}_g^j\omega$ belongs to $\mathbf{H}_p^{t,s}(R,C_0,C_1)$. Hence, $H(n,R,C_0,C_1)$ contains C^1 (in particular, it is not reduced to $\{0\}$).

To finish, we shall prove that the claim on the essential spectral radius of $\mathcal{L}_g$ holds on $\mathbf{H} = H(n,R,C_0,C_1)$, if C_1, n and R are large enough. If $\mathcal{M}$ is an operator acting on a Banach space E, we denote by $r_{\mathrm{ess}}(\mathcal{M},E)$ its essential spectral radius.

First claim: $r_{\mathrm{ess}}(\mathcal{L}_g, H(n,R,C_0,C_1)) \le r_{\mathrm{ess}}(\mathcal{L}_g^n, \mathbf{H}_p^{t,s}(R,C_0,C_1))^{1/n}$.

Let us admit this claim for the moment. Then, by (2.19), the essential spectral radius of $\mathcal{L}_g$ on $H(n, R, C_0, C_1)$ is at most

$$(C_\# N)^{1/N} (D_n^b)^{1/(pn)} \cdot (D_n^e)^{(1/n)(1-1/p)} \cdot \left\| g^{(n)} |\det DT^n|^{1/p} \max(\lambda_{u,n}^{-t}, \lambda_{s,n}^{-(t-|s|)}) \right\|_{L^\infty}^{1/n}.$$

Since $(C_\# N)^{1/N}$ tends to 1 when $N \to \infty$, this factor is not troublesome. However, we do not have Theorem 2.5 yet: In (2.6), there is a limit in n, while our last bound is for a fixed n. This is why we need to show the following statement:

Second claim: Let r be the limit in (2.6). *If C_1, n and R are large enough, we have $r_{\text{ess}}(\mathcal{L}_g^n, \mathbf{H}_p^{t,s}(R, C_0, C_1)) \leq r^n$.*

Putting together the first and second claims we deduce that the space $\mathbf{H} = H(n, R, C_0, C_1)$ satisfies the conclusion of Theorem 2.5 if C_1, n and R are large enough.

It remains to prove the two above claims. For this, we recall a characterization of the essential spectral radius of an operator $\mathcal{M}$ acting on a Banach space E.

(1) Let $\tau > 0$, assume that there exist a sequence $j(n) \to \infty$ and a sequence of compact operators $K_n : E \to E_n$ (for some Banach spaces E_n) such that $\left\|\mathcal{M}^{j(n)} w\right\|_E \leq \tau^{j(n)} \|w\|_E + \|K_n w\|_{E_n}$ for any $w \in E$ (or, equivalently, in a dense subset of E) and any large enough n. Then $r_{\text{ess}}(\mathcal{M}, E) \leq \tau$.
(2) Conversely, if $\tau > r_{\text{ess}}(\mathcal{M}, E)$, there exists a sequence of compact operators $K_n : E \to E$ such that, if n is large enough, $\|\mathcal{M}^n w\|_E \leq \tau^n \|w\|_E + \|K_n w\|_E$ for any $w \in E$.

The first assertion was proved by Hennion [Hen93] using a formula of Nussbaum. The second assertion follows from the spectral decomposition $\mathcal{M} = K + A$ where $KA = AK = 0$, K has finite rank (and corresponds to the eigenvalues of $\mathcal{M}$ of modulus $\geq \tau$), and the spectral radius of A is smaller than τ (just take $K_n = K^n$).

We prove the first claim. Let $\tau > r_{\text{ess}}(\mathcal{L}_g^n, \mathbf{H}_p^{t,s}(R, C_0, C_1))^{1/n}$. By Item 2, there exists a sequence of compact operators $K_{kn} : \mathbf{H}_p^{t,s}(R, C_0, C_1) \to \mathbf{H}_p^{t,s}(R, C_0, C_1)$ such that, for large enough k,

$$\left\|\mathcal{L}_g^{kn} \omega\right\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)} \leq \tau^{kn} \|\omega\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)} + \|K_{kn}\omega\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)}.$$

Therefore, for $\omega \in H(n, R, C_0, C_1)$,

$$\begin{aligned}
\left\|\mathcal{L}_g^{kn} \omega\right\|_{H(n,R,C_0,C_1)} &= \sum_{j=0}^{n-1} \left\|\mathcal{L}_g^{kn} \mathcal{L}_g^j \omega\right\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)} \\
&\leq \sum_{j=0}^{n-1} \tau^{kn} \left\|\mathcal{L}_g^j \omega\right\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)} + \left\|K_{kn} \mathcal{L}_g^j \omega\right\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)} \\
&= \tau^{kn} \|\omega\|_{H(n,R,C_0,C_1)} + \left\|\widetilde{K}_{kn}\omega\right\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)^n},
\end{aligned}$$

where the operator $\widetilde{K}_{kn}$ from $H(n, R, C_0, C_1)$ to $\mathbf{H}_p^{t,s}(R, C_0, C_1)^n$ is given by

$$\widetilde{K}_{kn}\omega = (K_{kn}\omega, K_{kn}\mathcal{L}_g\omega, \dots, K_{kn}\mathcal{L}_g^{n-1}\omega).$$

Since this operator is compact, Item 1 above gives that $r_{\text{ess}}(\mathcal{L}_g, H(n, R, C_0, C_1)) \leq \tau$, and thus the first claim.

Finally, we prove the second claim. The idea is to use Proposition 2.15 to go from $\mathbf{H}_p^{t,s}(R, C_0, C_1)$ to $\mathbf{H}_p^{t,s}(R', C_0, C_1')$ for large C_1' and R', use the good control on the essential spectral radius on $\mathbf{H}_p^{t,s}(R', C_0, C_1')$, and then return to $\mathbf{H}_p^{t,s}(R, C_0, C_1)$. Let C_1, n and R be as in the statement of the second claim. Consider $\tau > r$, and let us fix $C_1' > 2C_0$, k and R' large enough so that $r_{\text{ess}}(\mathcal{L}_g^{kn}, \mathbf{H}_p^{t,s}(R', C_0, C_1')) < \tau^{kn}$:

This is possible by Theorem 2.14. Therefore, by Item 2, for large j, there exists a compact operator $K_{jkn} : \mathbf{H}_p^{t,s}(R', C_0, C_1') \to \mathbf{H}_p^{t,s}(R', C_0, C_1')$ such that $\left\|\mathcal{L}_g^{jkn}\omega\right\|_{\mathbf{H}_p^{t,s}(R',C_0,C_1')} \le \tau^{jkn} \|\omega\|_{\mathbf{H}_p^{t,s}(R',C_0,C_1')} + \|K_{jkn}\omega\|_{\mathbf{H}_p^{t,s}(R',C_0,C_1')}$. By Proposition 2.15, we can choose m such that the operator $\mathcal{L}_g^{mn}$ sends $\mathbf{H}_p^{t,s}(R, C_0, C_1)$ to $\mathbf{H}_p^{t,s}(R', C_0, C_1')$ and $\mathbf{H}_p^{t,s}(R', C_0, C_1')$ to $\mathbf{H}_p^{t,s}(R, C_0, C_1)$ continuously, with a norm bounded by a constant that we denote by C. Then, for any $\omega \in \mathbf{H}_p^{t,s}(R, C_0, C_1)$,

$$\begin{aligned}\left\|\mathcal{L}_g^{(jk+2m)n}\omega\right\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)} &\le C \left\|\mathcal{L}_g^{(jk+m)n}\omega\right\|_{\mathbf{H}_p^{t,s}(R',C_0,C_1')} \\ &\le C\tau^{jkn} \left\|\mathcal{L}_g^{mn}\omega\right\|_{\mathbf{H}_p^{t,s}(R',C_0,C_1')} + \left\|K_{jkn}\mathcal{L}_g^{mn}\omega\right\|_{\mathbf{H}_p^{t,s}(R',C_0,C_1')} \\ &\le C^2\tau^{jkn} \|\omega\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)} + \left\|K_{jkn}\mathcal{L}_g^{mn}\omega\right\|_{\mathbf{H}_p^{t,s}(R',C_0,C_1')} .\end{aligned}$$

The operator $\widetilde{K}_{jkn} := K_{jkn}\mathcal{L}_g^{mn}$ is compact from $\mathbf{H}_p^{t,s}(R, C_0, C_1)$ to $\mathbf{H}_p^{t,s}(R', C_0, C_1')$. Therefore, Item 1 ensures that

$$r_{\mathrm{ess}}(\mathcal{L}_g^n, \mathbf{H}_p^{t,s}(R, C_0, C_1)) \le \liminf_{j\to\infty} (C^2\tau^{jkn})^{1/(jk+2m)} = \tau^n .$$

This ends the proof of the second claim and of the theorem. □

3. Invariance of the class of cone admissible local foliations

In order to prove the bounds necessary for Theorem 2.14, we need to check that the class of admissible foliations defined in Subsection 2.2 is invariant under the iteration of the map T^{-1} (viewed in charts). This is the purpose of the key Lemma 3.3 below, which says that if $\phi_m \in \mathcal{F}(m, \mathcal{C}^s, C_0, C_1)$ is an admissible foliation, then the chart ϕ' obtained by pulling it back by a diffeomorphism $\mathcal{T}^{-1}$ of $\mathbb{R}^d$, and reparameterizing to put it in standard form is still admissible *if the map $\mathcal{T}$ is sufficiently hyperbolic, $C^{1+\alpha}$, and satisfies a bunching condition* (see (3.1)). This fact is not surprising: It is well known (see e.g. the Hadamard-Perron arguments in [HK95, §6.2, §19]) that C^1 foliations remain C^1 after a graph transform if the transformation satisfies a bunching condition. However, the statement of Lemma 3.3 is a little involved because (in order to avoid exponential proliferation of the number of charts) we need to "glue together" all pulled back charts ϕ_m associated to a set $\mathcal{M}$ of "well-separated" points m. This must be done carefully, controlling the size of the domains of definition of the new chart ϕ' thus produced.

If the pullback of a foliation $\phi(x, y) = (F(x, y), y)$ under a map $\mathcal{T}$ is given in standard form by a map $\phi'(x, y) = (F'(x, y), y)$, this means that $\mathcal{T}^{-1} \circ \phi = \phi' \circ \mathbb{T}$ for some map $\mathbb{T}$ defined on a subset of $\mathbb{R}^d$, and sending stable leaves to stable leaves. This map $\mathbb{T}$ is needed to straighten $\mathcal{T}^{-1} \circ \phi$, which typically does not have the form $(x, y) \mapsto (F'(x, y), y)$. The map $\mathbb{T}$ corresponds to $\mathcal{T}^{-1}$ in the charts ϕ, ϕ', and it will be important to control well its smoothness and hyperbolicity. In particular, the following definition will be useful.

Definition 3.1. *For $C > 0$ let $D^1_{1+\alpha}(C)$ denote the set of C^1 diffeomorphisms Ψ defined on a subset of $\mathbb{R}^d$, sending stable leaves to stable leaves, and such that*

$$\max\bigl(\sup |D\Psi(x, y)|, \sup |D\Psi^{-1}(x, y)|, \sup_{x,y,y'} \frac{|D\Psi(x, y) - D\Psi(x, y')|}{|y - y'|^\alpha}\bigr) \le C .$$

Before we state Lemma 3.3, we need one more notation:

Definition 3.2. *Let $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ be extended cones (Definition 2.11). If an invertible matrix $M : \mathbb{R}^d \to \mathbb{R}^d$ sends $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ compactly, let $\lambda_u(M) = \lambda_u(M, \mathcal{C}, \widetilde{\mathcal{C}})$ be the least expansion under M of vectors in $\mathcal{C}^u$, and $\lambda_s(M) = \lambda_s(M, \mathcal{C}, \widetilde{\mathcal{C}})$ be the inverse of the least expansion under M^{-1} of vectors in $\widetilde{\mathcal{C}}^s$. Denote by $\Lambda_u(M) = \Lambda_u(M, \mathcal{C}, \widetilde{\mathcal{C}})$ and*

$\Lambda_s(M) = \Lambda_s(M, \mathcal{C}, \widetilde{\mathcal{C}})$ *the strongest expansion and contraction coefficients of* M *on the same cones.*

The key lemma can now be stated:

Lemma 3.3. *Let* $\mathcal{C}$ *and* $\widetilde{\mathcal{C}}$ *be extended cones, let* $\alpha \in (0,1]$ *and let* $\beta \in (0,\alpha)$. *For any large enough* C_0 *(depending on* $\mathcal{C}$ *and* $\widetilde{\mathcal{C}}$*) and any* $C_1 > 2C_0$, *there exist constants* C *(depending on* $\mathcal{C}$, $\widetilde{\mathcal{C}}$ *and* C_0*) and* ϵ *(depending on* $\mathcal{C}$, $\widetilde{\mathcal{C}}$, C_0 *and* C_1*) satisfying the following properties:*

Let $\mathcal{T}$ *be a* $C^{1+\alpha}$ *diffeomorphism of* $\mathbb{R}^d$ *with* $\mathcal{T}(0) = 0$ *and, setting* $M := D\mathcal{T}(0)$, *so that*

$$\left\|\mathcal{T}^{-1} \circ M - \mathrm{id}\right\|_{C^{1+\alpha}} \le \epsilon\,, \quad M \text{ sends } \mathcal{C} \text{ to } \widetilde{\mathcal{C}} \text{ compactly,}$$

$$\lambda_s(M)^{\alpha-\beta}\Lambda_u(M)^{1+\beta}\lambda_u(M)^{-1} < \epsilon\,, \qquad \lambda_u(M) > \epsilon^{-1}\,,\ \lambda_s(M)^{-1} > \epsilon^{-1}\,. \tag{3.1}$$

Let $\mathcal{M} \subset \mathbb{R}^d$ *be a finite set such that* $|m - m'| \ge C$ *for all* $m \neq m' \in \mathcal{M}$, *and consider any family of charts* $\{\phi_m \in \mathcal{F}(m, \widetilde{\mathcal{C}}^s, C_0, C_1) \mid m \in \mathcal{M}\}$.

Then, defining

$$\mathcal{M}' := \{m \in \mathcal{M} \mid B(m,d) \cap \mathcal{T}(B(0,d)) \neq \emptyset\}\,,$$

and setting $\Pi(x,y) = (x,0)$, *we have:*

(a) $|\Pi m - \Pi m'| \ge C_0$ *for all* $m \neq m'$ *in* $\mathcal{M}'$.

(b) There exist $\phi' \in \mathcal{F}(0, \mathcal{C}^s, C_0, C_1)$, *and diffeomorphisms* $\mathbb{T}_m$, *for* $m \in \mathcal{M}'$, *such that*

$$\mathcal{T}^{-1} \circ \phi_m = \phi' \circ \mathbb{T}_m \quad \text{ on } \phi_m^{-1}(B(m,d) \cap \mathcal{T}(B(0,d)))\,,\ \forall m \in \mathcal{M}'\,. \tag{3.2}$$

(c) For each $m \in \mathcal{M}'$, *we can write* $\mathbb{T}_m = \Psi \circ D^{-1} \circ \Psi_m$, *where*

- *The diffeomorphism* Ψ_m *is in* $D^1_{1+\alpha}(C)$, *its range contains* $B(\Pi m, C_0^{1/2})$, *and* $\Psi_m(\phi_m^{-1}(B(m,d))) \subset B(\Pi m, C_0^{1/2}/2)$.
- *The matrix* D *is block diagonal, of the form* $D = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ *with*

$$|Av| \ge C^{-1}\lambda_u(M)|v| \text{ and } |Bv| \le C\lambda_s(M)|v|\,.$$

- *The diffeomorphism* Ψ *is in* $D^1_{1+\alpha}(C)$, *its range contains* $B(0, C_0^{1/2})$.

Note that (c) implies in particular that each $\mathbb{T}_m$ sends stable leaves to stable leaves. Note also that if C_0 is large enough, then $\phi' \in \mathcal{F}(0, \mathcal{C}^s, C_0, C_1)$ implies $(\phi')^{-1}(B(0,d)) \subset B(0, C_0^{1/2}/2)$ (because $\|(\phi')^{-1}\|_{C^1} \le C_\#$ by Lemma 2.8).

Statements (b) and (c) are the main result of the lemma: (b) shows that the pullback of all the relevant charts ϕ_m can be glued together to form an admissible chart ϕ', while (c) gives an expression of $\mathbb{T}_m$, that is, $\mathcal{T}^{-1}$ in the charts ϕ_m, ϕ', as the composition of two well controlled diffeomorphisms Ψ, Ψ_m, and a matrix D with good hyperbolic properties. Statement (a), although an essential consequence of hyperbolicity, has a more technical nature: It is used in Step 2 of the proof of the lemma (when gluing foliations), and also later in the proof of Theorem 2.14. At the first reading, the reader can ignore the information on the ranges of Ψ and Ψ_m (but beware that they will be important in the proof of Theorem 2.14).

Remark 3.4. Composing with translations, we deduce a more general result from Lemma 3.3, replacing 0 by $\ell \in \mathbb{R}^d$, and allowing $\mathcal{T}(\ell) \neq \ell$: Just replace M by $D\mathcal{T}(\ell)$, the projection Π by $\Pi(x,y) = (x, y_{\mathcal{T}(\ell)})$, where $\mathcal{T}(\ell) = (x_{\mathcal{T}(\ell)}, y_{\mathcal{T}(\ell)})$, and assume that

$$\left\|(\mathcal{T}^{-1}[\cdot + \mathcal{T}(\ell)] - \ell) \circ D\mathcal{T}(\ell) - \mathrm{id}\right\|_{C^{1+\alpha}} \le \epsilon$$

and that $D\mathcal{T}(\ell)$ sends $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ compactly. One then uses the condition $B(m,d) \cap \mathcal{T}(B(\ell,d)) \neq \emptyset$ to define $\mathcal{M}'$. Of course, ϕ' is then in $\mathcal{F}(\ell, \mathcal{C}^s, C_0, C_1)$, equality

(3.2) holds on $\phi_m^{-1}(B(m,d)\cap\mathcal{T}(B(\ell,d)))$, and the range of Ψ contains $B(\ell,C_0^{1/2})$. Finally, we have $(\phi')^{-1}(B(\ell,d))\subset B(\ell,C_0^{1/2}/2)$.

Proof of Lemma 3.3. We shall write π_1 and π_2 for, respectively, the first and the second projection in $\mathbb{R}^d=\mathbb{R}^{d_u}\times\mathbb{R}^{d_s}$.

Step zero: Preparations. We shall write $C_\#$ and $\epsilon_\#$ for a large, respectively small, constant, depending only on $\mathcal{C},\widetilde{\mathcal{C}}$, that may vary from line to line. For the other parameters, we will always specify if they depend on C_0 or C_1.

The set $M(\mathbb{R}^{d_u}\times\{0\})$ is contained in $\widetilde{\mathcal{C}}^u$, hence uniformly transverse to $\{0\}\times\mathbb{R}^{d_s}$. Therefore, it can be written as a graph $\{(x,Px)\}$ for some matrix P with norm depending only on $\widetilde{\mathcal{C}}$. Let $Q(x,y)=(x,y-Px)$, so that QM sends $\mathbb{R}^{d_u}\times\{0\}$ to itself. In the same way, $M^{-1}(\{0\}\times\mathbb{R}^{d_s})$ is contained in $\mathcal{C}^s$, hence it is a graph $\{(P'y,y)\}$. Letting $Q'(x,y)=(x-P'y,y)$, the matrix $D=QM(Q')^{-1}$ leaves $\mathbb{R}^{d_u}\times\{0\}$ and $\{0\}\times\mathbb{R}^{d_s}$ invariant, i.e., it is block-diagonal, of the form $\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)$, and moreover $|Av|\geq C_\#^{-1}\lambda_u|v|$ and $|Bv|\leq C_\#\lambda_s|v|$ (since the matrices Q and Q', as well as their inverses, are uniformly bounded in terms of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$).

We can readily prove assertion (a) of the lemma. Let $m\in\mathcal{M}'$, there exists $z\in B(m,d)\cap\mathcal{T}(B(0,d))$. The set $Q\mathcal{T}(B(0,d))=DQ'(\mathcal{T}^{-1}M)^{-1}(B(0,d))$ is included in $\{(x,y)\mid |y|\leq C_\#\}$ for some constant $C_\#$ (the role of Q is important here). Since $Qz\in Q\mathcal{T}(B(0,d))$, we obtain $|\pi_2(Qz)|\leq C_\#$. Since $|z-m|\leq d$, we also have $|Qz-Qm|\leq C_\#$, hence $|\pi_2(Qm)|\leq C_\#$ (for a different constant $C_\#$). Since $Qm-\Pi m=(x_m,\pi_2(Qm))-(x_m,0)=(0,\pi_2(Qm))$, we obtain

$$|Qm-\Pi m|\leq C_\#\,. \tag{3.3}$$

Since the points $m\in\mathcal{M}'$ are far apart by assumption, the points Qm for $m\in\mathcal{M}'$ are also far apart, and it follows that the points Πm are also far apart. Increasing the distance between points in $\mathcal{M}'$, we can in particular ensure that $|\Pi m-\Pi m'|\geq C_0$ for any $m\neq m'\in\mathcal{M}'$, proving (a).

The strategy of the proof of the rest of the lemma is the following: We write

$$\mathcal{T}^{-1}=\mathcal{T}^{-1}M\cdot(Q')^{-1}\cdot D^{-1}\cdot Q\,. \tag{3.4}$$

We shall start from the partial foliation given by the maps ϕ_m for $m\in\mathcal{M}$, apply Q (Step 1) to obtain a new partial foliation at Qm, modify it via gluing (Step 2) to obtain a global foliation, and then push this foliation successively with D^{-1} (Step 3), $(Q')^{-1}$ (Step 4), and $\mathcal{T}^{-1}M$ (last step).

We shall use in this proof the spaces of local diffeomorphisms $\mathcal{D}(C_\#)$ and of matrix-valued functions $\mathcal{K}(C_\#)=\mathcal{K}^{\alpha,\beta}(C_\#)$ introduced in Appendix A. As in Remark A.6 of this appendix, we will write $\mathcal{K}(C_\#,A)$ for the functions defined on a set A and satisfying the inequalities defining $\mathcal{K}(C_\#)$ (A will sometimes be omitted when the domain of definition is obvious). The map ϕ_m belongs to $\mathcal{D}(C_\#)$ (see the proof of Lemma 2.8), and the matrix-valued function $D\phi_m$ belongs to $\mathcal{K}(C_\#,B(m,C_0))$ (boundedness of $D\phi_m$ is proved in Lemma 2.8, while the Hölder-like properties are given by (2.9)–(2.11)).

First step: Pushing the foliations with Q. We formulate in detail the construction in this first step (a version of Lemma 3.5 will be used also in the last step, replacing Q by $\mathcal{T}^{-1}M$, while steps 2-3-4 are much simpler).

Lemma 3.5. *(Notation as in Lemma 3.3 and Step 0 of its proof.) There exists a constant $C_\#$ such that, if C_0 is large enough and $C_1>2C_0$, for any $m=(x_m,y_m)\in\mathcal{M}'$ there exist two maps $\phi_m^{(1)}:B(\Pi m,C_0^{1/2})\to\mathbb{R}^d$ and $\Psi_m:B(m,C_0^{2/3})\to\mathbb{R}^d$ such that*

$$\phi_m^{(1)}\circ\Psi_m=Q\circ\phi_m \text{ on } \phi_m^{-1}(B(m,d))\,.$$

Moreover, Ψ_m *is a diffeomorphism in* $D^1_{1+\alpha}(C_\#)$ *whose range contains* $B(\Pi m, C_0^{1/2})$*, and* $\Psi_m(\phi_m^{-1}(B(m,d))) \subset B(\Pi m, C_0^{1/2}/2)$*. Finally,* $\phi_m^{(1)}(x,y) = (F_m^{(1)}(x,y), y)$ *on* $B(\Pi m, C_0^{1/2})$*, with* $F_m^{(1)}$ *a* C^1 *map so that* $F_m^{(1)}(x,0) = x$ *and* $DF_m^{(1)}$ *belongs to* $\mathcal{K}(C_\#, B(\Pi m, C_0^{1/2}))$*.*

Note that if $\mathcal{E}$ is the foliation given by $\phi_m(x,y) = (F_m(x,y), y)$, then by definition $\phi_m^{(1)}$ sends the stable leaves of $\mathbb{R}^d$ to the foliation $Q(\mathcal{E})$, i.e., $\phi_m^{(1)}$ is the standard parametrization of the foliation $Q(\mathcal{E})$.

Proof of Lemma 3.5. Fix $m = (x_m, y_m) \in \mathcal{M}'$. The map $Q \circ \phi_m$ does not qualify as $\phi_m^{(1)}$ for two reasons. First, $\pi_2 \circ Q \circ \phi_m(x,y)$ is generally not equal to y. Second, $\pi_1 \circ Q \circ \phi_m(x,0)$ is generally not equal to x. We shall use two maps $\Gamma^{(0)}$ and $\Gamma^{(1)}$ (sending stable leaves to stable leaves) to compensate for these two problems. The map $\Gamma^{(0)}$ will have the form $\Gamma^{(0)}(x,y) = (x, G(x,y))$ where for fixed x, the map $y \mapsto G(x,y)$ will be a diffeomorphism of the vertical leaf $\{x\} \times \mathbb{R}^{d_s}$, so that $\pi_2 \circ Q \circ \phi_m \circ \Gamma^{(0)}(x,y) = y$. In particular, $Q \circ \phi_m \circ \Gamma^{(0)}(x,0)$ is of the form $(L^{(1)}(x), 0)$, for some map $L^{(1)}$. Choosing $\Gamma^{(1)}(x,y) = ((L^{(1)})^{-1}(x), y)$ solves our second problem: the map

$$\phi_m^{(1)} := Q \circ \phi_m \circ \Gamma^{(0)} \circ \Gamma^{(1)}$$

satisfies both $\pi_2 \circ \phi_m^{(1)}(x,y) = y$ and $\pi_1 \circ \phi_m^{(1)}(x,0) = x$, as desired. Then, the map $\Psi_m = (\Gamma^{(0)} \circ \Gamma^{(1)})^{-1}$ sends stable leaves to stable leaves and $Q \circ \phi_m = \phi_m^{(1)} \circ \Psi_m$.

We shall now be more precise, justifying the existence of the maps mentioned above, and estimating their domain of definition, their range and their smoothness.

The map $\Gamma^{(0)}$. For fixed x, the map $y \mapsto G(x,y)$ should satisfy $\pi_2 \circ Q \circ \phi_m(x, G(x,y)) = y$, i.e., it should be the inverse to the map

$$L_x : y \mapsto \pi_2 \circ Q \circ \phi_m(x,y) = y - PF_m(x,y)\,, \tag{3.5}$$

where we denote $\phi_m(x,y) = (F_m(x,y), y)$. We claim that this map is invertible onto its image, and that there exists $\epsilon^0_\# > 0$ such that

$$|L_x(y') - L_x(y)| \geq \epsilon^0_\# |y' - y|\,, \quad \forall x \in B(x_m, C_0)\,, \quad \forall y, y' \in B(y_m, C_0)\,. \tag{3.6}$$

Indeed, fix $x \in B(x_m, C_0)$ and let $w = y' - y$. Writing $F(y) = F_m(x,y)$, we have

$$L_x(y') - L_x(y) = w - P \int_{t=0}^1 \partial_y F(y + tw) w \, dt\,. \tag{3.7}$$

Each vector $(\partial_y F(y+tw)w, w)$ belongs to $\widetilde{\mathcal{C}}^s$. Since this cone is convexly transverse to $\mathbb{R}^{d_u} \times \{0\}$, the set $\widetilde{\mathcal{C}}^s \cap (\mathbb{R}^{d_u} \times \{w\})$ is convex, hence

$$v_1 := \left(\int_{t=0}^1 \partial_y F(y + tw) w \, dt, w \right) \in \widetilde{\mathcal{C}}^s\,. \tag{3.8}$$

On the other hand, since the graph of P is included in $\widetilde{\mathcal{C}}^u$, $v_2 := (\int_{t=0}^1 \partial_y F(y + tw) w \, dt, P \int_{t=0}^1 \partial_y F(y+tw) w \, dt)$ belongs to $\widetilde{\mathcal{C}}^u$. Let $\epsilon^0_\# > 0$ be such that $B(v, \epsilon^0_\# |v|) \cap \widetilde{\mathcal{C}}^u = \emptyset$ for any $v \in \widetilde{\mathcal{C}}^s - \{0\}$. Since $v_1 \in \widetilde{\mathcal{C}}^s$ and $v_2 \in \widetilde{\mathcal{C}}^u$, we get $|v_1 - v_2| \geq \epsilon^0_\# |v_1|$. As v_1 and v_2 have the same first component, this gives $|\pi_2(v_1) - \pi_2(v_2)| \geq \epsilon^0_\# |v_1|$, i.e.,

$$\left| w - P \int_{t=0}^1 \partial_y F(y + tw) w \, dt \right| \geq \epsilon^0_\# |w|\,,$$

which implies (3.6) by (3.7).

The map $\Lambda^{(0)} : (x,y) \mapsto (x, L_x(y))$ is well defined on $B(m, C_0)$, its derivative is bounded by a constant $C_\#$, and its second component satisfies (3.6). Lemma A.1

(with x and y exchanged) shows that $\Lambda^{(0)} \in \mathcal{D}(C_\#)$ for some constant $C_\#$. In particular, $\Lambda^{(0)}$ admits an inverse $\Gamma^{(0)}$, which also belongs to $\mathcal{D}(C_\#)$.

By Lemma A.2, the range of $\Lambda^{(0)}$ (which coincides with the domain of definition of $\Gamma^{(0)}$) contains the ball $B(\Lambda^{(0)}(m), C_0/C_\#)$. Moreover, $\Lambda^{(0)}(m) = Qm$. By (3.3), we have $|Qm - \Pi m| \leq C_\#$, hence the domain of definition of $\Gamma^{(0)}$ contains $B(\Pi m, C_0/C_\# - C_\#)$. If C_0 is large enough, this contains $B(\Pi m, C_0^{2/3})$.

The map $\Gamma^{(1)}$. Consider $\phi_m^{(0)} := Q \circ \phi_m \circ \Gamma^{(0)}$. It is a composition of maps in $\mathcal{D}(C_\#)$, hence it also belongs to $\mathcal{D}(C_\#)$. Moreover, its restriction to $\mathbb{R}^{d_u} \times \{0\}$ has the form $(x, 0) \mapsto (L^{(1)}(x), 0)$. It follows that the map $L^{(1)}$ (defined on a subset of $\mathbb{R}^{d_u}$) also satisfies the inequalities defining $\mathcal{D}(C_\#)$. In particular, it is invertible, and we may define $\Gamma^{(1)}(x, y) = ((L^{(1)})^{-1}(x), y)$. This map belongs to $\mathcal{D}(C_\#)$. By construction, $\phi_m^{(1)} := Q \circ \phi_m \circ \Gamma^{(0)} \circ \Gamma^{(1)}$ can be written as $(F_m^{(1)}(x, y), y)$ with $F_m^{(1)}(x, 0) = x$.

We have $\phi_m^{(0)}(Qm) = Qm$. Since $|\Pi m - Qm| \leq C_\#$ by (3.3), and $\phi_m^{(0)}$ is Lipschitz, we obtain $|\phi_m^{(0)}(\Pi m) - \Pi m| \leq C_\#$, i.e., $|L^{(1)}(x_m) - x_m| \leq C_\#$. Since $L^{(1)} \in \mathcal{D}(C_\#)$, Lemma A.2 shows that $L^{(1)}(B(x_m, C_0^{2/3}))$ contains the ball $B(x_m, C_0^{2/3}/C_\# - C_\#)$. Therefore, it contains the ball $B(x_m, C_0^{1/2})$ if C_0 is large enough. Hence, the domain of definition of the map $\Gamma^{(1)}$ contains $B(\Pi m, C_0^{1/2})$. This shows that $\phi_m^{(1)}$ is defined on $B(\Pi m, C_0^{1/2})$.

The map Ψ_m. We can now define $\Psi_m = (\Gamma^{(0)} \circ \Gamma^{(1)})^{-1} = (L^{(1)}(x), L_x(y))$, so that $Q \circ \phi_m = \phi_m^{(1)} \circ \Psi_m$. We have seen that $\Psi_m \in \mathcal{D}(C_\#)$, hence $D\Psi_m$ and $D\Psi_m^{-1}$ are uniformly bounded. To show that $\Psi_m \in D^1_{1+\alpha}(C_\#)$, we should check that $|D\Psi_m(x, y) - D\Psi_m(x, y')| \leq C_\# |y - y'|^\alpha$. This follows directly from the construction and the corresponding inequality (2.9) for DF_m. Finally, since $\Psi_m \in \mathcal{D}(C_\#)$,

$$\Psi_m(\phi_m^{-1}(B(m, d))) \subset \Psi_m(B(m, C_\#)) \subset B(\Psi_m(m), C_\#) \,.$$

Since $Qm = \phi_m^{(1)}(\Psi_m(m))$ and $\Pi m = \phi_m^{(1)}(\Pi m)$, we get $|\Psi_m(m) - \Pi m| \leq C_\# |Qm - \Pi m| \leq C_\#$ by (3.3). Therefore, $\Psi_m(\phi_m^{-1}(B(m, d))) \subset B(\Pi m, C_\#)$, and this last set is included in $B(\Pi m, C_0^{1/2}/2)$ if C_0 is large enough.

The regularity of $DF_m^{(1)}$. To finish the proof, we should prove that $DF_m^{(1)}$ satisfies the bounds defining $\mathcal{K}(C_\#)$, for some constant $C_\#$ independent of C_0. Since $\phi_m^{(1)} = Q \circ \phi_m \circ \Gamma^{(0)} \circ \Gamma^{(1)}$, we have

$$D\phi_m^{(1)} = (DQ \circ \phi_m \circ \Gamma^{(0)} \circ \Gamma^{(1)}) \cdot (D\phi_m \circ \Gamma^{(0)} \circ \Gamma^{(1)}) \cdot (D\Gamma^{(0)} \circ \Gamma^{(1)}) \cdot D\Gamma^{(1)} \,. \tag{3.9}$$

Since $\mathcal{K}$ is invariant under multiplication (Proposition A.4), and under composition by Lipschitz maps sending stable leaves to stable leaves (Proposition A.5), it is sufficient to show that $D\phi_m$, $D\Gamma^{(0)}$, and $D\Gamma^{(1)}$ all satisfy the bounds defining $\mathcal{K}(C_\#)$. For $D\phi_m$, this follows from our assumptions (note that this is where (2.10)–(2.11) are used).

Since $\Gamma^{(0)} = (\Lambda^{(0)})^{-1}$, we have $D\Gamma^{(0)} = (D\Lambda^{(0)})^{-1} \circ \Gamma^{(0)}$. Since $D\Lambda^{(0)}$ is expressed in terms of DF_m, it belongs to $\mathcal{K}$. As $\mathcal{K}$ is invariant under inversion (Proposition A.4) and composition, we obtain $D\Gamma^{(0)} \in \mathcal{K}(C_\#)$.

Since $D\phi_m^{(1)}(x, 0) = \mathrm{id}$, it follows from (3.9) that, on the set $\{(x, 0)\}$, $D\Gamma^{(1)}$ is the inverse of the restriction of a function in $\mathcal{K}$, and in particular $D\Gamma^{(1)}(x, 0)$ is a β-Hölder continuous function of x, by (A.7). Since $D\Gamma^{(1)}(x, y)$ only depends on x, it follows that $D\Gamma^{(1)}$ belongs to $\mathcal{K}$. This concludes the proof of Lemma 3.5. □

We return to the proof of Lemma 3.3:
Second step: Gluing the foliations $\phi_m^{(1)}$ *together.*

Let $\gamma(x,y)$ be a C^∞ function equal to 1 on the ball $B(C_0^{1/2}/2)$, vanishing outside of $B(C_0^{1/2})$. Let $\phi_m^{(1)}(x,y) = (F_m^{(1)}(x,y),y)$ be a foliation defined by Lemma 3.5, and put

$$\phi_m^{(2)}(x,y) = (\gamma(x-x_m,y)(F_m^{(1)}(x,y)-x)+x,y)\,. \tag{3.10}$$

Then $\phi_m^{(2)}$ defines a foliation on the ball of radius $C_0^{1/2}$ around Πm, coinciding with $\phi_m^{(1)}$ on $B(\Pi m, C_0^{1/2}/2)$, and $\phi_m^{(2)}$ is equal to the identity on the boundary of $B(\Pi m, C_0^{1/2})$. By construction, $\phi_m^{(2)}(x,y) = (F_m^{(2)}(x,y),y)$ with $F_m^{(2)}(x,0) = x$. Moreover, $DF_m^{(2)}$ is expressed in terms of γ, $D\gamma$, $F_m^{(1)}$ and $DF_m^{(1)}$. All those functions belong to $\mathcal{K}(C_\#)$ (the first three functions are Lipschitz and bounded, hence in $\mathcal{K}(C_\#)$, while we proved in Lemma 3.5 that $DF_m^{(1)} \in \mathcal{K}(C_\#)$). Therefore, $DF_m^{(2)} \in \mathcal{K}(C_\#)$ by Proposition A.4.

We proved in (a) that the balls $B(\Pi m, C_0^{1/2})$ for $m \in \mathcal{M}'$ are disjoint, therefore all those foliations can be glued together (with the trivial vertical foliation outside of $\bigcup_{m\in\mathcal{M}'} B(\Pi m, C_0^{1/2})$), to get a single foliation parameterized by $\phi^{(2)} : \mathbb{R}^d \to \mathbb{R}^d$. We emphasize that this new foliation is not necessarily contained in the cone $Q(\widetilde{\mathcal{C}}^s)$, since the function γ contributes to the derivative of $\phi^{(2)}$. Nevertheless, it is uniformly transverse to the direction $\mathbb{R}^{d_u} \times \{0\}$, and this will be sufficient for our purposes. Let us write $\phi^{(2)}(x,y) = (F^{(2)}(x,y),y)$, where $F^{(2)}$ coincides everywhere with a function $F_m^{(2)}$ or with the function $(x,y) \mapsto x$. Since all the derivatives of those functions belong to $\mathcal{K}(C_\#)$, it follows that $DF^{(2)} \in \mathcal{K}(C_\#)$ (for some other constant $C_\#$, worse than the previous one due to the gluing). Since we will need to reuse this last constant, let us denote it by $C_\#^{(0)}$.

Third step: Pushing the foliation $\phi^{(2)}$ with D^{-1}. This step is very simple, although this is where (3.1) is needed: Define a new foliation by

$$F^{(3)}(x,y) = A^{-1}F^{(2)}(Ax,By), \quad \phi^{(3)}(x,y) = (F^{(3)}(x,y),y)\,, \tag{3.11}$$

so that $D^{-1}\phi^{(2)} = \phi^{(3)}D^{-1}$. The map $F^{(3)}$ satisfies $F^{(3)}(x,0) = x$. Moreover

$$\partial_x F^{(3)}(x,y) = A^{-1}(\partial_x F^{(2)})(Ax,By)A\,, \quad \partial_y F^{(3)}(x,y) = A^{-1}(\partial_y F^{(2)})(Ax,By)B\,.$$

In particular, if $|A^{-1}|$ and $|B|$ are small enough (which can be ensured by decreasing ϵ in (3.1)), we can make $\partial_y F^{(3)}$ arbitrarily small. Since $|B| \le 1 \le |A|$, it also follows that

$$\begin{aligned} |DF^{(3)}(x,y) - DF^{(3)}(x,y')| &\le |A^{-1}||A||DF^{(2)}(Ax,By) - DF^{(2)}(Ax,By')| \\ &\le |A^{-1}||A|C_\#^{(0)}|By-By'|^\alpha \le |A^{-1}||A|C_\#^{(0)}|B|^\alpha|y-y'|^\alpha\,. \end{aligned} \tag{3.12}$$

In the same way,

$$\begin{aligned} |DF^{(3)}(x,y) &- DF^{(3)}(x,y') - DF^{(3)}(x',y) + DF^{(3)}(x',y')| \\ &\le |A^{-1}||A||DF^{(2)}(Ax,By) - DF^{(3)}(Ax,By') \\ &\qquad - DF^{(3)}(Ax',By) + DF^{(3)}(Ax',By')| \\ &\le |A^{-1}||A|C_\#^{(0)}|Ax-Ax'|^\beta|By-By'|^{\alpha-\beta} \\ &\le |A^{-1}||A|C_\#^{(0)}|A|^\beta|B|^{\alpha-\beta}|x-x'|^\beta|y-y'|^{\alpha-\beta}\,. \end{aligned} \tag{3.13}$$

If the bunching constant ϵ in (3.1) is small enough (depending on C_1), we can ensure that the two last equations are bounded, respectively, by $|y-y'|^\alpha/(2C_1)$ and $|x-x'|^\beta|y-y'|^{\alpha-\beta}/(4C_0^2C_1)$, i.e., the map $F^{(3)}$ satisfies the requirements (2.9) and (2.11) for admissible foliations, with better constants that will be useful below.

Taking $y'=0$ in (3.13), we obtain

$$|DF^{(3)}(x,y)-DF^{(3)}(x',y)| \le |x-x'|^\beta |y|^{\alpha-\beta}/(4C_0^2C_1)+|DF^{(3)}(x,0)-DF^{(3)}(x',0)|.$$

Moreover, $\partial_x F^{(3)}(x,0)=\partial_x F^{(3)}(x',0)=\mathrm{id}$, so that

$$\begin{aligned}|DF^{(3)}(x,0)-DF^{(3)}(x',0)| &= |\partial_y F^{(3)}(x,0)-\partial_y F^{(3)}(x',0)|\\ &\le |A^{-1}||B||\partial_y F^{(2)}(Ax,0)-\partial_y F^{(2)}(Ax',0)|\\ &\le |A^{-1}||B|C_\#^{(0)}|Ax-Ax'|^\beta\\ &\le |A^{-1}||B|C_\#^{(0)}|A|^\beta |x-x'|^\beta\,.\end{aligned}$$

The quantity $|A^{-1}||B||A|^\beta$ is bounded by $C_\#\lambda_u^{-1}\lambda_s\Lambda_u^\beta$. Choosing ϵ small enough in (3.1), it can be made arbitrarily small. For $|y|\le C_0^2$, this yields

$$|DF^{(3)}(x,y)-DF^{(3)}(x',y)| \le |x-x'|^\beta/(2C_1)\,, \tag{3.14}$$

which is a small reinforcement of (2.10).

Fourth step: Pushing the foliation $\phi^{(3)}$ with $(Q')^{-1}$. Define a map $F^{(4)}(x,y)=F^{(3)}(x,y)+P'y$, and let $\phi^{(4)}(x,y)=(F^{(4)}(x,y),y)$. The corresponding foliation is the image of $\phi^{(3)}$ under $(Q')^{-1}$. Let us fix a cone $\mathcal{C}_1^s$ which sits compactly between $\mathcal{C}_0^s$ and $\mathcal{C}^s$. Since the graph $\{(P'y,y)\}$ is contained in $\mathcal{C}_0^s$, the foliation $F^{(4)}$ is contained in $\mathcal{C}_1^s$ if $\partial_y F^{(3)}$ is everywhere small enough. Moreover, the bounds of the previous step concerning $DF^{(3)}$ directly translate into the following bounds for $DF^{(4)}$, for all $x,x'\in\mathbb{R}^{d_u}$ and all $y,y'\in B(0,C_0^2)$:

$$|DF^{(4)}(x,y)-DF^{(4)}(x,y')| \le |y-y'|^\alpha/(2C_1)\,, \tag{3.15}$$

$$|DF^{(4)}(x,y)-DF^{(4)}(x',y)| \le |x-x'|^\beta/(4C_0^2C_1)\,, \tag{3.16}$$

$$\begin{aligned}|DF^{(4)}(x,y)-DF^{(4)}(x,y')-DF^{(4)}(x',y)+DF^{(4)}(x',y')|&\\ \le |x-x'|^\beta|y-y'|^{\alpha-\beta}/(2C_1)\,.&\end{aligned} \tag{3.17}$$

In particular, since $\partial_x F^{(4)}(x,0)=\mathrm{id}$, the bound (3.15) implies that $\partial_x F^{(4)}$ is bounded and has a bounded inverse on a ball of radius $C_1\ge 2C_0$.

Last step: Pushing the foliation $\phi^{(4)}$ with $\mathcal{T}^{-1}M$. Let $\mathcal{U}=\mathcal{T}^{-1}M$, and consider ϕ' the foliation obtained by pushing the foliation $\phi^{(4)}$ with $\mathcal{U}$. We claim that ϕ' belongs to $\mathcal{F}(0,\mathcal{C}^s,C_0,C_1)$, and that we can write $\mathcal{U}\circ\phi=\phi'\circ\Psi'$ for some $\Psi'\in D_1^{1+\alpha}(C_\#)$.

To prove this, we follow the arguments in the proof of Lemma 3.5 (with simplifications here since $\mathcal{U}$ is close to the identity). First, fix x and consider the map $L_x: y\mapsto \pi_2\circ\mathcal{U}\circ\phi^{(4)}(x,y)$. Writing $\mathcal{U}=\mathrm{id}+\mathcal{V}$ where $\|\mathcal{V}\|_{C^{1+\alpha}}\le\epsilon$, we have $L_x(y)=y+\pi_2\circ\mathcal{V}(F^{(4)}(x,y),y)$. Since $F^{(4)}$ is bounded in C^1 on the ball $B(0,2C_1)$, it follows that, if ϵ is small enough, then the restriction of L_x to the ball $B(0,2C_1)$ (in $\mathbb{R}^{d_s}$) is arbitrarily close to the identity. Therefore, its inverse is well defined, and we can set $\Gamma^{(0)}(x,y)=(x,L_x^{-1}(y))$. By construction, the map $\mathcal{U}\circ\phi^{(4)}\circ\Gamma^{(0)}(x,0)$ has the form $(L^{(1)}(x),0)$ for some function $L^{(1)}$, which is bounded in $C^{1+\beta}$ and arbitrarily close to the identity in C^1 if ϵ is small. Let $\Gamma^{(1)}(x,y)=((L^{(1)})^{-1}(x),y)$, then the map $\phi':=\mathcal{U}\circ\phi^{(4)}\circ\Gamma^{(0)}\circ\Gamma^{(1)}$ is defined on the set $\{(x,y)\mid |y|\le C_1\}$ (which contains $B(0,C_0)$), and it takes the form $\phi'(x,y)=(F'(x,y),y)$ for some function F' with $F'(x,0)=0$.

Since ϕ' is obtained by composing $\phi^{(4)}$ with diffeomorphisms arbitrarily close to the identity, it follows from (3.15)–(3.17) that F' satisfies (2.9)–(2.11). Indeed, the present analogue of (3.9) is

$$D\phi'=(D\mathcal{U}\circ\phi^{(4)}\circ\Gamma^{(0)}\circ\Gamma^{(1)})\cdot(D\phi^{(4)}\circ\Gamma^{(0)}\circ\Gamma^{(1)})\cdot(D\Gamma^{(0)}\circ\Gamma^{(1)})\cdot D\Gamma^{(1)}\,,$$

where $\Gamma^{(0)}$ and $\Gamma^{(1)}$ here satisfy the same properties as the maps with the same names in the proof of Lemma 3.5, and where $D\mathcal{U} \circ \phi^{(4)} \circ \Gamma^{(0)} \circ \Gamma^{(1)}$ is bounded and α-Hölder and thus belongs to $\mathcal{K}$ (see the remark before Proposition A.4). We may thus argue exactly as in the last step of the proof of Lemma 3.5.

Moreover, since $(\partial_y F^{(4)}(z)w, w)$ takes its values in the cone $\mathcal{C}_1^s$, it follows that $(\partial_y F'(z)w, w)$ lies in the cone $\mathcal{C}^s$ if $\mathcal{U}$ is close enough to the identity. Hence, the foliation defined by ϕ' is contained in $\mathcal{C}^s$. This shows that ϕ' belongs to $\mathcal{F}(0, \mathcal{C}^s, C_0, C_1)$.

Finally, the function $\Psi = (\Gamma^{(0)} \circ \Gamma^{(1)})^{-1}$ belongs to $D_1^{1+\alpha}(C_\#)$. This concludes the proof of Lemma 3.3. □

Remark 3.6. An inspection of the proof of Lemma 3.3 shows that one can obtain stronger conclusions: For any $C' > 0$, one can ensure that the final chart ϕ' is defined on a ball of radius C', and satisfies $|D\phi'(x,y) - D\phi'(x,y')| \le |y-y'|^\alpha / C'$, as follows. If the bunching and hyperbolicity conditions in (3.1) are large enough, the third step of the proof yields a chart $\phi^{(3)}$ with $|DF^{(3)}(x,y) - DF^{(3)}(x,y')| \le \delta |y-y'|^\alpha$ for arbitrarily small $\delta > 0$. Hence, in the inequality (3.15) regarding the map $F^{(4)}$ (which is defined on $\mathbb{R}^d$), the constant $2C_1$ can be replaced with an arbitrarily large constant, allowing an arbitrarily large domain of definition for ϕ'. The same observation holds for (3.16) and (3.17). This remark is the key to the proof of Proposition 2.15.

4. Results on the local spaces $H_p^{t,s}$.

4.1. **Basic facts on the local spaces $H_p^{t,s}$.** We start with reminders from [BG09].

The proof of Lemma 22 from [BG09] implies the following:

Lemma 4.1. *Let $t > 0$, $s < 0$ and $\widetilde{\alpha} > 0$ be real numbers with $t + |s| < \widetilde{\alpha}$. For any $p \in (1, \infty)$, there exists a constant $C_\#$ such that for any $C^{\widetilde{\alpha}}$ function $g : \mathbb{R}^d \to \mathbb{C}$,*

$$\|g \cdot \omega\|_{H_p^{t,s}} \le C_\# \|g\|_{C^{\widetilde{\alpha}}} \|\omega\|_{H_p^{t,s}} \ .$$

The following extension of a classical result of Strichartz (see [BG09, Lemma 23]) is the key to our results. It follows from Lemma 23 of [BG09] and a linear change of coordinates.

Lemma 4.2. *Let $1 < p < \infty$ and $1/p - 1 < s \le 0 \le t < 1/p$. Let $e_1, \dots, e_d$ be a basis of $\mathbb{R}^d$, such that $e_{d_u+1}, \dots, e_d$ form a basis of $\{0\} \times \mathbb{R}^{d_s}$. There exists a constant $C_\#$ (depending only on p, s, t and the norm of the matrix change of coordinate between $e_1, \dots, e_d$ and the canonical basis of $\mathbb{R}^d$) so that, for any subset U of $\mathbb{R}^d$ whose intersection with almost every line directed by a vector e_i has at most M connected components,*

$$\|1_U \omega\|_{H_p^{t,s}} \le C_\# M \|\omega\|_{H_p^{t,s}} \ .$$

The following is essentially Lemma 28 in [BG09].

Lemma 4.3 (Localization principle)**.** *Let $\mathbb{K}$ be a compact subset of $\mathbb{R}^d$. For each $m \in \mathbb{Z}^d$, consider a function η_m supported in $m + \mathbb{K}$, with uniformly bounded C^1 norm. For any $p \in (1, \infty)$ and t, $s \in \mathbb{R}$ with $|t| + |s| < 1$, there exists $C_\# > 0$ so that for each $\omega \in H_p^{t,s}$*

$$\left(\sum_{m \in \mathbb{Z}^d} \|\eta_m \omega\|_{H_p^{t,s}}^p \right)^{1/p} \le C_\# \|\omega\|_{H_p^{t,s}} \ .$$

Proof. Consider a compactly supported C^∞ function γ, equal to 1 on $\mathbb{K}$, and write $\gamma_m(z) = \gamma(z - m)$. Then $\eta_m = \eta_m \gamma_m$, and

$$\|\eta_m \omega\|_{H_p^{t,s}} = \|\eta_m \gamma_m \omega\|_{H_p^{t,s}} \le C_\# \|\gamma_m \omega\|_{H_p^{t,s}}$$

by Lemma 4.1. The result follows by applying [BG09, Lemma 28]. □

For any real number t and any $1 < p < \infty$, we set $H_p^t(X_0)$ to be the Sobolev-Triebel space defined as the distributions that have finite $H_p^t(\mathbb{R}^d)$ norm in any (fixed) smooth coordinate system.

As usual, a compact imbedding statement à la Arzelà-Ascoli will be used (recall that X_0 is compact):

Lemma 4.4. *Let $s < 0 < t$ with $t + |s| < 1$, and let $1 < p < \infty$. Assume that $t - |s| > -\beta$. Then, for any R, C_0, C_1, the space $\mathbf{H}_p^{t,s}(R, C_0, C_1)$ is continuously embedded in $H_p^{t-|s|}(X_0)$. In addition, we have the continuous embeddings*

$$\mathbf{H}_p^{t,s}(R, C_0, C_1) \subset \mathbf{H}_p^{t',s'}(R, C_0, C_1) \text{ if } t' \le t \text{ and } s' \le s\,. \tag{4.1}$$

Moreover, this inclusion is compact if $t' < t$.

Proof. Before proving the lemma, we start with a functional analytic preliminary, required because $t - |s|$ will be strictly negative in our application of the lemma. If $1/p + 1/p' = 1$ for $1 < p,\ p' < \infty$, and $r > 0$, then classical duality results (see e.g. [BG09, Lemma 20] and references therein) yield $(H_p^r)^* = H_{p'}^{-r}$. If G is a diffeomorphism of $\mathbb{R}^d$ then the dual operator L^* on H_p^r to $L(w') = w' \circ G$ is $w \mapsto |\det DG^{-1}| w \circ G^{-1}$. For $r \in [0, 1]$, H_p^r is invariant under the composition by a C^1 diffeomorphism G (since this is the case of $H_p^0 = L^p$, and H_p^1). By duality, H_p^{-r} is invariant by $w \mapsto |\det DG^{-1}| \cdot w \circ G^{-1}$. Therefore, Lemma 4.1 shows that H_p^{-r} is invariant under the composition with diffeomorphisms whose jacobian is C^β for some $\beta > r$.

We now turn to the proof of the lemma. In any admissible chart, the continuous embedding claim (4.1) follows from the definitions and properties of Triebel spaces, taking the supremum over all admissible charts. For the rest of the proof, let us fix R, C_0, C_1. To simplify notations, we will write $\mathbf{H}_p^{t,s}$ for $\mathbf{H}_p^{t,s}(R, C_0, C_1)$.

Consider now $s' \le s$ and $t' < t$. Fix also $t_0 < t$ with $t_0 - |s| > -\beta$. Since $H_p^{t,s}$ is included in $H_p^{t-|s|,0}$, it follows by taking the supremum over the admissible charts that $\mathbf{H}_p^{t,s}$ is included in $\mathbf{H}_p^{t-|s|,0}$. Moreover, for any admissible charts $\phi_1, \phi_2 \in \mathcal{F}(\zeta)$ for some ζ (recall (2.17)), the change of coordinates $\phi_2 \circ \phi_1^{-1}$ is C^1 and has a (uniformly) C^β Jacobian. It follows from the functional analytic preliminary that changing the system Φ of charts in the definition of the $\mathbf{H}_p^{t-|s|,0}$-norm gives equivalent norms. Therefore, $\mathbf{H}_p^{t-|s|,0}$ is isomorphic to the Triebel space $H_p^{t-|s|}(X_0)$. Since the inclusion of $H_p^{t-|s|}(X_0)$ in $H_p^{t_0-|s|}(X_0)$ is compact, it follows that the inclusion $\mathbf{H}_p^{t,s} \to \mathbf{H}_p^{t_0-|s|,0}$ is also compact.

Consider now a sequence $\omega_n \in \mathbf{H}_p^{t,s}$, with norms bounded by 1. To prove that the inclusion of $\mathbf{H}_p^{t,s}$ in $\mathbf{H}_p^{t',s'}$ is compact, it is sufficient to show that, for any ϵ, there exists a subsequence of ω_n along which

$$\limsup \|\omega_m - \omega_n\|_{\mathbf{H}_p^{t',s'}} \le 2\epsilon. \tag{4.2}$$

We can assume without loss of generality that ω_n converges in $\mathbf{H}_p^{t_0-|s|,0}$. Let $C(\epsilon)$ be such that any distribution ω on $\mathbb{R}^d$ satisfies

$$\|\omega\|_{H_p^{t',s'}} \le \epsilon \|\omega\|_{H_p^{t,s}} + C(\epsilon) \|\omega\|_{H_p^{t_0-|s|,0}}\,. \tag{4.3}$$

To prove that such a constant $C(\epsilon)$ exists, let us note that the kernel $a_{t',s'}$ defining the $H_p^{t',s'}$-norm is bounded by $\epsilon a_{t,s}$ outside of a compact set, where it is bounded by $C(\epsilon) a_{t_0-|s|,0}$ if $C(\epsilon)$ is large enough. Therefore, (4.3) follows from the Marcinkiewicz multiplier theorem (see e.g. [BG09, Theorem 21] or [Tri77, Theorem 2.4/2]).

Taking the supremum of the equation (4.3) over the admissible charts, we obtain

$$\|\omega_n - \omega_m\|_{\mathbf{H}_p^{t',s'}} \leq \epsilon \|\omega_n - \omega_m\|_{\mathbf{H}_p^{t,s}} + C(\epsilon) \|\omega_n - \omega_m\|_{\mathbf{H}_p^{t_0-|s|,0}} . \tag{4.4}$$

Since the quantity $\|\omega_n - \omega_m\|_{\mathbf{H}_p^{t_0-|s|,0}}$ converges to 0 when $n, m \to \infty$, this proves (4.2). □

The following lemma on partitions of unity is Lemma 32 from [BG09]:

Lemma 4.5. *Let t and s be arbitrary real numbers. There exists a constant $C_\#$ such that, for any distributions $v_1, \dots, v_l$ with compact support in $\mathbb{R}^d$, belonging to $H_p^{t,s}$, there exists a constant C depending only on the supports of the distributions v_i with*

$$\left\| \sum_{i=1}^{l} v_i \right\|_{H_p^{t,s}}^p \leq C_\# m^{p-1} \sum_{i=1}^{l} \|v_i\|_{H_p^{t,s}}^p + C \sum_{i=1}^{l} \|v_i\|_{H_p^{t-1,s}}^p , \tag{4.5}$$

where m is the intersection multiplicity of the supports of the v_i's, i.e., $m = \sup_{x \in \mathbb{R}^d} \mathrm{Card}\{i \mid x \in \mathrm{supp}(v_i)\}$.

4.2. **The effect of composition on the local space $H_p^{t,s}$.** In view of Theorem 2.5, we describe how the local spaces $H_p^{t,s}$ behave under composition with hyperbolic matrices and appropriate maps preserving the stable leaves.

The following lemma is a particular case of [BG09, Lemma 25].

Lemma 4.6. *For all $s < 0 < t$ and $t - |s| < 0$, for all $p \in (1, \infty)$, and every $t' < t$ there exists a constant $C_\#$ (depending only on t, s, p, t') so that the following holds: Let $D = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ be a block diagonal matrix such that $|Av| \geq \lambda_u |v|$ and $|Bv| \leq \lambda_s |v|$ for $\lambda_u > 1$ and $\lambda_s < 1$. Then there exists a constant C such that, for all $\omega \in H_p^{t,s}$,*

$$\left\|\omega \circ D^{-1}\right\|_{H_p^{t,s}} \leq C_\# |\det D|^{1/p} \max(\lambda_u^{-t}, \lambda_s^{-(t+s)}) \|\omega\|_{H_p^{t,s}} + C \|\omega\|_{H_p^{t',s}} .$$

Adapting the second part of the proof of [BG09, Lemma 25] gives:

Lemma 4.7. *Let $C > 0$, and let $-\alpha < s < 0 < t < 1$ with $\alpha t + |s| < \alpha$. There exists a constant $C' > 0$ so that for any $\Psi \in D^1_{1+\alpha}(C)$ whose range contains a ball $B(z, C_0^{1/2})$, and for any distribution $\omega \in H_p^{t,s}$ supported in $B(z, C_0^{1/2}/2)$, the composition $\omega \circ \Psi$ is well defined, and*

$$\|\omega \circ \Psi\|_{H_p^{t,s}} \leq C' \|\omega\|_{H_p^{t,s}} . \tag{4.6}$$

Proof. Without loss of generality, we may assume $z = \Psi^{-1}(z) = 0$. Let γ be a C^∞ function equal to 1 on $B(0, C_0^{1/2}/2)$ and vanishing outside of $B(0, C_0^{1/2})$. We want to show that the operator $\mathcal{M} : \omega \mapsto (\gamma\omega) \circ \Psi$ is bounded by C' as an operator from $H_p^{t,s}$ to itself. By interpolation, it is sufficient to prove this statement for $H_p^{1,0}$, for L^p, and for $H_p^{0,-\alpha}$. This is done in the second step of the proof of Lemma 25 in [BG09] – the result there is formulated for $C^{1+\alpha}$ diffeomorphisms, but a glance at the proof there indicates that the C^α regularity of the jacobian is only used along the stable leaves, in the argument for $H_p^{0,-\alpha}$, and the definition of $D^1_{1+\alpha}(C)$ ensures that the jacobian is indeed regular along stable leaves. □

5. Proof of the main theorem on piecewise cone hyperbolic maps

In this section, we prove Theorem 2.14 and Proposition 2.15.

We may fix once and for all a constant $C_0 > 1$ large enough so that the assumptions of Lemma 3.3 are satisfied for the finite set $\mathcal{C}_{i,j}$ of extended cones chosen in Section 2.3.

The following lemma implies Theorem 2.14 since the inclusion of $\mathbf{H}_p^{t,s}$ into $\mathbf{H}_p^{t',s}$ is compact for $s < 0 < t$ if $t' < t$, and $t + |s| < 1$, $t - |s| > -\beta$, by Lemma 4.4.

Lemma 5.1. *Let α, T, g, p be as in Theorem 2.5 and let $1/p-1 < s < 0 < t < 1/p$, with $\alpha|s| + t < \alpha$. For any $t' < t$ there is $C_\#$ so that, for any N, if C_1 is large enough, then for any large enough n which is a multiple of N, and for any large enough R, there exists D_n so that*

$$\left\|\mathcal{L}_g^n\omega\right\|^p_{\mathbf{H}_p^{t,s}(R,C_0,C_1)} \le D_n \left\|\omega\right\|^p_{\mathbf{H}_p^{t',s}(R,C_0,C_1)} + C_\#(C_\# N^p)^{n/N} D_n^b (D_n^e)^{p-1} \times \times \left\| |\det DT^n| \max(\lambda_{u,n}^{-t}, \lambda_{s,n}^{-(s+t)})^p |g^{(n)}|^p \right\|_{L^\infty} \left\|\omega\right\|^p_{\mathbf{H}_p^{t,s}(R,C_0,C_1)} . \tag{5.1}$$

Proof of Lemma 5.1. To simplify notation, we write $x \le_c y$ if $x \le y$ up to compact terms, i.e., terms which are controlled by $\left\|\omega\right\|_{\mathbf{H}_p^{t',s}(R,C_0,C_1)}$ for some $t' < t$. Note that if $t'' < t$ is such that $t'' < t'$, then an upper bound in terms of t'' trivially implies the upper bound for t' because $\left\|\omega\right\|_{H_p^{t'',s}} \le C \left\|\omega\right\|_{H_p^{t',s}}$. Conversely, an upper bound in terms of t' implies the upper bound for t'' because, for any $\epsilon > 0$, there exists a constant $C(\epsilon)$ so that for all v

$$\left\|\omega\right\|_{H_p^{t',s}} \le \epsilon \left\|\omega\right\|_{H_p^{t,s}} + C(\epsilon) \left\|\omega\right\|_{H_p^{t'',s}} .$$

(The above bound is proved just like (4.3).) We shall apply the above remark implicitly whenever we have a bound $x \le_c y$. This allows us to replace $t'' < 0$ by $0 < t' < t$ when invoking Lemmas 4.1 or 4.7.

Before starting the proof, let us describe the order in which we choose the constants. First, N is fixed in the statement (it will be used in the second step of the proof in order to apply Lemma 4.2). Then, we choose C_1 very large, in the second step below, so that the admissible charts ϕ_ζ are close enough to linear maps (C_1 depends on N). Then, we fix n to be some very large multiple of N, depending on C_1 (it should be large enough so that every branch of T^n is hyperbolic enough so that Lemma 3.3 applies). Finally, we choose R very large so that, at scale $1/R$, all the iterates of T up to time n look like linear maps, and all the boundaries of the sets we are interested in look like hyperplanes. For the presentation of the argument, we will start the proof with some values of C_1, n, R, and increase them whenever necessary, checking each time that C_1 does not depend on n, R, and that n does not depend on R, to avoid bootstrapping issues. We will denote by $C_\#$ a constant that does not depend on N, C_1, n, R, and may vary from line to line.

For every $\mathbf{i} \in I^n$, we fix a small neighborhood $\widetilde{O}_\mathbf{i}$ of $\overline{O_\mathbf{i}}$ such that $T_\mathbf{i}$ admits an extension to $\widetilde{O}_\mathbf{i}$ with the same hyperbolicity properties as the original $T_\mathbf{i}$. Reducing these sets if necessary, we can ensure that their intersection multiplicity is bounded by D_n^b, and that the intersection multiplicity of the sets $T_\mathbf{i}\widetilde{O_\mathbf{i}}$ is bounded by D_n^e.

For $\zeta = (i, j, m) \in \mathcal{Z}(R)$, let us write

$$A(\zeta) = A(\zeta, R) = (\kappa_\zeta^R)^{-1}(B(m, d)) \subset X .$$

The set $A(\zeta)$ is a neighborhood of q_ζ, of diameter bounded by $C_\# R^{-1}$, and containing the support of ρ_ζ.

Let us fix some system of charts Φ as in the Definition 2.12 of the $\mathbf{H}_p^{t,s}(R, C_0, C_1)$-norm. We want to estimate $\left\|\mathcal{L}_g^n\omega\right\|_\Phi$.

First step. The sets $\{T_\mathbf{i}^n(O_\mathbf{i}) \mid \mathbf{i} \in I^n\}$ have intersection multiplicity at most D_n^e. Writing[8] $\mathcal{L}_g^n\omega = \sum_\mathbf{i} 1_{T_\mathbf{i}^n O_\mathbf{i}}(g^{(n)}\omega) \circ T_\mathbf{i}^{-n}$, we get by Lemma 4.5 that for each

[8]Elements of L^∞ are defined almost everywhere, and the transfer operator is defined initially on L^∞, so the fact that $\bigcup_i O_i = X_0$ only modulo a zero Lebesgue measure set is irrelevant.

$\zeta \in \mathcal{Z}(R)$

$$\left\|(\rho_\zeta \cdot 1_{O_\zeta} \mathcal{L}_g^n \omega) \circ \Phi_\zeta\right\|_{H_p^{t,s}}^p \\ \le_c C_\#(D_n^e)^{p-1} \sum_{\mathbf{i}\in I^n} \left\|(\rho_\zeta 1_{O_\zeta} 1_{T_\mathbf{i}^n O_\mathbf{i}} (g^{(n)}\omega) \circ T_\mathbf{i}^{-n}) \circ \Phi_\zeta\right\|_{H_p^{t,s}}^p .$$

Summing over $\zeta \in \mathcal{Z}(R)$, we obtain

$$\left\|\mathcal{L}_g^n \omega\right\|_\Phi^p \le_c C_\#(D_n^e)^{p-1} \sum_{\zeta\in\mathcal{Z}(R),\mathbf{i}\in I^n} \left\|(\rho_\zeta 1_{O_\zeta} 1_{T_\mathbf{i}^n O_\mathbf{i}} (g^{(n)}\omega) \circ T_\mathbf{i}^{-n}) \circ \Phi_\zeta\right\|_{H_p^{t,s}}^p .$$

For $i \in I$, let $U_{i,j,2}$, $1 \le j \le N_i$, be arbitrary open sets covering a fixed neighborhood $\widetilde{O}_i^0$ of $\overline{O_i}$, such that $\overline{U_{i,j,2}} \subset U_{i,j,1}$ (they do not depend on n, R, or any other choice). For each $\zeta \in \mathcal{Z}(R)$, and $\mathbf{i} = (i_0, \dots, i_{n-1}) \in I^n$ such that $T_\mathbf{i}^n O_\mathbf{i}$ intersects $A(\zeta)$, the point $T_\mathbf{i}^{-n}(q_\zeta)$ belongs to $\widetilde{O}_{i_0}^0$ if R is large enough, we can therefore consider k such that it belongs to $U_{i_0,k,2}$. Then $\sum_{\ell\in\mathcal{Z}_{i_0,k}(R)} \rho_{i_0,k,\ell}$ is equal to 1 on a neighborhood of fixed size of $T_\mathbf{i}^{-n}(q_\zeta)$, so that $\sum_{\ell\in\mathcal{Z}_{i_0,k}(R)} \rho_{i_0,k,\ell} \circ T_\mathbf{i}^{-n}$ is equal to 1 on $A(\zeta)$ if R is large enough (depending on n but not on Φ or ζ). Since the intersection multiplicity of the supports of the $\rho_{i_0,k,\ell} \circ T_\mathbf{i}^{-n}$ is uniformly bounded, Lemma 4.5 gives, if R is large enough (uniformly in Φ, ζ, k, $\mathbf{i}$)

$$\left\|(\rho_\zeta 1_{O_\zeta} 1_{T_\mathbf{i}^n O_\mathbf{i}} (g^{(n)}\omega) \circ T_\mathbf{i}^{-n}) \circ \Phi_\zeta\right\|_{H_p^{t,s}}^p \\ \le_c C_\# \sum_{\ell\in\mathcal{Z}_{i_0,k}(R)} \left\|(\rho_\zeta 1_{O_\zeta} 1_{T_\mathbf{i}^n O_\mathbf{i}} (\rho_{i_0,k,\ell} \cdot g^{(n)}\omega) \circ T_\mathbf{i}^{-n}) \circ \Phi_\zeta\right\|_{H_p^{t,s}}^p .$$

Taking R large enough and summing over $\zeta \in \mathcal{Z}(R)$, $\mathbf{i} \in I^n$ and k in $\{1, \dots, N_{i_0}\}$ such that $T_\mathbf{i}^{-n}(q_\zeta) \in U_{i_0,k,2}$, we get (writing $\zeta' = (i_0, k, \ell) \in \mathcal{Z}(R)$)

$$\left\|\mathcal{L}_g^n \omega\right\|_\Phi^p \le_c C_\#(D_n^e)^{p-1} \sum_{\zeta,\mathbf{i},\zeta'} \left\|(\rho_\zeta 1_{O_\zeta} 1_{T_\mathbf{i}^n O_\mathbf{i}} (\rho_{\zeta'} \cdot g^{(n)}\omega) \circ T_\mathbf{i}^{-n}) \circ \Phi_\zeta\right\|_{H_p^{t,s}}^p , \tag{5.2}$$

where the sum is restricted to those $(\zeta, \mathbf{i}, \zeta')$ such that the support of $\rho_{\zeta'}$ is included in $\widetilde{O}_\mathbf{i}$, the support of ρ_ζ is included in $T_\mathbf{i}\widetilde{O}_\mathbf{i}$, and $O_{\zeta'} = O_{i_0}$ (this restriction will be implicit in the rest of the proof).

Second step: Getting rid of the characteristic function. We claim that, if R is large enough, then for any ζ, $\mathbf{i}$, ζ' as in the right-hand-side of (5.2)

$$\left\|(\rho_\zeta 1_{O_\zeta} 1_{T_\mathbf{i}^n O_\mathbf{i}} (\rho_{\zeta'} \cdot g^{(n)}\omega) \circ T_\mathbf{i}^{-n}) \circ \Phi_\zeta\right\|_{H_p^{t,s}}^p \\ \le C_\#(C_\# N^p)^{n/N} \left\|(\rho_\zeta (1_{O_{\zeta'}} \rho_{\zeta'} \cdot g^{(n)}\omega) \circ T_\mathbf{i}^{-n}) \circ \Phi_\zeta\right\|_{H_p^{t,s}}^p . \tag{5.3}$$

Note that $1_{T_\mathbf{i}^n O_\mathbf{i}} = 1_{T_\mathbf{i}^n O_\mathbf{i}} \cdot (1_{O_{\zeta'}} \circ T_\mathbf{i}^{-n})$. Hence, to prove this inequality, it is sufficient to show that the multiplications by $1_{O_\zeta} \circ \Phi_\zeta$ and by $1_{T_\mathbf{i}^n O_\mathbf{i}} \circ \Phi_\zeta$ act boundedly on $H_p^{t,s}$, with norms bounded respectively by $C_\#$ and $(C_\# N^p)^{n/N}$. We shall show the latter, the former is similar. Let $\kappa = n/N$, we decompose $\mathbf{i} = (i_0, \dots, i_{n-1})$ into subsequences of length N, as $(\mathbf{i}_0, \dots, \mathbf{i}_{\kappa-1})$. Then $1_{T_\mathbf{i}^n O_\mathbf{i}} = \prod_{j=0}^{\kappa-1} 1_{O_{\mathbf{i}_j}} \circ T_{\mathbf{i}_j \mathbf{i}_{j+1}\dots\mathbf{i}_{\kappa-1}}^{-(\kappa-j)N}$. Define a set $P_j = T_{\mathbf{i}_j \mathbf{i}_{j+1}\dots\mathbf{i}_{\kappa-1}}^{(\kappa-j)N}(O_{\mathbf{i}_j})$, it is therefore sufficient to show that each multiplication by $1_{P_j} \circ \Phi_\zeta$ acts boundedly on $H_p^{t,s}$, with norm at most $C_\# N^p$. Let us fix such a set $P = P_j$. Locally, its boundary is contained in the images of the boundaries of the sets O_i under iterates of the map T. Let $L > 0$ be such that the boundary of each O_i, $i \in I$, is made of at most L hypersurfaces, it follows that the boundary of P is made of at most LN hypersurfaces Q_h (which are all uniformly transverse to the stable cone).

We wish to use our transversality assumption to apply Lemma 4.2. Write $\zeta = (i,j,m)$. Since the support of $\rho_\zeta \circ (\kappa_\zeta^R)^{-1} = \rho_m$ is contained in the ball $B(m,d)$, it is sufficient to prove the bounded multiplier property for distributions supported in $\phi_\zeta^{-1}(B(m,d))$. In $B(m,d)$, the boundary of the set $\kappa_\zeta^R(P)$ is contained in $\bigcup \kappa_\zeta^R(Q_h)$. If R is large enough, all the hypersurfaces $\kappa_\zeta^R(Q_h)$ look like hyperplanes in $\mathbb{R}^d$.

We will need the following easy geometrical lemma.

Lemma 5.2. *For any $\delta > 0$, $\delta' > 0$ and $M > 0$, there exists $\epsilon > 0$ satisfying the following property. Consider M hyperplanes $H_1, \dots, H_M$ in $\mathbb{R}^d$, such that every H_j contains a d_u-dimensional subspace E_j making an angle at least δ with $\{0\} \times \mathbb{R}^{d_s}$. Then*

- *For any unit vector $f \in \mathbb{R}^d$, there exists a vector $e \in \mathbb{R}^d$ with $|e - f| \le \delta'$ making an angle at least ϵ with every H_j.*
- *For any unit vector $f \in \{0\} \times \mathbb{R}^{d_s}$, there exists a vector $e \in \{0\} \times \mathbb{R}^{d_s}$ with $|e - f| \le \delta'$ making an angle at least ϵ with every H_j.*

The first point is proved by arguing that the measure of the ϵ–neighborhood of H_j in the ball $B(f, \delta')$ tends to 0 when ϵ tends to 0. Therefore, if ϵ is small enough, there exists a vector e in $B(f, \delta')$ avoiding all those neighborhoods, hence satisfying the required conclusion. For the second point, we obtain in the same way a vector $e \in \{0\} \times \mathbb{R}^{d_s}$ with $|e - f| \le \delta'$ which is ϵ-transverse to $H_j \cap (\{0\} \times \mathbb{R}^{d_s})$ for $1 \le j \le M$. Since E_j in the assumptions is uniformly transverse to e, the result follows.

Let us fix $\delta' > 0$ so that any family $e_1, \dots, e_d$ which is δ'-close to the canonical orthonormal basis $(f_1, \dots, f_d)$ of $\mathbb{R}^d$ is still a basis, and the matrices of the coordinate changes are bounded by a constant $C_\#$.

The pullback of every hypersurface $\kappa_\zeta^R(Q_h)$ under the differential $D\phi_\zeta(m)$ is very close to an hyperplane in $\mathbb{R}^d$. Applying the lemma with $M = LN$, we therefore obtain vectors $e_1, \dots, e_d$ which are δ'-close to an orthonormal basis of $\mathbb{R}^d$, such that $e_{d_u+1}, \dots, e_d$ form a basis of $\{0\} \times \mathbb{R}^{d_s}$, and which make everywhere an angle at least ϵ with the hypersurfaces $\kappa_\zeta^R(Q_h)$, for some $\epsilon > 0$ depending solely on N.

Consider now a straight line directed by one of the vectors e_l. Its image under ϕ_ζ is not anymore a straight line. However, if ϕ_ζ is very close to a linear map (which is true if C_1 is large enough), then it will almost be a straight line. In particular, its direction will deviate by at most $\epsilon/2$, hence it will be transverse to the hypersurface $\kappa_\zeta^R(Q_h)$, and it will intersect it in at most one point.

We have proved that, if C_1 is large enough, then any line S directed by one of the vectors e_l intersects each boundary hypersurface of $\Phi_\zeta^{-1}(P)$ in at most one point. Since $\Phi_\zeta^{-1}(P)$ has at most NL boundary hypersurfaces, S intersects $\Phi_\zeta^{-1}(P)$ along at most NL connected components. Therefore, Lemma 4.2 (together with our assumption that $1/p - 1 < s < 0 < t < 1/p$) implies that the multiplication by the characteristic function of this set acts boundedly on $H_p^{t,s}$, with a norm bounded by $C_\# NL$. This proves (5.3).

Combining (5.3) with (5.2), we get

(5.4)
$$\left\|\mathcal{L}_g^n \omega\right\|_\Phi^p \le_c C_\# (C_\# N^p)^{n/N} (D_n^e)^{p-1} \sum_{\zeta, \mathbf{i}, \zeta'} \left\| (\rho_\zeta (1_{O_{\zeta'}} \rho_{\zeta'} \cdot g^{(n)} \omega) \circ T_{\mathbf{i}}^{-n}) \circ \Phi_\zeta \right\|_{H_p^{t,s}}^p .$$

Third step: Using the composition lemma. The right hand side of (5.4) involves a sum over ζ' and ζ, and has therefore too many terms. In this step, we shall use Lemma 3.3, to pull the charts Φ_ζ back at time $-n$, and glue some of the pulled-back charts together to get rid of the summation over ζ.

Let us partition $\mathcal{Z}(R)$ into finitely many subsets $\mathcal{Z}^1,\dots,\mathcal{Z}^E$ such that $\mathcal{Z}^e$ is included in one of the sets $\mathcal{Z}_{i,j}(R)$, and $|m-m'|\geq C(C_0)$ whenever $(i,j,m)\neq(i,j,m')\in\mathcal{Z}^e$, where $C(C_0)$ is the constant C constructed in Lemma 3.3 (it only depends on C_0). The number E may be chosen independently of n.

We shall prove the following: For any $\zeta'\in\mathcal{Z}(R)$, any $\mathbf{i}\in I^n$ (such that the support of $\rho_{\zeta'}$ is included in $\widetilde{O}_{\mathbf{i}}$ and $O_{\zeta'}=O_{i_0}$) and any $1\leq e\leq E$, there exists an admissible chart $\Phi'=\Phi'_{\zeta',\mathbf{i},e}\in\mathcal{F}(\zeta')$ such that

(5.5)
$$\sum_{\zeta\in\mathcal{Z}^e}\left\|(\rho_\zeta(1_{O_{\zeta'}}\rho_{\zeta'}\cdot g^{(n)}\omega)\circ T_{\mathbf{i}}^{-n})\circ\Phi_\zeta\right\|_{H_p^{t,s}}^p\leq_c C_\#\chi_n\left\|(1_{O_{\zeta'}}\rho_{\zeta'}\cdot\omega)\circ\Phi'_{\zeta',\mathbf{i},e}\right\|_{H_p^{t,s}}^p\,,$$

where

$$\chi_n=\left\||\det DT^n|\max(\lambda_{u,n}^{-t},\lambda_{s,n}^{-(s+t)})^p|g^{(n)}|^p\right\|_{L^\infty}\,. \tag{5.6}$$

As always, the sum on the left hand side of (5.5) is restricted to those values of ζ such that the support of ρ_ζ is included in $T_{\mathbf{i}}\widetilde{O}_{\mathbf{i}}$

Let us fix ζ', $\mathbf{i}$ and e as above, until the end of the proof of (5.5). All the objects we shall now introduce shall depend on these choices, although we shall not make this dependence explicit to simplify the notations. Let i,j be such that $\mathcal{Z}^e\subset\mathcal{Z}_{i,j}(R)$, and let $\mathcal{M}=\{m\mid(i,j,m)\in\mathcal{Z}^e\}$. Since the points in $\mathcal{M}$ are distant of at least $C(C_0)$, Lemma 3.3 will apply.

Increasing R, we can ensure that the map

$$\mathcal{T}:=\kappa_{i,j}^R\circ T_{\mathbf{i}}^n\circ(\kappa_{\zeta'}^R)^{-1}$$

is arbitrarily close to its differential $M=D\mathcal{T}(\ell)$ at $\ell:=\kappa_{\zeta'}(q_{\zeta'})$, i.e., the map $(\mathcal{T}^{-1}[\cdot+\mathcal{T}(\ell)]-\ell)\circ M$ is close to the identity in $C^{1+\alpha}$, say on the ball $B(0,2d)$. Moreover, recalling the notation from the beginning of Section 3, the matrix M sends $\mathcal{C}_{\zeta'}$ to $\mathcal{C}_{i,j}$ compactly, and

$$C_\#\geq\lambda_u(M,\mathcal{C}_{\zeta'},\mathcal{C}_{i,j})/\lambda_u^{(n)}(q_{\zeta'})\geq C_\#^{-1}\,, \tag{5.7}$$

with similar inequalities for λ_s and Λ_u. Since T is uniformly hyperbolic and satisfies the bunching conditions (2.3) and (2.4), we can ensure by taking n large enough that M satisfies (3.1) for the constant $\epsilon=\epsilon(C_0,C_1)$ constructed in Lemma 3.3. By Lemma A.3, since the map $(\mathcal{T}^{-1}[\cdot+\mathcal{T}(\ell)]-\ell)\circ M$ is close to the identity on $B(0,2d)$, there exists a diffeomorphism of $\mathbb{R}^d$, close to the identity and coinciding with this map on $B(0,d)$. Composing with M^{-1} and translating, we obtain an extension of $\mathcal{T}^{-1}$, coinciding with $\mathcal{T}^{-1}$ on $B(\mathcal{T}(\ell),d)$, and still denoted by $\mathcal{T}^{-1}$. Taking R large enough, we can ensure that $\left\|(\mathcal{T}^{-1}[\cdot+\mathcal{T}(\ell)]-\ell)\circ M-\mathrm{id}\right\|_{C^{1+\alpha}}\leq\epsilon(C_0,C_1)$.

We may therefore apply Lemma 3.3 (see also Remark 3.4), and we obtain a block diagonal matrix D, a chart ϕ' around ℓ, and diffeomorphisms Ψ_m, Ψ such that, for any m in the set $\mathcal{M}'$ of those elements in $\mathcal{M}$ for which $\rho_\zeta\cdot\rho_{\zeta'}\circ T_{\mathbf{i}}^{-n}$ is nonzero,

$$\mathcal{T}^{-1}\circ\phi_\zeta=\phi'\circ\Psi\circ D^{-1}\circ\Psi_m \tag{5.8}$$

on the set where $(\rho_\zeta\cdot\rho_{\zeta'}\circ T_{\mathbf{i}}^n)\circ\Phi_\zeta$ is nonzero.

Writing $\omega'=(1_{O_{\zeta'}}\rho_{\zeta'}\cdot g^{(n)}\omega)\circ(\kappa_{\zeta'}^R)^{-1}$, we have (recall that (i,j) is fixed so that $\mathcal{Z}^e\subset\mathcal{Z}_{i,j}(R)$)

$$\begin{aligned}\sum_{\zeta\in\mathcal{Z}^e}&\left\|(\rho_\zeta(1_{O_{\zeta'}}\rho_{\zeta'}\cdot g^{(n)}\omega)\circ T_{\mathbf{i}}^{-n})\circ\Phi_\zeta\right\|_{H_p^{t,s}}^p\\&=\sum_{m\in\mathcal{M}'}\left\|\rho_m\circ\phi_{i,j,m}\cdot\omega'\circ\mathcal{T}^{-1}\circ\phi_{i,j,m}\right\|_{H_p^{t,s}}^p\\&=\sum_{m\in\mathcal{M}'}\left\|(\rho_m\circ\phi_{i,j,m}\circ\Psi_m^{-1}\cdot\omega'\circ\phi'\circ\Psi\circ D^{-1})\circ\Psi_m\right\|_{H_p^{t,s}}^p\,.\end{aligned}$$

Using the notations and results of Lemma 3.3, the terms in this last equation are of the form $v \circ \Psi_m$, where v is a distribution supported in $\Psi_m(\phi_{i,j,m}^{-1}(B(m,d))) \subset B(\Pi m, C_0^{1/2}/2)$. Since the range of Ψ_m contains $B(\Pi m, C_0^{1/2})$, and since $\alpha t + |s| < \alpha$, Lemma 4.7 gives $\|v \circ \Psi_m\|_{H_p^{t,s}} \leq C_\# \|v\|_{H_p^{t,s}}$, yielding a bound

$$C_\# \sum_{m \in \mathcal{M}'} \left\| \rho_m \circ \phi_{i,j,m} \circ \Psi_m^{-1} \cdot \omega' \circ \phi' \circ \Psi \circ D^{-1} \right\|_{H_p^{t,s}}^p .$$

The functions $\rho_m \circ \phi_{i,j,m} \circ \Psi_m^{-1}$ have a bounded C^1 norm, and are supported in the balls $B(\Pi m, C_0^{1/2}/2)$, whose centers are distant by at least C_0, by Lemma 3.3 (a). Therefore, by Lemma 4.3, the last expression is bounded by

$$C_\# \left\| \omega' \circ \phi' \circ \Psi \circ D^{-1} \right\|_{H_p^{t,s}}^p .$$

We may apply Lemma 4.6 to the composition with D^{-1} (to obtain an improvement in the $H_p^{t,s}$ norm, up to compact terms). Since ω' is supported in $B(\ell, C_0^{1/2}/2)$ while the range of Ψ contains $B(\ell, C_0^{1/2})$ (by Lemma 3.3), Lemma 4.7 implies that the composition with Ψ is bounded. Summing up, we obtain

$$\begin{aligned}(5.9)\quad \sum_{\zeta \in \mathcal{Z}^e} & \left\| (\rho_\zeta (1_{O_{\zeta'}} \rho_{\zeta'} \cdot g^{(n)} \omega) \circ T_{\mathbf{i}}^{-n}) \circ \Phi_\zeta \right\|_{H_p^{t,s}}^p \\ & \leq_c C_\# \chi_n^{(0)}(q_{\zeta'}) \left\| (1_{O_{\zeta'}} \rho_{\zeta'} \cdot g^{(n)} \omega) \circ (\kappa_{\zeta'}^R)^{-1} \circ \phi' \right\|_{H_p^{t,s}}^p ,\end{aligned}$$

where

$$\chi_n^{(0)}(q_{\zeta'}) = (|\det DT^n| \max(\lambda_{u,n}^{-t}, \lambda_{s,n}^{-(s+t)})^p)(q_{\zeta'}) .$$

Let $\nu > 0$. Since $(\kappa_{\zeta'}^R)^{-1}$ contracts by a factor $1/R$, we can ensure by increasing R that the C^γ norm of $g^{(n)} \circ (\kappa_{\zeta'}^R)^{-1}$ on $B(\ell, d)$ is bounded by $C_\# |g^{(n)}(q_{\zeta'})| + \nu$ (recall that, by assumption, g belongs to C^γ for some $\gamma > t + |s|$). The term ν here is necessary when $|g|$ is not bounded away from 0. Choosing ν small enough, we can ensure that $(|g^{(n)}(q_{\zeta'})| + \nu)^p \chi_n^{(0)}(q_{\zeta'}) \leq 2\chi_n$. Hence, (5.9) and Lemma 4.1 yield

$$\begin{aligned}\sum_{\zeta \in \mathcal{Z}^e} & \left\| (\rho_\zeta (1_{O_{\zeta'}} \rho_{\zeta'} \cdot g^{(n)} \omega) \circ T_{\mathbf{i}}^{-n}) \circ \Phi_\zeta \right\|_{H_p^{t,s}}^p \\ & \leq_c C_\# \chi_n \left\| (1_{O_{\zeta'}} \rho_{\zeta'} \omega) \circ (\kappa_{\zeta'}^R)^{-1} \circ \phi' \right\|_{H_p^{t,s}}^p .\end{aligned}$$

This concludes the proof of (5.5). Summing over all possible values of ζ', $\mathbf{i}$ and, e, we obtain

$$(5.10)\quad \left\| \mathcal{L}_g^n \omega \right\|_\Phi^p \leq_c C_\# (C_\# N^p)^{n/N} (D_n^e)^{p-1} \chi_n \sum_{\zeta', \mathbf{i}} \sum_{e=1}^E \left\| (1_{O_{\zeta'}} \rho_{\zeta'} \omega) \circ \Phi'_{\zeta', \mathbf{i}, e} \right\|_{H_p^{t,s}}^p .$$

Fourth step: Conclusion. The right hand side of (5.10) is essentially of the form $\|\omega\|_{\Phi'}^p$ for some family of admissible charts Φ', with the difference that to a point $q_{\zeta'}$ for $\zeta' \in \mathcal{Z}(R)$ correspond several admissible charts around it. Since E is independent of n, the number of those charts around $q_{\zeta'}$ is at most $C_\# \cdot \mathrm{Card}\{\mathbf{i} \mid \widetilde{O}_{\mathbf{i}} \cap A(\zeta') \neq \emptyset\}$. If R is large enough, we can ensure that this quantity is bounded by the intersection multiplicity of the sets $\widetilde{O}_{\mathbf{i}}$, which is at most D_n^b by construction. Therefore, we obtain

$$\left\| \mathcal{L}_g^n \omega \right\|_\Phi^p \leq_c C_\# (C_\# N^p)^{n/N} (D_n^e)^{p-1} D_n^b \chi_n \left\| \omega \right\|_{\mathbf{H}_p^{t,s}(R, C_0, C_1)}^p . \qquad \square$$

Proof of Proposition 2.15. Remark 3.6 shows that the charts ϕ' we constructed in the third step of the proof of Lemma 5.1 can be defined on larger balls, and with better bounds. In particular, these new charts will be admissible when looked at a scale R' and with a smoothness constant C_1', for any $R/2 \le R' \le 2R$ and $C_1/2 \le C_1' \le 2C_1$. The proof of Lemma 5.1 therefore gives the following statement:

For any large enough C_1 (say $C_1 \ge C_1^{(0)}$), for any large enough n (say $n \ge n^{(0)}(C_1)$), and for any large enough R (say $R \ge R^{(0)}(n, C_1)$), then for any $R' \in [R/2, 2R]$ and $C_1' \in [C_1/2, 2C_1]$, the operator $\mathcal{L}_g^n$ maps continuously $\mathbf{H}_p^{t,s}(R, C_0, C_1)$ to $\mathbf{H}_p^{t,s}(R', C_0, C_1')$.

It follows that, for any $C_1 \ge C_1^{(0)}$ and $R \ge R^{(0)}(n^{(0)}(C_1), C_1)$, and for any $C_1' \ge C_1^{(0)}$ and $R' \ge R^{(0)}(n^{(0)}(C_1'), C_1')$, there exists an integer n such that $\mathcal{L}_g^n$ maps $\mathbf{H}_p^{t,s}(R, C_0, C_1)$ to $\mathbf{H}_p^{t,s}(R', C_0, C_1')$. Moreover, if n' is large enough, $\mathcal{L}_g^{n'}$ maps $\mathbf{H}_p^{t,s}(R, C_0, C_1)$ to itself. Writing a large enough integer N as $n' + n$, we get that $\mathcal{L}_g^N$ maps $\mathbf{H}_p^{t,s}(R, C_0, C_1)$ to $\mathbf{H}_p^{t,s}(R', C_0, C_1')$. □

Appendix A. Calculus for some classes of maps

This appendix groups some straightforward results about classes of maps $\mathcal{D}$ and $\mathcal{K}$ which appear in the proofs of Lemma 2.8 and Lemmas 3.3–3.5 (together with an easy result, which is useful for the proof of Lemma 5.1).

A.1. **The class $\mathcal{D}$.** For $C_\# > 0$, let us denote by $\mathcal{D}(C_\#)$ the class of C^1 maps f defined on an open subset of $\mathbb{R}^d$, satisfying

$$C_\#^{-1}|z - z'| \le |f(z) - f(z')| \le C_\#|z - z'|, \tag{A.1}$$

for any z, z' in the domain of definition of f. It follows that f is a local diffeomorphism, and that $\|Df\| \le C_\#$, $\left\|(Df)^{-1}\right\| \le C_\#$.

Lemma A.1. *Assume that $f(x, y) = (g(x, y), y)$ is defined on a set $A_1 \times A_2$ where A_1 and A_2 are convex, that $|Dg| \le C$, and that $|g(x, y) - g(x', y)| \ge C^{-1}|x - x'|$ for some $C > 0$. Then $f \in \mathcal{D}(C_\#)$, for some constant $C_\#$ depending only on C.*

Proof. Since the second coordinate of $f(x, y)$ is equal to y, while the derivative of f is bounded by C, we have

$$|y - y'| \le |f(x, y) - f(x', y')| \le C_\#^1(|x - x'| + |y - y'|)\,, \tag{A.2}$$

for some constant $C_\#^1$ depending only on C. This proves the (trivial) upper bound in (A.1).

Consider now two points $z = (x, y), z' = (x', y') \in A_1 \times A_2$. If $|y - y'| \ge C^{-1}|x - x'|/(2C_\#^1)$, we have in particular $|y - y'| \ge \epsilon_\#^2|z - z'|$ for some $\epsilon_\#^2$, and we get from (A.2) that $|f(z) - f(z')| \ge \epsilon_\#^2|z - z'|$. Otherwise,

$$\begin{aligned}|f(x, y) - f(x', y')| &\ge |f(x, y) - f(x', y)| - |f(x', y) - f(x', y')| \\ &\ge C^{-1}|x - x'| - C_\#^1|y - y'| \ge C^{-1}|x - x'|/2\,.\end{aligned}$$

This proves the lower bound in (A.1) in all cases. □

Lemma A.2. *Let $f \in \mathcal{D}(C_\#)$, and assume that the domain of definition of f contains a ball $B(z, r)$. Then the range of f contains $B(f(z), r/C_\#)$.*

Proof. Let $r' < r$, and consider $A = f(B(z, r')) \cap B(f(z), r'/C_\#)$. Since f is a local diffeomorphism, this is an open subset of $B(f(z), r'/C_\#)$. Moreover, if $|z' - z| = r'$, then $f(z')$ does not belong to $B(f(z), r'/C_\#)$, since $|f(z') - f(z)| \ge |z' - z|/C_\# = r'/C_\#$. Therefore, A is also equal to $f(\overline{B(z', r)}) \cap B(f(z), r'/C_\#)$. This is a closed subset of $B(f(z), r'/C_\#)$, since $f(\overline{B(z', r)})$ is compact.

Finally, A is open and closed in $B(f(z), r'/C_\#)$. By connectedness, it coincides with this whole ball. In particular, the range of f contains $B(f(z), r'/C_\#)$. Letting r' tend to r, we conclude the proof. □

Let us also mention the following easy result, which is useful for the proof of Lemma 5.1.

Lemma A.3. *Let $\alpha \in (0,1]$ and let $f : B(0,1) \to \mathbb{R}^d$ be a diffeomorphism such that $\|f - \mathrm{id}\|_{C^{1+\alpha}}$ is small enough. Then there exists a diffeomorphism $\widetilde{f}$ of $\mathbb{R}^d$, coinciding with f on $B(0,1/2)$, and such that $\left\|\widetilde{f} - \mathrm{id}\right\|_{C^{1+\alpha}} \leq C_\# \|f - \mathrm{id}\|_{C^{1+\alpha}}$, for some universal constant $C_\#$ depending only on the dimension d.*

Proof. Let us write, for $z \in B(0,1)$, $f(z) = z + \psi(z)$ with $\|\psi\|_{C^{1+\alpha}}$ small. We may define the required extension $\widetilde{f}$ of f by $\widetilde{f}(z) = z + \gamma(z)\psi(z)$ where γ is C^∞, equal to 1 on $B(0,1/2)$ and supported in $B(0,1)$. If $\|\psi\|_{C^{1+\alpha}}$ is small enough, then $\langle D\widetilde{f}(z)v, v\rangle \geq |v|^2/2$ for any point z and any vector v. Integrating this inequality, it follows that $|\widetilde{f}(z) - \widetilde{f}(z')| \geq |z - z'|/2$. Therefore, $\widetilde{f}$ belongs to the class $\mathcal{D}(2)$. By Lemma A.2, it is surjective, hence it is a diffeomorphism of $\mathbb{R}^d$. □

A.2. **The class $\mathcal{K}$.** Let us fix $\alpha \in (0,1]$ and $\beta \in (0,\alpha)$. We denote by $\mathcal{K} = \mathcal{K}^{\alpha,\beta}$ the class of matrix-valued functions K on $\mathbb{R}^d$ such that, for some constant C and for all $x, x' \in \mathbb{R}^{d_u}$ and all $y, y' \in \mathbb{R}^{d_s}$,

$$|K(x,y)| \leq C\,, \tag{A.3}$$

$$|K(x,y) - K(x',y)| \leq C|x - x'|^\beta\,, \tag{A.4}$$

$$|K(x,y) - K(x,y')| \leq C|y - y'|^\alpha\,, \tag{A.5}$$

$$|K(x,y) - K(x',y) - K(x,y') + K(x',y')| \leq C|x - x'|^\beta |y - y'|^{\alpha-\beta}\,. \tag{A.6}$$

If $K \in \mathcal{K}$, we write $\|K\|$ for the the smallest C satisfying the inequalities above. We write $\mathcal{K}(C)$ for the functions in $\mathcal{K}$ with $\|K\| \leq C$.

For instance, any bounded α-Hölder continuous function K belongs to $\mathcal{K}$ (to obtain (A.6), treat separately the cases $|x - x'| \leq |y - y'|$ and $|x - x'| > |y - y'|$). Note also that if K is C^1 then the left-hand-side of (A.6) can be rewritten as $|\int_y^{y'} \partial_{y'-y}K(x,t) - \partial_{y'-y}K(x',t)\,dt|$, i.e., it is a finite-difference-type expression for $\partial_x\partial_y K$.

Proposition A.4. *A function in $\mathcal{K}$ satisfies*

$$|K(x,y) - K(x',y')| \leq 3\,\|K\|\,(|x - x'| + |y - y'|)^\beta. \tag{A.7}$$

If $K, K' \in \mathcal{K}$, then $K + K' \in \mathcal{K}$, with $\|K + K'\| \leq \|K\| + \|K'\|$. Moreover, $KK' \in \mathcal{K}$, with $\|KK'\| \leq 6\,\|K\|\,\|K'\|$. Finally, if K is everywhere invertible and $|K^{-1}| \leq h$ for some finite number h, then $K^{-1} \in \mathcal{K}$ and $\left\|K^{-1}\right\| \leq 5\max(1,h^3)\max(1,\|K\|^3)$.

Proof. Notice first that we have

$$|K(x,y) - K(x,y')| \leq 2\,\|K\|\,|y - y'|^{\alpha-\beta}\,. \tag{A.8}$$

Indeed, this follows from (A.5) if $|y - y'| \leq 1$, and from (A.3) if $|y - y'| > 1$. This inequality also holds if $|y - y'|^{\alpha-\beta}$ is replaced with $|y - y'|^\beta$ (with the same proof). Therefore, by (A.4),

$$\begin{aligned} |K(x,y) - K(x',y')| &\leq |K(x,y) - K(x',y)| + |K(x',y) - K(x',y')| \\ &\leq \|K\|\,|x - x'|^\beta + 2\,\|K\|\,|y - y'|^\beta \leq 3\,\|K\|\max(|x - x'|, |y - y'|)^\beta\,. \end{aligned}$$

(A.7) follows.

Consider now $K, K' \in \mathcal{K}$. It is trivial that $\|K + K'\| \le \|K\| + \|K'\|$. We turn to KK'. Let us write a, b, c, d for $K(x, y), K(x', y), K(x, y'), K(x', y')$. Similarly, we use a', b', c', d' for K'. The inequality (A.3) for KK' is trivial, (A.4) follows from the equality $aa' - bb' = a(a' - b') + (a - b)b'$, and (A.5) is similar. For (A.6), we use the identity

$$aa'-bb'-cc'+dd' = c(a'-b'-c'+d')+(a-b-c+d)d'+(a-c)(a'-b')+(a-b)(b'-d'),$$

and the bounds for $a - c$, $a' - b'$, $a - b$ and $b' - d'$ given by (A.4) and (A.8). This concludes the proof for KK'.

Finally, assume $|K^{-1}| \le h$. Then (A.3) holds for K^{-1}. Moreover, (A.4) follows from the equality $|a^{-1} - b^{-1}| = |a^{-1}(b - a)b^{-1}| \le h^2|a - b|$. (A.5) is similar. For (A.6), we use the identity

$$\begin{aligned} a^{-1} - b^{-1} - c^{-1} + d^{-1} &= a^{-1}(b + c - a - d)b^{-1} \\ &\quad + a^{-1}(c - a)c^{-1}(d - c)b^{-1} + c^{-1}(d - c)b^{-1}(d - b)d^{-1}, \end{aligned}$$

and the bounds (A.4) and (A.8). □

We recall that the subsets $\{x\} \times \mathbb{R}^{d_s}$ of $\mathbb{R}^d$ are called "stable leaves" of $\mathbb{R}^d$ in this article.

Proposition A.5. *Let $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ send stable leaves to stable leaves, and assume that its best Lipschitz constant L is finite. Then, for $K \in \mathcal{K}$, the function $K \circ \Psi$ also belongs to $\mathcal{K}$, and $\|K \circ \Psi\| \le 3 \max(1, L)\, \|K\|$.*

Proof. The inequality (A.3) is trivial for $K \circ \Psi$. For (A.4), we write using (A.7)

$$\begin{aligned} |K \circ \Psi(x, y) - K \circ \Psi(x', y)| &\le 3\, \|K\|\, d(\Psi(x, y), \Psi(x', y))^\beta \\ &\le 3\, \|K\|\, L^\beta d((x, y), (x', y))^\beta \le 3\, \|K\| \max(1, L)|x - x'|^\beta. \end{aligned}$$

(A.5) for $K \circ \Psi$ follows from (A.5) for K and from the fact that Ψ sends stable leaves to stable leaves and is Lipschitz continuous.

We turn to (A.6). We write $\Psi(x, y) = (x_1, y_1)$, $\Psi(x, y') = (x_1, y_1')$, $\Psi(x', y) = (x_2, y_2)$ and $\Psi(x', y') = (x_2, y_2')$.

Assume first $|y - y'| \le |x - x'|$. Then

$$\begin{aligned} &|K(x_1, y_1) - K(x_1, y_1') - K(x_2, y_2) + K(x_2, y_2')| \\ &\qquad \le |K(x_1, y_1) - K(x_1, y_1')| + |K(x_2, y_2) - K(x_2, y_2')| \\ &\qquad \le \|K\|\, |y_1 - y_1'|^\alpha + \|K\|\, |y_2 - y_2'|^\alpha \le 2\, \|K\|\, L^\alpha |y - y'|^\alpha. \end{aligned}$$

Since $L^\alpha \le \max(1, L)$ and $|y - y'|^\alpha \le |x - x'|^\beta |y - y'|^{\alpha-\beta}$, this is the desired conclusion. Assume now $|x - x'| \le |y - y'|$. Then

$$\begin{aligned} &|K(x_1, y_1) - K(x_1, y_1') - K(x_2, y_2) + K(x_2, y_2')| \\ &\qquad \le |K(x_1, y_1) - K(x_1, y_1') - K(x_2, y_1) + K(x_2, y_1')| \\ &\qquad\qquad + |K(x_2, y_2) - K(x_2, y_1)| + |K(x_2, y_2') - K(x_2, y_1')|. \\ &\qquad \le \|K\|\, |x_1 - x_2|^\beta |y_1 - y_1'|^{\alpha-\beta} + \|K\|\, |y_2 - y_1|^\alpha + \|K\|\, |y_2' - y_1'|^\alpha. \end{aligned} \tag{A.9}$$

Since Ψ is Lipschitz continuous, we have $|x_1 - x_2| \le L|x - x'|$ and $|y_1 - y_1'| \le L|y - y'|$. Moreover,

$$|y_2 - y_1| \le d((x_1, y_1), (x_2, y_2)) = d(\Psi(x, y), \Psi(x', y)) \le Ld((x, y), (x', y)) = L|x - x'|.$$

Since $|x - x'| \le |y - y'|$, we obtain $|y_2 - y_1|^\alpha \le L^\alpha |x - x'|^\alpha \le \max(1, L)|x - x'|^\beta |y - y'|^{\alpha-\beta}$. Moreover, $|y_2' - y_1'|$ satisfies a similar inequality. Finally, (A.9) is bounded by $3\, \|K\| \max(1, L)|x - x'|^\beta |y - y'|^{\alpha-\beta}$. This concludes the proof. □

Remark A.6. If A_1 and A_2 are convex subsets of, respectively, $\mathbb{R}^{d_u}$ and $\mathbb{R}^{d_s}$, we can define analogously a space $\mathcal{K}(C, A_1 \times A_2)$ of matrix-valued functions defined on $A_1 \times A_2$ and satisfying (A.3)–(A.6). The previous results also hold for this space, with the same proofs, up to the following small modification: In Proposition A.5, if K is defined on $A_1 \times A_2$, we need to require that Ψ be defined on $A_1' \times A_2'$ with $\Psi(A_1' \times A_2') \subset A_1 \times A_2$. Successive applications of the proposition in the proof of Lemma 3.5 will require stronger conditions. The careful reader is invited to check that this does not cause any problems in the proof of Lemma 3.5.

Appendix B. Convex transversality

We prove the claims made after Definition 2.1. Consider the cone $\{(x, y) \mid |x| \leq |Ay|\}$ for some nonzero linear map A. We should prove that, for any vector space E so that $C \cap E = \{0\}$, the set $C \cap (E + w)$ is convex for all $w \in \mathbb{R}^d$.

Proof. Pick z_1, z_2 in $C \cap (E + w)$, we want to show that the segment $[z_1, z_2]$ is included in $C \cap (E + w)$. The line directed by $z_0 := z_2 - z_1$ is contained in E, so $z_0 = (x_0, y_0) \notin C$, i.e., $|Ay_0|^2 < |x_0|^2$.

Let $D = \{(x_1 + tx_0, y_1 + ty_0), t \in [0, 1]\}$ be the segment between $z_1 = (x_1, y_1)$ and z_2. The leading coefficient of the polynomial $\Phi(t) := |x(t)|^2 - |Ay(t)|^2 = |x_1 + tx_0|^2 - |Ay_1 + tAy_0|^2$ is $|x_0|^2 - |Ay_0|^2 > 0$. Therefore, the set $\{t \mid \Phi(t) \leq 0\}$ is convex, i.e., $C \cap D$ is convex. Since z_1 and z_2 belong to $C \cap D$, we find $D \subset C \cap D$, as desired. □

Appendix C. A more general setting

For the sake of simplicity, we have formulated all our results for the transfer operator associated to a map. However, it turns out that the same proof applies to a wider class of operators, which would formally correspond to the transfer operators of multivalued maps. This kind of generalized transfer operators has been studied in one dimension in [BR96].

In our main result, we also assumed that the continuity domains of the stable and unstable cones coincide with the domains of definition of the branches of the map. Although this assumption is quite natural, it plays no role in the proof, and can therefore be removed.

These remarks lead to the following general setting, which turns out to be useful for many applications (see the comments after the statement of Theorem C.1). We consider finitely many subsets $(O_i)_{i \in I}$ of a manifold X (that may not be disjoint, and may not cover everything), with compact closure, and maps $T_i : O_i \to X$ such that T_i admits a $C^{1+\alpha}$ extension to a neighborhood of $\overline{O_i}$, for some $\alpha \in (0, 1]$. Consider also finitely many disjoint open subsets $(\Pi_e)_{e \in E}$, covering almost all X, and assume that on each of these subsets are given two convexly transverse cones $\mathcal{C}_e^{(u)}(q)$ and $\mathcal{C}_e^{(s)}(q)$ in the tangent space $\mathcal{T}_q X$, depending continuously on $q \in \Pi_e$ and which extend continuously up to the boundary of Π_e.

The following transversality conditions are needed. For the domains Π_e, we require transversality with the stable cones at time 0: the boundary of each set Π_e is a finite union of hypersurfaces $P_{e,k}$ such that, for all $q \in P_{e,k}$, the tangent space $\mathcal{T}_q P_{e,k}$ is transverse to $\mathcal{C}_e^{(s)}(q)$. For the domains O_i, we only require transversality at time 1 (i.e., in the image): the boundary of each set O_i is a finite union of hypersurfaces $K_{i,k}$ such that, for all $q \in K_{i,k}$ and all e such that $T_i(q) \in \overline{\Pi_e}$, the cone $\mathcal{C}_e^{(s)}(T_i(q))$ is transverse to $\mathcal{T}_{T_i(q)}(T_i(K_{i,k}))$.

We will need hyperbolicity: for each $q \in \overline{O_i} \cap \overline{\Pi_e} \cap T_i^{-1}(\overline{\Pi_{e'}})$, then $DT_i(q)\mathcal{C}_e^{(u)}(q) \subset \mathcal{C}_{e'}^{(u)}(T_i(q))$, and there exists $\lambda_{i,u}(q) > 1$ (independent of e, e') such that

$$|DT_i(q)v| \geq \lambda_{i,u}(q)|v|\,, \forall v \in \mathcal{C}_e^{(u)}(q)\,.$$

Moreover, for each $q \in \overline{O_i} \cap \overline{\Pi_e} \cap T_i^{-1}(\overline{\Pi_{e'}})$, then $DT_i^{-1}(T_i(q))\mathcal{C}_{e'}^{(s)}(T_i(q)) \subset \mathcal{C}_e^{(s)}(q)$, and there exists $\lambda_{i,s}(q) \in (0,1)$ (independent of e, e') such that

$$|DT_i^{-1}(T_i(q))v| \geq \lambda_{i,s}^{-1}(q)|v|\,, \forall v \in \mathcal{C}_{e'}^{(s)}(T_i(q))\,.$$

For $\mathbf{i} \in I^n$, we define $O_{\mathbf{i}}$ and $T_{\mathbf{i}}$ as in Paragraph 2.1, and we also define the complexities D_n^b and D_n^e at the beginning and at the end, and the best expansion and contraction coefficients $\lambda_{\mathbf{i},u}(q)$ and $\lambda_{\mathbf{i},s}(q)$. In this generalized setting, we obtain the following variant of Theorem 2.5:

Theorem C.1. *Let T satisfy the piecewise hyperbolicity and transversality conditions just given. Assume that the bunching conditions* (2.3) *and* (2.4) *are satisfied for some parameters α, β, and consider parameters p, s, t satisfying* (2.5)*. Then there exists a space $\mathbf{H}$ of distributions on X with the following properties.*

Consider functions $(g_i)_{i\in I}$, defined on O_i and admitting a C^γ extension to its closure for some $\gamma > t + |s|$. Define an operator $(\mathcal{L}_g\omega)(q) = \sum_{T_i(q')=q} g_i(q')\omega(q')$. Then this operator acts on $\mathbf{H}$. Moreover, its essential spectral radius on $\mathbf{H}$ is at most the limit when n tends to infinity of

$$(D_n^b)^{\frac{1}{pn}} \cdot (D_n^e)^{\frac{1}{n}\left(1-\frac{1}{p}\right)} \cdot \sup_{\mathbf{i}=(i_0,\dots,i_{n-1})} \left\| g_{\mathbf{i}}^{(n)} |\det DT_{\mathbf{i}}^n|^{\frac{1}{p}} \max(\lambda_{\mathbf{i},u}^{-t}, \lambda_{\mathbf{i},s}^{-(t-|s|)}) \right\|_{L^\infty(O_{\mathbf{i}})}^{\frac{1}{n}},$$

where we set $g_{\mathbf{i}}^{(n)}(q) = \prod_{k=0}^{n-1} g_{i_k}(T^k_{(i_0,\dots,i_{k-1})}(q))$, for $n \geq 1$.

In the case of a single-valued map, and when the sets Π_e and O_i coincide, this theorem reduces to Theorem 2.5. However, this extension is useful is many cases. For instance, if there is a single cone field (i.e., $\Pi_1 = X$), then the transversality condition is only on the images $T(O_i)$, it is therefore weaker than the condition in Definition 2.3 (we already mentioned this fact and its relevance for Sinai billiards in Remark 2.4). Another interest of Theorem C.1 is that the class of operators studied there is closed under time reversal. Indeed, for all functions ω_1, ω_2, we have

$$\text{(C.1)} \qquad \int \omega_1 \mathcal{L}_g \omega_2 \,\mathrm{dLeb} = \sum_i \int_{T_i(O_i)} \omega_1 \cdot (g_i\omega_2) \circ T_i^{-1} \,\mathrm{dLeb} = \sum_i \int_{O_i} (|\det DT_i| g_i \cdot \omega_1 \circ T_i) \cdot \omega_2 \,\mathrm{dLeb}\,.$$

Therefore, the adjoint of $\mathcal{L}_g$ is the operator $\omega \mapsto \sum_i 1_{O_i} \operatorname{Jac}(T_i) g_i \cdot \omega \circ T_i$, to which Theorem C.1 also applies (if transversality with the unstable cones is satisfied). It is sometimes more convenient to apply the theorem in this direction, since its statement is not completely symmetric with respect to the stable and unstable directions. An important particular case, which will appear in Proposition D.3 and its Corollary D.4, and which is useful when studying e.g. Lozi maps, is when $g_i = |\det DT_i|^{-1}$ for all i, and the O_i form a partition of X_0. In this case, the dual operator is just $\mathcal{M}(\omega) = \omega \circ T$.

Sketch of proof of Theorem C.1. The proof of Theorem 2.5 applies almost directly to yield Theorem C.1, we should only modify slightly the charts and the norm to take into account the fact that the sets Π_e and O_i do not coincide, by introducing an additional dependency on e.

More precisely, for every i, e, we can consider as in Subsection 2.3 charts $\kappa_{i,e,j}$ (for $1 \leq j \leq N_{i,e}$) whose domains of definitions $U_{i,e,j,0}$ cover a neighborhood of

$\overline{\Pi_e} \cap \overline{O_i}$, and extended cones $\mathcal{C}_{i,e,j}$ such that, wherever $\kappa_{i',e',j'} \circ T_i \circ \kappa_{i,e,j}^{-1}$ is defined, its differential sends $\mathcal{C}_{i,e,j}$ to $\mathcal{C}_{i',e',j'}$ compactly.

Let $U_{i,e,j,1}$ be a subset with compact closure of $U_{i,e,j,0}$ such that the sets $U_{i,e,j,1}$ ($1 \leq j \leq N_{i,e}$) still cover $\overline{\Pi_e} \cap \overline{O_i}$. We can then define sets $\mathcal{Z}_{i,e,j}(R)$ and $\mathcal{Z}(R) = \{(i,e,j,m) \mid m \in \mathcal{Z}_{i,e,j}(R)\}$ as in (2.15) and (2.16). For $\zeta = (i,e,j,r) \in \mathcal{Z}(R)$, let $\Pi_\zeta = \Pi_e$. We can then follow line by line the discussion in Subsection 2.3, define a norm

$$\|\omega\|_\Phi = \left(\sum_{\zeta \in \mathcal{Z}(R)} \left\| (\rho_\zeta(R) \cdot 1_{\Pi_\zeta} \omega) \circ \Phi_\zeta \right\|_{H_p^{t,s}}^p \right)^{1/p}$$

for any system of charts Φ, and finally put $\|\omega\|_{\mathbf{H}_p^{t,s}(R,C_0,C_1)} = \sup_\Phi \|\omega\|_\Phi$.

The proof of Theorem 2.14 still works in this context, with trivial notational modifications (one should replace 1_{O_ζ} and $1_{O_{\zeta'}}$ by 1_{Π_ζ} and $1_{\Pi_{\zeta'}}$, and insert a characteristic function $1_{\Pi_{\zeta'}}$ in (5.2)). The transversality of the boundary of Π_e with the stable cone is used at the beginning of the second step to show that the multiplication by 1_{Π_ζ} is bounded on $H_p^{t,s}$, while the transversality of the boundary of the image of O_i with this cone is used to show the same multiplier property for $1_{T_{\mathbf{i}}^n O_{\mathbf{i}}}$.

Finally, the result follows from the analogue of Theorem 2.14, by the arguments of Subsection 2.4. □

Appendix D. Physical measures

In this appendix, we discuss the existence of physical measures, combining our main result Theorem 2.5 (or its extension Theorem C.1), with Theorem 33 of [BG09]. The discussion is essentially straightforward once the above results are given, apart from a more subtle point: one should check that the possible physical measures would give no mass to the discontinuity set of T.

Let us first give a convenient definition:

Definition D.1. *Let T be a measurable map on an open subset X_0 with compact closure of a manifold. A* physical description *of T is a finite number of probability measures $\mu_1, \dots, \mu_l$ which are T-invariant and ergodic, and disjoint sets $A_1, \dots, A_l$ such that $\mu_i(A_i) = 1$, $\mathrm{Leb}(A_i) > 0$, $\mathrm{Leb}(X_0 \setminus \bigcup_{i=1}^l A_i) = 0$ and, for every $x \in A_i$ and every function $f \in C^0(X_0)$, we have $\frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \to \int f \, d\mu_i$. Moreover, for every i, there exist an integer k_i and a decomposition $\mu_i = \mu_{i,1} + \dots + \mu_{i,k_i}$ such that T sends $\mu_{i,j}$ to $\mu_{i,j+1}$ for $j \in \mathbb{Z}/k_i\mathbb{Z}$, and the probability measures $k_i \mu_{i,j}$ are mixing for T^{k_i}.*

We could strengthen the requirements by requiring that the measures $\mu_{i,j}$ are exponentially mixing for T^{k_i} and Hölder observables, and that all kinds of statistical limit theorems (central limit theorem, strong invariance principle, etc.) are satisfied. These additional properties will also hold in the examples below.

Consider now a piecewise hyperbolic map T. We will deal with a true (i.e., single-valued) map T, but we will not necessarily assume that the continuity domains of the cone families coincide with the continuity domains of T, as in Appendix C.

We give two results, corresponding to the application of our main theorems in forward or backward time.

Proposition D.2. *Let T be a piecewise $C^{1+\alpha}$ hyperbolic map on a domain X_0 with compact closure in a manifold X, such that*

- *the boundaries of the continuity domains of the cone families are transverse to the stable cones,*

- *the images under T of the boundaries of the continuity domains of T are transverse to the stable cones.*

Assume, for some $\beta \in (0, \alpha)$, the bunching condition

$$\sup_{\mathbf{i} \in I^n,\, q \in \overline{O}_{\mathbf{i}}} \frac{\lambda^{(n)}_{\mathbf{i},s}(q)^{\alpha-\beta} \Lambda^{(n)}_{\mathbf{i},u}(q)^{1+\beta}}{\lambda^{(n)}_{\mathbf{i},u}(q)} < 1 .$$

Assume also that, for some parameters $p \in (1, \infty)$ and $t, s \in \mathbb{R}$ with

$$1/p - 1 < s < 0 < t < 1/p , \quad -\beta < t - |s| < 0 , \quad \alpha t + |s| < \alpha ,$$

we have for some n

$$\text{(D.1)} \quad (D^b_n)^{1/(pn)} \cdot (D^e_n)^{(1/n)(1-1/p)} \cdot \left\| |\det DT^n|^{1/p-1} \max(\lambda^{-t}_{u,n}, \lambda^{-(t-|s|)}_{s,n}) \right\|^{1/n}_{L^\infty} < 1.$$

Then T admits a physical description.

Proposition D.3. *Let T be a piecewise $C^{1+\alpha}$ hyperbolic map on a domain X_0 with compact closure in a manifold X, such that*

- *the boundaries of the continuity domains of the cone families are transverse to the unstable cones,*
- *the preimages under T of the boundaries of the continuity domains of T are transverse to the unstable cones.*

Assume, for some $\beta \in (0, \alpha)$, the bunching condition

$$\sup_{\mathbf{i} \in I^n,\, q \in \overline{O}_{\mathbf{i}}} \frac{\lambda^{(n)}_{\mathbf{i},u}(q)^{\alpha-\beta} \Lambda^{(n)}_{\mathbf{i},s}(q)^{1+\beta}}{\lambda^{(n)}_{\mathbf{i},s}(q)} > 1 .$$

Assume also that, for some parameters $p \in (1, \infty)$ and $t, s \in \mathbb{R}$ with

$$1/p - 1 < s < 0 < t < 1/p , \quad -\beta < |s| - t < 0 , \quad \alpha|s| + t < \alpha ,$$

we have for some n

$$\text{(D.2)} \quad (D^b_n)^{1/(pn)} \cdot (D^e_n)^{(1/n)(1-1/p)} \cdot \left\| |\det DT^n|^{1/p-1} \max(\lambda^{|s|-t}_{u,n}, \lambda^{|s|}_{s,n}) \right\|^{1/n}_{L^\infty} < 1.$$

Then T admits a physical description.

In both propositions, if D^b_n and D^e_n grow subexponentially fast and $\det DT \equiv 1$, the limit in (D.1) and (D.2) is < 1 for any valid choice of parameters p, s, t. If $\det DT \not\equiv 1$, one should choose the parameters more carefully, as in the next corollary.

Corollary D.4. *If D^b_n and D^e_n grow subexponentially fast, $d_s = 1$ and the transversality conditions of Proposition D.3 are satisfied, then T admits a physical description.*

Proof. Since $d_s = 1$, we can fix $\beta > 0$ such that the bunching condition of Proposition D.3 is satisfied. Then the limit in (D.2) is < 1 if p is very close to 1, $t = \beta/2$ and $s = 1/p - 1 + \epsilon$ for some very small ϵ: the easy computation is the same as in [BG09, Example 3]. Therefore, Proposition D.3 gives the result. □

We state here the slightly stronger version of [BG09, Theorem 33] that we shall need to prove the two propositions above:

Theorem D.5. *Let T be a nonsingular measurable map on an open subset X_0 with compact closure in a manifold X. Let us define its transfer operator $\mathcal{L}$ by $\int_{X_0} \mathcal{L}u \cdot v \,\mathrm{dLeb} = \int_{X_0} u \cdot v \circ T \,\mathrm{dLeb}$ whenever v is bounded and measurable. It is given by $\mathcal{L}u(x) = \sum_{Ty=x} |\det DT(y)|^{-1} u(y)$.*

Let H_0 be a vector subspace of $L^\infty(\mathrm{Leb})$, endowed with a (possibly non–complete) norm $\|\cdot\|$. Assume that

(1) *There exist $\alpha > 0$ and $C > 0$ such that, for any $u \in H_0$ and $f \in C^\alpha(X)$, then $fu \in H_0$ and $\|fu\| \leq C \|f\|_{C^\alpha} \|u\|$.*
(2) *There exists $C > 0$ such that, for any $u \in H_0$, $\left|\int u \,\mathrm{dLeb}\right| \leq C \|u\|$.*
(3) *The transfer operator $\mathcal{L}$ associated to T sends H_0 to itself, and satisfies $\|\mathcal{L}u\| \leq C \|u\|$. Therefore, $\mathcal{L}$ admits a continuous extension to the completion H of H_0 (still denoted by $\mathcal{L}$). We assume that the essential spectral radius of this extension is < 1, and that the iterates of $\mathcal{L}$ are uniformly bounded.*
(4) *There exist $f_0 \in H_0$ taking its values in $[0, 1]$ and $N_0 > 0$ such that $f_0 = 1$ on $T^{N_0}(X_0)$.*
(5) *For any $u \in H$ which is a limit of nonnegative functions $u_n \in H_0$ and for which there exists a measure μ_u such that $\langle u, g \,\mathrm{dLeb}\rangle = \int g \, d\mu_u$ for any C^α function g, then the measure μ_u gives zero mass to the discontinuity set of T.*

Then T admits a physical description.

In the fifth point, $\langle u, g \,\mathrm{dLeb}\rangle$ is defined as follows. A function $u \in H_0$ can be multiplied by g and then integrated against Lebesgue measure. Those operations are continuous for the norm (by the first and second assumption), and therefore extend to H.

Theorem D.5 is stronger than [BG09, Theorem 33] for the following reasons:

- We do not assume that the space H is a space of distributions, i.e., there may be elements $u \in H$ with $\langle u, g \,\mathrm{dLeb}\rangle = 0$ for any C^∞ function g. The space H_0 used in the proof of Proposition D.2 is a space of distributions, but this is not clear for the space H_0 used in the proof of Proposition D.3 (it would be true if C^1 were dense in $\mathbf{H}$, but we do not know if this holds). This is why we had to abstain from using this assumption in Theorem D.5.
- The conclusion "T admits a physical description" gives the convergence of Birkhoff sums for all continuous functions, while [BG09, Theorem 33] obtains such a convergence only for functions in the closure of H_0 for the C^0 norm.

We next show how to reduce Theorem D.5 to [BG09, Theorem 33].

Proof of Theorem D.5. We first deal with the second issue, that [BG09, Theorem 33] proves the convergence of Birkhoff sums only for functions in the closure of H_0 in the C^0 norm. In fact, the proof in [BG09] gives this convergence for any countable family of functions in H_0. Let g_n be a family of C^α functions, dense in C^0. We obtain the convergence of Birkhoff sums for all the functions $g_n f_0$, since they all belong to H_0 by assumption. Moreover, for all $k \geq N_0$, $(g_n f_0) \circ T^k = g_n \circ T^k$. Therefore, the convergence of Birkhoff sums also holds for all the functions g_n. Since they are dense in C^0, this concludes the proof.

Let us now deal with the first problem, that H is not necessarily a space of distributions. Let $G \subset H$ be the problematic subspace, i.e., $G = \{u \in H \mid \langle u, g \,\mathrm{dLeb}\rangle = 0 \text{ for all } g \in C^\alpha\}$. If $G = \{0\}$, the results of [BG09] directly apply, otherwise we have to eliminate it. We can not work directly with the quotient space H/G, since it is possible that G is not invariant under $\mathcal{L}$. On the other hand, for $|\lambda| = 1$, let $E_\lambda \subset H$ be the eigenspace of $\mathcal{L}$ for the eigenvalue λ, then $F_\lambda = E_\lambda \cap G$ is invariant under $\mathcal{L}$. All the arguments in [BG09] then apply on $H/\bigoplus F_\lambda$ (modulo straightforward adjustments). $\square$

Proof of Proposition D.2. By Theorem 2.5, under the assumptions of the proposition, we may construct a Banach space $\mathbf{H}$ (of distributions) on which the essential spectral radius of $\mathcal{L}$ (as defined in the statement of Theorem D.5) is < 1. To simplify notations, we will pretend that $\mathbf{H}$ is the space $\mathbf{H}_p^{t,s}(R, C_0, C_1)$, and not the more complicated space constructed using (2.20).

We wish to apply Theorem D.5 to $H_0 = \mathbf{H} \cap L^\infty(\mathrm{Leb})$, to obtain the conclusion of the proposition. The first four assumptions of this theorem are trivial, but the fifth one should be checked more carefully. The norm in $\mathbf{H}$ is a supremum of norms along admissible charts. Let us fix one such chart, and consider $\widetilde{H}$ the space obtained by using only the norm in this chart. This space is not interesting from the dynamical point of view (it is not invariant under $\mathcal{L}$), but $\mathbf{H}$ is continuously contained in $\widetilde{H}$. Moreover, [BG09, Lemma 34] shows that, if an element $u \in \widetilde{H}$ satisfies $\langle u, g \,\mathrm{dLeb}\rangle = \int g \, d\mu_u$ for some nonnegative measure μ_u, then μ_u gives zero mass to the discontinuities of T. Since $\mathbf{H}$ is smaller than $\widetilde{H}$, this readily implies the same result for $\mathbf{H}$. □

Proof of Proposition D.3. Consider the operator $\mathcal{M}u = u \circ T$. This operator is obtained locally by composing with hyperbolic maps, therefore we may apply Theorem C.1 to it (under suitable transversality assumptions, that are exactly those of Proposition D.3) – one should simply be careful with notations, since stable and unstable directions are exchanged. The assumption (D.2) ensures that the essential spectral radius of $\mathcal{M}$ on the space $\mathbf{H}$ constructed in Theorem C.1 (for the parameters $p' = p/(p-1)$, $s' = -t$ and $t' = -s$) is < 1. Moreover, since $\mathbf{H}$ is a space of distributions, one may prove as in the first step of the proof of [BG09, Theorem 33] that there is no eigenvalue of modulus > 1 and no Jordan block for the eigenvalues of modulus 1, i.e., the iterates of $\mathcal{M}$ on $\mathbf{H}$ are uniformly bounded. As above, we will pretend that $\mathbf{H} = \mathbf{H}_{p'}^{t',s'}(R, C_0, C_1)$ to simplify notations.

Define a (possibly infinite) norm $\|\cdot\|$ (dual to the $\mathbf{H}$–norm) on $L^\infty(\mathrm{Leb})$ by

$$\|u\| = \sup_{v \in \mathbf{H} \cap L^\infty(\mathrm{Leb}),\ \|v\|_{\mathbf{H}} \leq 1} \left| \int uv \,\mathrm{dLeb} \right|, \tag{D.3}$$

and let H_0 be the set of elements of $L^\infty(\mathrm{Leb})$ with $\|u\| < \infty$. Since the dual of $\mathcal{M}$ is $\mathcal{L}$ (as defined in the statement of Theorem D.5), it follows that $\mathcal{L}$ leaves H_0 invariant, that its essential spectral radius on the completion of H_0 is < 1, and that the iterates of $\mathcal{L}$ are uniformly bounded.

We wish to apply Theorem D.5 to this space H_0, to conclude the proof. As above, the first four conditions of this theorem are easily checked, but we should be more careful for the last one.

For any hypersurface Q bounding a domain O_i, consider a decreasing sequence $K_n(Q)$ of neighborhoods of Q, with sides parallel to Q (in local coordinate charts), and converging to Q. It follows from the argument in the second step of the proof of Lemma 5.1 that the $\mathbf{H}_{p'}^{t',s'}(R, C_0, C_1)$ norm of $1_{K_n(Q)}$ in any admissible chart is uniformly bounded. Therefore, $\left\|1_{K_n(Q)}\right\|_{\mathbf{H}} \leq C$ for some constant C independent of n. The same argument even shows that $1_{K_n(Q)}$ is uniformly bounded in the space $\mathbf{H}_{p'}^{t'',s'}(R, C_0, C_1)$ if $t'' \in (t', 1/p')$. By Lemma 4.4, this space is compactly included in $\mathbf{H}$, therefore the sequence $1_{K_n(Q)}$ is compact in $\mathbf{H}$. Any of its cluster values has to be 0 as a distribution (since $\mathrm{Leb}(K_n(Q)) \to 0$). Since $\mathbf{H}$ is a space of distributions, it follows that all the cluster values of $1_{K_n(Q)}$ are 0, hence $1_{K_n(Q)}$ tends to 0 in $\mathbf{H}$.

Let K_n be the (finite) union of the $K_n(Q)$ for all boundary hypersurfaces of the sets O_i. Then K_n contains the discontinuity set of T in its interior, and 1_{K_n} tends to 0 in $\mathbf{H}$.

Consider u in the completion H of H_0, such that u is a limit of nonnegative functions u_m, and such that, for some measure μ_u, we have $\langle u, g \, \mathrm{dLeb}\rangle = \int g \, d\mu_u$ for any C^α function g. Consider a C^α function g such that $0 \le g \le 1_{K_n}$. We have $\langle u_m, g\rangle \le \langle u_m, 1_{K_n}\rangle$ since u_m is a nonnegative function. Letting m tend to infinity, we get $\langle u, g\rangle \le \langle u, 1_{K_n}\rangle \le \|u\| \, \|1_{K_n}\|_{\mathbf{H}}$. Choosing g equal to 1 on the discontinuity set of T, we get $\mu_u(\mathrm{Disc}\, T) \le \|u\| \, \|1_{K_n}\|_{\mathbf{H}}$. Since this quantity tends to 0 when $n \to \infty$, this concludes the proof. □

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