THERMODYNAMIC FORMALISM FOR PIECEWISE EXPANDING MAPS IN FINITE DIMENSION

VIVIANE BALADI[(1),(2)] AND ROBERTO CASTORRINI[(3)]

ABSTRACT. For $\bar{\alpha} > 1$ and $\alpha \in (0, \bar{\alpha}]$, we study weighted transfer operators associated to a piecewise expanding $\mathcal{C}^{\bar{\alpha}}$ map T on a compact manifold of dimension $d \geq 1$, and a piecewise $\mathcal{C}^{\alpha}$ weight g, acting on Sobolev spaces. We bound the essential spectral radius in terms of a topological pressure for a subadditive potential. Under a new small boundary pressure condition, we improve the estimate by establishing a variational principle for piecewise expanding maps and subadditive potentials.

1. Introduction

1.1. **Functional Approach to Ergodic Properties.** For M a connected compact Riemannian manifold and $T : M \to M$, the *functional analytic* approach to statistical properties of the dynamics T consists in finding a Banach space $\mathcal{B}$ of functions or distributions on M such that the *(Ruelle) transfer operator*

$$\mathcal{L}_{T,g}\varphi(x) = \sum_{Ty=x} g(y)\varphi(y)\,, \quad x \in M\,,$$

weighted by a suitable function $g : M \to \mathbb{C}$, and defined initially on a subset of measurable functions $\varphi : M \to \mathbb{C}$, extends to a bounded operator on $\mathcal{B}$ on which its *essential spectral radius*[1] is smaller than its spectral radius ("quasicompactness").

Date: February 26, 2024, to appear DCDS.

2020 *Mathematics Subject Classification.* Primary 37C30; Secondary 37C83.

Key words and phrases. Transfer operator. Piecewise expanding map. Subadditive thermodynamic formalism. Topological pressure.

It is a pleasure to thank Y. Guedes-Bonthonneau, J. Buzzi, M. Demers, D.-J. Feng, M. Misiurewicz, T. Persson, D. Smania, I.P. Toth, and, especially, M. Jézéquel, for helpful comments. Part of this work was done while VB was visiting Lund University on a Knut and Alice Wallenberg fellowship in 2021, ITS-ETHZ Zurich in 2022 and 2023, and the Mittag Leffler Institute in 2023. This work was started when RC was working at LPSM, CNRS, Paris. We are grateful to the referees for a careful reading. This research is partially supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 787304). RC is partially supported by the research project PRIN 2017S35EHN of the Italian Ministry of Education and Research. RC acknowledges membership to the GNFM/INDAM..

[1] The radius of the smallest disc outside of which the spectrum of $\mathcal{L}_{T,g}$ consists of isolated eigenvalues of finite multiplicity. Although the essential spectral radius depends on $\mathcal{B}$, these eigenvalues in general do not, see e.g. [29, Thm 2.3].

If g is positive and T is mixing, the spectral picture can sometimes be strengthened as follows: The transfer operator has a positive maximal eigenvalue, which is the exponential $e^{P(\log g)}$ of the topological pressure of $\log g$. This eigenvalue is simple, and the rest of the spectrum is contained in a smaller disc. This "spectral gap" often implies existence, uniqueness and decay of correlations (for suitable observables) of the equilibrium state of $\log g$, i.e. the invariant measure maximising $h_\mu + \int \log g \mathrm{d}\mu$ (where h_μ is the Kolmogorov entropy). For $g = |\det DT|^{-1}$, we have in many cases that $e^{P_{\mathrm{top}}(\log g)} = 1$, and the equilibrium state of $\log g$ is the physical (SRB) measure.

Another desirable goal (besides finding a Banach space on which the essential spectral radius is small) is to relate the isolated eigenvalues of the transfer operator with the poles of a dynamical zeta function defined by assuming that $\mathrm{Fix}\, T^n = \{x \in M \mid T^n(x) = x\}$ is finite for each fixed n, and setting (in the sense of formal power series) $\zeta_{T,g}(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix}\, T^n} g^{(n)}(x)$. We hope that the Milnor–Thurston kneading operator approach of [13] (see [14] or [3, §3.2] for an implementation to smooth dynamics in arbitrary dimension) can be applied to piecewise expanding or piecewise hyperbolic dynamics in arbitrary dimension.

1.2. **(Piecewise) Expanding Case.** For *expanding and piecewise expanding* maps T (with smallest expansion denoted by $\lambda > 1$), the relevant $\mathcal{B}$ is a space of functions. In the *smooth* expanding case, the pioneering bounds of Ruelle [38] on the essential spectral radius, taking $\mathcal{B}$ the space of Hölder functions, were shown to be optimal by Gundlach–Latushkin [31], who reformulated them using a variational (thermodynamic) expression. The *piecewise* expanding theory is fairly complete in one-dimension, usually taking $\mathcal{B}$ the set BV of functions of bounded variation (piecewise monotonicity is enough there, see e.g. [2, 13]).

For *higher dimensional* piecewise expanding dynamics, quasicompactness (and even ergodic properties such as existence of the SRB measure) can fail [45, 18] if one does not make *additional assumptions on the "complexity at the beginning"* $D^b(T)$ (also called "entropy multiplicity," see (2.7)): The works [10, 11, 41, 35] require some version of $D^b(T) < \log \lambda$ ("hyperbolicity beats complexity at the beginning") to bound the essential spectral radius. Cowieson [24] proved that $D^b(T) = 0$ for T in an open and dense subset of piecewise $\mathcal{C}^{\bar\alpha}$ expanding maps, and that $D^b(T) = 0$ implies a spectral gap for the operator associated to $g = |\det DT|^{-1}$ acting on $\mathcal{B} = BV$ (see [42] for a different choice of $\mathcal{B}$). For arbitrary piecewise $\mathcal{C}^\alpha$ weights g, in any dimension, Thomine [41], inspired by [10], obtained[2] a bound (see (2.9)) on the essential spectral radius on classical Sobolev spaces $\mathcal{B} = \mathcal{H}^t_p$, for $1 < p < \infty$ and $0 < t < \min\{\alpha, 1/p\}$. Even if $D^b(T) = 0$ and $g = |\det DT|^{-1}$, Thomine's bound ensures quasicompactness only if either T satisfies some pinching condition (for example if T is a β transformation) or p is close enough to 1, in order to control the exponential growth of the number

[2] Using a tower construction, Buzzi et al. [22, 21] had previously found mild additional conditions implying exponential decay of correlations for Hölder observables.

of preimages of T^n (the "complexity at the end", see Remark 2.4). Liverani [35, (3)] found sufficient conditions ensuring a spectral gap[3] on $\mathcal{B} = BV$ for $d \geq 1$ and piecewise expanding maps having possibly infinitely many domains of continuity and (controlled) blowup of derivatives, with $g = |\det DT|^{-1}$. The only results linking the poles of the zeta function to the spectrum of $\mathcal{L}_{T,g}$ for higher dimensional[4] piecewise expanding maps were obtained by Buzzi and Keller [20], for piecewise affine maps and $g = |\det DT|^{-1}$.

1.3. **(Piecewise) Hyperbolic Case.** The pioneering work [15] introduced *anisotropic* Banach spaces $\mathcal{B}$ of *distributions,* adapted to Anosov diffeomorphisms T. These spaces can be classified in two categories (see [4]): *Geometric spaces,* using integration over stable submanifolds, and *micro-local spaces,* using the Fourier transform. The nature of geometric norms (taking a supremum over a class of submanifolds) does not seem to be amenable to the Milnor–Thurston approach. In the *smooth* hyperbolic case, the best known estimate for the essential spectral radius is obtained for micro-local spaces by *thermodynamic formalism* techniques, as a variational expression for a subadditive topological pressure [14].

Many physically relevant models, such as dispersive billiards are uniformly hyperbolic, but only piecewise smooth. The geometric approach [15, 30] has been used to study the SRB measure of piecewise hyperbolic maps with controlled complexity in dimension two ([26]), but also the SRB measure and other equilibrium states of Sinai billiard maps and flows ([28, 7, 8, 9, 6]). It has recently been extended to the random Lorentz gas, via Birkhoff cones [27]. Estimates on the essential spectral radius for micro-local spaces were[5] obtained ([10, 11]) for weighted piecewise hyperbolic surface maps. (The results there do not apply to Sinai billiards, for which the derivative is unbounded.) We are not aware of any result linking the poles of dynamical zeta functions with the spectrum of transfer operators for piecewise hyperbolic maps.

A modification $\mathcal{U}_p^{t,s}$ of the micro-local spaces of [11] suitable in the piecewise smooth setting has been proposed in [4, 5]. Jézéquel, observed that, even if $D^b(T) = 0$ and $g = |\det DT|^{-1}$, the bound on the essential spectral radius of [4, Thm 4.1] may not imply quasicompactness: For a linear automorphism T of the two-torus with expanding eigenvalue $\Lambda > 1$, the essential spectral radius of the operator for $|\det DT|^{-1} \equiv 1$ acting on the space $\mathcal{U}_p^{t,s}$ from [4] is bounded by $r_0(t,s,p) = \frac{\Lambda}{\Lambda^{1/p}} \max\{\Lambda^{-t}, \Lambda^{t+s}\}$ (the factor Λ comes from a naive use of "complexity at the end"). To ensure that characteristic functions are bounded multipliers [5, Thm 3.1], we must take $-1+1/p < s < -t < 0$, so that $r_0(t,s,p) > 1$. Our hope is that a "thermodynamic" control of the complexity at the end (using fragmentation and reconstitution, as below) will replace $r_0(t,s,p)$ by $r(t,s,p) =$

[3] The assumption [35, (3)] that "hyperbolicity beats complexity" is similar to requiring $D^b_{\{\nu_n\}} < 0$, see (2.21).

[4] In the more general setting of non-degenerate entropy expanding maps, Buzzi [19, §6] found a domain of meromorphy for $\zeta_{T,g}(z)$ if $g \equiv 1$ (but no spectral interpretation of the poles).

[5] Even if $D^b(T) = 0$, [10, 11] do not always give quasicompactness if $g \neq |\det DT|^{-1}$.

$\frac{\Lambda^{\tilde p}}{\Lambda^{1/p}} \max\{\Lambda^{-t}, \Lambda^{t+s}\}$ for some $\tilde p \in (0,1)$ (probably $(p-1)/p$, in view of (2.9) and [11, Thm 2.5]), allowing parameters for which $r(t,s,p) < 1$.

1.4. **Outline of the Results.** We consider the toy-model case of weighted piecewise expanding maps and classical (isotropic) Sobolev spaces $\mathcal{H}^t_p$, just like in [41], but thermodynamic estimates replace the "complexity at the end": Our first main result, Theorem 2.10, gives an unconditional bound (2.22) on the essential spectral radius in terms of the topological pressure of a subadditive potential related to $\log |g|$. (We recover the optimal bound [12, 34] in dimension one. We improve on Thomine's bound [41] in the generic small boundary entropy case $D^b(T) = 0$. If $g = |\det DT|^{-1}$, our bound is analogous to Liverani's bound [35, Thm 1, Lemma 3.1] for the essential spectral radius. Our bound coincides with Gundlach–Latushkin's [31] bound on Hölder spaces if the map and weight are smooth. See Remarks 2.12, 2.13, 2.15, 2.16, 2.19.) Assuming small boundary pressure, a variational principle of Buzzi–Sarig [23] allows us to reformulate (2.22) in Corollary 2.11.

Next, Theorem 2.17 generalises this additive variational principle [23] to a class of subadditive potentials (subadditive potentials appear naturally in dynamics, for example $\log |\det DT|$ in dimension two or higher, see below — our results are the piecewise smooth analogue of [14, §3], see also [3, App. B]). Combining Theorem 2.17 with Theorem 2.10 yields Corollary 2.18, which gives the variational expression (2.29) for the bound (2.22), under a new subadditive small boundary pressure condition. Our results are strongest in the SRB case $g = |\det DT|^{-1}$, letting $1/p > t$ both tend to 1.

One of the features of our approach is fragmentation-reconstitution Lemma 3.7, which allows us to conveniently use a zoom for arbitrary values of our parameter p. We hope that our results lay the groundwork for the implementation of the "ultimate" micro-local Banach space $\mathcal{U}^{t,s}_p$ from [4, 5] in the setting of piecewise hyperbolic systems, giving also information on zeta functions.

1.5. **Outline of the Paper.** The paper is organised as follows: In Section 2, after defining our class of piecewise $\mathcal{C}^{\bar\alpha}$ expanding maps T and piecewise $\mathcal{C}^\alpha$ weights g, we state our two main results: Theorem 2.10 on the essential spectral radius of the weighted transfer operator $\mathcal{L}_g$ and Theorem 2.17 on the subadditive variational principle. We state and prove Corollary 2.11 and Corollary 2.18, which follow from Theorem 2.10 and, respectively, (2.18) and Theorem 2.17, and give conditional variational expressions for the bound on the essential spectral radius. In Section 3, we establish Theorem 2.10. For this, we prove the key Lasota–Yorke inequality, Proposition 3.8, in §3.3 and exploit it in §3.4. Then, Section 4 contains the (independent) proof of Theorem 2.17, adapting [23], using symbolic dynamics and a variational principle of Cao–Feng–Huang [25] for continuous subadditive potentials.

2. Setting, Definitions, and Precise Statement of Results

2.1. **Piecewise Expanding Maps.** Throughout, M is a compact connected $\mathcal{C}^\infty$ Riemannian manifold of dimension $d < \infty$, and $\mathcal{T}_x$ denotes the tangent space of M at $x \in M$. If M has a boundary we let $\widetilde{M}$ be a compact connected $\mathcal{C}^\infty$ Riemannian manifold of dimension $d < \infty$ containing the union of M and a small neighbourhood of its boundary, otherwise we take $\widetilde{M} = M$. For noninteger $\beta > 1$, we denote by $\mathcal{C}^\beta$ those $\mathcal{C}^{[\beta]}$ maps whose partial derivatives of order $[\beta]$ are $(\beta - [\beta])$-Hölder. If a map F is invertible on a set E, we write $F^{-1}|_E = (F|_E)^{-1}$, abusing notation. Fixing real numbers $\bar\alpha > 1$ and $0 < \alpha \le \bar\alpha$, we introduce our object of study:

Definition 2.1. *A map $T : M \to M$ is called* piecewise ($\mathcal{C}^{\bar\alpha}$) expanding *if there exists a finite set of pairwise disjoint open sets $\mathcal{O} = \{O_i\}_{i\in I}$, covering Lebesgue almost all M, such that each ∂O_i is a finite union of $\mathcal{C}^1$ compact hypersurfaces with boundaries, and moreover, for each $i \in I$ there exists a neighbourhood $\widetilde{O}_i$ of $\overline{O}_i$ in $\widetilde{M}$ and a $\mathcal{C}^{\bar\alpha}$ diffeomorphism $\widetilde{T}_i : \widetilde{O}_i \to T_i(\widetilde{O}_i) \subset \widetilde{M}$ such that $T|_{O_i} = \widetilde{T}_i|_{O_i}$, and, setting $\lambda_i(x) = \inf_{v\in\mathcal{T}_xM\setminus\{0\}} \frac{\|D_x\widetilde{T}_i v\|}{\|v\|}$ for $i \in I$ and $x \in \widetilde{O}_i$,*

$$\lambda = \inf_{i\in I} \inf_{x\in\widetilde{O}_i} \lambda_i(x) > 1\,. \tag{2.1}$$

Remark 2.2. *Using a Taylor series, our assumption implies that, for any $\lambda' \in (1, \lambda)$, there exists $\epsilon' > 0$ such that, refining $\mathcal{O}$ to a finite collection $\mathcal{O}' = \{O'_i\}_{i\in I'}$ (such that each $\partial O'_i$ is a finite union of $\mathcal{C}^1$ compact hypersurfaces with boundaries) of pairwise disjoint open sets of diameter smaller than ϵ' covering Lebesgue almost all M, we have*

$$d(T(x), T(y)) \ge \lambda' d(x, y), \quad \forall x, y \in O'_i\,,\ \forall i\,.$$

From now on, we assume that such a refinement has been done, using the notation λ, $\mathcal{O}$, O_i, I, for λ', $\mathcal{O}'$, O'_i, I'.

For T as in Definition 2.1 (and Remark 2.2), we introduce, for $n \ge 1$ and $\mathbf{i} = (i_0, \ldots, i_{n-1}) \in I^n$, the n-cylinder $O_{\mathbf{i}}$ by

$$O_{\mathbf{i}} = O_{(i_0,\ldots,i_n)} = \bigcap_{k=0}^{n-1} T^{-k} O_{i_k}\,.$$

Note that for each $n \ge 1$ and almost every $x \in M$, there exists a unique $\mathbf{i} \in I^n$ such that $x \in O_{\mathbf{i}}$. The corresponding (mod-0) partition into n-cylinders $O_{\mathbf{i}}$ is denoted by $\mathcal{O}^{(n)}$ (so that $\mathcal{O} = \mathcal{O}^{(1)}$). We set $\operatorname{diam}(\mathcal{O}^{(n)}) = \max_{O_{\mathbf{i}}\in\mathcal{O}^{(n)}} \operatorname{diam}(O_{\mathbf{i}})$, so that[6] $\operatorname{diam}(\mathcal{O}^{(n)}) \le \lambda^{-n} \operatorname{diam}(M)$.

[6]Recall Remark 2.2, and proceed inductively on n, noting that $\bigcap_{k=0}^{n-1} T^{-k} O_{i_k} = O_{i_0} \bigcap T^{-1}(\bigcap_{k=1}^{n-1} T^{-(k-1)} O_{i_k})$.

For $n \geq 1$ and $\mathbf{i} \in I^n$ with $O_{\mathbf{i}} \neq \emptyset$, the map $\widetilde{T}^n_{\mathbf{i}} = \widetilde{T}_{i_{n-1}} \circ \cdots \circ \widetilde{T}_{i_0}$ is defined in the neighbourhood $\widetilde{O}_{\mathbf{i}} = \bigcap_{k=0}^{n-1} \widetilde{T}^{-k}_{(i_0,\ldots,i_{k-1})} \widetilde{O}_{i_k}$ of $\overline{O}_{\mathbf{i}}$ (we put $\overline{O}_{\mathbf{i}} = \emptyset$ if $O_{\mathbf{i}} = \emptyset$). Setting

$$\partial \mathcal{O} = \cup_i \partial O_i \,, \quad \mathcal{S}_{\mathcal{O}} = \cup_{k \geq 0} T^{-k}(\partial \mathcal{O}) \,, \tag{2.2}$$

(note that $\mathcal{S}_{\mathcal{O}}$ has zero Lebesgue measure), we put

$$\nu_n(x) = \frac{1}{\inf_{\|v\|=1} \|D_x T^n v\|} \,, \; x \in M \setminus \mathcal{S}_{\mathcal{O}} \,, \tag{2.3}$$

$$\tilde{\nu}_{n,\mathbf{i}}(y) = \|D_{T^n_{\mathbf{i}} y} (\widetilde{T}^n_{\mathbf{i}})^{-1}\| \,, \; y \in \widetilde{O}_{\mathbf{i}} \,, \; \mathbf{i} \in I^n; \tag{2.4}$$

$$\tilde{\nu}_n(x) = \sup_{\mathbf{i} \in I^n : x \in \widetilde{O}_{\mathbf{i}}} \tilde{\nu}_{n,\mathbf{i}}(x) \in [\nu_n(x), \lambda^{-n}] \,, \; x \in M \,.$$

The function ν_n is submultiplicative (multiplicative if $d = 1$). We set

$$\nu_*(x) = \lim_{n \to \infty} \nu_n(x)^{1/n} \,, \, x \in M \setminus \mathcal{S}_{\mathcal{O}} \,. \tag{2.5}$$

For $1 < p < \infty$ and $t \geq 0$, we denote by $\mathcal{H}^t_p = \mathcal{H}^t_p(M)$ the standard Sobolev space on M (see Section 3.1). We write $r_{\mathrm{ess}}(\mathcal{L}|_{\mathcal{B}})$ for the essential spectral radius of a bounded operator $\mathcal{L}$ on a Banach space $\mathcal{B}$. For a fixed piecewise expanding map T, a function $f : M \to \mathbb{C}$ is called piecewise continuous if $f|_{O_i}$ extends continuously to $\widetilde{O}_i$, and f is called piecewise $\mathcal{C}^\alpha$ if $f|_{O_i}$ is $\mathcal{C}^\alpha$. If f is piecewise $\mathcal{C}^\alpha$, it is easy to see that each $f|_{O_i}$ admits a $\mathcal{C}^\alpha$ extension $\tilde{f}_i$ (with[7] the same Hölder constant C_i) to $\widetilde{O}_i$ for each $i \in \{1, .., I\}$, and we set

$$f^{(n)}(x) = \prod_{k=0}^{n-1} f(T^k(x)) \,, x \in M \,, \; \tilde{f}^{(n)}_{\mathbf{i}} = \prod_{k=0}^{n-1} \tilde{f}_{i_k}(\widetilde{T}^k_{\mathbf{i}}(x)) \,, \, \mathbf{i} \in I^n \,, \, x \in \widetilde{O}_{\mathbf{i}} \,. \tag{2.6}$$

We can now define the transfer operator:

Definition 2.3. *Let $T : M \to M$ be piecewise $C^{\bar{\alpha}}$ expanding. Let $g : M \to \mathbb{C}$ be piecewise $\mathcal{C}^\alpha$. Fix $1 \leq p \leq \infty$. The transfer operator $\mathcal{L}_g : L_p(M) \to L_p(M)$ is*

$$\mathcal{L}_g(\varphi)(x) = \mathcal{L}_{T,g}(\varphi)(x) = \sum_{y : T(y) = x} g(y)\varphi(y), \quad x \in M \,, \; \varphi \in L_p(M) \,.$$

Next, for $\mu \in \mathrm{Erg}(T)$ (the set of ergodic T-invariant probability measures on M), let $h_\mu = h_\mu(T)$ be its Kolmogorov entropy and $\chi_\mu(DT)$ the smallest Lyapunov exponent of the linear cocycle DT. If $d = 1$, the Birkhoff ergodic theorem gives $\chi_\mu(DT) = \int \log |T'| \, d\mu = \int \log |\det DT| \, d\mu$.

Finally, the *asymptotic complexity at the beginning (or entropy multiplicity, see e.g.* [16]*)* $D^b = h_{\mathrm{mult}}$ of T (and $\mathcal{O}$) is

$$D^b = D^b(T) = \lim_{n \to \infty} \frac{1}{n} \log D^b_n(T) \,, \tag{2.7}$$

[7] It is enough to consider real-valued f in charts. Set $\tilde{f}_i(x) = \inf_{y \in O_i} \{f(y) + C_i d(x,y)^\alpha\}$.

where the *n-complexity at the beginning* $D^b_n(T)$ of T (and $\mathcal{O}$) is

$$D^b_n = D^b_n(T) = \max_{x\in M} \# \left\{ \mathbf{i} = (i_0, \dots, i_{n-1}) \mid \overline{O_{\mathbf{i}}} \ni x \right\} \,,\ n \ge 1 \,. \tag{2.8}$$

Remark 2.4 (Complexity at the end D^e)**.** *The works* [10, 11, 41] *also use* complexity at the end

$$D^e_n = D^e_n(T) = \max_{x\in M} \#\{\mathbf{i} = (i_0, \dots, i_{n-1}) \mid x \in \overline{T^n(O_{\mathbf{i}})}\} \,,\ n \ge 1 \,,$$
$$D^e = D^e(T) = \lim_{n\to\infty} \frac{1}{n} \log D^e_n > 0 \,.$$

For $Tx = 2x \mod 1$ on $[0,1]$, we have $D^e_n(T) = 2^n$, see Remarks 2.6 and 2.15, and Lemma 2.8 for more about complexity at the end. Thomine's bound [41] *is*

$$r_{ess}(\mathcal{L}_g|_{\mathcal{H}^t_p}) \le \lim_{n\to\infty} \left(D^b_n(T)^{\frac{1}{p}} \, D^e_n(T)^{\frac{p-1}{p}} \, \sup \left| g^{(n)} |\det DT^n|^{\frac{1}{p}} \nu^t_n \right| \right)^{1/n} \,. \tag{2.9}$$

Choosing $p > 1$ close to 1 allows to control the contribution of D^e_n, but such an exponent p increases the contribution of $|\det DT^n|^{1/p}$. In our estimates, the complexity at the end will be implicit in the topological pressure P^*_{top}.

2.2. **Pressure P^*_{top} and Boundary Pressure.** We define the pressure of subadditive sequences for a piecewise expanding map T, generalising the pressure[8] $P^*_{\mathrm{top}}(T, \log f, E)$ (for $E \subset M$ and $f : M \to \mathbb{R}^+_*$) studied e.g. by Buzzi–Sarig [23].

Definition 2.5 (Pressure of a Subadditive Potential)**.** *A submultiplicative sequence for the piecewise $\mathcal{C}^{\bar\alpha}$ expanding map T is a sequence $\{f_n : M \to \mathbb{R}^+ \mid n \ge 1\}$ of bounded functions with $f_{m+n}(x) \le f_m(T^n(x)) \cdot f_n(x)$ for all $m, n \ge 1$. For $E \subset M$ measurable, and $\{f_n\}$ submultiplicative, the topological pressure of T and (the subadditive potential) $\{\log f_n \mid n \ge 1\}$ on E is*[9]

$$P^*_{\mathrm{top}}(T, \{\log f_n \mid n \ge 1\}, E) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}\in I^n : E\cap \bar O_{\mathbf{i}} \ne \emptyset} \sup_{\overline{O_{\mathbf{i}}}} f_n \in [-\infty, \infty) \,. \tag{2.10}$$

We write $P^*_{\mathrm{top}}(\{\log f_n\}, E)$ and $P^*_{\mathrm{top}}(\{\log f_n\})$ when the meaning is clear. If $f_n = f^{(n)}$ is multiplicative, we just write $P^*_{\mathrm{top}}(\log f, E)$ and $P^*_{\mathrm{top}}(\log f)$. The *topological entropy of T on a measurable set $E \subset M$* is $P^*_{\mathrm{top}}(T, 0, E)$.

For all $q \ge 1$, we have the trivial bound

$$e^{\frac{P^*_{\mathrm{top}}(0,E)}{q}} \cdot \lim_{n\to\infty} (\inf f_n)^{1/n} \le e^{\frac{P^*_{\mathrm{top}}(\{q \log f_n\},E)}{q}} \le e^{\frac{P^*_{\mathrm{top}}(0,E)}{q}} \cdot \lim_{n\to\infty} (\sup f_n)^{1/n} \,. \tag{2.11}$$

[8] The partition $\mathcal{O}$ is a topological generator, but T is not continuous, so classical results do not apply. We do not relate here P^*_{top} to pressure defined via open covers, separated sets, or spanning sets. See [8, 9] for entropy and pressure via natural topological generators for billiards.

[9] The limit exists in $\mathbb{R} \cup \{-\infty\}$ by subadditivity.

Remark 2.6 (Comparing Pressure with Complexity)**.** *Note that $e^{D^e(T)}$ is just the spectral radius of $\mathcal{L}_1$ on L_∞: Indeed, we have*

$$\max \mathcal{L}_1^n(1) = e^{D^e_n(T)} \le \exp \#\{\mathbf{i} \in I^n \mid O_{\mathbf{i}} \neq \emptyset\}\,. \text{ Thus, } D^e(T) \le P^*_{\mathrm{top}}(0)\,. \tag{2.12}$$

For the complexity at the beginning, we have

$$D^b(T) \le P^*_{\mathrm{top}}(0, \partial\mathcal{O})\,, \tag{2.13}$$

(Indeed, setting $P^*_{n,\mathrm{top}}(0, \partial\mathcal{O}) = \#\{\mathbf{i} \in I^n \mid \partial\mathcal{O} \cap \bar{O}_{\mathbf{i}} \neq \emptyset\}$, we have, $D^b_1(T) \le P^*_{1,\mathrm{top}}(0, \partial\mathcal{O})$ and $D^b_n(T) \le \max\{D^b_{n-1}(T), P^*_{n,\mathrm{top}}(0, \partial\mathcal{O})\}$ if $n \ge 2$.)

In the other direction, using that $\partial\mathcal{O}$ has codimension one, we have by [17, Prop. 5.2] *(condition (A2) there is satisfied) that*

$$P^*_{\mathrm{top}}(-\log|\det DT|, \partial\mathcal{O}) \le -\log\lambda + D^b(T)\,. \tag{2.14}$$

Set $\Lambda_0 = 1$, and,[10] *for $d \ge 2$,*

$$\Lambda_{d-1} = \exp \limsup_{n\to\infty} \frac{1}{n} \log \max_{x,V} \| \wedge^{d-1} (DT^n_x)|_V \|\,, \tag{2.15}$$

where the maximum ranges over $(d-1)$-dimensional subspaces V of $\mathcal{T}_x M$. If T is piecewise affine, then [16, Prop. 4] *implies that*

$$\begin{aligned} P^*_{\mathrm{top}}(\log f, \partial\mathcal{O}) &\le \sup \log f + P^*_{\mathrm{top}}(0, \partial\mathcal{O}) \\ &\le \sup \log f + \log \Lambda_{d-1} + D^b(T)\,. \end{aligned} \tag{2.16}$$

Definition 2.7 (Small Boundary Pressure)**.** *Let T be piecewise $\mathcal{C}^{\bar\alpha}$ expanding. A submultiplicative sequence $\{f_n : M \to \mathbb{R}^+ \mid n \ge 1\}$, satisfies the small boundary pressure condition if*

$$P^*_{\mathrm{top}}(T, \{\log f_n\}, \partial\mathcal{O}) < P^*_{\mathrm{top}}(T, \{\log f_n\})\,. \tag{2.17}$$

For *multiplicative sequences* $f_n = f^{(n)}$, associated to a piecewise $\mathcal{C}^\alpha$ function $f : M \to \mathbb{R}^+$ with $\inf f > 0$, Buzzi and Sarig ([23, Thm 1.3]) showed

$$\begin{aligned} &P^*_{\mathrm{top}}(T, \log f, \partial\mathcal{O}) < P^*_{\mathrm{top}}(T, \log f) \implies \\ &\qquad P^*_{\mathrm{top}}(\log f) = \sup_{\mu \in \mathrm{Erg}(T)} \Big\{ h_\mu(T) + \int \log f \mathrm{d}\mu \Big\}\,. \end{aligned} \tag{2.18}$$

They also showed that small boundary pressure implies that there are finitely many measures realising the supremum, and that, if T is strongly topologically mixing, this maximum is uniquely attained. In a previous work, Buzzi [17, Thm A] had established (2.18) for $g = |\det DT|^{-1}$, showing also that

$$\begin{aligned} &P^*_{\mathrm{top}}(T, -\log|\det DT|, \partial\mathcal{O}) < P^*_{\mathrm{top}}(T, -\log|\det DT|) \implies \\ &\qquad\qquad P^*_{\mathrm{top}}(-\log|\det DT|) = 0\,. \end{aligned} \tag{2.19}$$

Theorem 2.17 below generalises (2.18) to certain subadditive potentials.

A useful consequence of (2.18) is the following lemma:

[10] $\wedge^k(A)|_V$ denotes the quantity by which the k-dimensional volume of V is multiplied by the linear map A.

Lemma 2.8 (Small Boundary Pressure Implies $P^*_{\mathrm{top}}(T,0) \le D^e(T)$)**.** *Let T be piecewise $\mathcal{C}^{\bar\alpha}$ expanding, let $g : M \to \mathbb{R}$ be piecewise $\mathcal{C}^\alpha$ with $\inf g > 0$. If $P^*_{\mathrm{top}}(T, \log g, \partial\mathcal{O}) < P^*_{\mathrm{top}}(T, \log g)$, then $P^*_{\mathrm{top}}(T, \log g)$ is bounded by the logarithm of the spectral radius of $\mathcal{L}_g$ on L_∞. (In particular, if $P^*_{\mathrm{top}}(T, 0, \partial\mathcal{O}) < P^*_{\mathrm{top}}(T, 0)$ then $D^e(T) = P^*_{\mathrm{top}}(T, 0)$ by the equality in (2.12).)*

The fact that $P^*_{\mathrm{top}}(T, \log g)$ is bounded by the logarithm of the spectral radius of $\mathcal{L}_g$ on L_∞ for $g > 0$ is well known if $d = 1$ (see [2, Thm 3.3]).

Proof. In view of (2.18), for any $\delta > 0$ there exists $\mu_\delta \in \mathrm{Erg}(T)$ such that $P^*_{\mathrm{top}}(T, \log g) \le h_{\mu_\delta}(T) + \int \log g \, d\mu_\delta + \delta$. The claim thus follows from an application of Rohlin's formula. See e.g. [2, pp. 160–161]. □

2.3. **The Essential Spectral Radius (Theorem 2.10 and Corollary 2.11).** We introduce the *weighted n-complexity at the beginning* $D^b_n(T, \bar f)$ of T and a nonnegative function $\bar f$:

$$D^b_n(\log \bar f) = D^b_n(T, \log \bar f) = \sup_{x\in M} \sum_{\mathbf{i}\in I^n | \overline{O_\mathbf{i}} \ni x} \sup_{\overline{O_\mathbf{i}}} \bar f\,,\ n \ge 1\,, \tag{2.20}$$

and we set, for a submultiplicative sequence of nonnegative functions f_n,

$$D^b(\{\log f_n\}) = D^b(T, \{\log f_n\}) = \lim_{n\to\infty} \frac{1}{n} \log D^b_n(T, \log f_n)\,. \tag{2.21}$$

If $f_n \equiv 1$ for all n, we recover $D^b_n(T)$ and $D^b(T)$ from (2.8), (2.7).

We have the following generalisation of (2.13):

Lemma 2.9. *We have $D^b(T, \{\log f_n\}) \le P^*_{\mathrm{top}}(\{\log f_n\}, \partial\mathcal{O})$.*

Proof. Setting $P^*_{n,\mathrm{top}}(\log \bar f, \partial\mathcal{O}) = \sum_{\mathbf{i}\in I^n | \partial\mathcal{O}\cap \bar O_\mathbf{i} \ne \emptyset} \sup_{\overline{O_\mathbf{i}}} \bar f$, we have

$$D^b_1(T, \log \bar f) \le P^*_{1,\mathrm{top}}(\log \bar f, \partial\mathcal{O})$$

and (recalling also that $f_n \le (f_1 \circ T^{n-1}) \cdot f_{n-1}$),

$$D^b_n(T, \log f_n) \le \max\{D^b_{n-1}(T, \log f_n), P^*_{n,\mathrm{top}}(\log f_n, \partial\mathcal{O})\}\,,\ \forall n \ge 2\,. \quad \Box$$

The first main result follows. (It is proved in §3.4.)

Theorem 2.10 (Spectral and Essential Spectral Radius)**.** *Let $T : M \to M$ be piecewise $\mathcal{C}^{\bar\alpha}$ expanding and recall ν_n from (2.3). Let $g : M \to \mathbb{C}$ be piecewise $\mathcal{C}^\alpha$. For all $p \in (1,\infty)$ and $t \in (0, \min\{1/p, \alpha\})$, the operator $\mathcal{L}_g$ on $L_p(M)$ restricts boundedly[11] to $\mathcal{H}^t_p(M)$, with essential spectral radius there bounded by*

$$R^{t,p}_*(g) = \exp\Big(\frac{D^b(T)}{p} + \frac{p-1}{p} P^*_{\mathrm{top}}\big(\{\frac{p}{p-1} \log\big(|g^{(n)}| \cdot |\det DT^n|^{\frac{1}{p}} \cdot \nu^t_n\big)\}\big)\Big). \tag{2.22}$$

[11] See §3.1 for the definition of $\mathcal{H}^t_p(M)$.

Moreover, for $s \in [0,t]$, the essential spectral radius of $\mathcal{L}_g|_{\mathcal{H}^t_p(M)}$ is bounded by[12]

$$R^{t,p}_{*,s}(g) = \exp\Bigg(\frac{D^b(T, \{\log \nu^s_n\})}{p} + \frac{p-1}{p} P^*_{\mathrm{top}}(\{\frac{p}{p-1} \log(|g^{(n)}| \cdot |\det DT^n|^{\frac{1}{p}} \cdot \nu^{t-s}_n)\}) \Bigg) . \tag{2.23}$$

Finally, the spectral radius of $\mathcal{L}_g$ on $\mathcal{H}^0_p(M) = L_p(M)$ is bounded by $R^{0,p}_*(g)$.

Note that by (2.11), we have (similar bounds can be written for $R^{t,p}_{*,s}(g)$)

$$\begin{aligned} R^{t,p}_*(g) &\le e^{\frac{D^b(T)}{p} + \frac{p-1}{p} P^*_{\mathrm{top}}(\frac{p}{p-1}\log|g|)} \lim_{n\to\infty} \| |\det DT^n|^{\frac{1}{p}} \nu^t_n \|^{\frac{1}{n}}_{L_\infty} \\ &\le e^{\frac{D^b(T)}{p} + \frac{p-1}{p} P^*_{\mathrm{top}}(0)} \lim_{n\to\infty} \| g^{(n)} \cdot |\det DT^n|^{\frac{1}{p}} \nu^t_n \|^{\frac{1}{n}}_{L_\infty} . \end{aligned} \tag{2.24}$$

The Ruelle inequality is the property that

$$\sup_{\mu \in \mathrm{Erg}(T)} \{ h_\mu(T) - \int \log |\det DT| \mathrm{d}\mu \} \le 0 . \tag{2.25}$$

By (2.18) and (2.19), small boundary pressure for $f_n = |\det DT^n|^{-1}$ implies the Ruelle inequality. See [1, (1)] for a "large image" condition (called quasi-Markovianity there) which ensures the Ruelle inequality.

We state a corollary of (2.22) (the reader is invited to derive variational bounds from (2.23), Lemma 2.9):

Corollary 2.11. *In the setting of Theorem 2.10, set $f = (|g||\det DT|^{1/p})^{p/(p-1)}$. Assume that the Ruelle inequality holds. If $P^*_{\mathrm{top}}(T, \log f, \partial\mathcal{O}) < P^*_{\mathrm{top}}(T, \log f)$ then*

$$\begin{aligned} R^{t,p}_*(g) &\le \nu^t_* \cdot e^{\frac{D^b(T)}{p}} \exp\big(\sup_{\mu \in \mathrm{Erg}(T)} \{ \frac{p-1}{p} h_\mu(T) + \int \log(|g||\det DT|^{1/p}) \mathrm{d}\mu \} \big) \\ &\le \nu^t_* \cdot e^{\frac{D^b(T)}{p}} \exp\big(\sup_{\mu \in \mathrm{Erg}(T)} \{ \int \log(|g||\det DT|) \mathrm{d}\mu \} \big) . \end{aligned} \tag{2.26}$$

Clearly, $\sup_{\mu \in \mathrm{Erg}(T)} \{ \int \log(|g||\det DT|) \mathrm{d}\mu \} \le \lim_{n\to\infty} (\|g^{(n)}|\det DT^n|\|_{L_\infty})^{1/n}$.

Proof of Corollary 2.11. The first bound follows from (2.18) and (2.22). To show the second one, use Ruelle's inequality and proceed as for [3, (2.27)]. □

We list some comments about the unconditional result Theorem 2.10.

[12]The term $D^b(T, \{\log \nu^s_n\})/p$ in (2.23) can be replaced by $P^*_{\mathrm{top}}(\{\log \nu^s_n\}, \partial O)/p$,applying Lemma 2.9 to $f_n = \nu^s_n$.

Remark 2.12 (The case $d = 1$)**.** *If $d = 1$, then $D^b(T) = 0$, and $|\det DT^n| = |DT^n| = \nu_n^{-1}$ so that (2.24) is*

$$\le e^{\frac{p-1}{p} P^*_{\mathrm{top}}(0)} \lim_{n\to\infty} \left\| g^{(n)} |DT^n|^{\frac{1}{p}-t} \right\|^{1/n} . \tag{2.27}$$

Letting $p \to 1$, $t \to 1$ in (2.27) (or in (2.9)), we recover the (sometimes optimal [34]) bound $\lim_n \sup |g^{(n)}|^{1/n}$ from [12] for the essential spectral radius on BV.

Remark 2.13 (Spectral Gap if $g = |\det DT|^{-1}$)**.** *For $g = |\det DT|^{-1}$ the dual of $\mathcal{L}_g$ fixes Lebesgue measure, and Lebesgue measure belongs to the dual of any $\mathcal{H}^t_p(M)$ with $t \ge 0$. In addition, the norm of $\|\mathcal{L}_g\|_{L_1(M)} \le 1$, with $\mathcal{H}^t_p(M) \subset L_1(M)$. Hence, if $r_{ess}(\mathcal{L}_{|\det DT|^{-1}}|_{\mathcal{H}^t_p}) < 1$, for some $0 < t < 1/p$, then the spectral radius of $\mathcal{L}_g$ on $\mathcal{H}^t_p$ is equal to one and standard arguments imply that*[13] *T has finitely many ergodic absolutely continuous invariant probability measures, with densities in $\mathcal{H}^t_p(M)$, the union of whose ergodic basins has full measure (see e.g.* [10, Thm 33] *or* [35, Thm 1]*). Each of these measures is exponentially mixing (up to a finite period) for $\mathcal{C}^v$ Hölder observables if $v > t$. Taking t (and thus p) close enough to 1, our bound* (2.24) *for $r_{ess}(\mathcal{L}_{|\det DT|^{-1}}|_{\mathcal{H}^t_p})$ is strictly smaller than one if $D^b(T) = 0$. More generally, if $\sigma = \exp(D^b(T, \{\nu^s_n\})) < 1$ for some $s \le t$ then* (2.23) *is bounded by σ for t and p close enough to 1, which is comparable to Liverani's bound from* [35, (3), Lemma 3.1].

Remark 2.14 (Spectral Gap if $g \ge 0$)**.** *The spectral radius of $\mathcal{L}_{T,g}$ on L_∞ is $\le P^*_{\mathrm{top}}(\log|g|)$ (similarly as for* (2.12)*). Although $\mathcal{H}^t_p$ is not included in L_∞ if $t < 1/p$, we conjecture, in view of* [23, Thm 1.2]*, that, if $g \ge 0$ and there exist $t < \min\{1/p, \alpha\}$ such that $r_{ess}(\mathcal{L}_g|_{\mathcal{H}^t_p}) < P^*_{\mathrm{top}}(\log g) = R^{0,\infty}_*(g)$, then the spectral radius of $\mathcal{L}_{T,g}$ on $\mathcal{H}^t_p$ is $P^*_{\mathrm{top}}(\log g)$. In particular, combining the maximal eigenvectors of $\mathcal{L}_g$ and its dual should then give another construction for the equilibrium states of Buzzi–Sarig* [23]*, with the additional perk of exponential decay of correlations for suitable observables.*

Remark 2.15 (Comparing (2.22) with Thomine's bound (2.9))**.** *Our bound* (2.24) *is less than or equal to* (2.9) *as soon as $D^e(T) \ge P^*_{\mathrm{top}}(0)$. By Lemma 2.8, this holds under the small boundary entropy condition $P^*_{\mathrm{top}}(T, 0, \partial\mathcal{O}) < P^*_{\mathrm{top}}(T, 0)$. For T a multidimensional β-transformation (i.e. a piecewise affine T, with DT constant) on $[0,1]^d$, it is known* [16, Thm 1, Lemma 1] *that $D^b(T) = 0$ and $P^*_{\mathrm{top}}(0) = h_{top} = \sum_{i=1}^d \xi_i = \log|\det DT|$, for $0 < \xi_1 \le \cdots \le \xi_d$ the Lyapunov exponents of T. By* [20, Thm 1]*, we have $D^e(T) = \lim_{n\to\infty} \max \mathcal{L}^n_{T,1}(1) \ge \exp(\sum_{i=1}^d \xi_i)$. Thus, we recover Thomine's bound* (2.9) *since* (2.24) *gives*

$$r_{ess}(\mathcal{L}_{|\det DT|^{-1}}|_{\mathcal{H}^t_p}) \le e^{-t\xi_1 + \frac{p-1}{p}\sum_{i=1}^d \xi_i + \frac{1-p}{p}\sum_{i=1}^d \xi_i} = e^{-t\xi_1} .$$

[13] See also the comment after (2.18) for the existence of equilibrium states for general $g > 0$ under small boundary pressure.

Our bound (2.22) *can be strictly smaller than* (2.9) *(we expect that this holds generically if $D^e(T) = P^*_{\rm top}(0)$): Take $d = 1$ and*[14] *T continuous, with $g = |\det DT|^{-1}$. By the variational principle* [23, Thm 3.1] *for $-\log|\det DT|$ (using* (2.14) *with $D^b(T) = 0$), there exists μ_{SRB} such that*[15]

$$\begin{aligned}\frac{p-1}{p} P^*_{\rm top}(\frac{p}{p-1}\log g + \frac{1}{p-1}\log|\det DT|) \\ &= \frac{p-1}{p}\big(h_{\mu_{SRB}} - \int \log|\det DT| d\mu_{SRB}\big) \\ &\le \frac{p-1}{p}\big(P^*_{\rm top}(0) - \int \log|\det DT| d\mu_{SRB}\big) .\end{aligned}$$

If T has a fixed point with Lyapunov exponent $< \int \log|\det DT| d\mu_{SRB}$ then

$$\int \log|\det DT|^{-1+1/p} d\mu_{SRB} < \lim_n \frac{1}{n}\log\sup|\det DT^n|^{-1+1/p} .$$

Remark 2.16 (Comparing (2.22) with Gundlach–Latushkin)**.** *If T is $\mathcal{C}^{\bar\alpha}$ and g is $\mathcal{C}^\alpha$ on M, the condition $t < 1/p$ can be lifted, and we get the optimal bound*

$$\exp \sup_{\mu\in{\rm Erg}(T)} \left(h_\mu(T) + \int \log|g| {\rm d}\mu - t\chi_\mu(DT) \right)$$

from [31] *for $\mathcal{B} = \mathcal{C}^\alpha$ by*[16] *letting $p \to \infty$ in* (2.22)*.*

2.4. **Variational Principle (Theorem 2.17 and Corollary 2.18).** Our second main result (proved in Section 4) is a variational principle for certain subadditive potentials, generalising (2.18):

Theorem 2.17 (Variational Principle)**.** *Let T be piecewise $\mathcal{C}^{\bar\alpha}$ expanding, let*[17] *$G : M \to \mathbb{C}$ be piecewise $\mathcal{C}^\alpha$, and let $t \ge 0$. If the small boundary pressure condition* (2.17) *holds for $f_n = f_{n,t} = |G^{(n)}| \cdot \nu_n^t$ (recall* (2.3)*), then*

$$\sup_{\mu\in{\rm Erg}(T)} \big\{ h_\mu(T) + \int \log|G| {\rm d}\mu - t\chi_\mu(DT) \big\} = P^*_{\rm top}(\{\log f_n\}) .$$

In addition, the supremum above is attained.

Define

$$f_{n,t,p} = |g^{(n)}| \cdot |\det DT^n|^{\frac{1}{p}} \cdot \nu_n^t \, , \, t \ge 0 \, , \, p \ge 1 \, , \; n \ge 1 \, . \tag{2.28}$$

Theorem 2.17 and the proof of Theorem 2.10 allow us to show the following corollary in §3.4 see the proof of Corollary 2.11 for the last claim):

[14] Then $P^*_{\rm top}(0) = D^e(T)$. Moreover, $P^*_{\rm top}(0) = h_{\rm top}(T) > 0$: Taking the O_i to be maximal monotonicity intervals, $\#\{\mathbf{i} \in I^n \mid O_{\mathbf{i}} \ne \emptyset\}$ is the lap-number lap_n of T, and $\lim_{n\to\infty} n^{-1}\log lap_n = h_{\rm top}(T)$.

[15] Note that the left hand-side is equal to zero.

[16] In the smooth case $D^b(T) = 0$, and the variational principle (2.18) for the subadditive potential $\log\big(|g^{(n)}| \cdot |\det DT^n|^{\frac{1}{p}} \cdot \nu_n^t\big)$ holds [3, App. B].

[17] Our application is $G = (g \cdot |\det DT|^{\frac{1}{p}})^q$.

Corollary 2.18 (Variational Expression for the Essential Spectral Radius). *Let T, g, p, t, be as in Theorem 2.10. Assume that the Ruelle inequality* (2.25) *holds. If T satisfies the small boundary pressure condition* (2.17) *for $f_n = f^q_{n,t,p}$ with $q \in [1, \frac{p}{p-1}]$, and $0 \le t < \min\{1/p, \alpha\}$, then the spectral radius of $\mathcal{L}_g$ on $\mathcal{H}^0_p = L_p$ is bounded by $R^{0,p,q}(g)$, the essential spectral radius of $\mathcal{L}_g$ on $\mathcal{H}^t_p$ is bounded by*

$$\begin{aligned} R^{t,p,q}_*(g) = \exp\Big(&\frac{D^b(T)}{p} \\ &+ \sup_{\mu \in \mathrm{Erg}(T)} \big\{ \frac{h_\mu(T)}{q} + \int \log(|g| |\det DT|^{\frac{1}{p}}) \mathrm{d}\mu - t\chi_\mu(DT) \big\} \Big) . \end{aligned} \tag{2.29}$$

In addition, we have

$$R^{t,p,q}_*(g) \le \exp\big(\frac{D^b(T)}{p}\big) \cdot \lim_{n\to\infty} \big\| g^{(n)} |\det DT^{(n)}|^{1/p+1/q} \, \nu^t_n \big\|^{1/n}_{L_\infty} . \tag{2.30}$$

If $D^b(T) = 0$ and (2.17) holds for $f^q_{n,t,p}$ with $q = p/(p-1)$, the bound (2.29) reduces to the bound in [3, Thm 2.15] for smooth expanding maps.

Note that $1/p + 1/q \in [1, 1 + 1/p]$. If $q = p/(p-1)$ then $1/p + 1/q = 1$, and, comparing (2.30) to Thomine's bound (2.9), we see that the complexity at the end has disappeared, at the cost of a higher weight on $|\det DT|$.

Remark 2.19 (Small Boundary Pressure Condition (2.17)). *The bound* (2.17) *for $f^q_{n,t,p}$ holds*[18] *if*

$$\begin{aligned} q\big(&\log[\sup(|g| |\det DT|^{1/p} \lim_n (\sup \nu^t_n)^{1/n})] - \log\big[\inf(|g| |\det DT|^{1/p}) \lim_n (\inf \nu^t_n)^{1/n}\big]\big) \\ &< P^*_{\mathrm{top}}(0) - P^*_{\mathrm{top}}(0, \partial\mathcal{O}) . \end{aligned} \tag{2.31}$$

Indeed, by (2.11), *we find*

$$e^{\frac{P^*_{\mathrm{top}}(\{q \log f_{n,t,p}\})}{q}} \ge e^{\frac{P^*_{\mathrm{top}}(0)}{q}} \lim_{n\to\infty} \inf[|g^{(n)}| \cdot |\det DT^n|^{\frac{1}{p}} \nu^t_n]^{1/n} ,$$

while (2.11) *gives*

$$e^{\frac{P^*_{\mathrm{top}}(\{q \log f_{n,t,p}\}, \partial\mathcal{O})}{q}} \le e^{\frac{P^*_{\mathrm{top}}(0, \partial\mathcal{O})}{q}} \lim_{n\to\infty} \sup[|g^{(n)}| \cdot |\det DT^n|^{\frac{1}{p}} \nu^t_n]^{1/n} .$$

In the piecewise affine case, (2.16) *gives $P^*_{\mathrm{top}}(0, \partial\mathcal{O}) \le D^b(T) + \log \Lambda_{d-1}$ (recall* (2.15)*). If $d \ge 2$,* (2.31) *holds for β-transformations T and $g = |\det DT|^{-1}$ with $0 < t < 1/p$, for arbitrary $q \ge 1$. Indeed, $D^b(T) = 0$ and $P^*_{\mathrm{top}}(0) = \sum_{i=1}^d \xi_i \equiv \log|\det DT|$, with ν_n a constant function, and $\lim_n \nu_n^{1/n} = \exp(-\xi_1)$, so that* (2.31) *reads $\sum_{i=2}^d \xi_i < \sum_{i=1}^d \xi_i$.*

---

[18]Cf. the condition $\sup \log|g| - \inf \log|g| < P^*(0)$ from the pioneering work [33].

3. Proof of Theorem 2.10 on the Essential Spectral Radius

3.1. **Sobolev Spaces.** For $p \in (1, \infty)$ and $t \in \mathbb{R}$, define local Sobolev spaces by

$$H^t_p = H^t_p(\mathbb{R}^d) = \{u \in L_p(\mathbb{R}^d) \mid \|u\|_{H^t_p(\mathbb{R}^d)} = \|\mathbb{F}^{-1}((1 + |\xi|^2)^{t/2} \cdot \mathbb{F}u)\|_{L_p(\mathbb{R}^d)} < \infty\} ,$$

where $\mathbb{F}$ is the Fourier transform. If $t \geq 0$ then $H^t_p(\mathbb{R}^d)$ is the closure of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions for the norm $\|u\|_{H^t_p(\mathbb{R}^d)}$, see e.g. [43, Thm 3.2/2, Rk 3.2/2].

Recall $\widetilde{M}$ from the beginning of §2.1. To patch the local spaces together, we will use charts and partitions of unity:

Definition 3.1 (Admissible Charts and Partition of Unity for T)**.** *For a piecewise $\mathcal{C}^{\bar{\alpha}}$ expanding map T, we define admissible charts to be a finite system of $\mathcal{C}^\infty$ local charts $\{(V_\omega, \kappa_\omega)\}_{\omega \in \Omega}$, where each V_ω is open in $\widetilde{M}$ and such that $\overline{V}_\omega \subset \widetilde{O}_{i(\omega)}$, for some $i(\omega) \in I$, with $M \subset \cup_\omega V_\omega$, and where each $\kappa_\omega : V_\omega \to \mathbb{R}^d$ is a diffeomorphism onto its image. We define an admissible partition of unity to be a $\mathcal{C}^\infty$ partition of unity $\{\theta_\omega : \widetilde{M} \to [0, 1]\}_{\omega \in \Omega}$ such that the support of θ_ω is contained in V_ω.*

Note that if M has a boundary, the boundary cutoff will be performed through the characteristic functions of the O_i.

Definition 3.2 ($\mathcal{H}^t_p(M)$)**.** *Let T be piecewise $\mathcal{C}^{\bar{\alpha}}$ expanding and let κ_ω and θ_ω be as in Definition 3.1. For $1 < p < \infty$ and $t < 1/p$, let $\mathcal{H}^t_p = \mathcal{H}^t_p(M)$ be defined by*

$$\mathcal{H}^t_p(M) = \{\varphi \in L_p(M) \mid \|\varphi\|_{H^t_p(M)} = \sum_{\omega \in \Omega} \left\|(\theta_\omega \cdot \varphi) \circ \kappa_\omega^{-1}\right\|_{H^t_p(\mathbb{R}^d)} < \infty\}.$$

By [39], the definition above makes sense if M has a boundary (in that case $\kappa_\omega(\mathrm{supp}\,(\theta_\omega))$ has a boundary for some ω) since $t < 1/p$. Changing the system of charts or the partition of unity produces equivalent norms (see [40, I.5, I.6]). It is easy to see that $\mathcal{H}^t_p(M)$ is the closure of $\mathcal{C}^v(M)$ for the norm $\|\varphi\|_{H^t_p(M)}$ for $v > t$.

3.2. **Toolbox. Zoomed Norms.** We collect results used for the Lasota–Yorke inequality. We shall use the following localisation to zoom into smaller scales.

Lemma 3.3 (Localisation [44, Thm 2.4.7(ii)])**.** *Fix τ in the set $\mathcal{C}^\infty_0(\mathbb{R}^d, [0, 1])$ of compactly supported C^∞ functions from $\mathbb{R}^d$ to $[0, 1]$. For $x \in \mathbb{R}^d$ and $m \in \mathbb{Z}^d$, set $\tau_m(x) = \tau(x + m)$. For any $p \in (1, \infty)$ and $t \in \mathbb{R}^+$ there exists $C_{t,p,\tau} < \infty$ such that*

$$\left(\sum_{m \in \mathbb{Z}^d} \|\tau_m u\|^p_{H^t_p} \right)^{\frac{1}{p}} \leq C_{t,p,\tau} \|u\|_{H^t_p} \, , \ \forall u \in H^t_p(\mathbb{R}^d) \, . \tag{3.1}$$

In addition, if $\sum_{m \in \mathbb{Z}^d} \tau_m(x) = 1$ for all x, then there exists $C_{t,p,\tau} < \infty$ such that

$$\|u\|_{H^t_p} \leq C_{t,p,\tau} \left(\sum_{m \in \mathbb{Z}^d} \|\tau_m u\|^p_{H^t_p} \right)^{\frac{1}{p}} , \ \forall u \text{ such that: } \forall m \, , \ \tau_m u \in H^t_p(\mathbb{R}^d) \, . \tag{3.2}$$

For any measurable set O and every $p \in (1,\infty)$ we have $\|\mathbf{1}_O \varphi\|_{L_p} \le \|\varphi\|_{L_p}$. The next result is the reason behind the constraint $t < 1/p$:

Lemma 3.4 (Characteristic Functions as Bounded Multipliers [39, Cor. II.4.2])**.** *For any $-1 + 1/p < t < 1/p < 1$ there exists $C_{t,p}$ such that, for any $L \ge 1$, and every measurable set $O \subset \mathbb{R}^d$ whose intersection with almost every line parallel to some coordinate axis has at most L connected components, we have*

$$\|\mathbf{1}_O \varphi\|_{H^t_p} \le C_{t,p} L \|\varphi\|_{H^t_p}\,, \ \forall \varphi \in H^t_p(\mathbb{R}^d)\,.$$

We will be able to use Lemma 3.4 when composing with the iterate T^n of a piecewise expanding map in view of the following result.

Lemma 3.5 ([41, Lemma 5.1])**.** *Recall Definitions 2.1 and 3.1. Let L_0 be the maximal number of smooth boundary components of the O_i. For any $n \ge 1$, $\mathbf{i} \in I^n$, $x \in \overline{O}_{\mathbf{i}}$, and for any $\omega \in \Omega$ such that $x \in \operatorname{supp} \theta_\omega$, there exist a neighbourhood O' of x and an orthogonal matrix A such that the intersection of $A(\kappa_\omega(O' \cap O_{\mathbf{i}}))$ with almost any line parallel to a coordinate axis has at most $L_0 n$ components.*

The next result is crucial (for $F = \mathrm{id}$, it plays the part of a Leibniz bound):

Lemma 3.6 (Local Lasota–Yorke Bound [3, Lemma 2.21])**.** *Let $d \ge 1$. For each $0 \le t < \alpha$ there exists c_t such that for any $p \in (1,\infty)$, any open $U \subset \mathbb{R}^d$ any $F : U \to \mathbb{R}^d$ extending to a bilipschitz $\mathcal{C}^{\bar\alpha}$ diffeomorphism of $\mathbb{R}^d$ with*

$$\sup_{\mathbb{R}^d} |\det DF| \le 2 \sup_U |\det DF|\,, \tag{3.3}$$

and any $\mathcal{C}^\alpha$ function $f : \mathbb{R}^d \to \mathbb{C}$ supported in a compact set $K \subset U$, we have

$$\|f \cdot (\varphi \circ F)\|_{H^0_p} \le \sup_K |f| \sup_U |\det DF|^{-1/p} \|\varphi\|_{H^0_p}\,, \ \forall \varphi \in H^0_p(\mathbb{R}^d) = L_p(\mathbb{R}^d)\,,$$

and for any $t' < t$ there is $C_{t,t',p}(f,F)$ such that, for any $\varphi \in \mathcal{H}^t_p$ supported in K,

$$\|f \cdot (\varphi \circ F)\|_{H^t_p} \le c_t \sup_K |f| \sup_U \|DF\|^t \sup_U |\det DF|^{-1/p} \|\varphi\|_{H^t_p} + C_{t,t',p}(f,F) \|\varphi\|_{H^{t'}_p}\,.$$

The intersection multiplicity of a family of subsets of $\mathbb{R}^d$ is the maximal number of sets having nonempty intersection. The intersection multiplicity of a partition of unity of $\mathbb{R}^d$ is the intersection multiplicity of the family formed by taking the supports of the maps in the partition. With this terminology, we recall the *fragmentation-reconstitution* lemma, at the core of our local computations.

Lemma 3.7 (Fragmentation and Reconstitution [3, Lemmas 2.26 and 2.27])**.** *Let $1 < p < \infty$ and $t \ge 0$, and let $K \subset \mathbb{R}^d$ be compact. For any $t' \in \mathbb{Z}$ there exists a constant $C > 0$ such that, for any partition of unity $\{\theta_j\}_{j=1}^J$ of K with intersection multiplicity β, there exist finite constants $C_\theta, \tilde{C}_\theta > 0$ (depending on the θ_j only* *through their supports) such that*

(3.4)
$$\|\sum_{j=1}^{J}\theta_j w\|_{H_p^t} \le \beta^{\frac{p-1}{p}}(\sum_{j=1}^{J}\|\theta_j w\|^p_{H_p^t})^{1/p} + \tilde C_\theta \sum_{j=1}^{J}\|\theta_j w\|_{H_p^{t'}} \text{ (fragmentation)},$$

(3.5)
$$(\sum_{j=1}^{J}\|\theta_j w\|^p_{H_p^t})^{1/p} \le C\beta^{1/p}\sup_{1\le j\le J}\|\theta_j w\|_{H_p^t} + C_\theta \sum_{j=1}^{J}\|\theta_j w\|_{H_p^{t'}} \text{ (reconstitution)}.$$

In addition, if $t = 0$, then[19] *we may take $C_\theta = \tilde C_\theta = 0$.*

Finally, for $p \in (1,\infty)$ and $t \ge 0$, following Thomine [41], define a *zoomed norm* for any increasing sequence $r_n > 1$ (chosen in the proof of Proposition 3.8) by

$$(3.6)\quad R_n(x) = r_n \cdot x\,, \forall x \in \mathbb{R}^d\,,\ n \in \mathbb{Z}_+\,;\ \|\varphi\|_{r_n,t,p} = \sum_{\omega\in\Omega}\left\|(\theta_\omega\varphi)\circ\kappa_\omega^{-1}\circ R_n^{-1}\right\|_{H_p^t}\,.$$

The zoomed norm $\|\varphi\|_{r_n,t,p}$ defined above is equivalent to $\|\cdot\|_{\mathcal{H}_p^t}$. It is used for example when applying Lemma 3.5 and Lemma 3.6 below.

3.3. **A Global Lasota–Yorke Inequality.**

We will prove the Lasota–Yorke inequality by combining the zoomed norm with the fragmentation-reconstitution techniques from §3.2, to obtain a thermodynamic factor in front of the strong norm. Recall the sequence $f_{n,t,p} = |g^{(n)}|\cdot|\det DT^n|^{\frac{1}{p}}\cdot\nu_n^t$ from (2.28).

Proposition 3.8 (Lasota–Yorke Bound)**.** *Fix $p \in (1,\infty)$, and $0 < t < \min\{\frac{1}{p},\alpha\}$. Then there exists $C_{t,p}$ such that, for any T and g as in Theorem 2.10,*

$$\|\mathcal{L}_g^n\varphi\|_{L_p} \le C_{t,p}(D_n^b)^{\frac{1}{p}}\Big(\sum_{\mathbf{i}\in I^n}\sup_{O_{\mathbf{i}}} f_{n,0,p}^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}\cdot\|\varphi\|_{L_p}\,,\ \forall\varphi\in L_p(M)\,,\ \forall n\ge 1\,,$$

and, in addition, for each T, there exists an increasing[20] *sequence $\{r_n = r_n(T)\}$ such that, for each g and each $t' < t$, there exists $C_n = C_{n,t,t',p}(g)$ such that*

$$\|\mathcal{L}_g^n\varphi\|_{r_n,t,p} \le C_{t,p}n(D_n^b)^{\frac{1}{p}}\Big(\sum_{\mathbf{i}\in I^n}\sup_{O_{\mathbf{i}}} f_{n,t,p}^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}\cdot\|\varphi\|_{r_n,t,p}$$
$$(3.7)\qquad\qquad + C_n\|\varphi\|_{r_n,t',p}\,,\ \forall\varphi\in\mathcal{H}_p^t\,,\ \forall n\ge 1\,.$$

Proof. We prove (3.7). (The bound on $L_p(M)$ follows from a simplification of the argument for (3.7), using the L_p bound in Lemma 3.6 and the last claim of Lemma 3.7. In particular, the zoom is not needed.) For $n \ge 1$ and each $\mathbf{i} \in I^n$,

[19] This follows from the Hölder inequality. See the proof of [3, Lemmas 2.26 and 2.27].
[20] The sequence r_n is independent of t and p.

select a $\mathcal{C}^\infty$ function $\theta_{\mathbf{i}}$ on M such that $\operatorname{supp}\theta_{\mathbf{i}} \subset \widetilde{O}_{\mathbf{i}}$ and $\theta_{\mathbf{i}} \equiv 1$ on $O_{\mathbf{i}}$. Then we have[21]

$$\mathcal{L}_g^n \varphi(x) = \sum_{\mathbf{i}\in I^n} (\theta_{\mathbf{i}} \tilde g_{\mathbf{i}}^{(n)} \mathbf{1}_{O_{\mathbf{i}}} \varphi) \circ \widetilde{T}_{\mathbf{i}}^{-n}(x)\,, \text{ for Lebesgue a.e. } x \in M\,. \tag{3.8}$$

Recall Definition 3.1. For $\omega \in \Omega$, we have

$$\big(\theta_\omega \mathcal{L}_g^n \varphi\big) \circ \kappa_\omega^{-1} \circ R_n^{-1} = \sum_{\mathbf{i}\in I^n} \bigg(\theta_\omega \big[(\theta_{\mathbf{i}} \tilde g_{\mathbf{i}}^{(n)} \mathbf{1}_{O_{\mathbf{i}}} \varphi) \circ \widetilde{T}_{\mathbf{i}}^{-n}\big]\bigg) \circ \kappa_\omega^{-1} \circ R_n^{-1}\,. \tag{3.9}$$

Since $\varphi = \sum_{\omega'\in\Omega} \theta_{\omega'}\varphi$, the triangle inequality followed by the fragmentation bound (3.4) in Lemma 3.7 applied for fixed $\mathbf{i}$ to the partition of unity $\{\theta_{\omega'}\}$ gives constants C (depending on the intersection multiplicity) and $\tilde C_{\{\theta_{\omega'}\}}$ such that

$$\begin{aligned} &\| \big(\theta_\omega \mathcal{L}_g^n \varphi\big) \circ \kappa_\omega^{-1} \circ R_n^{-1} \|_{H_p^t} \\ &\le C \sum_{\mathbf{i}\in I^n} \bigg(\sum_{\omega'} \| \big(\theta_\omega \big[(\theta_{\mathbf{i}} \tilde g_{\mathbf{i}}^{(n)} \mathbf{1}_{O_{\mathbf{i}}} \theta_{\omega'} \varphi) \circ \widetilde{T}_{\mathbf{i}}^{-n}\big]\big) \circ \kappa_\omega^{-1} \circ R_n^{-1} \|_{H_p^t}^p \bigg)^{1/p} \\ &\quad + \tilde C_{\{\theta_{\omega'}\}} \sum_{\mathbf{i}\in I^n} \sum_{\omega'} \| \big(\theta_\omega \big[(\theta_{\mathbf{i}} \tilde g_{\mathbf{i}}^{(n)} \mathbf{1}_{O_{\mathbf{i}}} \theta_{\omega'} \varphi) \circ \widetilde{T}_{\mathbf{i}}^{-n}\big]\big) \circ \kappa_\omega^{-1} \circ R_n^{-1} \, \|_{H_p^{t'}}\,. \end{aligned} \tag{3.10}$$

We focus on the first double sum in the right hand-side (the second is similar). For τ_m such that $\sum_m \tau_m \equiv 1$ as in the localisation Lemma 3.3, set

$$\varphi_{\omega'}^{m,n} = (\tau_m \circ R_n \circ \kappa_{\omega'}) \cdot (\theta_{\omega'} \varphi)\ , \quad m \in \mathbb{Z}^d\,, \ \text{ so that } \theta_{\omega'}\varphi = \sum_{m\in\mathbb{Z}^d} \varphi_{\omega'}^{m,n}\,. \tag{3.11}$$

Since $\theta_{\omega'}$ is compactly supported, only a finite number of terms in the above sum are nonzero. In addition, the functions $(\tau_m \circ R_n \circ \kappa_{\omega'})_{m\in\mathbb{Z}^d}$ have finite intersection multiplicity β. Thus, for each $\mathbf{i}$ and ω', the fragmentation bound (3.4) in Lemma 3.7 (applied to the partition of unity τ_m) estimates the pth power of the term for $(\mathbf{i}, \omega')$ in (3.10) by (using $(|a| + |a'|)^p < 2^{p-1}(|a|^p + |a'|^p)$)

$$\begin{aligned} &C_{t,p} \beta^{(p-1)/p} \sum_{m\in\mathbb{Z}^d} \| \bigg(\theta_\omega [(\theta_{\mathbf{i}} \tilde g_{\mathbf{i}}^{(n)} \mathbf{1}_{O_{\mathbf{i}}} \varphi_{\omega'}^{m,n}) \circ \widetilde{T}_{\mathbf{i}}^{-n}] \bigg) \circ \kappa_\omega^{-1} \circ R_n^{-1} \|_{H_p^t}^p \\ &\quad + \tilde C_{t,p,\tau\circ R_m} \sum_{m\in\mathbb{Z}^d} \| \bigg(\theta_\omega [(\theta_{\mathbf{i}} \tilde g_{\mathbf{i}}^{(n)} \mathbf{1}_{O_{\mathbf{i}}} \varphi_{\omega'}^{m,n}) \circ \widetilde{T}_{\mathbf{i}}^{-n}] \bigg) \circ \kappa_\omega^{-1} \circ R_n^{-1} \|_{H_p^{t'}}^p\,. \end{aligned} \tag{3.12}$$

We focus on the first term above. The set of indices $\mathbf{i} \in I^n$ for which the term in (3.12) corresponding to a fixed pair (ω', m) is non zero is contained in the set

$$J^n = J^n(\omega', m) = \{\mathbf{i} \in I^n : O_{\mathbf{i}} \cap \operatorname{supp} \varphi_{\omega'}^{m,n} \ne \emptyset\}. \tag{3.13}$$

Since R_n expands by a factor r_n, while the size of the supports of the functions τ_m is uniformly bounded, taking r_n large enough, we can guarantee that $\operatorname{supp}\varphi_{\omega'}^{m,n}$ is small enough such that $\#J^n(\omega', m) \le D_n^b$ for each ω' and m.

---

[21]Since $\mathrm{Leb}(\partial\mathcal{O}) = 0$ and $\mathcal{L}_g$ acts on a subset of $L_p(M)$, the formula (3.8) is legitimate.

For ω, ω', and $\mathbf{i} \in J^n$ such that $\widetilde{T}_{\mathbf{i}}^{-n}(V_\omega) \cap (V_{\omega'} \cap O_{\mathbf{i}}) \neq \emptyset$, setting

$$\varphi_{\omega' \mathbf{i}}^{m,n} = \tau_m \cdot \big([\mathbf{1}_{O_{\mathbf{i}}} \cdot (\theta_{\omega'} \varphi)] \circ \kappa_{\omega'}^{-1} \circ R_n^{-1}\big)$$
$$F = R_n \circ \kappa_{\omega'} \circ \widetilde{T}_{\mathbf{i}}^{-n} \circ \kappa_\omega^{-1} \circ R_n^{-1}\,, \;\; f = \tilde{\tau}_m \cdot \big(\big(\theta_\omega [\tilde{\theta}_{\omega'} \theta_{\mathbf{i}} \tilde{g}_{\mathbf{i}}^{(n)}] \circ \widetilde{T}_{\mathbf{i}}^{-n}\big) \circ \kappa_\omega^{-1} \circ R_n^{-1}\big)\,,$$

for $\mathcal{C}^\infty$ functions $\tilde{\theta}_{\omega'} : \widetilde{M} \to [0,1]$, $\tilde{\tau}_m : \mathbb{R}^d \to [0,1]$ with

$$\theta_{\omega'}(x) = \tilde{\theta}_{\omega'}(x)\theta_{\omega'}(x)\,,\ \forall x \in \widetilde{M}\,, \quad \tilde{\tau}_m(u)\tau_m(u) = \tau_m(u)\,,\ \forall u \in \mathbb{R}^d\,,$$

we have

$$(\theta_\omega[(\theta_{\mathbf{i}} \tilde{g}_{\mathbf{i}}^{(n)} \mathbf{1}_{O_{\mathbf{i}}} \varphi_{\omega'}^{m,n}) \circ \widetilde{T}_{\mathbf{i}}^{-n}]) \circ \kappa_\omega^{-1} \circ R_n^{-1} = f \cdot (\varphi_{\omega' \mathbf{i}}^{m,n} \circ F)\,. \tag{3.14}$$

Increasing r_n if needed (and choosing $\tilde{\tau}_m$ with a small enough support), the map $F = F(n, \mathbf{i}, \omega, \omega')$ satisfies the condition (3.3) of Lemma 3.6, if we take for U the intersection of $R_n(\kappa_\omega(\widetilde{T}_{\mathbf{i}}^n(\widetilde{O}_{\mathbf{i}} \cap V_{\omega'})))$ with the interior of the support of $\tilde{\tau}_m$. Then, Lemma 3.6 applied to the map F, the weight $f = f(n, m, \mathbf{i}, \omega, \omega')$, which is $\mathcal{C}^\alpha$ and supported on U, and the test function $\varphi_{\omega' \mathbf{i}}^{m,n}$ gives constants C_n (depending on r_n) and C_t such that, recalling $\tilde{\nu}_{n,\mathbf{i}}^t$ from (2.4), and, setting $\Theta_{\mathbf{i}} = \operatorname{supp} \theta_{\mathbf{i}}$,

$$\begin{aligned} &\|f \cdot (\varphi_{\omega' \mathbf{i}}^{m,n} \circ F)\|_{H_p^t} \\ &\le C_t \sup_{\Theta_{\mathbf{i}}} |\tilde{g}_{\mathbf{i}}^{(n)}| \sup_{\Theta_{\mathbf{i}}} |\det D\widetilde{T}_{\mathbf{i}}^n|^{\frac{1}{p}} \sup_{\Theta_{\mathbf{i}}} \tilde{\nu}_{n,\mathbf{i}}^t \|\varphi_{\omega' \mathbf{i}}^{m,n}\|_{H_p^t} + C_n \|\varphi_{\omega' \mathbf{i}}^{m,n}\|_{H_p^{t'}}\,. \end{aligned} \tag{3.15}$$

Next, using bounded distortion for uniformly expanding maps (see e.g. [36, III.1]), there exists $C > 0$ such that for all n and $\mathbf{i}$,

$$\sup_{\Theta_{\mathbf{i}}}(|\tilde{g}_{\mathbf{i}}^{(n)}|) \sup_{\Theta_{\mathbf{i}}}(|\det D\widetilde{T}_{\mathbf{i}}^n|^{1/p}) \le C \inf_{\Theta_{\mathbf{i}}}(|\tilde{g}_{\mathbf{i}}^{(n)}\| \det D\widetilde{T}_{\mathbf{i}}^n|^{1/p})\,.$$

Finally, using $(\inf a) \cdot (\sup b) \le \sup(a \cdot b)$, we have

$$\sup_{\Theta_{\mathbf{i}}} |\tilde{g}_{\mathbf{i}}^{(n)}| \sup_{\Theta_{\mathbf{i}}} |\det D\widetilde{T}^n|^{\frac{1}{p}} \sup_{\Theta_{\mathbf{i}}} \tilde{\nu}_{n,\mathbf{i}}^t \le C \sup_{\Theta_{\mathbf{i}}}(|\tilde{g}_{\mathbf{i}}^{(n)}|| \det D\widetilde{T}_{\mathbf{i}}^n|^{\frac{1}{p}} \tilde{\nu}_{n,\mathbf{i}}^t)\,. \tag{3.16}$$

Setting $\Sigma_{\mathbf{i}} = \{(\omega', m) \mid \mathbf{i} \in J^n(\omega', m)\}$ and

$$\Xi_{n,\mathbf{i}} = \sup_{\Theta_{\mathbf{i}}}[|\tilde{g}_{\mathbf{i}}^{(n)}|| \det D\widetilde{T}_{\mathbf{i}}^n|^{\frac{1}{p}} \tilde{\nu}_{n,\mathbf{i}}^t]\,, \tag{3.17}$$

by (3.15), (3.14), (3.16), and the Minkowski inequality, we have

$$\begin{aligned} &\sum_{\mathbf{i} \in I^n} \Big(\sum_{(\omega', m) \in \Sigma_{\mathbf{i}}} \|(\theta_{\mathbf{i}} \tilde{g}_{\mathbf{i}}^{(n)}) \circ \widetilde{T}_{\mathbf{i}}^{-n} \circ \kappa_\omega^{-1} \circ R_n^{-1} \cdot \varphi_{\omega' \mathbf{i}}^{m,n} \circ F\|_{H_p^t}^p \Big)^{\frac{1}{p}} \\ &\le C_{t,p} \sum_{\mathbf{i}} \Xi_{n,\mathbf{i}} \Big(\sum_{(\omega', m) \in \Sigma_{\mathbf{i}}} \|\varphi_{\omega' \mathbf{i}}^{m,n}\|_{H_p^t}^p \Big)^{\frac{1}{p}} + C_n \sum_{\mathbf{i}} \Big(\sum_{(\omega', m) \in \Sigma_{\mathbf{i}}} \|\varphi_{\omega' \mathbf{i}}^{m,n}\|_{H_p^{t'}}^p \Big)^{\frac{1}{p}}\,. \end{aligned} \tag{3.18}$$

Let us estimate the first term above. For all $q \geq 1$, the[22] Hölder inequality gives

$$\sum_{\mathbf{i} \in I^n} \Xi_{n,\mathbf{i}} \Big(\sum_{(\omega',m) \in \Sigma_{\mathbf{i}}} \|\varphi^{m,n}_{\omega' \mathbf{i}}\|^p_{H^t_p} \Big)^{\frac{1}{p}}$$

$$\leq \Big(\sum_{\mathbf{i}} \Xi^q_{n,\mathbf{i}} \Big)^{\frac{1}{q}} \Big(\sum_{\mathbf{i}} \Big(\sum_{(\omega',m) \in \Sigma_{\mathbf{i}}} \|\varphi^{m,n}_{\omega' \mathbf{i}}\|^p_{H^t_p} \Big)^{\frac{q}{(q-1)p}} \Big)^{\frac{q-1}{q}} . \tag{3.19}$$

To estimate $\|\varphi^{m,n}_{\omega' \mathbf{i}}\|_{H^t_p}$, we argue as in [41] to get rid of the characteristic functions $\mathbf{1}_{O_{\mathbf{i}}}$. On the one hand, if the support of $\varphi^{n,m}_{\omega}$ is small enough, which is guaranteed if r_n is large enough, then Lemma 3.5 provides a neighbourhood O' of this support and a matrix A such that the intersection of $R_n(A(\kappa_{\omega'}(O' \cap O_{\mathbf{i}})))$ with almost any line parallel to a coordinate axis has at most $L_0 n$ connected components. Hence, since[23] $t < 1/p$, Lemma 3.4 applied to multiplication by $\mathbf{1}_{O' \cap O_{\mathbf{i}}} \circ \kappa^{-1}_{\omega'} \circ A^{-1} \circ R^{-1}_n$, using that A is orthogonal and commutes with R_n, implies, for all $\mathbf{i} \in J^n$,

$$\|(\mathbf{1}_{O_{\mathbf{i}}} \circ \kappa^{-1}_{\omega'} \circ R^{-1}_n) \cdot v\|_{H^t_p} \leq C_{t,p} L_0 n \|v\|_{H^t_p} \, , \forall v \, . \tag{3.20}$$

Thus, we obtain

$$\|\varphi^{m,n}_{\omega' \mathbf{i}}\|_{H^t_p} \leq C_{t,p} L_0 n \|\tau_m \cdot (\theta_{\omega'} \varphi) \circ \kappa^{-1}_{\omega'} \circ R^{-1}_n\|_{H^t_p} \, , \forall \mathbf{i} \in J^n \, . \tag{3.21}$$

We can now estimate the second factor in the right hand-side of (3.19). Using (3.21) and (3.1) from the localisation Lemma 3.3, we have, for any $q \in [1, \frac{p}{p-1}]$,

$$\Bigg(\sum_{\mathbf{i} \in I^n} \Bigg(\sum_{(\omega',m) \in \Sigma_{\mathbf{i}}} \|\varphi^{m,n}_{\omega' \mathbf{i}}\|^p_{H^t_p} \Bigg)^{\frac{q}{(q-1)p}} \Bigg)^{\frac{q-1}{q}} \tag{3.22}$$

$$\leq C_{t,p} n \Bigg(\sum_{\mathbf{i}} \Big(\sum_{(\omega',m) \in \Sigma_{\mathbf{i}}} \|\tau_m \cdot (\theta_{\omega'} \varphi) \circ \kappa^{-1}_{\omega'} \circ R^{-1}_n\|^p_{H^t_p} \Big)^{\frac{q}{(q-1)p}} \Bigg)^{\frac{q-1}{q}}$$

$$\leq C_{t,p} n \Bigg(\Big(\sum_{\omega',m} D^b_n \cdot \sup_{\mathbf{i} \in J^n(\omega',m)} \|\tau_m \cdot (\theta_{\omega'} \varphi) \circ \kappa^{-1}_{\omega'} \circ R^{-1}_n\|^p_{H^t_p} \Big)^{\frac{q}{(q-1)p}} \Bigg)^{\frac{q-1}{q}}$$

$$\leq C_{t,p} n (D^b_n)^{1/p} \Big(\sum_{\omega'} \|(\theta_{\omega'} \varphi) \circ \kappa^{-1}_{\omega'} \circ R^{-1}_n\|^p_{H^t_p} \Big)^{\frac{1}{p}} , \tag{3.23}$$

exchanging the sums over $\mathbf{i}$ and (ω', m) (which is legitimate since $\frac{q}{(q-1)p} \geq 1$) and using $\# J^n(\omega', m) \leq D^b_n$ for all m and ω' in the penultimate line. Combining

[22]We use the standard notation $(\sum_i |a_i|^{q'})^{1/q'} = \sup_i |a_i|$ if $q' = \infty$.

[23]This is the only place where this assumption is used.

(3.23), (3.19), and (3.18) with (3.10), we obtain, for any $q \in [1, \frac{p}{p-1}]$,

$$\| \big(\theta_\omega \mathcal{L}_g^n \varphi\big) \circ \kappa_\omega^{-1} \circ R_n^{-1} \|_{H_p^t} \le C_n \sum_{\omega'} \|(\theta_{\omega'}\varphi) \circ \kappa_{\omega'}^{-1} \circ R_n^{-1}\|_{H_p^{t'}}$$
$$+ C_{t,p} n (D_n^b)^{1/p} \big(\sum_{\mathbf{i} \in I^n} \Xi_{n,\mathbf{i}}^q \big)^{\frac{1}{q}} \big(\sum_{\omega'} \|(\theta_{\omega'}\varphi) \circ \kappa_{\omega'}^{-1} \circ R_n^{-1}\|_{H_p^t}^p \big)^{\frac{1}{p}} .$$

Hence, the reconstitution bound (3.5) in Lemma 3.7 applied to the partition of unity $\theta_{\omega'}$ gives

$$\| \big(\theta_\omega \mathcal{L}_g^n \varphi\big) \circ \kappa_\omega^{-1} \circ R_n^{-1} \|_{H_p^t} \le C_n \sup_{\omega' \in \Omega} \|(\theta_{\omega'}\varphi) \circ \kappa_{\omega'}^{-1} \circ R_n^{-1}\|_{H_p^{t'}}$$
$$+ C_{t,p} n (D_n^b)^{1/p} \big(\sum_{\mathbf{i} \in I^n} \Xi_{n,\mathbf{i}}^q \big)^{\frac{1}{q}} \sum_{\omega' \in \Omega} \|(\theta_{\omega'}\varphi) \circ \kappa_{\omega'}^{-1} \circ R_n^{-1}\|_{H_p^t} . \tag{3.24}$$

Since Ω is finite and the map $q \mapsto \|v\|_{\ell^q}$ is strictly decreasing on $[1, \infty)$ for any fixed $v \in \mathbb{R}^D$, this implies (3.7) by (3.17) and the definition (3.6) of $\| \cdot \|_{r_n,t,p}$. (Replacing $\Theta_{\mathbf{i}}$ by $O_{\mathbf{i}}$, $\tilde g_{\mathbf{i}}^{(n)}$ by $g^{(n)}$, $\det D\widetilde{T}_{\mathbf{i}}^n$ by $\det DT^n$, and $\tilde\nu_{n,\mathbf{i}}$ by ν_n in (3.17) costs a factor $(1+\epsilon)^n$, for arbitrarily small $\epsilon > 0$, up to taking $\theta_{\mathbf{i}}$ with small enough support.) □

3.4. **Proof of Theorem 2.10 and Corollary 2.18.**

Proof of Theorem 2.10. The claim on the spectral radius on $L_p = \mathcal{H}_p^0$ is an obvious consequence on the L_p bound in Proposition 3.8.

For the bound on the essential spectral radius, since each norm $\| \cdot \|_{r_{n_0},t,p}$ is equivalent to $\| \cdot \|_{\mathcal{H}_p^t}$, and since the inclusion of $\mathcal{H}_p^t$ in $L_p(M)$ is compact for $t > 0$, the Lasota–Yorke bound in Proposition 3.8 implies, by a result of Hennion [32, Cor. 1], that

$$r_{\mathrm{ess}}(\mathcal{L}_g|_{\mathcal{H}_p^t}) \le \rho = e^{\frac{D^b(T)}{p}} \lim_{n \to \infty} \Bigg(\sum_{\mathbf{i} \in I^n} \sup_{O_{\mathbf{i}}} \big(|g^{(n)}| |\det DT^n|^{\frac{1}{p}} \nu_n^t \big)^{\frac{p}{p-1}} \Bigg)^{\frac{p-1}{pn}} . \tag{3.25}$$

(Indeed, fix $n_0 \ge 1$ very large, so that the limit in (3.25) is almost attained, decompose $n = \ell n_0 + n'$, with $\ell \ge 0$ and $0 \le n' < n_0$, and apply Proposition 3.8 inductively to bound $\mathcal{L}_g^n$ for the norm $\| \cdot \|_{r_{n_0},t,p}$, which is equivalent to the norm of $\mathcal{H}_p^t(M)$.) Then (3.25) implies the bound (2.22) for the essential spectral radius, by definition of P_{top}^*.

To show (2.23), we modify the proof of Proposition 3.8 as follows: In the definition (3.17) of $\Xi_{n,\mathbf{i}}$, we replace t by $t - s$. In (3.18), (3.19), and (3.22), we replace $\varphi_{\omega',\mathbf{i}}^{m,n}$ by $\sup_{\Theta_{\mathbf{i}}} \tilde\nu_{n,\mathbf{i}}^s \varphi_{\omega',\mathbf{i}}^{m,n}$. In the line after (3.22), we insert $(\sup_{\Theta_{\mathbf{i}}} \tilde\nu_{n,\mathbf{i}}^s)^p$ before the factor $\|\tau_m \cdot ...\|_{H_p^t}^p$. In (3.23) and the line above it, we replace D_n^b by $(1+\epsilon)^n D_n^b(\{\nu_n^s\})$ (with ϵ as in the end of the proof of Proposition 3.8). □

Proof of Corollary 2.18. Since (3.24) holds for any $q \in [1, \frac{p}{p-1}]$, the conclusion of Proposition 3.8 holds replacing $\big(\sum_{\mathbf{i}\in I^n} \sup_{O_{\mathbf{i}}} f_{n,t,p}^{\frac{p}{p-1}}\big)^{\frac{p-1}{p}}$ by the infimum of

$$\big(\sum_{\mathbf{i}\in I^n} \sup_{O_{\mathbf{i}}} f_{n,t,p}^{q}\big)^{\frac{1}{q}}$$

over such q. Thus (2.22) in Theorem 2.10 holds replacing $\frac{p-1}{p} P^*_{\mathrm{top}}(\{\frac{p}{p-1}\log f_{n,t,p}\})$ by the infimum over such q of $\frac{1}{q}P^*_{\mathrm{top}}(\{q \log f_{n,t,p}\})$. The bound (2.29) follows from this version of (2.22) combined with Theorem 2.17 with $G = g|\det DT|^{\frac{1}{p}}$: We have, for $q \in [1, \frac{p}{p-1}]$,

$$\begin{aligned}
\log r_{\mathrm{ess}}(\mathcal{L}_g|_{\mathcal{H}^t_p}) - \frac{D^b(T)}{p} &\le \frac{1}{q}P^*_{\mathrm{top}}(q\log(|G^{(n)}|\nu^t_n)) \\
&= \frac{1}{q}\sup_{\mu\in \mathrm{Erg}(T)}\left\{h_\mu(T) + \int q\log|G|\mathrm{d}\mu - tq\chi_\mu(DT)\right\} \\
&= \sup_{\mu\in \mathrm{Erg}(T)}\left\{\frac{1}{q}h_\mu(T) + \int \log|g(\det DT)^{\frac{1}{p}}|\mathrm{d}\mu - t\chi_\mu(DT)\right\}.
\end{aligned}$$

The final claim is shown just like the second bound of Corollary 2.11. □

4. Proof of Theorem 2.17 on the Subadditive Variational Principle

We show Theorem 2.17 in §4.3, adapting the proof[24] of [23, Thm 3.1(i)] to subadditive potentials. For this, we first state and prove a key proposition about measures with $\mu(\mathcal{S}_\mathcal{O}) > 0$ in §4.1 and next recall in §4.2 the symbolic dynamics of a piecewise expanding map and a variational principle of Cao–Feng–Huang.

4.1. Measures Giving Nonzero Mass to $\mathcal{S}_\mathcal{O}$.

The proof of Theorem 2.17 is based on the following proposition, inspired from [23, Prop. 3.1]:

Proposition 4.1. *Let T be a piecewise $\mathcal{C}^{\bar\alpha}$ expanding map. Recall $\mathcal{S}_\mathcal{O}$ from* (2.2). *For each $\mu \in \mathrm{Erg}(T)$ such that $\mu(\mathcal{S}_\mathcal{O}) > 0$, every $t \ge 0$, and each piecewise $\mathcal{C}^\alpha$ function $G : M \to \mathbb{R}^+_*$, we have*

$$h_\mu(T) + \int \log G\mathrm{d}\mu - t\chi_\mu(DT) \le P^*_{\mathrm{top}}(\{\log(G^{(n)}\nu^t_n)\}, \mathcal{S}_\mathcal{O}).$$

Our proof of the proposition uses the following lemma (see [14], [3, Lemma B.3]):

Lemma 4.2. *Let T be a piecewise $\mathcal{C}^{\bar\alpha}$ expanding map, and let $G : M \to \mathbb{R}$ be piecewise continuous. Then, for any measurable set $E \subset M$ and any $t \ge 0$,*

$$P^*_{\mathrm{top}}(T, \{\log(|G^{(n)}|\nu^t_n) \mid n \ge 1\}, E) = \lim_{m\to\infty}\frac{1}{m}P^*_{\mathrm{top}}(T^m, \log(|G^{(m)}|\nu^t_m), E).$$

[24]Applying directly [23, Thm 1.3] and using Lemma 4.2 would also give Theorem 2.17, along the lines of [3, Lemma B.6]. (Note that small boundary pressure of T^m for $\log f_{m,t}$ for all large enough m is equivalent to the condition in Theorem 2.17 by Lemma 4.2.)

Proof. The limit in the right-hand side exists in $\mathbb{R}\cup\{-\infty\}$ by submultiplicativity. By definition, for each $\epsilon > 0$, there exists $m \geq 1$ such that

$$P^*_{\mathrm{top}}(T, \{\log(|G^{(n)}|\nu^t_n) \mid n \geq 1\}, E) + \epsilon \geq \frac{1}{m} \log \sum_{\mathbf{i}\in I^m | E\cap O_{\mathbf{i}}\neq\emptyset} \sup_{O_{\mathbf{i}}}(|G^{(m)}|\nu^t_m)$$
$$\geq \frac{1}{m} P^*_{\mathrm{top}}(T^m, \log(|G^{(m)}|\nu^t_m), E)\,.$$

(we used that the limit defining $P^*_{\mathrm{top}}(T^m, \log f, E)$ is an infimum). Thus

$$P^*_{\mathrm{top}}(T, \{\log(|G^{(n)}|\nu^t_n) \mid n \geq 1\}, E) \geq \lim_{m\to\infty} \frac{1}{m} P^*_{\mathrm{top}}(T^m, \log(|G^{(m)}|\nu^t_m), E)\,.$$

The other inequality follows from submultiplicativity, which gives, for any $m > 0$,

$$\begin{aligned} P^*_{\mathrm{top}}(T, \{\log(|G^{(n)}|\nu^t_n) \mid n \geq 1\}, E) &= \lim_{k\to\infty} \frac{1}{mk} \log \sum_{\mathbf{i}\in I^{mk} | E\cap O_{\mathbf{i}}\neq\emptyset} \sup_{O_{\mathbf{i}}}(|G^{(mk)}|\nu^t_{mk}) \\ &\leq \frac{1}{m} \lim_{k\to\infty} \frac{1}{k} \log \sum_{\mathbf{i}\in (I^m)^k | E\cap O_{\mathbf{i}}\neq\emptyset} \sup_{O_{\mathbf{i}}}(|G^{(m)}|^{(k)}\nu^{tk}_m) \\ &= \frac{1}{m} P^*_{\mathrm{top}}(T^m, \log(|G^{(m)}|\nu^t_m), E)\,. \end{aligned}$$

□

Proof of Proposition 4.1. We may assume $\inf G > 0$ because, if G_k is a sequence of piecewise continuous functions such that $\inf_M G_k > 0$, with $G_k \geq G_{k+1} \geq |G|$ for all k, and $\lim_{k\to\infty} \|G_k - |G|\|_{L_\infty(M)} = 0$, then for any measurable set E, applying our definitions to the sequences $f_n = \log(G^{(n)}\nu^t_n)$ and (for a fixed k) $f'_n = \log(G^{(n)}_k \nu^t_n)$,

$$\lim_{k\to\infty} P^*_{\mathrm{top}}(T, \{\log(G^{(n)}_k \nu^t_n)\}, E) = P^*_{\mathrm{top}}(T, \{\log(|G^{(n)}|\nu^t_n)\}, E)\,,\ \forall t \geq 0\,. \tag{4.1}$$

To show (4.1), it suffices to show that for all $t \geq 0$ (see[25] [3, Lemma B.4])

$$\lim_{k\to\infty} P^*_{\mathrm{top}}(T, \{\log(G^{(n)}_k \nu^t_n) \mid n \geq 1\}, E) \leq P^*_{\mathrm{top}}(T, \{\log(|G^{(n)}|\nu^t_n) \mid n \geq 1\}, E)\,.$$

Fix E and t. For any $\epsilon > 0$, there exists $m = m(\epsilon)$ large enough such that

$$P^*_{\mathrm{top}}(T, \{\log(|G^{(n)}|\nu^t_n) \mid n \geq 1\}, E) + \epsilon \geq \frac{1}{m} \log \sum_{\mathbf{i}\in I^m | E\cap O_{\mathbf{i}}\neq\emptyset} \sup_{O_{\mathbf{i}}}(|G^{(m)}|\nu^t_m)\,.$$

Then take $k_0(m)$ such that for all $k \geq k_0$

$$\frac{1}{m} \log \sum_{\mathbf{i}\in I^m | E\cap O_{\mathbf{i}}\neq\emptyset} \sup_{O_{\mathbf{i}}}(G^{(m)}_k \nu^t_m) \leq \frac{1}{m}\big(\epsilon + \log \sum_{\mathbf{i}\in I^m | E\cap O_{\mathbf{i}}\neq\emptyset} \sup_{O_{\mathbf{i}}}(|G^{(m)}|\nu^t_m)\big)\,.$$

[25] There are typos in the proof there: G_n should be replaced by G (twice) in the third line of that proof, and $Q_*(T, G, \lambda^{(*)}, \mathcal{W}) + 2\epsilon$ should be $Q_*(T, G, \lambda^{(*)}, \mathcal{W}, m) + \epsilon$ in the 5th line.

We conclude the proof of (4.1) by submultiplicativity: For all $k \geq k_0$,

$$P^*_{\mathrm{top}}(T, \{\log(G_k^{(n)}\nu_n^t) \mid n \geq 1\}, E) \leq P^*_{\mathrm{top}}(T, \{\log(|G^{(n)}|\nu_n^t) \mid n \geq 1\}, E) + 2\epsilon\,.$$

So let us assume that $\inf |G| > 0$. By definition,

$$(G^{(n)}\nu_n^t)(x) = \exp\left(\sum_{k=0}^{n-1} \log G(T^k(x)) - t \log \inf_{\|v\|=1} \|D_x T^n(v)\|\right)\,.$$

We introduce a generalisation of (2.6), setting, for $n, m \geq 1$,

$$f^{(n,m)}(x) = \prod_{k=0}^{n-1} f(T^{km}(x))\,,\ x \in M\,.$$

Fix $\epsilon > 0$. First, by Lemma 4.2 there exist $m_0 \geq 1$ and a sequence $n_0(m) \geq 1$ such that (using the convention that the supremum of any function over the empty set is zero)

$$\begin{aligned}\sum_{\mathbf{i}\in I^{nm}} \sup_{O_{\mathbf{i}}}(G^{(mn)}(\nu_m^t)^{(n,m)}) &\leq e^{n\left(P^*_{\mathrm{top}}(T^m, \log(G^{(m)}\nu_m^t), \mathcal{S}_{\mathcal{O}}) + \epsilon\right)}\,,\ \forall m \geq 1\,, \forall n \geq n_0(m),\\ &\leq e^{nm\left(P^*_{\mathrm{top}}(\{\log(G^{(k)}\nu_k^t\}, \mathcal{S}_{\mathcal{O}}) + 2\epsilon\right)}\,,\ \forall m \geq m_0\,, n \geq n_0(m).\end{aligned} \tag{4.2}$$

Next, for any $\mu \in \mathrm{Erg}(T)$, Oseledec's theorem [46] implies

$$\lim_{m\to\infty} \frac{t}{m} \log \nu_m(x) = -t\chi_\mu(DT)\,, \qquad \text{for } \mu \text{ almost every } x\,. \tag{4.3}$$

Thus, by the Birkhoff ergodic theorem, there exists a set $R \subset M$, with $\mu(R) > 1 - \frac{\mu(\mathcal{S}_{\mathcal{O}})}{2}$, and there exists an integer $m_0 \leq m_1(\epsilon) < \infty$ such that

$$(G^{(m_1)}\nu_{m_1}^t)(x) \geq e^{m_1\left(\int \log G \mathrm{d}\mu - t\chi_\mu(DT) - \epsilon\right)}\,,\ \forall x \in R\,. \tag{4.4}$$

Therefore

$$(G^{(m_1)}\nu_{m_1}^t)^{(n,m_1)}(x) \geq e^{nm_1\left(\int \log G \mathrm{d}\mu - t\chi_\mu(DT) - \epsilon\right)}\,,\ \forall x \in R\,, \forall n \geq 1\,. \tag{4.5}$$

For each $n \geq 1$, define

$$K_n = \{\mathbf{i} \in I^{nm_1} \mid O_{\mathbf{i}} \cap R \cap \mathcal{S}_{\mathcal{O}} \neq \emptyset\}\,.$$

Since $\mathcal{K}_n = \cup_{\mathbf{i}\in K_n} O_{\mathbf{i}}$ contains $R \cap \mathcal{S}_{\mathcal{O}}$, we have $\inf_n \mu(\mathcal{K}_n) \geq \mu(R \cap \mathcal{S}_{\mathcal{O}}) > \frac{\mu(\mathcal{S}_{\mathcal{O}})}{2} > 0$. Next, since $\log G$ is[26] piecewise Hölder, and $\mathrm{diam}(\mathcal{O}^{(n)}) \leq \mathrm{diam}(M)/\lambda^n$, there exist $C_G < \infty$ and $n_1(\epsilon) \geq 1$ such that, for all $n \geq n_1$ and all $\mathbf{i} \in I^{nm_1}$,

$$|\log G^{(nm_1)}(x) - \log G^{(nm_1)}(y)| \leq C_G(\mathrm{diam}(\mathcal{O}))^\alpha \leq n\frac{\epsilon}{2}\,,\ \forall x, y \in O_{\mathbf{i}}\,, \tag{4.6}$$

and, in addition since ν_{m_1} is Hölder on $O_{\mathbf{j}}$ for each $\mathbf{j} \in I^{m_1}$, we have for all $n \geq n_1$,

$$t|\log(\nu_{m_1})^{(n,m_1)}(x) - \log(\nu_{m_1})^{(n,m_1)}(y)| \leq n\frac{\epsilon}{2}\,,\ \forall x, y \in O_{\mathbf{i}}\,,\ \forall \mathbf{i} \in I^{nm_1}\,. \tag{4.7}$$

[26]We do not see why piecewise uniform continuity of ϕ suffices for [23, (5)].

It follows from (4.6–4.7) and (4.5) that, for all $n \ge n_1$,

$$\sum_{\mathbf{i}\in K_n} \sup_{x\in O_{\mathbf{i}}} (G^{(nm_1)}(\nu^t_{m_1})^{(n)})(x) \ge \sum_{\mathbf{i}\in K_n} e^{-n\epsilon} \sup_{x\in O_{\mathbf{i}}\cap R} (G^{(m_1)}\nu^t_{m_1})^{(n)}(x) \tag{4.8}$$
$$\ge \#K_n \cdot e^{nm_1\left(\int \log G \mathrm{d}\mu - t\chi_\mu(DT) - 2\epsilon\right)} .$$

Therefore, since $m_1 \ge m_0$, recalling (4.2) we have, for all $n \ge \max\{n_0(m_0), n_1\}$,

$$\#K_n \le C e^{nm_1\left(P^*_{\mathrm{top}}(\{\log(G^{(n)}\nu^t_n)\},\mathcal{S}_{\mathcal{O}})+2\epsilon\right)} \cdot e^{-nm_1\left(\int \log G \mathrm{d}\mu - t\chi_\mu(DT) - 2\epsilon\right)} . \tag{4.9}$$

Rudolph's formula for the entropy (see [37, §5.1, §5.10]) says that if $\mu \in \mathrm{Erg}(T)$ then, for any fixed $\gamma \in (0,1)$ and any finite generator, denoting by K'_ℓ the minimal cardinality of a collection of ℓ-cylinders whose union has measure at least γ, we have $h_\mu(T) = \liminf_{\ell\to\infty} \frac{1}{\ell}\log K'_\ell$. Therefore, taking $\mathcal{O}$ as a generator and $\gamma = \frac{\mu(\mathcal{S}_{\mathcal{O}})}{2}$, we have $\#K_n \ge K'_{nm_1}$ so that

$$\begin{aligned} h_\mu(T) &\le \liminf_{n\to\infty} \frac{1}{nm_1} \log \#K_n \\ &\le P^*_{\mathrm{top}}(\{\log(G^{(n)}\nu^t_n)\},\mathcal{S}_{\mathcal{O}}) + 2\epsilon - \int \log G d\mu + t\chi_\mu(DT) + 2\epsilon , \end{aligned}$$

where we used (4.9) for the second inequality. To conclude, let $\epsilon \to 0$. □

4.2. **Symbolic Dynamics. Continuous Subadditive Variational Principle.** We use the symbolic dynamics for a piecewise $\mathcal{C}^{\bar\alpha}$ expanding map T from [23, Beginning of §3]: Set

$$\Sigma_0(T) = \{\mathbf{i}_\infty = (i_0, i_1, \dots) \in I^{\mathbb{Z}_+} \mid \exists x \in M \mid T^n x \in O_{i_n} , \forall n \ge 0\} ,$$

and let $\Sigma(T)$ be the closure of $\Sigma_0(T)$ in $I^{\mathbb{Z}_+}$ for the product topology of the discrete topology on I. (This topology is compatible with the distance $\mathrm{dist}(\mathbf{i}_\infty, \mathbf{j}_\infty) = 2^{-n}$, where $n(\mathbf{i}_\infty, \mathbf{j}_\infty) = \min\{k \in \mathbb{Z}_+ \mid \mathbf{i}^k_\infty \ne \mathbf{j}^k_\infty\}$, where $\mathbf{i}^k_\infty = (i_0, i_1, \dots, i_{k-1}) \in I^k$, and with the convention $\min\emptyset = \infty$.) Let $\sigma : \Sigma(T) \to \Sigma(T)$ be the left-shift on $\Sigma(T)$.

By compactness of M, and since T is piecewise expanding, for each $\mathbf{i}_\infty \in \Sigma(T)$ there exists a unique $x \in M$ such that $\cap_{n\ge1} \overline{O}_{\mathbf{i}^n_\infty} = \{x\}$. This defines a map $\pi : \Sigma(T) \to M$ by $\pi(\mathbf{i}_\infty) = \cap_{n\ge1}\overline{O}_{\mathbf{i}^n_\infty}$. Setting $\mathcal{S}^\pi_{\mathcal{O}} = \pi^{-1}(\mathcal{S}_{\mathcal{O}})$ and $\partial\mathcal{O}^\pi = \pi^{-1}(\partial\mathcal{O})$, it is easy to check that the restriction

$$\pi_* : \Sigma(T) \setminus \bigcup_{k\ge0} \sigma^{-k}(\partial\mathcal{O}^\pi) = \Sigma(T) \setminus \mathcal{S}^\pi_{\mathcal{O}} \to M \setminus \bigcup_{k\ge0} T^{-k}(\partial\mathcal{O}) = M \setminus \mathcal{S}_{\mathcal{O}}$$

is measurable, and bijective, with measurable inverse, and we have $\pi_* \circ \sigma = T \circ \pi_*$. Note that σ is a continuous transformation of the compact metric space $\Sigma(T)$.

Next, given a function $f : M \to \mathbb{R}$, we define $f^\pi : \Sigma(T) \to \mathbb{R}$ by

$$f^\pi(\mathbf{i}_\infty) = \lim_{m\to\infty} \inf_{O_{\mathbf{i}^m_\infty}} f . \tag{4.10}$$

Then we have

$$f^\pi(\mathbf{i}_\infty) = f(\pi(\mathbf{i}_\infty)) , \quad \forall \mathbf{i}_\infty : \pi(\mathbf{i}_\infty) \notin \partial\mathcal{O} . \tag{4.11}$$

Moreover, for each $\alpha > 0$ there exists α' such that if f is piecewise α-Hölder, respectively continuous, on M then f^π is α'-Hölder, respectively continuous, on $\Sigma(T)$. If $\phi : \Sigma(T) \to \mathbb{R}_+^*$ is continuous then the topological pressure $P_{\mathrm{top}}(\sigma, \log \phi)$ is well-defined. If f is such that $f^\pi : \Sigma(T) \to \mathbb{R}_+^*$ is continuous, then

$$P_{\mathrm{top}}(\sigma, \log f^\pi) = P^*_{\mathrm{top}}(T, \log f) .$$

More generally, if $f_n : M \to \mathbb{R}_+^*$ is a submultiplicative sequence of functions with each f_n^π continuous, it is easy to see that

$$P_{\mathrm{top}}(\sigma, \{\log f_n^\pi\}) = P^*_{\mathrm{top}}(T, \{\log f_n\}) , \tag{4.12}$$

where the topological pressure in the left-hand side is defined using (n, ϵ)-separated sets for continuous transformations of compact metric spaces in [25, p. 640], or in the case of left-shift σ (see [25, §4, p. 649]) using cylinders.

The topological entropy of σ is finite. Thus, for a submultiplicative sequence of continuous functions $\phi_n : \Sigma(T) \to \mathbb{R}_+$, the variational principle in [25, Thm 1.1 and §4] says that (the limit below exists by subadditivity)

$$P_{\mathrm{top}}(\sigma, \{\log \phi_n\}) = \sup_{\mu_\sigma \in \mathrm{Erg}(\sigma)} \{h_{\mu_\sigma}(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n \mathrm{d}\mu_\sigma\} . \tag{4.13}$$

4.3. **Proof of Theorem 2.17.**

Proof of Theorem 2.17. Fix $t \geq 0$. In Step I, we shall prove that[27]

$$\sup_{\mu \in \mathrm{Erg}(T)} h_\mu(T) + \int \log |G| \mathrm{d}\mu - t\chi_\mu(DT) \leq \log P^*_{\mathrm{top}}(\{\log(G^{(n)}\nu_n^t)\}) .$$

In Step II, we shall find $\mu_{0,t} \in \mathrm{Erg}(T)$ with

$$h_{\mu_{0,t}}(T) + \int \log |G| \mathrm{d}\mu_{0,t} - t\chi_{\mu_{0,t}}(DT) = \log P^*_{\mathrm{top}}(\{\log(G^{(n)}\nu_n^t)\}) .$$

Both steps will use Proposition 4.1.

We can assume $\inf |G| > 0$ by (4.1). We associate the sequence of continuous functions $\{\log f^\pi_{n,t}\}$ to $\{\log f_{n,t} = \log(|G^{(n)}|\nu_n^t)\}$ via (4.10).

We start with Step I. Let $\mu \in \mathrm{Erg}(T)$. Assume first that $\mu(\mathcal{S}_\mathcal{O}) = 0$. Then,

$$\pi : (\Sigma(T), \mu \circ \pi, \sigma) \to (M, \mu, T) ,$$

(for the Borel σ algebras of $\Sigma(T)$, M) is a measure-theoretic isomorphism. Thus

$$h_\mu(T) + \int \log |G| d\mu - t\chi_\mu(DT) = h_{\mu\circ\pi^{-1}}(\sigma) + \lim_n \frac{1}{n} \int \log f^\pi_{n,t} \mathrm{d}\mu \circ \pi^{-1} .$$

---

[27]This does not require small boundary pressure.

Next, by the variational principle (4.13),

$$h_{\mu\circ\pi^{-1}}(\sigma) + \lim_n \frac{1}{n}\int \log f^{\pi}_{n,t} \mathrm{d}\mu\circ\pi^{-1} \le P_{\mathrm{top}}(\sigma, \{\log f^{\pi}_{n,t}\}) .$$

Therefore, since $P_{\mathrm{top}}(\sigma, \{\log f^{\pi}_{n,t}\}) = P^*_{\mathrm{top}}(T, \{\log f_{n,t}\})$ by (4.12), we have

$$h_\mu(T) + \int \log |G| \mathrm{d}\mu - t\chi_\mu(DT) \le P^*_{\mathrm{top}}(T, \{\log f_{n,t}\}) .$$

Finally, if $\mu(\mathcal{S}_{\mathcal{O}}) > 0$, then Proposition 4.1 gives

$$h_\mu(T) + \int \log |G| \mathrm{d}\mu - t\chi_\mu(DT) \le P^*_{\mathrm{top}}(T, \{\log f_{n,t}\}, \mathcal{S}_{\mathcal{O}}) \le P^*_{\mathrm{top}}(T, \{\log f_{n,t}\}) .$$

We move to Step II. Since σ is expansive and $\Sigma(T)$ is compact, the functions

$$\mu_\sigma \mapsto h_{\mu_\sigma}(\sigma^n) \text{ and } \mu_\sigma \mapsto \lim_n \frac{1}{n}\int \log f^{\pi}_{n,t} \, \mathrm{d}\mu_\sigma , \qquad \mu_\sigma \in \mathrm{Erg}(\sigma)$$

are upper semi-continuous (indeed $-\int \lim_n \frac{1}{n}\log \nu^{\pi}_n \, \mathrm{d}\mu > 0$ is lower semi-continuous, as it is the smallest Lyapunov exponent $\chi_\mu(DT)$, see e.g. [46, Lemma 9.1], hence $\lim_n \frac{1}{n}\int \log \nu^{\pi}_n \, \mathrm{d}\mu < 0$ and $\lim_n \frac{1}{n}\int \log f^{\pi}_{n,t} \, \mathrm{d}\mu$ are upper semi-continuous). Therefore, there exists $\mu_0 \in \mathrm{Erg}(\sigma)$ with

$$\begin{aligned} h_{\mu_0}(\sigma) + \lim_n \frac{1}{n}\int \log f^{\pi}_{n,t} \, \mathrm{d}\mu_0 &= \sup_{\mu_\sigma\in \mathrm{Erg}(\sigma)} h_{\mu_\sigma}(\sigma) + \lim_n \frac{1}{n}\int \log f^{\pi}_{n,t} \, \mathrm{d}\mu_\sigma \\ &= P_{\mathrm{top}}(\sigma, \{\log f^{\pi}_{n,t}\}) , \end{aligned}$$

where the second inequality follows from the variational principle (4.13).

Setting $\mu_{0,t} = \mu_0 \circ \pi^{-1}$, suppose that $\mu_{0,t}(\mathcal{S}_{\mathcal{O}}) > 0$, so that $\mu_0(\pi^{-1}(\mathcal{S}_{\mathcal{O}})) > 0$. Then Proposition 4.1 and the small boundary condition (2.17) would imply

$$\begin{aligned} P_{\mathrm{top}}(\sigma, \{\log f^{\pi}_{n,t}\}) = h_{\mu_0}(\sigma) + \lim_n \frac{1}{n}\int \log f^{\pi}_{n,t} \mathrm{d}\mu_0 &\le P_{\mathrm{top}}(\sigma, \{\log f^{\pi}_{n,t}\}, \pi^{-1}\partial\mathcal{O}) \\ &\le P^*_{\mathrm{top}}(T, \{\log f_{n,t}\}, \partial\mathcal{O}) < P^*_{\mathrm{top}}(T, \{\log f_{n,t}\}) . \end{aligned}$$

This would contradict $P^*_{\mathrm{top}}(T, \{\log f_{n,t}\}) = P_{\mathrm{top}}(\sigma, \{\log f^{\pi}_{n,t}\})$. Thus, $\mu_{0,t}(\mathcal{S}_{\mathcal{O}}) = 0$ and, arguing as in Step I, we have

$$h_{\mu_{0,t}}(T) + \lim_n \frac{1}{n}\int \log f_{n,t} \mathrm{d}\mu_{0,t} = h_{\mu_0}(\sigma) + \lim_n \frac{1}{n}\int \log f^{\pi}_{n,t} \mathrm{d}\mu_0 .$$

The right-hand side is $P_{\mathrm{top}}(\sigma, \{\log f^{\pi}_{n,t}\}) = P^*_{\mathrm{top}}(T, \{\log f_{n,t}\})$, and we conclude. □

References

[1] J.F Alves and A. Pumariño, *Entropy formula and continuity of entropy for piecewise expanding maps,* Ann. Inst. H. Poincaré, Anal. non lin. **38** (2021) 91–108.

[2] V. Baladi, *Positive Transfer Operators and Decay of Correlations,* Advanced Series in Nonlinear Dynamics, **16,** World Scientific (2000).

[3] V. Baladi, *Dynamical Zeta Functions and Dynamical Determinants for Hyperbolic Maps. A Functional Approach,* Springer Ergebnisse (2018).

[4] V. Baladi, *The quest for the ultimate anisotropic Banach space,* J. Stat. Phys. **166** (2017) 525–557. (*Corrections and complements,* **170** (2018) 1242–1247).

[5] V. Baladi, *Characteristic functions as bounded multipliers on anisotropic spaces,* Proc. Amer. Math. Soc. **146** (2018) 4405–4420.

[6] V. Baladi, J. Carrand, and M. Demers, *Measure of maximal entropy for finite horizon Sinai billiard flows,* to appear Ann. H. Lebesgue.

[7] V. Baladi, M.F. Demers, and C. Liverani, *Exponential decay of correlations for finite horizon Sinai billiard flows,* Invent. Math. **211** (2018) 39–177.

[8] V. Baladi and M.F. Demers, *On the measure of maximal entropy for finite horizon Sinai billiard maps,* JAMS **33** (2020) 381–449.

[9] V. Baladi and M.F. Demers, *Thermodynamic formalism for dispersing billiards,* J. Mod. Dynam. **18** (2022) 415–493.

[10] V. Baladi and S. Gouëzel, *Good Banach spaces for piecewise hyperbolic maps via interpolation,* Ann. Inst. H. Poincaré, Analyse non lin. **26** (2009) 1453–1481.

[11] V. Baladi and S. Gouëzel, *Banach spaces for piecewise cone hyperbolic maps,* J. Modern Dynam. **4** (2010) 91–135.

[12] V. Baladi and G. Keller, *Zeta functions and transfer operators for piecewise monotone transformations,* Comm. Math. Phys. **127** (1990) 459–477.

[13] V. Baladi and D. Ruelle, *Sharp determinants,* Invent. Math. **123** (1996) 553–574.

[14] V. Baladi and M. Tsujii, *Dynamical determinants and spectrum for hyperbolic diffeomorphisms,* In Probabilistic and Geometric Structures in Dynamics, pp. 29–68, Contemp. Math. **469** Amer. Math. Soc. (2008).

[15] M. Blank, G. Keller, and C. Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps,* Nonlinearity **15** (2002) 1905–1973.

[16] J. Buzzi, *Intrinsic ergodicity of affine maps in* $[0,1]^d$*,* Monatsh. Math. **124** (1997) 97–118.

[17] J. Buzzi, *Thermodynamical formalism for piecewise invertible maps: absolutely continuous invariant measures as equilibrium states* (Seattle, 1999), 749–783, Proc. Sympos. Pure Math. **69** Amer. Math. Soc., Providence (2001).

[18] J. Buzzi, *No or infinitely many a.c.i.p. for piecewise expanding* C^r *maps in higher dimensions,* Comm. Math. Phys. **222** (2001) 495–501.

[19] J. Buzzi, *Puzzles of quasi-finite type, zeta functions and symbolic dynamics for multidimensional maps,* Ann. Inst. Fourier **60** (2010) 801–852.

[20] J. Buzzi and G. Keller, *Zeta functions and transfer operators for multidimensional piecewise affine and expanding maps,* Ergodic Theory Dynam. Systems **21** (2001) 689–716.

[21] J. Buzzi and V. Maume-Deschamps, *Decay of correlations for piecewise invertible maps in higher dimensions,* Israel J. Math. **131** (2002) 203–220.

[22] J. Buzzi, F. Paccaut, and B. Schmitt, *Conformal measures for multidimensional piecewise invertible maps,* Ergodic Theory Dynam. Systems **21** (2001) 1035–1049.

[23] J. Buzzi and O. Sarig, *Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps,* Ergodic Theory Dynam. Systems **23** (2003) 1383–1400.

[24] W.J. Cowieson, *Absolutely continuous invariant measures for most piecewise smooth expanding maps,* Ergodic Theory Dynam. Systems **22** (2002) 1061–1078.

[25] Y.-L. Cao, D.-J. Feng, and W. Huang, *The thermodynamic formalism for sub-additive potentials,* Discrete Cont. Dyn. Syst. **20** (2008) 639–657.

[26] M.F. Demers and C. Liverani, *Stability of statistical properties in two-dimensional piecewise hyperbolic maps,* Trans. Amer. Math. Soc. **360** (2008) 4777–4814.

[27] M.F. Demers and C. Liverani, *Projective cones for sequential dispersing billiards,* Comm. Math. Phys. **401** (2023) 841–923.

[28] M.F. Demers and H.-K. Zhang, *Spectral analysis for the transfer operator for the Lorentz gas,* J. Modern Dynam. **5** (2011) 665–709.

[29] S. Gouëzel, *Spectre du flot géodésique en courbure négative,* d'après F. Faure et M. Tsujii, Astérisque **380** pp. 325–353, Séminaire Bourbaki. Vol. 2014/2015 (2016).
[30] S. Gouëzel and C. Liverani, *Banach spaces adapted to Anosov systems,* Ergodic Theory Dynam. Systems **26** (2006) 189–217.
[31] V.M. Gundlach and Y.A. Latushkin, *A sharp formula for the essential spectral radius of the Ruelle transfer operator on smooth and Hölder spaces,* Ergodic Theory Dynam. Systems **23** (2003) 175–191.
[32] H. Hennion, *Sur un théorème spectral et son application aux noyaux lipchitziens,* Proc. Amer. Math. Soc. **118** (1993) 627–639.
[33] F. Hofbauer and G. Keller, *Equilibrium states for piecewise monotonic transformations,* Ergodic Theory Dynam. Systems **2** (1982) 23–43.
[34] G. Keller, *On the rate of convergence to equilibrium in one-dimensional systems,* Comm. Math. Phys. **96** (1984) 181–193.
[35] C. Liverani, *Multidimensional expanding maps with singularities: a pedestrian approach,* Ergodic Theory Dynam. Systems **33** (2013) 168–182.
[36] R. Mañé, *Ergodic Theory and Differentiable Dynamics,* Springer Ergebnisse (1987).
[37] D. Rudolph, *Fundamentals of Measurable Dynamics,* Oxford Press (1990).
[38] D. Ruelle, *An extension of the theory of Fredholm determinants,* Inst. Hautes Etudes Sci. Publ. Math. **72** (1990) 175–193 (1991).
[39] R. Strichartz, *Multipliers on fractional Sobolev spaces,* J. Math. Mech. **16** (1967) 1031–1060.
[40] M.E. Taylor, *Pseudo Differential Operators,* Princeton University Press (1981).
[41] D. Thomine, *A spectral gap for transfer operators of piecewise expanding maps,* Discrete Cont. Dynam. Systems (A) **30** (2011) 917–944.
[42] B. Saussol, *Absolutely continuous invariant measures for multidimensional expanding maps,* Israel J. Math. **116** (2000) 223–248.
[43] H. Triebel, *General function spaces III (spaces $B^{g(x)}_{p,q}$ and $F^{g(x)}_{p,q}$, $1 < p < \infty$: basic properties),* Analysis Math. **3** (1977) 221–249.
[44] H. Triebel, *Theory of function spaces II,* Birkhäuser, Basel (1992).
[45] M. Tsujii, *Piecewise expanding maps on the plane with singular ergodic properties,* Ergodic Theory Dynam. Systems **20** (2000) 1851–1857.
[46] M. Viana, *Lectures on Lyapunov Exponents,* Cambridge Studies in Advanced Math. Cambridge University Press (2014).

(1) Sorbonne Université and Université Paris Cité, CNRS, Laboratoire de Probabilités, Statistique et Modélisation, F-75005 Paris, France

(2) Institute for Theoretical Studies, ETH, 8092 Zürich, Switzerland
Email address: `baladi@lpsm.paris`

(3) Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy
Email address: `roberto.castorrini@dm.unipi.it`