# THE QUEST FOR
# THE ULTIMATE ANISOTROPIC BANACH SPACE

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*Dedicated to David Ruelle and Yasha Sinai on their 80th birthdays.*

Abstract. We present a new scale $\mathcal{U}_p^{t,s}$ ($s < -t < 0$ and $1 \le p < \infty$) of anisotropic Banach spaces, defined via Paley–Littlewood, on which the transfer operator $\mathcal{L}_g \varphi = (g \cdot \varphi) \circ T^{-1}$ associated to a hyperbolic dynamical system $T$ has good spectral properties. When $p = 1$ and $t$ is an integer, the spaces are analogous to the "geometric" spaces $\mathcal{B}^{t,|s+t|}$ considered by Gouëzel and Liverani [26]. When $p > 1$ and $-1 + 1/p < s < -t < 0 < t < 1/p$, the spaces are somewhat analogous to the geometric spaces considered by Demers and Liverani [16]. In addition, just like for the "microlocal" spaces defined by Baladi–Tsujii [10] (or Faure–Roy–Sjöstrand [19]), the transfer operator acting on $\mathcal{U}_p^{t,s}$ can be decomposed into $\mathcal{L}_{g,b} + \mathcal{L}_{g,c}$, where $\mathcal{L}_{g,b}$ has a controlled norm while a suitable power of $\mathcal{L}_{g,c}$ is nuclear. This "nuclear power decomposition" enhances the Lasota–Yorke bounds and makes the spaces $\mathcal{U}_p^{t,s}$ amenable to the kneading approach of Milnor–Thurson [34] (as revisited by Baladi–Ruelle [8, 9, 2]) to study dynamical determinants and zeta functions.

## 1. Introduction

The goal of this note is to briefly present the various types of anisotropic Banach spaces available in the dynamical systems literature, highlighting their strengths and weaknesses, and to propose a new "microlocal" scale $\mathcal{U}_p^{t,s}$ which could address the shortcomings of the existing spaces. We next explain what we mean by this.

Let $T$ be[1] a transitive $C^r$ Anosov diffeomorphism on a connected compact Riemann manifold $M$, with $r > 1$. For a complex-valued $C^{r-1}$ weight function $g$ on $M$, define the Ruelle transfer operator by

$$\mathcal{L}_g \varphi = (g \cdot \varphi) \circ T^{-1} \,. \tag{1}$$

Here, $\varphi$ can be a function, for example in $L_1(dm)$ or $L_2(dm)$, with $dm$ normalised Lebesgue measure. It is however essential to let $\mathcal{L}_g$ act on Banach or Hilbert spaces of *distributions* in order to obtain a spectrum with dynamical relevance. For

*Date*: October 28, 2016.

I express deep thanks to S. Gouëzel for the argument in Appendix A (I am sole responsible for the introduction of any mistakes therein) and for sharing his ideas very generously over all these years. Thanks also to R. Anton for comments, and to A. Adam for finding several typos in a draft version. I am very grateful to W. Sickel and O. Besov for references and comments on Besov and Triebel–Lizorkin spaces with mixed Lebesgue norms. I thank the three anonymous referees for comments and questions which helped me to significantly improve the presentation.

[1] Most of the present article can be generalised to transitive locally maximal compact hyperbolic sets $\Lambda$, assuming sometimes also that $\Lambda$ is a (transitive) attractor for $T$. See [27, 2].

$g = |\det DT|^{-1}$, the transfer operator is just the Perron–Frobenius operator

$$\mathcal{P}\varphi = \frac{\varphi}{|\det DT|} \circ T^{-1}$$

of the Anosov diffeomorphism $T$. Since the pioneering work of Blank–Keller–Liverani [14] at the turn of the century, it has been established[2] that the spectrum of $\mathcal{P}$ on a suitable Banach space $\mathcal{B}$ of anisotropic distributions gives information on the Sinai-Ruelle-Bowen (SRB) measure: The spectral radius is equal to 1, where the maximal eigenvalue is simple, while the corresponding eigenvector is in fact a Radon measure $\mu$, which is just the SRB measure $\mu$ of $T$. The rest of the spectrum lies in a disc of radius strictly smaller than 1, which allows proving the exponential rate of decay of the correlations

$$\int \varphi(\psi \circ T^k) \mathrm{d}\mu - \int \varphi \mathrm{d}\mu \int \psi \mathrm{d}\mu$$

(as $k \to \infty$) for Hölder observables $\psi$ and $\varphi$. An important step towards establishing these facts is the obtention of an upper bound $\rho_{ess} < 1$ for the essential spectral radius of $\mathcal{P}$ on $\mathcal{B}$. Exponential mixing of the SRB measure had of course been obtained previously by Ruelle [37] (see also [15, (1.26)]) who worked with a transfer operator associated to a symbolic model for the dynamics (via Markov partitions), and used ideas from statistical mechanics. The importance of the early contributions of Ruelle (and Sinai, see e.g. [43]) cannot be stressed enough, and many fundamental results were obtained using the ideas they imported from statistical mechanics and the tools of Markov partitions. However, the introduction of transfer operators acting on a Banach space $\mathcal{B}$ of anisotropic distributions allowed to better exploit the $C^r$ smoothness of $T$. For example, working with an enhanced version, due to Gouëzel and Liverani [26], of the anisotropic Banach space from [14], Liverani [32] proved that, when $g = |\det DT|^{-1}$, the dynamical determinant

$$d_g(z) = \exp - \left( \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n(x)=x} \frac{\prod_{k=0}^{n-1} g(T^k(x))}{|\det(\mathrm{id} - DT_x^{-n})|} \right) \tag{2}$$

admits an analytic extension to a disc of radius $R_\zeta > 0$ (this had been established previously by Kitaev [30], for a larger value $R_\zeta$, by other methods), where its zeroes are exactly the inverses of the eigenvalues of modulus $\geq R_\zeta^{-1}$ of $\mathcal{L}_g$ acting on $\mathcal{B}$ (this was new). In addition, Liverani showed that $R_\zeta$ can be made arbitrarily large if $r = \infty$. This represented significant progress with respect to the pioneering results of Ruelle [39] and Pollicott [36] on dynamical zeta functions and dynamical determinants for hyperbolic diffeomorphisms. (See also [33] and [11] for enhancements of Liverani's result on $d_g(z)$.) The anisotropic spaces are also convenient to establish statistical and stochastic stability as well as linear response [4] (simplifying the proofs of Ruelle [40, 28]), and they could be used towards the theory of extreme values [6].

A palette of Banach spaces of anisotropic distributions appropriate for hyperbolic dynamics (without the need for Markov partitions nor assuming differentiability of the dynamical foliations) have been introduced by dynamicists and semi-classical analysts since the pioneering paper [14]. These spaces come in scales parametrised by two real numbers $v < 0$ and $t > 0$, and, setting aside the spaces related to

[2] For the Gibbs states associated to operators $\mathcal{L}_g$, weighted by general positive $g$, we refer to [27] and [11, 2].

the classical anisotropic spaces of Triebel [47], they can be roughly classified in two groups: In the first, “geometric” group [14, 26], the norm of $\varphi$ is obtained by taking a supremum of averages (of derivatives $\partial^{\vec{t}}$ of $\varphi$, for integer $t = |\vec{t}|$, integrated against $C^{|v|}$ test functions) over a class of admissible leaves (with tangent vectors in stable cones for $T$). Hölder versions of this space exist [16, 17] for small $0 < t < 1$ and $|v| < 1$. The *shortcomings* of this approach are that the bounds for of fractional $0 < t < 1$ are cumbersome to obtain, and that (since the kneading approach of [9, 11] is not available) the relation between the eigenvalues of the transfer operator and the zeroes of a dynamical determinant can only be obtain in a reduced domain. In the second group [10], the norm in charts of $\varphi$ is the $L_p$ average (for $1 < p < \infty$) of $\Delta^{t,v}(\varphi)$, where the operator $\Delta^{t,v}$ interpolates smoothly between $(\mathrm{id} + \Delta)^{v/2}$ (in stable cones in the cotangent space) and $(\mathrm{id} + \Delta)^{t/2}$ (in unstable cones in the cotangent space), where $\Delta$ is the Laplacian. This second “microlocal” (or pseudodifferential, or Sobolev) approach is seductive since it allows using an array of powerful techniques, in particular to study the dynamical determinant, especially in the Hilbert case $p = 2$. It was embraced by the semiclassical community [19]. Its *main shortcoming* is that it does not seem to be amenable to the study of piecewise smooth systems (Appendix A, but see also Footnote 22). We mention here that the recent proof [3] of exponential decay of correlations for Sinai billiard flows was obtained by using the Hölder variant of the “geometric” spaces in the first group [16, 17].

In this paper, we shall propose a new “microlocal” scale $\mathcal{U}_p^{t,s}$. We expect that the *kneading operator* strategy [11] can be implemented with this new norm, allowing the study of dynamical determinants in larger domains than those accessible via the “geometric” approach. More importantly, we believe that, contrarily to existing “microlocal” spaces in the literature (see Appendix A), the spaces $\mathcal{U}_p^{t,s}$ can be used for *piecewise smooth systems* (see Remark 3.9). In particular, we hope that the new spaces can be used to obtain good bounds (in the spirit of the variational principle type bound (25)) for the essential spectral radius of piecewise hyperbolic maps in any dimension, if the boundaries of the smoothness domains satisfy some transversality assumption with the stable cones, but without assuming bunching conditions. We hope they can allow relating the eigenvalues of the transfer operator $\mathcal{L}_g$ with poles of the weighted dynamical zeta function of a piecewise hyperbolic map (in any dimension), adapting the kneading approach of [9, 11]. We expect that similar spaces can also be introduced for piecewise hyperbolic flows, allowing improvement of the results of [7]. Finally, spaces of the type $\mathcal{U}_p^{t,s}$ can perhaps be used also for piecewise hyperbolic systems with billiard-type singularities in any dimension. (For piecewise smooth dynamics, suitable assumptions relating complexity and hyperbolicity will be needed, and we hope that thermodynamic expressions like (25) will allow a formulation in terms of “pressure of the unstable Jacobian on the boundary.”)

The anisotropic spaces $\mathcal{U}_p^{t,s}$ introduced below are based on Besov spaces with different regularity exponents $s < 0 < t$, but replacing the spatial averaging $L_p(\mathbb{R}^d)$ by $\sup_\Gamma L_p(\Gamma)$, for a suitable set of smooth submanifolds $\Gamma$. If the supremum were taken over the leaves of a smooth foliation (e.g. $\mathbb{R}^{d_s} \times \{x_{d_u}\}$), this would be an instance of a *mixed (Lebesgue) norm* anisotropic Besov space (see [13], see [29] for invariance under diffeomorphisms, and see also [12, 35]). However[3] we must take in our definition below the supremum over leaves $\Gamma$ ranging over a set which does *not* form a foliation. Because of this difference, we will not use any mixed norms results in the proof of the Lasota–Yorke estimates (they may be useful to show that characteristic functions are bounded multipliers, see Remark 3.9).

A caveat is in order here: *Even if the leaves* $\Gamma$ *are all horizontal* $d_s$*-dimensional planes, the mixed norm* $\sup_\Gamma L_2(\Gamma)$ *is not associated to a scalar product.* The norms $\mathcal{U}_p^{t,s}$ introduced below also suffer from this handicap.

The paper is organised as follows: In Section 2, we present three types of existing anisotropic spaces (the classical Triebel spaces, the geometric spaces *à la* Liverani, and the microlocal spaces) including their strong and weak points. In Section 3.1, we explain the motivation for the new spaces $\mathcal{U}_p^{t,s}$, with a recap of the shortcoming of the existing spaces. Section 3.2 contains Definition 3.3 of $\mathcal{U}_p^{t,s}$, and Section 3.3 is devoted to comments on this definition (Remark 3.9 there indicates why the new space could be used for piecewise smooth dynamics). Section 4 contains Theorem 4.1 which says that, if the stable dimension $d_s = 1$, then the essential spectral radius of the transfer operator $\mathcal{L}_g$ on $\mathcal{U}_1^{t,s}$ satisfies the same sharp bounds as those obtained in [11]. (In Remark 4.5 we sketch a proof if $d_s \geq 2$.) The proof of Theorem 4.1 hinges on the key Lasota–Yorke Lemma 4.2. We explain in Remark 4.6 why this lemma should also imply a nuclear power decomposition. Appendix A contains the argument of Gouëzel showing that multiplication by characteristic functions is not a bounded multiplier on the spaces of [10]. In Appendix B we compare (heuristically) the spaces $\mathcal{U}_1^{t,s}$ with the spaces $\mathcal{B}^{t,|t+s|}$ of [26]. Finally, Appendix C contains some technical material regarding integration by parts, proper support, and comparison with classical spaces.

## 2. A short tour in the jungle of anisotropic Banach spaces

In this section, we briefly describe the three types of existing anisotropic spaces used for discrete-time $C^r$ hyperbolic dynamics with $1 < r \leq \infty$, listed in chronological order:

- In §2.1, Triebel-type "foliated" spaces, where invariant differentiable foliations, or invariant classes of foliations (assuming bunching), are used. (This — classical — type was not discussed in the introduction.)
- In §2.2 "geometric" spaces due to Liverani and co-authors (Blank, Demers, Gouëzel, Keller), where strictly invariant *cones in the tangent space* are used to define admissible leaves. (This type belongs to the "geometric" group mentioned in the introduction.)
- In §2.3 "microlocal" spaces due to Tsujii and co-authors (Baladi, Faure), where strictly invariant *cones in the cotangent space,* are used, via Fourier transforms and pseudo-differential operators. (This type belongs to the "microlocal" group mentioned in the introduction.) The approach used by the semi-classical community, see e.g. [19], is essentially the same.

Before we describe these three types of spaces, two observations should be made:

First, a remarkable feature of the new spaces $\mathcal{U}_p^{t,s}$ introduced in the following section will involve cones *both in the tangent and in the cotangent space:* In the

[3]Note also that our anisotropic regularity exponents have different signs.

tangent space for the definition and proofs, and in the cotangent spaces (only) in the proofs.

Second, the situation of real-analytic hyperbolic dynamics is rather different, and we shall not discuss it in this note. We just mention that the transfer operators are then compact (in fact nuclear or trace class) when acting on suitable Banach (or Hilbert) spaces, and that, *very roughly*, the analogues of the foliation-spaces in §2.1 are those introduced by Ruelle [38] and Fried [22], the analogues of the geometric spaces of §2.2 are those of Rugh [41] and Fried [23], while the microlocal spaces of §2.3 are analogous to those of Faure and Roy [18].

We now move to the definitions of the three types of spaces. As a preparation for §2.1 and §2.3, we first recall the Fourier space description of the classical scale of *isotropic* Sobolev spaces: For $d \geq 1$ and $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, we write $x\xi$ for the scalar product of $x$ and $\xi$. Then the Fourier transform $\mathbb{F}$ and its inverse $\mathbb{F}^{-1}$ are defined on the space [42] of *rapidly decreasing functions* $\varphi, \psi \in \mathcal{S}$ by

$$\mathbb{F}(\varphi)(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi}\varphi(x)\mathrm{d}x\,, \quad \xi \in \mathbb{R}^d\,, \tag{3}$$

$$\mathbb{F}^{-1}(\psi)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi}\psi(\xi)\mathrm{d}\xi\,, \quad x \in \mathbb{R}^d\,, \tag{4}$$

and then extended to the space [42] of *temperate distributions* $\varphi, \psi \in \mathcal{S}'$ as usual. For $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, and suitable $a : \mathbb{R}^{2d} \to \mathbb{R}$ and $\varphi : \mathbb{R}^d \to \mathbb{C}$, we shall use the notation

$$a^{Op}(\varphi)(x) = \mathbb{F}^{-1}(a(x,\cdot)\cdot\mathbb{F}(\varphi))(x)\,, \quad x \in \mathbb{R}^d\,.$$

(We say that $a^{Op}$ is the operator associated to the "symbol" $a$.) Note that *if the function $a$ only depends on $\xi$* then

$$a^{Op}(\varphi) = (\mathbb{F}^{-1}a) * \varphi\,, \tag{5}$$

which implies $\|a^{Op}\varphi\|_{L_p} \leq \|\mathbb{F}^{-1}a\|_1\|\varphi\|_{L_p}$ for all $n$, by Young's inequality in $L_p$ for $1 \leq p \leq \infty$.

The classical Sobolev spaces $H^t_p$ for $1 < p < \infty$ and $t \in \mathbb{R}$ can be defined by

$$H^t_p(\mathbb{R}^d) := (\mathrm{id} + \Delta)^{-t/2}(L_p(\mathbb{R}^d))\,, \tag{6}$$

(using fractional powers [44]) or, equivalently, as the Banach space of those distributions in $\mathcal{S}'$ so that

$$\|\varphi\|_{H^t_p} := \|((1+|\xi|^2)^{t/2})^{Op}(\varphi)\|_{L_p(\mathbb{R}^d)} < \infty\,.$$

It is known that $\mathcal{S}$ is dense in $H^t_p(\mathbb{R}^d)$, so that $H^t_p(\mathbb{R}^d)$ coincides with the closure of $\mathcal{S}$ for the norm $\|\varphi\|_{H^t_p}$.

Finally, multiplier results imply that the norm $\|\varphi\|_{H^t_p}$ is equivalent to the following Paley–Littlewood norm: Fix a $C^\infty$ function $\chi : \mathbb{R}_+ \to [0,1]$ with

$$\chi(x) = 1, \quad \text{for } x \leq 1, \qquad \chi(x) = 0, \quad \text{for } x \geq 2. \tag{7}$$

Define $\psi_n : \mathbb{R}^d \to [0,1]$ for $n \in \mathbb{Z}_+$, by $\psi_0(\xi) = \chi(\|\xi\|)$, and

$$\psi_n(\xi) = \chi(2^{-n}\|\xi\|) - \chi(2^{-n+1}\|\xi\|)\,, \quad n \geq 1\,. \tag{8}$$

Note that $\sup_n \|\mathbb{F}^{-1}\psi_n\|_{L_1} < \infty$ and for every multi-index $\beta$, there exists a constant $C_\beta$ such that

$$\|\partial^\beta\psi_n\|_{L_\infty} \leq C_\beta 2^{-n|\beta|}\,, \qquad \forall\, n \geq 0\,. \tag{9}$$

Then, setting, $\varphi_n = \psi_n^{Op}\varphi$ the Paley–Littlewood norm which is equivalent [42, §2.1] with $\|\varphi\|_{H_p^t}$ is given by

$$\Big\| \big(\sum_{n\ge 0} 4^{tn}|\varphi_n|^2\big)^{1/2} \Big\|_{L_p(\mathbb{R}^d)} .$$

The norm above is a Triebel–Lizorkin-type norm: We first take an $\ell^2$ norm over the indices $n$ and then the $L_p$ norm over the space $\mathbb{R}^d$. The Besov(–Hölder–Zygmund–Lipschitz) scales have a Paley–Littlewod description of Besov type, taking first the spatial $L_p$ norm and then an $\ell^\infty$ norm over indices. There are other variants of the Besov and Triebel–Lizorkin scales [42].

We conclude with the obvious remark that if $p = 2$ then $H_2^t(\mathbb{R}^d)$ is a Hilbert space, otherwise $H_p^t(\mathbb{R}^d)$ is only a (complex) Banach space.

We now return to the Anosov situation. Let $\mathbb{R}^d = \mathbb{R}^{d_s} \times \mathbb{R}^{d_u}$, with $d_s \ge 1$ and $d_u \ge 1$ the stable and unstable dimensions of our Anosov diffeomorphism $T$. Let also $\lambda_s < 1 < \lambda_u$ be the weakest asymptotic contraction and weakest asymptotic expansion of $T$. Using an adapted Mather metric, we can assume that the expansion of $D_xT$ along $E^u_x$ is stronger than $\lambda_u$, while its contraction along $E^s_x$ is stronger than $\lambda_s$, and that the angle between $E^s_x$ and $E^u_x$ is everywhere arbitrarily close to $\pi/2$. We proceed with definitions of the main existing scales of anisotropic spaces in the following subsections, *emphasizing that the anisotropic spaces will involve positive regularity in the unstable directions of $T$ and negative regularity in the stable directions of $T$.*

2.1. **Triebel-type "foliated" spaces** [1, 4, 5, 7]**.** Denote the full Laplacian by $\Delta$ and the stable and unstable "foliated" Laplacians by $\Delta_s = \sum_{j=1}^{d_s} \partial^2_{x_j}$ and $\Delta_u = \sum_{j=d_s+1}^{d} \partial^2_{x_j}$. Triebel spaces such as

$$H_p^{t,s}(\mathbb{R}^d) := (\mathrm{id} + \Delta)^{-t/2}(\mathrm{id} + \Delta_s)^{-s/2}(L_p(\mathbb{R}^d)) , \tag{10}$$

for $1 < p < \infty$ and $t, s \in \mathbb{R}$ have been well studied [47, 48]. The choices $s < -t < 0$ will be natural for us. The Triebel spaces[4] $H_p^{t,s}$, as well as the similar spaces

$$(\mathrm{id} + \Delta)^{-t/2}(\mathrm{id} + \Delta_u)^{-u/2}(L_p(\mathbb{R}^d)) \tag{11}$$

(with $t < 0$ fractional global derivatives and $t + u > 0$ unstable derivatives) and

$$(\mathrm{id} + \Delta_s)^{-s/2}(\mathrm{id} + \Delta_u)^{-u/2}(L_p(\mathbb{R}^d)) \tag{12}$$

(with $s < 0$ fractional stable derivatives and $u > 0$ fractional unstable derivatives) all have a definition using the Fourier transform. For $H_p^{t,s}$, we have

$$\|\varphi\|_{H_p^{t,s}} \simeq \|((1 + |\xi|^2 + |\eta|^2)^{t/2}(1 + |\eta|^2)^{s/2})^{Op}(\varphi)\|_{L_p(\mathbb{R}^d)} ,$$

where $\simeq$ means that the norms are equivalent, and where $\xi \in \mathbb{R}^{d_u}$ and $\eta \in \mathbb{R}^{d_s}$.

Finally, $\mathcal{S}$ is dense in each of the spaces just described [47], and each space has a Paley–Littlewood description, by standard multiplier results.

Such Triebel spaces can be used for transfer operators under the (very strong) assumption that at least one of the dynamical foliations of $T$ (stable or unstable) is at least $C^{1+\epsilon}$. To define, e.g., $H_p^{t,s}(M)$, assuming that the stable foliation of $T$ is $C^{1+\epsilon}$, consider a finite system of $C^{1+\epsilon}$ local charts $\{(V_\omega, \kappa_\omega)\}_{\omega\in\Omega}$, that is, a cover $\mathcal{V} = \{V_\omega\}$ of $M$ by open subsets $V_\omega$ and diffeomorphisms $\kappa_\omega : U_\omega \to V_\omega$

[4] Note that the total fractional derivative in the stable directions is $t + s$ and not $s$.

such that $M \subset \cup_\omega V_\omega$, and $U_\omega$ is a bounded open subset of $\mathbb{R}^d$ for each $\omega \in \Omega$, assuming in addition that for each small enough local stable leaf $W = W^s_x$ of $T$, the image $\kappa_\omega^{-1}(W \cap V_\omega)$ is horizontal, that is a subset of $\mathbb{R}^{d_s} \times \{y_u(x)\}$ for some fixed $y_u(x) \in \mathbb{R}^{d_u}$. Letting $\{\theta_\omega\}$ be a $C^\infty$ finite partition of unity for $M$ subordinate to the cover $\mathcal{V}$, the Banach space $H^{t,s}_p(M)$ is then defined to be the closure of $C^\infty(M)$ for the norm

$$\sum_\omega \|(\theta_\omega \varphi) \circ \kappa_\omega\|_{H^{t,s}_p} .$$

Transfer operators acting on anisotropic Banach spaces based on $H^{t,s}_p$ were first studied in [1] (under the stronger assumption that the foliations be $C^\infty$, see [4] for $C^{1+\epsilon}$ foliations). A modification $\widetilde{H}^{s,t}_p(M)$ of the space allows working in more generality, replacing the differentiability assumption on the foliations by a bunching condition on the Lyapunov exponents [5, 7]. The idea is to consider a class of foliations admissible with respect to stable cones (the class – but not the individual foliations — being invariant under the dynamics).

**Upper bound for the essential spectral radius of $\mathcal{P}$:** When $T$ is an Anosov diffeomorphism *satisfying bunching conditions* [5, (2.3)–(2.4)], the results of [4, 5] imply, for the modified norm described above:

$$\rho_{ess}(\mathcal{P}|_{\widetilde{H}^{s,t}_p(M)}) \le \limsup_{n\to\infty} (\sup |\det DT^n|^{1/p-1})^{1/n} \max\{\lambda_u^{-t}, \lambda_s^{-t-s}\} ,$$

where $s < -t < 0$ with $t - s < r - 1$. For the variant given by (11), we get, under suitable bunching conditions,

$$\limsup_{n\to\infty} (\sup |\det DT^n|^{1/p-1})^{1/n} \max\{\lambda_u^{-t-u}, \lambda_s^{-t}\} ,$$

where $u - t < r - 1$. If both stable and unstable foliations are differentiable, we get for the Triebel space given by (12),

$$\limsup_{n\to\infty} (\sup |\det DT^n|^{1/p-1})^{1/n} \max\{\lambda_u^{-u}, \lambda_s^{-s}\} ,$$

where $u - s < r - 1$. These bounds give the best results when $p \to 1$. See also [4, 5, 2] for more general weighted operators $\mathcal{L}_g$.

**Advantages:** For $p = 2$ we get a Hilbert space. Strichartz proved [45] that $H^t_p(\mathbb{R}^d)$ is invariant under multiplication by characteristic functions of domains $E$ with piecewise smooth boundaries if $-1+1/p < t < 1/p$. This property is inherited [4] by $H^{t,s}_p$ if $-1 + 1/p < s \le t < 1/p$, as long as the boundary of $E$ satisfies some transversality condition with respect to the stable foliation (and similarly for (11) and (12), as well as the variants in [5, 7], mutatis mutandis). This allows the study of piecewise cone hyperbolic systems satisfying bunching (as well as complexity and transversality) conditions [4, 5, 7].

**Limitations:** The bunching condition is a strong limitation especially in high dimensions. Also, the spaces in [5, 7] do not seem adapted[5] to study systems such as discrete-time billiards, where the derivatives of the map may (and do) blow up at the boundaries of the smoothness domains.

[5] The compact embedding lemma causes problems, since it makes it necessary to require dynamically invariant bounds on the Jacobians of the charts which trivialise the admissible foliations.

2.2. **Cones in the tangent space: The "geometric" spaces $\mathcal{B}^{t,v}$ of Gouëzel–Liverani.** The idea for these spaces was introduced in [14] and perfected in [26, 27] (in particular the averages over the whole manifold used in [14] were replaced there by averages over admissible stable leaves, as in [31], and as described below). We first recall the notion of admissible stable leaves from Gouëzel and Liverani [26, §3]. For $\kappa > 0$, we define the stable cone at $x \in V$ by

$$\mathcal{C}^s(x) = \{w_1 + w_2 \in T_xM \mid w_1 \in E^s(x)\,, w_2 \perp E^s(x)\,, \|w_2\| \le \kappa \|w_1\|\}\ .$$

If $\kappa > 0$ is small enough then $D_xT^{-1}(\mathcal{C}^s(x) \setminus \{0\})$ lies in the interior of $\mathcal{C}^s(T(x))$, and $D_xT^{-1}$ expands the vectors in $\mathcal{C}^s(x)$ by $\lambda_s^{-1}$.

**Definition 2.1** (Admissible charts)**.** There exist an integer $N$, real numbers $\epsilon_\omega \in (0,1)$, and $C^r$ coordinate charts $\kappa_\omega$ defined on $(-\epsilon_\omega, \epsilon_\omega)^d \subset \mathbb{R}^d$, such that $M$ is covered by the open sets $\big(\kappa_\omega((-\epsilon_\omega/2, \epsilon_\omega/2)^d)\big)_{\omega=1\ldots N}$, and the following conditions hold: $D\kappa_\omega(0)$ is an isometry, $D\kappa_\omega(0) \cdot \big(\mathbb{R}^{d_s} \times \{0\}\big) = E^s(\kappa_\omega(0))$, and the $C^r$-norms of $\kappa_\omega$ and its inverse are bounded by $1 + \kappa$.

Pick $c_\omega \in (\kappa, 2\kappa)$ such that the cone in charts

$$\mathcal{C}^s_\omega = \{w_1 + w_2 \in \mathbb{R}^d \mid w_1 \in \mathbb{R}^{d_s} \times \{0\}, w_2 \in \{0\} \times \mathbb{R}^{d_u}, \|w_2\| \le c_\omega \|w_1\|\}$$

satisfies $D_x\kappa_\omega(\mathcal{C}^s_\omega) \supset \mathcal{C}^s(\kappa_\omega(x))$ and $D_{\kappa_\omega(x)}T^{-1}(D\kappa_\omega(x)\mathcal{C}^s_\omega) \subset \mathcal{C}^s(T^{-1}(\kappa_\omega(x)))$ for any $x \in (-\epsilon_\omega, \epsilon_\omega)^d$. Let $G_\omega(C_0)$ be the set of graphs of $C^r$ maps $\gamma : U_\gamma \to (-\epsilon_\omega, \epsilon_\omega)^{d_u}$ defined on a subset $U_\gamma$ of $(-\epsilon_\omega, \epsilon_\omega)^{d_s}$, with $|D\gamma| < c_\omega$ and $|\gamma|_{C^r} \le C_0$. (In particular, the tangent space to the graph of $\gamma$ belongs to the interior of the cone $\mathcal{C}^s_\omega$.) Uniform hyperbolicity of $T$ implies (see [26, Lemma 3.1]) that if $C_0$ is large enough, then there exists $C_0' < C_0$ such that, for any $\Gamma \in G_\omega(C_0)$ and any $\omega$, the set $\kappa_\omega^{-1}(T^{-1}(\kappa_\omega(\Gamma)))$ is included in $G_\omega(C_0')$.

**Definition 2.2** (Admissible graphs and admissible stable leaves)**.** An admissible graph is a $C^r$ map $\gamma$ defined on a ball $\bar B(w, K_1\delta) \subset (-2\epsilon_\omega/3, 2\epsilon_\omega/3)^{d_s}$ for small enough $\delta > 0$ and large enough $K_1$, taking its values in $(-2\epsilon_\omega/3, 2\epsilon_\omega/3)^{d_u}$ with range(id, $\gamma$) $\in G_\omega(C_0)$. An admissible stable leaf is $\Gamma = \kappa_\omega \circ (\mathrm{id}, \gamma)(\bar B(w, \delta))$ where $\gamma : \bar B(w, K_1\delta) \to \mathbb{R}^{d_u}$ is an admissible graph on $B_\omega := (-2\epsilon_\omega/3, 2\epsilon_\omega/3)^{d_s}$.

Let $t \ge 1$ be an integer, and let $v > 0$ be real, with $t + v < r - 1$. The definition of the norm of $\mathcal{B}^{t,v}$ in coordinates (see [26, Lemma 3.2]) is then

$$\|\varphi\|_{t,v} = \max_{\substack{0 \le t' \le t \\ t' \in \mathbb{Z}}} \max_{\substack{|\vec t| = t' \\ 1 \le \omega \le N}} \sup_\gamma \sup_{|\phi|_{C^{v+t'}} \le 1} \int_{B(w,\delta)} [\partial^{\vec t}(\varphi \circ \kappa_\omega)] \circ (\mathrm{id}, \gamma) \cdot \phi \, dm_{d_s}\,, \tag{13}$$

where the test function $\phi$ is compactly supported in $\bar B(w, \delta)$, the measure $dm_{d_s}$ is Lebesgue measure on $\mathbb{R}^{d_s}$, and $\gamma$ ranges over admissible graphs on $B_\omega$. Define $\mathcal{B}^{t,v}$ to be the closure of $C^{r-1}(M)$ for the norm $\|\varphi\|_{t,v}$. (In [26], the parameter $t$ was noted $p$ while $v$ was noted $q$.)

**Upper bound for the essential spectral radius of $\mathcal{P}$ and $\mathcal{L}_g$:** If $r > 2$, Gouëzel and Liverani show [26, 27]

$$\rho_{ess}(\mathcal{P}|_{\mathcal{B}^{t,v}}) \le \max\{\lambda_u^{-t}, \lambda_s^v\}\,, \quad \rho_{ess}(\mathcal{L}_g|_{\mathcal{B}^{t,v}}) \le e^{P_{top}(\log(|g| \det DT|_{E_s}))} \max\{\lambda_u^{-t}, \lambda_s^v\}$$

under the constraints ($t$ is an integer and $v$ is real)

$$1 \le t < (r-1) - v < r - 1\,.$$

(See also [27] for operators $\mathcal{L}_g$ with more general weights.)

**Advantages:** One of the strong points of the approach above using admissible leaves is that the norm can be modified to accommodate systems with singularities, including discrete and continuous-time billiards. We refer to [16, 17, 3].

**Limitations:** There is no Hilbert space in these scales. The kneading approach to dynamical determinants is not available and is replaced by other methods inspired from D. Dolgopyat's thesis [33, 24]. Unfortunately, these methods give a value for $R_\zeta$ which is of the order of $\rho_{ess}^{-1/2}$.

Since $t > 0$ must be an integer, the regularity assumption on $T$ is $C^r$ for $r > 2$ and the constraint on $v$ is $v < r - 1 - t \le r - 2$. The thermodynamic analysis in [11, §3] giving the sharp bound (25) (see also [2]) is not available for these spaces.

The analogues of the spaces for piecewise smooth systems [16, 17, 3] are not very easy to handle (stable and unstable norms must be handled separately, and the unstable norm involves Hölder quotients for $t < 1$) and have only been implemented in dimension two for maps and three for flows.

2.3. **Cones in the cotangent space: "Microlocal" spaces** [10, 11, 19]**.** We focus on the space $W^{t,s}_{p,\dagger}$ from [10]. We need some notation. A cone in $\mathbb{R}^d$ is a subset which is invariant under scalar multiplication. For two cones $\mathbf{C}$ and $\mathbf{C}'$ in $\mathbb{R}^d$, we write $\mathbf{C} \Subset \mathbf{C}'$ if $\overline{\mathbf{C}} \subset \text{interior}(\mathbf{C}') \cup \{0\}$. We say that a cone $\mathbf{C}$ is $d'$-dimensional if $d' \ge 1$ is the maximal dimension of a linear subset of $\mathbf{C}$.

**Definition 2.3.** A cone pair is $\mathbf{C}_\pm = (\mathbf{C}_+, \mathbf{C}_-)$, where $\mathbf{C}_+$ and $\mathbf{C}_-$ are closed cones in $\mathbb{R}^d$, with nonempty interiors, of respective dimensions $d_u$ and $d_s$ and so that $\mathbf{C}_+ \cap \mathbf{C}_- = \{0\}$. A cone system is a quadruple

$$\Theta = (\mathbf{C}_\pm, \varphi_+, \varphi_-)\,, \qquad \varphi_- = 1 - \varphi_+\,,$$

with $\mathbf{C}_\pm = (\mathbf{C}_+, \mathbf{C}_-)$ a cone pair and $\varphi_\pm : \mathbf{S}^{d-1} \to [0,1]$ two $C^\infty$ functions on the unit sphere $\mathbf{S}^{d-1}$ in $\mathbb{R}^d$ satisfying

$$\varphi_+(\xi) = 1 \text{ if } \xi \in \mathbf{S}^{d-1} \cap \mathbf{C}_+, \qquad \varphi_+(\xi) = 0 \text{ if } \xi \in \mathbf{S}^{d-1} \cap \mathbf{C}_-.$$

Introduce for real numbers $t$ and $v$ the functions

$$\Psi_{t,\Theta_+}(\xi) = (1 + \|\xi\|^2)^{t/2} \varphi_+\left(\frac{\xi}{\|\xi\|}\right) \quad \text{and} \quad \Psi_{v,\Theta_-}(\xi) = (1 + \|\xi\|^2)^{v/2} \varphi_-\left(\frac{\xi}{\|\xi\|}\right).$$

For a cone system $\Theta$, a compact set $K \subset \mathbb{R}^d$ with nonempty interior, and $\varphi \in C^\infty(K)$, we define norms for $1 < p < \infty$ and $v \le 0 \le t$ by

$$\|\varphi\|_{W^{\Theta,t,v}_{p,\dagger}} = \|\Psi^{Op}_{t,\Theta_+}(\varphi)\|_{L_p} + \|\Psi^{Op}_{v,\Theta_-}(\varphi)\|_{L_p}\,. \tag{14}$$

We next give the local definition of one of the spaces[6] introduced in [10]:

**Definition 2.4** (Anisotropic Sobolev spaces $W^{\Theta,t,v}_{p,\dagger}(K)$ in $\mathbb{R}^d$)**.** For a cone system $\Theta$, a compact set $K \subset \mathbb{R}^d$ with nonempty interior, $1 \le p \le \infty$ and $v \le 0 \le t$, let $W^{\Theta,t,v}_{p,\dagger}(K)$ be the completion of $C^\infty(K)$ with respect to $\|\cdot\|_{W^{\Theta,t,v}_{p,\dagger}}$.

[6]There are two other variants of the norms given in [10], $W^{\Theta,t,v}_{p,\dagger\dagger}$ and $W^{\Theta,t,v}_{p}$. For the present purposes we need not enter into details. We just mention that the three norms are related, but not equivalent, that most of the work is done with $W^{\Theta,t,v}_{p}$, which is given in Paley–Littlewood form, and that the notation in [10] involved a $*$ that we chose to discard. See [10, App. A].

**Definition 2.5** (Admissible charts and partition of unity for $T$)**.** Admissible charts and partition of unity for $T$ are: A finite system of $C^\infty$ local charts $\{(V_\omega, \kappa_\omega)\}_{\omega\in\Omega}$, with open subsets $V_\omega \subset M$, and diffeomorphisms $\kappa_\omega : U_\omega \to V_\omega$ such that $M \subset \cup_\omega V_\omega$, and $U_\omega$ is a bounded open subset of $\mathbb{R}^d$ for each $\omega \in \Omega$, together with a finite $C^\infty$ partition of unity $\{\theta_\omega\}$ for $M$, subordinate to the cover $\mathcal{V} = \{V_\omega\}$.

**Definition 2.6** (Admissible cone systems for $T$)**.** Since $T$ is Anosov, we may choose local charts indexed by a finite set $\Omega$ as in Definition 2.5, and cone pairs $\{\mathbf{C}_{\omega,\pm} = (\mathbf{C}_{\omega,+}, \mathbf{C}_{\omega,-})\}_{\omega\in\Omega}$, so that the following conditions hold[7]:

- If $x \in V_\omega$, the cone $(D\kappa_\omega^{-1})^*_x(\mathbf{C}_{\omega,+})$ contains the ($d_u$-dimensional) normal subspace of $E^s(x)$, and the cone $(D\kappa_\omega^{-1})^*_x(\mathbf{C}_{\omega,-})$ contains the ($d_s$-dimensional) normal subspace of $E^u(x)$.
- If $V_{\omega'\omega} = T(V_\omega) \cap V_{\omega'} \neq \emptyset$, the $C^r$ map corresponding to $T^{-1}$ in charts,
$$F = \kappa_\omega^{-1} \circ T^{-1} \circ \kappa_{\omega'} : \kappa_{\omega'}^{-1}(V_{\omega'\omega}) \to U_\omega \,,$$
extends to a bilipschitz $C^1$ diffeomorphism of $\mathbb{R}^d$ so that, using $A^{tr}$ to denote the transposition of a matrix $A$,
$$DF_x^{tr}(\mathbb{R}^d \setminus \mathbf{C}_{\omega,+}) \Subset \mathbf{C}_{\omega',-} \,, \qquad \forall x \in \mathbb{R}^d \,.$$
(We say that $F$ is *cone hyperbolic* from $\mathbf{C}_{\omega,\pm}$ to $\mathbf{C}_{\omega',\pm}$.)
- In addition, there exists, for each $x, y$, a linear transformation $\mathbb{L}_{xy}$ satisfying $(\mathbb{L}_{xy})^{tr}(\mathbb{R}^d \setminus \mathbf{C}_+) \Subset \mathbf{C}'_-$ and $\mathbb{L}_{xy}(x-y) = F(x) - F(y)$. (We say that $F$ is *regular cone hyperbolic* from $\mathbf{C}_{\omega,\pm}$ to $\mathbf{C}_{\omega',\pm}$.)

The anisotropic spaces introduced[8] in [11] and in [19] are variants of the spaces $W^{t,v}_{p,\dagger}$. (The semiclassical approach [19] takes $p=2$ and uses "escape functions," which play the role of our cone systems.)

**Upper bound for the essential spectral radius of $\mathcal{P}$:**
$$\rho_{ess}(\mathcal{P}|_{W^{t,v}_p}) \le \limsup_{m\to\infty} (\sup |\det DT^m|^{-1+1/p})^{1/m} \max\{\lambda_u^{-t}, \lambda_s^{-v}\} \,.$$
The constraints are $v < 0 < t < r - 1 + v$, and we get the best results when $p \to 1$. (The bound in [10] is in fact slightly more favorable.)

Besov versions $C^{t,v}_*$ of the spaces are also considered in [10]. The bound for the essential spectral radius of $\mathcal{P}$ on $C^{t,v}_*$ is $\le \max\{\lambda_u^{-t}, \lambda_s^{-v}\}$, for the same constraints $s < 0 < t < r - 1 + v$.

For the variant of the Banach space constructed in [11], a sharper bound is obtained for $\rho_{ess}(\mathcal{P}|_{W^{t,v}_p})$
$$\exp \sup_{\mu\in \mathrm{Erg}\,(T)} \Big\{ h_\mu(T) + \chi_\mu\left((\det(DT|_{E^u})^{-1})\right) \\ + \max\big\{t\chi_\mu(DT^{-1}|_{E^u}), |v|\chi_\mu(DT|_{E^s})\big\}\Big\} \,,$$
where $\mathrm{Erg}\,(T)$ denotes the set of $T$-invariant ergodic Borel probability measures, $h_\mu(T)$ denotes the metric entropy of $(\mu, T)$, and $\chi_\mu(A) \in \mathbb{R} \cup \{-\infty\}$ is the largest

[7] $\mathbf{C}_{\omega,\pm}$ are locally constant cone fields in the cotangent bundle $T^*\mathbb{R}^d$, so that the conditions are expressed with respect to normal subspaces.

[8] Note that [11] uses both cones in tangent and cotangent space, but the averaging over admissible leaves does not play the same role there as in [26, 16] or as in the definition of $\mathcal{U}^{t,s}_p$ below.

Lyapunov exponent of a linear cocycle $A$ over $T$. (For general operators $\mathcal{L}_g$ the bound from [11] is stated below in (25).)

**Advantages:** The bound (25) for the essential spectral radius $\rho_{ess}$ of $\mathcal{L}_g$ on the spaces of [11] is the sharpest known. (The proof uses thermodynamic sums via suitable partitions of unity and fragmentation–reconstruction lemmas.)

The nuclear power decomposition obtained in [11, 2] allows implementing the kneading operator approach to obtain the sharpest known estimate for $R_\zeta$, of the order of $\rho_{ess}^{-1}$ (as in [30]) for the radius of holomorphy of the weighted dynamical determinant (2).

For $p = 2$ we get a Hilbert space.

The variants introduced by the semi-classical community (following the work of Faure–Roy–Sjöstrand, [19, 20]) have led to spectacular results on hyperbolic flows which are beyond the scope of the present paper.

**Limitations:** Multiplication by the characteristic function of a domain (however smooth the boundary of that domain, and even if its boundary is transversal to the cones) is in general not a bounded operator on the spaces $W^{t,s}_{p,\dagger}$ from [10] (see Appendix A). This fact, which was first noticed by Gouëzel [25], is a serious obstruction to study piecewise smooth systems. The other spaces in [10, 11, 19] also appear to suffer from this limitation.

Note also that the Leibniz[9] bounds for the spaces in [10, 11] require different cone systems in the left-hand and right-hand sides, see e.g. the proof of [10, Prop. 7.2] or [2].

We end with the limitations of the semi-classical variant of the spaces [19]: The pseudodifferential tools used there only work if $r$ is large enough, depending on $d$. Also, the thermodynamic sums leading to the good bound (25) obtained in [11] for the essential spectral radius are not explicitly available there.

## 3. A Paley–Littlewood avatar of the Demers–Gouëzel–Liverani spaces: $\mathcal{U}^{t,s}_p$

3.1. **Motivation.** In this section, we give a "microlocal" (Paley–Littlewood) definition of spaces $\mathcal{U}^{t,s}_p$ with $s < -t < 0$ which are inspired by the "geometric" spaces (see Appendix B) $\mathcal{B}^{t,|s+t|}$ from [26] discussed in §2.2.

Before defining the new spaces, we list the advantages of the new scale with respect to the existing ones:

- Compared to the Triebel (foliation) norms [1, 4, 5] presented in §2.1 the advantage is that, since we replace the foliations by "free" admissible leaves and use mixed Lebesgue-norms, we do not need bunching assumptions[10] and we can also hope to study piecewise hyperbolic systems, even with billiard-type singularities. Indeed, when iterating, we handle the global derivative ($(\mathrm{id} + \Delta)^{t/2}$ with $t > 0$) and the foliated derivative (of the type

[9] A Leibniz bound is a bound on the norm of $f\varphi$, for smooth enough $f$, in terms of the norm of $\varphi$ and the derivatives or modulus of continuity of $f$.

[10] Iterating Triebel anisotropic spaces $H^{t,s}$ via admissible charts, even with a mixed norm — supremum over verticals of an $L_p$ norm over horizontals — requires bunching assumptions [5] to obtain invariance of charts if the stable foliation of $T$ is not smooth, and also control of Jacobians, not available for Sinai billiards.

$(\mathrm{id}+\Delta_s)^{s/2}$, with $s<0$, along admissible stable leaves of $T$) almost separately (except for the use of (35) to couple wave packets for $\mathbb{R}^d$ and for a stable leaf $\Gamma$ in the proof of Sublemma 4.4). (See also Remark 3.8.)

- With respect to the geometric norms $\mathcal{B}^{t,|v|}$ discussed in §2.2 the advantage is that we may now consider all real parameters $t>0$, while Gouëzel–Liverani [26, 27] were limited [11] to integer $t\geq 1$. This gives sharper bounds, also in view of the possibility of using thermodynamic sums as in [11].

  Also, since the decomposition of the transfer operators given in the Lasota–Yorke Lemma 4.2 (see Remark 4.6) is of "nuclear power" type, we expect that we can carry out the kneading operator arguments of Milnor–Thurston [34] as revisited in [9] and, especially, [11] (see also [2]). This would allow improving on the results of Liverani et al. [33] (and the results from the semiclassical community, which often require large differentiability in large dimension) on the dynamical determinant (2), also potentially for piecewise smooth systems and for continuous-time dynamics (flows) especially in high dimension or low regularity.
- With respect to the microlocal norms from [10, 11, 19] discussed in §2.3 (see Appendix A), the advantage is that, for $p>1$, $t<1/p$, and $s>-1+1/p$, we may hope to work with spaces $\mathcal{U}^{t,s}_p$ in piecewise smooth hyperbolic situations (like in [16] or [17], see Remark B.2) and piecewise hyperbolic systems with billiard-type singularities like [17, 3]. (See Remark 3.9.)

  Linear response was recently obtained [6] for hyperbolic systems and some discontinuous observables by using spaces $\mathcal{B}^{t,|s+t|}$ from [26], and we may hope to also prove this result by using $\mathcal{U}^{t,s}_p$.

  Other positive aspects with respect to the spaces of [10, 11] could be a more straightforward Leibniz inequality, see the comment after Corollary 4.3, and a more direct [2] proof of the relation between maximal eigenvectors and Gibbs states for general positive weights $g$, in particular a better understanding of induced measures on quasi-unstable leaves [27].

We end by mentioning that both the definition of the flat trace [11, 2] (which is an ingredient of the kneading operator argument) and the Dolgopyat estimates [7] (for flows) are essentially norm-independent.

### 3.2. Paley–Littlewood definition of $\mathcal{U}^{t,s}_p$.

We shall use the cone systems $\Theta$ from Definition 2.3. The other key ingredient is adapted[12] from [11]:

**Definition 3.1** (Fake stable leaves). Let $\mathbf{C}_+$ be a cone, and let $C_{\mathcal{F}}>1$. Let $\mathcal{F}(\mathbf{C}_+,C_{\mathcal{F}})$ (also noted simply $\mathcal{F}(\mathbf{C}_+)$ or $\mathcal{F}$ when the meaning is clear) be the set of all $C^r$ (embedded) submanifolds $\Gamma\subset\mathbb{R}^d$, of dimension $d_s$, with $C^r$ norms of submanifold charts bounded by $C_{\mathcal{F}}$, and so that the straight line connecting any two distinct points in $\Gamma$ is normal to a $d_u$-dimensional subspace contained in $\mathbf{C}_+$.

If $F$ is regular cone hyperbolic from $\mathbf{C}_\pm$ to $\mathbf{C}'_\pm$ (recall Definition 2.6) then, assuming in addition[13] that the extension of $F$ to $\mathbb{R}^d$ is $C^r$ there exists $C_{\mathcal{F}}<\infty$ so that this extension maps each element of $\mathcal{F}(\mathbf{C}_+)$ to an element of $\mathcal{F}(\mathbf{C}'_+)$.

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[11] Demers–Liverani [16] only consider two-dimensional systems and require not very handy Hölder-type ratios to handle regularity $t<1$, see also [17, 3].

[12] The submanifolds $\Gamma$ there were only assumed to be $C^1$ and the condition on $C_{\mathcal{F}}$ was absent.

[13] This is possible in the application, up to taking smaller charts.

We need some notation in view of performing dyadic decompositions in Fourier space. We may assume that $E_- := \mathbb{R}^{d_s} \times \{0\}$ is included in $\mathbf{C}_-$, and we denote by $\pi = \pi_-$ the orthogonal projection from $\mathbb{R}^d$ to the quotient $\mathbb{R}^{d_s}$ and by $\pi_\Gamma$ its restriction to $\Gamma$. Our assumption on $\mathcal{F}$ implies that $\pi_\Gamma : \Gamma \to \mathbb{R}^{d_s}$ is a $C^r$ diffeomorphism onto its image with a $C^r$ inverse. Letting $\pi_+$ be the projection from $\mathbb{R}^d$ to the quotient $\mathbb{R}^d \setminus E_- = \mathbb{R}^{d_u}$, we have that $\Gamma$ is the graph of the $C^r$ map

$$\gamma = \pi_+ \circ \pi_\Gamma^{-1} : \mathbb{R}^{d_s} \cap \pi_-(\Gamma) \to \mathbb{R}^{d_u} \,, \tag{15}$$

and the $C^r$ norm of $\gamma$ is bounded by a universal scalar multiple of $C_\mathcal{F}$.

**Definition 3.2** (Isotropic norm on stable leaves)**.** Fix $\mathbf{C}_\pm$ so that $\mathbb{R}^{d_s} \times \{0\}$ is included in $\mathbf{C}_-$. Let $\Gamma \in \mathcal{F}(\mathbf{C}_+)$ and let $\varphi$ be continuous and compactly supported. For $w \in \Gamma \subset \mathbb{R}^d$, we set

$$\psi_{\ell_s}^{Op(\Gamma)}(\varphi)(w) = \frac{1}{(2\pi)^{d_s}} \int_{z \in \mathbb{R}^{d_s}} \int_{\eta_s \in \mathbb{R}^{d_s}} e^{i(\pi_\Gamma(w) - z)\eta_s} \psi_{\ell_s}^{(d_s)}(\eta_s) \varphi(\pi_\Gamma^{-1}(z)) \mathrm{d}\eta_s \mathrm{d}z \,, \tag{16}$$

where $\psi_k^{(d_s)} : \mathbb{R}^{d_s} \to [0,1]$ is defined as in (8). For every $1 \le q \le \infty$, $1 \le p \le \infty$, and $-(r-1) < s < r-1$, define an auxiliary isotropic norm on $C^0(\Gamma)$ as

$$\|\varphi\|_{p,q,\Gamma}^s = \left( \sum_{\ell_s \in \mathbb{Z}_+} \left( 2^{\ell_s s} \| \psi_{\ell_s}^{Op(\Gamma)}(\varphi) \|_{L_p(\mu_\Gamma)} \right)^q \right)^{1/q} , \tag{17}$$

where $\mu_\Gamma$ is the Riemann volume on $\Gamma$ induced by the standard metric on $\mathbb{R}^d$. When $q = \infty$, we sometimes just write

$$\|\varphi\|_{p,\Gamma}^s = \|\varphi\|_{p,\infty,\Gamma}^s = \sup_{\ell_s \in \mathbb{Z}_+} 2^{\ell_s s} \| \psi_{\ell_s}^{Op(\Gamma)}(\varphi) \|_{L_p(\mu_\Gamma)} \,.$$

Note that (17) is just the classical $d_s$-dimensional Besov norm[14] $B_{p,q}^s$ of $\varphi|_\Gamma$ in the chart given by $\pi_\Gamma^{-1}$:

$$\|\varphi\|_{p,q,\Gamma}^s = \|\varphi \circ \pi_\Gamma^{-1}\|_{B_{p,q}^s(\mathbb{R}^{d_s})} \,.$$

*We are considering admissible leaves on the manifolds like Liverani et al.* [26, 16]*, so for all practical purposes the cones live in the tangent space and not in the cotangent space. To prove Lasota–Yorke estimates, however, it will be crucial to also use cones in the cotangent space, see* (35). (The reader was already warned in Footnote 8 that the analogy with the norms [11] is misleading and superficial.)

We next give[15] the definition of the local space:

**Definition 3.3** (The local space $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)$)**.** Let $K \subset \mathbb{R}^d$ be a non-empty compact set. For a cone pair $\mathbf{C}_\pm = (\mathbf{C}_+, \mathbf{C}_-)$ so that $\mathbb{R}^{d_s} \times \{0\}$ is included in $\mathbf{C}_-$, a constant $C_\mathcal{F} \ge 1$, and real numbers, $1 \le p \le \infty$, $t$, and $s$, define for a $C^\infty$ function $\varphi$ supported in $K$,

$$\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}} = \sup_{\Gamma \in \mathcal{F}(\mathbf{C}_+, C_\mathcal{F})} \sup_{\ell \in \mathbb{Z}_+} 2^{\ell t} \| \psi_\ell^{Op}(\varphi) \|_{p,\Gamma}^s \,. \tag{18}$$

Set $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)$ to be the completion of $C^\infty(K)$ with respect to $\|\cdot\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}}$.

Our first observation is the following lemma:

[14] See [42, §2.1, Def. 2] for a definition of the classical Besov norm $B_{p,q}^s$.
[15] The definition below can be compared to the norm in [11], but the norms are not equivalent.

**Lemma 3.4** (Comparing $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)$ with classical spaces)**.** *Assume $s < -t < 0$. For any $u > t$, there exists a constant $C = C(u, K)$ such that $\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}} \le C\|\varphi\|_{C^u}$ for all $\varphi \in C^\infty(K)$. For any $u > |t+s|$, the space $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)$ is contained in the space of distributions of order $u$ supported on $K$.*

The proof of Lemma 3.4 is given in Appendix C. Lemma 3.4 implies the following statement (as in the proof of [11, Lemma 4.21], see also [2, Chapter 5]):

**Lemma 3.5** (Approximation by finite rank operators)**.** *Let $K \subset \mathbb{R}^d$ be compact, let $s \le -t \le 0$, and let $\mathbf{C}_\pm$ and $\mathbf{C}'_\pm$ be arbitrary cone pairs. For each $v > 0$ and every $\phi \in C^\infty(K)$, there exist a constant $C_v$ and, for all integers $n_1 \ge n_0 \ge 1$, an operator $\mathcal{T}_{n_1} : \mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K) \to \mathcal{U}_p^{\mathbf{C}'_\pm,t,s}(K)$ of rank at most $2^{d(n_1+5)}$, so that the operator $\mathcal{R}_{n_0}$ defined by (32) satisfies*

$$\|(\mathcal{R}_{n_0} - \mathcal{T}_{n_1})\varphi\|_{\mathcal{U}_p^{\mathbf{C}'_\pm,t,s}(K)} \le C_v 2^{-dvn_1}\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)}\,.$$

We now define the global space $\mathcal{U}_p^{t,s}$:

**Definition 3.6** (Anisotropic spaces $\mathcal{U}_p^{t,s}$ on $M$)**.** Fix $C^\infty$ charts $\kappa_\omega : V_\omega \to \mathbb{R}^d$ and a partition of unity $\theta_\omega$ as in Definitions 2.5 and 2.6. Fix real numbers $s$ and $t$. The Banach space $\mathcal{U}_p^{t,s}$ is the completion of $C^\infty(M)$ for the norm

$$\|\varphi\|_{\mathcal{U}_p^{t,s}(T)} := \max_{\omega\in\Omega} \|(\theta_\omega \cdot \varphi) \circ \kappa_\omega\|_{\mathcal{U}_p^{\mathbf{C}_{\omega,\pm},t,s}}\,.$$

In Appendix B, we discuss why the anisotropic spaces $\mathcal{U}_1^{t,s}$ are analogues of the (Blank–Keller–)Gouëzel–Liverani [14, 26, 27] spaces $\mathcal{B}^{t,|s+t|}$ for integer $t$. Since not only $s$, but also $t$, can be taken arbitrarily close to zero, the spaces $\mathcal{U}_p^{t,s}$ are also somewhat similar to the Demers–Liverani spaces of [16] when $p > 1$ and $-1+1/p < s < -t < 0 < t < 1/p$. (But see Remark B.2.)

### 3.3. Comments on the definition of $\mathcal{U}_p^{t,s}$.

*Remark* 3.7 (Choice of the parameter $q$). For any $\epsilon > 0$, any $s$, $p$, and any $q' \le q \le \infty$, the Besov spaces on $\mathbb{R}^d$ satisfy the bounded inclusions $B^{s+\epsilon}_{p,\infty} \subset B^s_{p,q'} \subset B^s_{p,q}$, see [42, §2.2.1]. Denoting the Triebel-Lizorkin scale by $F^s_{p,q'}$, it is also well known [42, §2.2.2] that

(19)
$$\|\varphi\|_{B^s_{p,q}} \le C\|\varphi\|_{F^s_{p,q'}} \text{ if } \max(p,q') \le q\,, \qquad \|\varphi\|_{F^s_{p,q}} \le C\|\varphi\|_{B^s_{p,q}} \text{ if } q \le \min(p,q')\,.$$

In particular,

$$\|\varphi\|_{B^s_{p,\infty}} \le C\|\varphi\|_{F^s_{p,2}}\,, \forall p\,, \tag{20}$$

where [42, §2.1.2] $F^s_{p,2}(\mathbb{R}^{d_s}) = H^s_p(\mathbb{R}^{d_s})$. The case $q \ne \infty$ can be handled by slightly changing the value of $s$. In particular, if $s < 0$,

$$\|\varphi\|_{B^s_{p,q}} \le C\|\varphi\|_{F^0_{p,2}} = C\|\varphi\|_{L_p}\,, \forall p, q\,.$$

Instead of taking $q = \infty$ in the norm $\|\cdot\|^s_{p,q,\Gamma}$, one could consider two parameters $1 < q < \infty$ and $1 < q' < \infty$:

$$\Big(\sum_{\ell\in\mathbb{Z}_+} (2^{\ell t}\|\psi_\ell^{Op}(\varphi)\|^s_{p,q,\Gamma})^{q'}\Big)^{1/q'}\,,$$

but in view of the first paragraph of this remark, we expect that this would just make the computations more painful without any benefit. Also, since it is convenient to take the supremum over $\Gamma$ at the very end of Definition 3.3, the choice $q = \infty$ is most compatible with a Besov norm. (See however Appendix B.)

*Remark* 3.8 (Comparison with mixed (Lebesgue) anisotropic Besov norms). Setting for fixed $x_+ \in \mathbb{R}^{d_u}$

$$(\mathbb{F}_-(\varphi))_{x_+}(\xi_-) = \int_{\mathbb{R}^{d_s}} e^{-ix_-\xi_-}\varphi(x_-, x_+)\mathrm{d}x_- \,, \quad \xi_- \in \mathbb{R}^{d_s}\,,$$

and

$$\mathbb{F}_-^{-1}(\psi_{x_+})(x_-, x_+) = \frac{1}{(2\pi)^{d_s}} \int_{\mathbb{R}^{d_s}} e^{ix_-\xi_-}\psi_{x_+}(\xi_-)\mathrm{d}\xi_- \,, \quad x_- \in \mathbb{R}^{d_s}\,,$$

it is easy to see that for any fixed $x_+ \in \mathbb{R}^{d_u}$ and $\Gamma = \mathbb{R}^{d_s} \times \{x_+\}$,

$$\psi_{\ell_s}^{Op(\Gamma)}\psi_\ell^{Op}\varphi(x_-, x_+) = \mathbb{F}_-^{-1}\big[\psi_{\ell_s}(\xi_-)\big(\mathbb{F}_- \circ \mathbb{F}^{-1}(\psi_\ell(\xi)(\mathbb{F}\varphi))\big)\big](x_-, x_+)\,. \tag{21}$$

Considering the set $\Sigma$ of horizontal leaves $\mathbb{R}^{d_s} \times \{x_+\}$, the formula (21) implies

$$\sup_\ell \sup_{\ell_s} 2^{\ell t} 2^{\ell_s s} \sup_{\Gamma \in \Sigma} \|\psi_{\ell_s}^{Op(\Gamma)}\psi_\ell^{Op}\varphi\|_{L_p(\Gamma)} = \sup_\ell 2^{\ell t} \sup_{\Gamma\in\Sigma} \|\psi_\ell^{Op}\varphi\|_{B^s_{p,\infty}(\Gamma)}\,. \tag{22}$$

The left-hand side above is an anisotropic mixed Besov norm $B^{s,t}_{(\infty,p),(\infty,\infty)}$ where the norm $L_p(\mathbb{R}^d)$ is replaced by $\sup_{x_+\in\mathbb{R}^{d_u}} L_p(\mathbb{R}^{d_s} \times \{x_+\})$. Such mixed (Lebesgue) norm spaces have been studied [13, 29], and they satisfy the expected compact embedding and interpolation properties. The right-hand side in (22) is similar to $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}$, except that we restrict to $\Sigma$ instead of considering all $\Gamma \in \mathcal{F}(\mathbf{C}_+)$. Now, for each $\Gamma \in \mathcal{F}$, we can construct a $C^r$ foliation of manifolds parallel to $\Gamma$ (obtained by trivial translations) by recalling (15) and setting

$$\Phi_\Gamma(x_-, x_+) = (x_-, \gamma(x_-) + x_+)\,, \tag{23}$$

noting that $\Phi_\Gamma$ maps the horizontal hyperplane through the origin $\mathbb{R}^{d_s} \times \{0\}$ to $\Gamma$, and $\Phi_\Gamma$ maps each horizontal $\mathbb{R}^{d_s} \times \{x_+\}$ to a parallel leaf $\Gamma_{x_+}$. Note also that the jacobian of the holonomy $x_+ \mapsto \gamma(x_-) + x_+$ is constant equal to 1. Each leaf $\Gamma_{x_+}$ also belongs to $\mathcal{F}(\mathbf{C}_+)$, up to taking smaller chart neighbourhoods. Using $\Phi_\Gamma$ as a straightening chart for the parallel foliation, and noting that $\gamma_\Gamma$ satisfies uniform bounds by definition of $\mathcal{F}$, we have argued that the norms

$$\sup_{\Gamma\in\mathcal{F}} \|\varphi \circ \Phi_\Gamma\|_{B^{t,s}_{(\infty,p),(\infty,\infty)}} \tag{24}$$

and $\|\varphi\|_{\mathcal{U}_p^{t,s}}$ are similar. Beware however that when proving the Lasota–Yorke bound we should use $\mathcal{U}_p^{t,s}$, and not the equivalent norm $\sup_\Gamma \|\varphi \circ \Phi_\Gamma\|_{B^{t,s}_{(\infty,p),(\infty,\infty)}}$. In other words, working with $\mathcal{U}_p^{t,s}$ is the key to bypassing invariance of charts under the dynamics (this invariance caused difficulties in [5, 7]). However, the theory of mixed anisotropic Besov norms can perhaps be used to obtain other properties (see e.g. Remark 3.9).

*Remark* 3.9 (Piecewise smooth systems). In the application to transfer operators of $C^r$ Anosov diffeomorphisms, we take $-(r-1) < s < -t < 0$. In view of considering piecewise smooth hyperbolic maps, we conjecture that multiplication by the characteristic function of a domain $E$ with piecewise smooth boundary (satisfying

[4, 5] a suitable transversality condition with respect to the cone $\mathbf{C}_-$) is a bounded multiplier on $\mathcal{U}_p^{t,s}$ if

$$-1 + 1/p < s < -t < 0 < t < 1/p\,.$$

We sketch a possible argument involving interpolation (another strategy would be to use paraproducts as in [42, §4.6.3]).

Recall (see e.g. [42, Thm 4.6.3/1]) that for any $1 \le q \le \infty$, multiplication by the characteristic function of a half-plane in $\mathbb{R}^n$ is a bounded multiplier on the Besov space $B^s_{p,q}(\mathbb{R}^n)$ if $\frac{1}{p} - 1 < s < \frac{1}{p}$. For $t = 0$ and $-1 + 1/p < s < 0$, we may apply this bounded multiplier property on each $B^s_{p,\infty}(\mathbb{R}^{d_s})$. (Assuming that the number of connected components of $E \cap \Gamma$ is uniformly bounded: this is the transversality condition.)

For $s = 0$ and $0 < t < 1/p$, take a sequence of leaves $\Gamma_n$ tending to the supremum realising the norm (18) of $\chi_E \varphi$. For each leaf $\Gamma_n$, we can construct a $C^r$ foliation of leaves in $\mathcal{F}(\mathbf{C}_+)$ parallel to $\Gamma_n$ (obtained by trivial translations), see (23). Then, the supremum over the leaves of this foliation of the supremum over $\ell$ in (18) is similar in spirit to a mixed Besov [13] norm , where $\sup_{x_+} \|\varphi(\cdot, x_+)\|_{B^0_{p,\infty}(\mathbb{R}^{d_s})}$ replaces $\|\varphi\|_{L_p(\mathbb{R}^d)}$ in $B^t_{p,\infty}(\mathbb{R}^d)$. So we can hope that the bounded multiplier property extends to the case $s = 0$.

In view of the known interpolation results [13, §30], we can hope that interpolating between the cases $t = 0$ and $s = 0$ would give the desired bound for each fixed $\Gamma_n$ (as in [4, Lemma 23]).

As a final comment, note that in [16], [17], or [3], the fact that the systems are only piecewise smooth is not[16] handled by showing that multiplication by characteristic functions of suitable domains $E$ is a bounded operator on the space. Instead, the authors use a $t$-Hölder quotient in the transversal (i.e. unstable) direction, where the leaves $\Gamma$ must be "comparable," i.e., both lie in a single domain $E$ where smoothness (including bounded distortion) holds.

## 4. Bounding the essential spectral radius of $\mathcal{L}_g$ on $\mathcal{U}_1^{t,s}$

In this section, we prove the following result:

**Theorem 4.1** (Essential spectral radius of $\mathcal{L}_g$ on $\mathcal{U}_1^{t,s}$)**.** *If $d_s = 1$ then the essential spectral radius of the transfer operator $\mathcal{L}_g(\varphi) = (g \cdot \varphi) \circ T^{-1}$ enjoys the same upper bound when acting on $\mathcal{U}_1^{t,s}$ as on the space $\mathcal{C}^{t,v}$ from [11] with $v = t + s$, that is:*

$$\exp \sup_{\mu \in \mathrm{Erg}\,(T)} \Big\{ h_\mu(T) + \int \log |g \det(DT|_{E^s})|\, d\mu + \max\big\{ t\chi_\mu(DT^{-1}|_{E^u}), |t+s|\chi_\mu(DT|_{E^s})\big\} \Big\}\,. \tag{25}$$

The bound (25) is the best known [11, 2, 30] estimate on the essential spectral radius in the hyperbolic case. The new norm $\mathcal{U}_1^{t,s}$ is thus at least as good as the norm from [11] if $d_s = 1$. We believe that Theorem 4.1 also holds if $d_s > 1$: Remark 4.5 in §4.2 contains the ideas needed for a proof. We refrain from spelling this proof out in full detail, in order to keep the length of this note within reasonable bounds.

[16] Lemma 3.7 of [17] shows that such characteristic functions *belong* to the space, which is in general a weaker statement.

4.1. **The local Lasota–Yorke Lemma 4.2.** The key ingredient for the proof of Theorem 4.1 is a Lasota–Yorke lemma. We need some notation. Let $F$ be a $C^r$ diffeomorphism defined on an open subset of $\mathbb{R}^d$ containing a compact set $K$. Assume that $F$ is regular cone hyperbolic from a cone pair $\mathbf{C}$ to a cone pair $\mathbf{C}'$. We use the notation

$$\|F\|_+ = \sup_{x\in K} \sup_{\substack{\xi\neq 0\\ DF_x^{tr}(\xi)\notin \mathbf{C}'_-}} \frac{\|DF_x^{tr}(\xi)\|}{\|\xi\|}, \tag{26}$$

$$\|F\|_- = \inf_{x\in K} \inf_{\substack{\xi\neq 0\\ \xi\notin \mathbf{C}_+}} \frac{\|DF_x^{tr}(\xi)\|}{\|\xi\|}, \|F\|_{--} = \sup_{x\in K} \sup_{\substack{\xi\neq 0\\ \xi\notin \mathbf{C}_+}} \frac{\|DF_x^{tr}(\xi)\|}{\|\xi\|}, \tag{27}$$

and

$$|\det(DF|_{\mathbf{C}_+^\perp})|(x) := \inf_{L^\perp\subset \mathbf{C}_+} |\det(DF|_L)|(x), \tag{28}$$

where $\inf_{L^\perp\subset\mathbf{C}_+}$ denotes the infimum over all $d_s$-dimensional subspaces $L\subset\mathbb{R}^d$ with normal subspace contained in $\mathbf{C}_+$, and $\det(DF|_L)(x)$ is the expansion factor of the linear mapping $DF_x : L \to DF_x(L)$, with respect to the volume induced by the Riemannian metric on each $d_s$-dimensional linear subspace.

The key lemma follows:

**Lemma 4.2** (Local Lasota–Yorke estimate)**.** *Let $\mathbf{C}$ and $\mathbf{C}'$ be two cone pairs. and let $K\subset\mathbb{R}^d$ be compact. For any $-(r-1) < s < -t < 0$ there exists $C>0$ so that for every $C^{r-1}$ function $f$ supported in the interior of $K$ and every $C^r$ diffeomorphism $F$ defined on an open subset $U$ of $\mathbb{R}^d$ containing $K$ which is regular cone hyperbolic from $\mathbf{C}_\pm$ to $\mathbf{C}'_\pm$, and such that $\|F\|_-\geq 1$, the following holds: let $\phi\in C^\infty$ be supported in $K$ and $\equiv 1$ on the support of $f$. Set*

$$\mathcal{M}(\varphi) = f\cdot(\varphi\circ F),$$

*then there exists a decomposition $\mathcal{M} = \mathcal{M}_b + \mathcal{M}_c = \phi\mathcal{M}_b + \phi\mathcal{M}_c$ so that, denoting*

$$C(F,\Gamma,s) = |s|\|F^{-1}|_{F(\Gamma)}\|_{C^r}(1+\max\{\|F\|_-^s, \|F\|_-^{-1}\}), \tag{29}$$

*we have*

$$\|\mathcal{M}_b\varphi\|_{\mathcal{U}_p^{\mathbf{C}'_\pm,t,s}} \leq \nu_b\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}} \text{ where} \tag{30}$$
$$\nu_b := C\frac{C(F,\Gamma,s)\|f\circ F^{-1}\|_{C^{r-1}(F(\Gamma))}\|F\|_+^t + \sup|f|\|F\|_-^s\|F\|_{--}^t}{\inf|\det(DF|_{(\mathbf{C}'_+)^\perp})|^{1/p}},$$

*and $\phi\mathcal{M}_c$ is a compact operator from $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(F(K))$ to $\mathcal{U}_p^{\mathbf{C}'_\pm,t,s}(K)$ so that, in addition, for any $\delta>0$, there exists a constant $C_{F,f,\delta}$ so that for any $n_0\geq 1$*

$$\|(\phi-\mathcal{R}_{n_0})\mathcal{M}_c\varphi\|_{\mathcal{U}_p^{\mathbf{C}'_\pm,t,s}} \leq C_{F,f,\delta}2^{-(r-1-\delta-t)n_0}\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}}, \tag{31}$$

*where*

$$\mathcal{R}_{n_0}(\varphi) = \phi\cdot\sum_{n\leq n_0}\psi_n^{Op}(\varphi). \tag{32}$$

Remark 4.6 below explains why the above Lasota–Yorke lemma can probably be enhanced to give a "nuclear power decomposition."

We end this subsection with a Leibniz bound:

**Lemma 4.3** (Leibniz bound on $\mathcal{U}_p^{t,s}$)**.** *Let $r > 1$, and let $-r+1 < s < -t < 0$. If $f : \mathbb{R}^d \to \mathbb{C}$ is $C^{r-1}$ and supported in a compact set $K$ and if $\mathbb{R}^d \setminus \mathbf{C}_+ \Subset \mathbf{C}'_-$, then for all $\varphi \in \mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)$, we have*

$$\|f\varphi\|_{\mathcal{U}_p^{\mathbf{C}'_\pm,t,s}(K)} \le C\|f\|_{C^{r-1}}\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)} .$$

We expect that $\mathcal{M}$ is a bounded operator even if $F$ is not cone hyperbolic, and that the Leibniz inequality above also holds without the conditions on $\mathbf{C}_\pm$ and $\mathbf{C}'_\pm$.

4.2. **Introducing cones — Sublemma 4.4.** In the proof of Lemma 4.2, it will be necessary to distinguish the frequencies in the cotangent space which are in $(DF^{tr})^{-1}(\mathbb{R}^d \setminus \mathbf{C}_+)$. Towards this goal, recalling the function $\chi$ from (7), and letting $\xi \in \mathbb{R}^d$, define $\tilde\psi_0(\xi) = \chi(2^{-1}\|\xi\|)$ and

$$\tilde\psi_\ell(\xi) \quad = \quad \chi(2^{-\ell-1}\|\xi\|) - \chi(2^{-\ell+2}\|\xi\|) , \qquad \ell \ge 1. \tag{33}$$

Note that the $\tilde\psi_\ell$ satisfy (9) and, in addition, $\tilde\psi_\ell(\xi) = 1$ if $\xi \in \mathrm{supp}(\psi_\ell)$ (where the functions $\psi_\ell$, with "thinner supports," giving a partition of unity were defined in (8)). Next, for $\sigma \in \{+,-\}$, write

$$\psi_{\Theta,\ell,\sigma}(\xi) = \psi_\ell(\xi)\varphi_\sigma\left(\frac{\xi}{\|\xi\|}\right) , \ \tilde\psi_{\Theta,\ell,\sigma}(\xi) = \tilde\psi_\ell(\xi)\varphi_\sigma\left(\frac{\xi}{\|\xi\|}\right) . \tag{34}$$

We claim[17] that there exists a constant $C < \infty$ so that for all $\ell$ and all $\varphi$

$$\|\psi_\ell^{Op}(\varphi)\|^s_{p,\Gamma} \le \|\tilde\psi^{Op}_{\Theta,\ell,+}\psi_\ell^{Op}(\varphi)\|^s_{p,\Gamma} + \|\tilde\psi^{Op}_{\Theta,\ell,-}\psi_\ell^{Op}(\varphi)\|^s_{p,\Gamma} \le 2C\|\psi_\ell^{Op}(\varphi)\|^s_{p,\Gamma} . \tag{35}$$

The first inequality is just the triangle inequality since $\psi_\ell = (\varphi_+ + \varphi_-)\tilde\psi_\ell\psi_\ell$. For the second inequality, it is enough to show that for $\sigma = \pm$ and all $\varphi$

$$\sup_\ell \|\tilde\psi^{Op}_{\Theta,\ell,\sigma}(\tilde\varphi)\|^s_{p,\Gamma} \le C\|\tilde\varphi\|^s_{p,\Gamma} . \tag{36}$$

The bound (36) is a consequence of the easily proved fact that (see e.g. [10])

$$\sup_{(\ell,\sigma)} \|\mathbb{F}^{-1}(\tilde\psi_{\Theta,\ell,\sigma})\|_{L_1(\mathbb{R}^d)} < \infty , \tag{37}$$

together with the following version of Young's inequality (which can be proved like [11, Lemma 4.2], see also [2, Chapter 5], by using that any translation $\Gamma + x$ of $\Gamma \in \mathcal{F}$ also belongs to $\mathcal{F}$):

$$\|\hat\psi * \varphi\|^s_{p,\Gamma} \le \|\hat\psi\|_{L_1(\mathbb{R}^d)} \sup_{x\in\mathbb{R}^d} \|\varphi\|^s_{p,\Gamma+x} \le \|\hat\psi\|_{L_1(\mathbb{R}^d)} \sup_{\tilde\Gamma\in\mathcal{F}} \|\varphi\|^s_{p,\tilde\Gamma} . \tag{38}$$

In the sequel, we shall sometimes abusively neglect to insert the operators $\tilde\psi^{Op}_\ell$ or $\tilde\psi^{Op}_{\Theta,\ell,\sigma}$, to simplify notation. (In view of Young's inequality (38) and the almost orthogonality property $\psi_n^{Op} \circ \tilde\psi_\ell^{Op} \equiv 0$ if $|n-\ell| > 5$, this does not create problems.)

The proof of Lemma 4.2 will be based on the following sublemma:

**Sublemma 4.4.** *Let $1 \le p < \infty$, let $-(r-1) < s < 0$, and let $\Theta$, $\Theta'$ and $\mathcal{F}$ be fixed. Then there exists $C$ so that for any $F$, $f$, and $\mathcal{M}$ as in Lemma 4.2, there exists $m_0$ so that for all $n \ge m_0$, all $\Gamma \in \mathcal{F}(\mathbf{C}_+)$, and all $\varphi$,*

$$\|\psi^{Op}_{\Theta',n,-}\mathcal{M}(\varphi)\|^s_{p,\Gamma} \le C\sup_K |f| \frac{\|F\|^s_-}{\inf|\det(DF|_{(\mathbf{C}'_+)^\perp})|^{1/p}} \sup_{\tilde\Gamma\in\mathcal{F}(\mathbf{C}_+)} \|\varphi\|^s_{p,\tilde\Gamma} , \tag{39}$$

[17] Compare to [10, (A.5)] where the situation was a bit different.

*and, in addition, for all* $\Gamma \in \mathcal{F}(\mathbf{C}_+)$*, and all* $\varphi$*, recalling* (29),

$$\|\mathcal{M}(\varphi)\|^s_{p,\Gamma} \le C \frac{C(F,\Gamma,s)\|f\circ F^{-1}\|_{C^{r-1}(F(\Gamma))}}{\inf|\det(DF|_{(\mathbf{C}'_+)^\perp})|^{1/p}} \|\varphi\|^s_{p,F(\Gamma)}\,. \tag{40}$$

Postponing the proofs of Lemma 4.2 and Sublemma 4.4 to §4.3, we next prove the theorem:

*Proof of Theorem 4.1.* If the local map $F$ is $T^{-m}$ in charts, where $T$ is a $C^r$ Anosov diffeomorphism, the bound on $\mathcal{M}_b$ in Lemma 4.2 can be enhanced, as we explain next. First, if $m$ is large enough and $K$ is small enough (the latter follows from taking suitable $m$-dependent partitions of unity, as part of our pedestrian "microlocal" approach), we may assume in addition that $F^{-1}$ is cone hyperbolic from $\mathbf{C}'_\pm$ to $\mathbf{C}_\pm$ and, recalling (29), that

$$|\det(DF|_{\mathbf{C}_+^\perp})| > 1\,,\ \|F\|_{--} \ge \|F\|_- > 1\,,\ \|F\|_+ < 1\,,\ C(F,\Gamma,s) \le 2\,. \tag{41}$$

Since $d_s = 1$, we may in addition ensure that $\|F\|_-/\|F\|_{--} \le 1$ be arbitrarily close to 1, by taking $K$ sufficiently small (via suitably refined partitions of unity). The factor in the right-hand side of (30) in Lemma 4.2 can then be improved to

$$\nu_b := C\frac{\sup_\Gamma \|f\circ F^{-1}\|_{C^{r-1}(F(\Gamma))}\|F\|_+^t + \sup|f|\|F\|_-^{s+t}}{\inf|\det(DF|_{(\mathbf{C}'_+)^\perp})|^{1/p}}\,. \tag{42}$$

Finally, if $F_m = T^{-m}$ and $f_m(x) = \prod_{j=0}^{m-1}(g(T^{-j}(x)))$, it is not difficult to see that for any $\Gamma$

$$\limsup_{m\to\infty}\left(\frac{\|f_m\circ F_m^{-1}\|_{C^{r-1}(F_m(\Gamma))}}{\sup|f_m|}\right)^{1/m} \le 1\,. \tag{43}$$

(Use that all partial derivatives of $F_m^{-1} = T^m$ along the admissible stable leaf $F_m(\Gamma)$ are bounded by $C\lambda_s^m$.) We may thus replace $\|f\circ F^{-1}\|_{C^{r-1}(F(\Gamma))}$ by $C\sup|f|$ in the bound (42). If $p=1$, we claim that this is sufficient to get the claimed bound (25) on the essential spectral radius when $d_s = 1$: Indeed, we may proceed exactly as in [2, Chapter 5, Proof of Thm 5.1] (see also [11]) using Hennion's theorem, and suitable charts to get bounds by thermodynamic sums (see [2, Appendix B]) via partitions of unity (adapted to $T^m$). We refer to [2, Chapter 5, Proof of Thm 5.1] for details. We just mention here that, in the present case, the "fragmentation lemma" (used to expand along a partition of unity) is just the triangle inequality, while the "reconstitution lemma" (used to regroup the terms from a partition of unity) is the trivial inequality $\sum|a_k e_k| \le (\sum|a_k|)\sup|e_k|$ combined with the following[18] variant of Corollary 4.3: If the $\theta_k$ are smooth functions, then $\sup_k\|\theta_k\varphi\|_{\mathcal{U}_1^{t,s}}$ may be bounded by $\|\varphi\|_{\mathcal{U}_1^{t,s}}\cdot\sup_k\|\theta_k\|_{C^0}$ plus a term which can be included in the compact term of the decomposition $\mathcal{L}_g^m$ arising from Lemma 4.2. □

*Remark* 4.5 (The case $d_s > 1$). If $d_s > 1$, assuming for simplicity that $F$ has $d_s$ distinct Lyapunov exponents, we introduce $d_s+1$ cones $\{\mathbf{C}_+, \mathbf{C}_-^{(1)}, \ldots, \mathbf{C}_-^{(d_s)}\}$, satisfying appropriate strict invariance properties, an associated cone system $\Theta_{d_s} = (\mathbf{C}_+, \varphi_+, \mathbf{C}_-^{(j)}, \varphi_-^{(j)}, j = 1,\ldots,d_s)$, and a partition of unity $\varphi_+ + \varphi_- = 1$, with $\varphi_- = \sum_{j=1}^{d_s}\varphi_-^{(j)}$. Considering the partition of unity $\psi^{(Op)}_{\Theta_{d_s},n,+} + \sum_{j=1}^{d_s}(\psi^{(j)}_{\Theta_{d_s},n,-})^{(Op)} = \mathrm{id}$

[18] This variant follows from Lemma 4.2 applied to $F = \mathrm{id}$, using appropriate cones.

generalising (34), and adapting the proofs of Lemma 4.2 and Sublemma 4.4, replaces $\|F\|_-^s \|F\|_{--}^t$ in $\nu_b$ from (30) by $\sum_{j=1}^{d_s} (\|F\|_-^{(j)})^s (\|F\|_{--}^{(j)})^t$ where

$$\|F\|_-^{(1)} = \inf_{x \in K} \inf_{\substack{\xi \neq 0 \\ \xi \notin \mathbf{C}_+}} \frac{\|DF_x^{tr}(\xi)\|}{\|\xi\|}, \|F\|_{--}^{(1)} = \sup_{x \in K} \sup_{\substack{\xi \neq 0 \\ \xi \notin \mathbf{C}_+}} \frac{\|DF_x^{tr}(\xi)\|}{\|\xi\|},$$

and, for $j \geq 2$,

$$\|F\|_-^{(j)} = \inf_{x \in K} \inf_{\substack{\xi \neq 0 \\ \xi \notin (\mathbf{C}_+ \cup_{k=1}^{j-1} \mathbf{C}_-^{(k)})}} \frac{\|DF_x^{tr}(\xi)\|}{\|\xi\|}, \|F\|_{--}^{(j)} = \sup_{x \in K} \sup_{\substack{\xi \neq 0 \\ \xi \notin (\mathbf{C}_+ \cup_{k=1}^{j-1} \mathbf{C}_-^{(k)})}} \frac{\|DF_x^{tr}(\xi)\|}{\|\xi\|}.$$

Just like in the proof of Theorem 4.1 for $d_s = 1$, we can make $\|F\|_-^{(j)}$ as close as desired to $\|F\|_{--}^{(j)}$, so that (42) (and thus the bound from Theorem 4.1 on the essential spectral radius) should also hold if $d_s > 1$.

4.3. **Proving Sublemma 4.4 and Lemma 4.2.** We first prove the lemma and then the sublemma (both proofs will use the modified Young inequality (38)):

*Proof of Lemma 4.2.* We shall use (35). We need more notation: For $m_0 \geq 1$ fixed large enough[19], depending on $F$, $\mathcal{F}$, and $s$, in Sublemma 4.4, we say that $(\ell, \tau) \hookrightarrow (n, \sigma)$ if (exactly) one of the following conditions holds:

- $(\tau, \sigma) = (+, +)$ and $2^n \leq \|F\|_+ 2^{\ell+4}$,
- $(\tau, \sigma) = (-, -)$ and $2^{m_0} \leq 2^n \leq 2^{\ell+4} \|F\|_{--}$,
- $(\tau, \sigma) = (+, -)$ and $2^{m_0} \leq 2^n \leq 2^{\ell+4} \|F\|_{--}$.

Otherwise, we write $(\ell, \tau) \not\hookrightarrow (n, \sigma)$. (This is a variant of the notion used in [10, 11].)

By the definition of $\not\hookrightarrow$ and by cone hyperbolicity, there exists an integer $N(F) > 0$ such that, if $(\ell, \tau) \not\hookrightarrow (n, \sigma)$ and $\max\{n, \ell\} \geq N(F)$, we have

$$d(\mathrm{supp}(\psi_{\Theta', n, \sigma}), DF_x^{tr}(\mathrm{supp}(\tilde{\psi}_{\Theta, \ell, \tau}))) \geq 2^{\max\{n, \ell\} - N(F)} \quad \text{for } x \in \mathrm{supp}(f). \tag{44}$$

We decompose $\mathcal{M} = \mathcal{M}_b + \mathcal{M}_c$ where

$$\mathcal{M}_b \varphi = \sum_{(n, \sigma)} \psi_{\Theta', n, \sigma}^{Op} \sum_{(\ell, \tau) \hookrightarrow (n, \sigma)} \mathcal{M}(\psi_{\Theta, \ell, \tau}^{Op} \varphi),$$

and

$$\mathcal{M}_c \varphi = \sum_{(n, \sigma)} \psi_{\Theta', n, \sigma}^{Op} \sum_{(\ell, \tau) \not\hookrightarrow (n, \sigma)} \mathcal{M}(\psi_{\Theta, \ell, \tau}^{Op} \varphi).$$

We first prove the bound (30) on $\mathcal{M}_b$. Fix $\Gamma$ and $(n, \sigma)$. We want to estimate

$$2^{nt - \ell t} \|\psi_{\Theta', n, \sigma}^{Op} \sum_{(\ell, \tau) \hookrightarrow (n, \sigma)} \mathcal{M}(\psi_{\Theta, \ell, \tau}^{Op} \varphi)\|_{p, \Gamma}^s.$$

If $\sigma = +$, we have for any $(\ell, \tau) \hookrightarrow (n, +)$ that $\tau = +$, and, since $t > 0$, the definition ensures $2^{nt} \leq C \|F\|_+^t 2^{\ell t}$. This implies $\sum_{(\ell, +) \hookrightarrow (n, +)} 2^{nt - \ell t} \leq C \|F\|_+^t$ and by (40) from Sublemma 4.4 and (36), we obtain the term with $\|F\|_+^t$ in (30).

If $\sigma = -$ and $\tau = -$, then since $t > 0$, it follows that for any $(\ell, -) \hookrightarrow (n, -)$

$$2^{nt} \leq C \|F\|_{--}^t 2^{\ell t}. \tag{45}$$

If $\sigma = -$ and $\tau = +$, since $t > 0$, it follows that for any $(\ell, +) \hookrightarrow (n, -)$

$$2^{nt} \leq C \|F\|_{--}^t 2^{\ell t}. \tag{46}$$

[19] The constant $C_{F, f, \delta}$ in (31) depends on $m_0$ but the constant in (30) does not.

So, if $\sigma = -$ then, by (39) from the Sublemma, we get the term with $\|F\|_-^s \|F\|_{--}^t$ in (30) (if $m_0$ is large enough).

Recall that for $k \in \mathbb{Z}_+^*$ the $k$-th approximation number of a bounded operator $\mathcal{Q} : \mathcal{B} \to \mathcal{B}'$ between Banach spaces is

$$a_k(\mathcal{Q}) = \inf\{\|\mathcal{Q} - \mathcal{R}\|_{\mathcal{B}\to\mathcal{B}'} \mid \mathrm{rank}\,(\mathcal{R}) < k\}\,. \tag{47}$$

Clearly, $\lim_{k\to\infty} |a_k(\mathcal{Q})| = 0$ implies that $\mathcal{Q}$ is compact. Using the bound (31) and Lemma 3.5 to control the approximation numbers of $\phi\mathcal{M}_c$ (as in [11] and [2]) implies the compactness claim on $\phi\mathcal{M}_c$.

It remains to show the bound (31) on $\mathcal{M}_c$. For this, we shall use integration by parts as in [10, 11]: Recalling the functions $\tilde\psi_\ell$ from (33), we claim that it is enough to show that if $(\ell,\tau) \not\hookrightarrow (n,\sigma)$ then

$$\|\psi^{Op}_{\Theta',n,\sigma}(\mathcal{M}\,\tilde\psi^{Op}_{\Theta,\ell,\tau}\tilde\varphi)\|^s_{p,\Gamma} \le \sup_{\tilde\Gamma\in\mathcal{F}(\mathbf{C}_+)} C_{F,f} 2^{-(r-1)\max\{n,\ell\}} \|\tilde\varphi\|^s_{p,\tilde\Gamma}\,. \tag{48}$$

Indeed, since $\varphi_{\Theta,\ell,\tau} = \tilde\psi^{Op}_{\Theta,\ell,\tau}\psi^{Op}_{\Theta,\ell,\tau}\varphi$, we find for any $\Gamma \in \mathcal{F}(\mathbf{C}'_+)$ and any $m_1 \ge 10$, using (48),

$$\begin{aligned}
&\sup_{(n,\sigma)} 2^{nt} \sum_{(\ell,\tau)\not\hookrightarrow(n,\sigma)} \sum_{n'>m_1} \|\psi^{Op}_{n'}(\psi^{Op}_{\Theta',n,\sigma}\mathcal{M}\varphi_{\Theta,\ell,\tau})\|^s_{p,\Gamma}\\
&\qquad\le \sup_{(n,\sigma)\,,\,n\ge m_1-5} \quad 2^{nt} \sum_{(\ell,\tau)\not\hookrightarrow(n,\sigma)} \sum_{n'>m_1} \|\psi^{Op}_{\Theta',n,\sigma} * (\mathcal{M}(\tilde\psi^{Op}_{\Theta,\ell,\tau}\varphi_{\Theta,\ell,\tau}))\|^s_{p,\Gamma}\\
&\qquad\le \sup_{\tilde\Gamma} C_{F,f} \sup_{(n,\sigma)\,,\,n\ge m_1-5} \sum_{(\ell,\tau)} 2^{-(r-1)\max\{n,\ell\}} 2^{(n-\ell)t} 2^{t\ell} \|\varphi_{\Theta,\ell,\tau}\|^s_{p,\tilde\Gamma}\\
&\qquad\le \sup_{\tilde\Gamma} C_{F,f} \sup_{n\ge m_1-5} (n+C) 2^{(t-(r-1))n} \sup_{(\ell,\tau)} 2^{t\ell} \|\varphi_{\Theta,\ell,\tau}\|^s_{p,\tilde\Gamma}\,.
\end{aligned} \tag{49}$$

Thus, using Corollary 4.3 in order to take into account[20] the factor $\phi$ (this is legitimate since the proof of Corollary 4.3 does not use anything beyond (48) in the present proof), we get for any $\varphi$ supported in $F(K)$ that

$$\|\phi\cdot\big(\sum_{n>m_1} \psi^{Op}_n(\mathcal{M}_c\varphi)\big)\|_{\mathcal{U}_p^{\mathbf{C}'_\pm,t,s}(K)} \le C_{F,f,\delta} 2^{-(r-1-t-\delta)m_1} \|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(F(K))}\,, \tag{50}$$

for any $\delta > 0$ and any $m_1 \ge 1$ (the case $1 \le m_1 < 10$ is trivial), by the definition of $\mathcal{M}_c$. The estimate (49) also gives that $\mathcal{M}_c$ is bounded from $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}$ to $\mathcal{U}_p^{\mathbf{C}'_\pm,t,s}$.

To prove (48), we use (38) together with integration by parts: Since (48) is obvious when $\max\{n,\ell\} < N(F)$, we shall assume $\max\{n,\ell\} \ge N(F)$. We have

$$\psi^{Op}_{\Theta',n,\sigma}(\mathcal{M}\,\tilde\psi^{Op}_{\Theta,\ell,\tau}\varphi)(x) = (2\pi)^{-2d} \int V^{\ell,\tau}_{n,\sigma}(x,y)\cdot\varphi\circ F(y)|\det DF(y)|\mathrm{d}y,$$

where

$$V^{\ell,\tau}_{n,\sigma}(x,y) = \int e^{i(x-w)\xi + i(F(w)-F(y))\eta} f(w)\psi_{\Theta',n,\sigma}(\xi)\tilde\psi_{\Theta,\ell,\tau}(\eta)\mathrm{d}w\mathrm{d}\xi\mathrm{d}\eta\,. \tag{51}$$

[20]We should use here the cone hyperbolicity assumption to insert intermediate cones here, for simplicity we disregard this operation.

Since $\|\varphi\circ F\|^s_{p,\Gamma}\le C(F)\|\varphi\|^s_{p,F(\Gamma)}$, the bound (48) follows if we show that there exists $C_{F,f}$ such that for all $(\ell,\tau)\not\hookrightarrow(n,\sigma)$ and all $1<p\le\infty$ the integral operator

$$H^{\ell,\tau}_{n,\sigma}: v\mapsto \int V^{\ell,\tau}_{n,\sigma}(\cdot,y)v(y)\mathrm{d}y$$

satisfies

$$\|H^{\ell,\tau}_{n,\sigma}(v)\|^s_{p,\Gamma}\le C_{F,f}\cdot 2^{-(r-1)\max\{n,\ell\}}\sup_{\tilde\Gamma}\|v\|^s_{p,\tilde\Gamma}\,.$$

Defining the integrable function $b:\mathbb{R}^d\to\mathbb{R}_+$ by

$$b(x)=1 \quad \text{if } \|x\|\le 1,\qquad b(x)=\|x\|^{-d-1}\quad \text{if } \|x\|>1, \tag{52}$$

we set for $m>0$

$$b_m:\mathbb{R}^d\to\mathbb{R},\qquad b_m(x)=2^{dm}\cdot b(2^m x)\,, \tag{53}$$

so that $\|b_m\|_{L_1}=\|b\|_{L_1}$. The required estimate on $H^{\ell,\tau}_{n,\sigma}$ then follows if we show

$$|V^{\ell,\tau}_{n,\sigma}(x,y)|\le C_{F,f}2^{-(r-1)\max\{n,\ell\}}\cdot b_{\min\{n,\ell\}}(x-y)\,, \tag{54}$$

for some $C_{F,f}>0$ and all $(\ell,\tau)\not\hookrightarrow(n,\sigma)$. Indeed, as the right hand side of (54) is written as a function of $x-y$, we can apply (38). Finally, (54) can be proved by integrating (51) by parts $(r-1)$ times with respect to $w$ in the sense of Appendix C and using (44), just like in [10, 11]. □

The Leibniz bound is now straightforward:

*Proof of Lemma 4.3.* The claim is an immediate consequence of (30) and the bound (48) in the first part of the proof of Lemma 4.2. □

It remains to prove the sublemma:

*Proof of Sublemma 4.4.* Since $-(r-1)<s<0$, the bound (40) is not difficult to prove, using e.g. the fact that $B^s_{p,\infty}(\mathbb{R}^{d_s})$ is the dual of little Besov space $b^{|s|}_{p/(p-1),1}(\mathbb{R}^{d_s})$, and is left to the reader.

Fix $\phi$, smooth, compactly supported and $\equiv 1$ on the support of $f$. To prove (39), we shall show that there exists a constant $C$ (depending only on the cone systems, and on the support and the $C^r$ norm of $\phi$) and for any $\delta>0$, there exists a constant $C_{F,f,\delta}$ so that, for any $\tilde\Gamma\in\mathcal{F}$ and any $C^\infty$ function $\varphi$ on $\mathbb{R}^d$, there exists a decomposition

$$\mathcal{M}(\varphi)(w)=\mathcal{M}_{b,\tilde\Gamma}(\varphi)(w)+\mathcal{M}_{c,\tilde\Gamma}(\varphi)(w)\,,\quad \forall w\in\tilde\Gamma\,, \tag{55}$$

so that

$$\|\phi\mathcal{M}_{b,\tilde\Gamma}(\varphi)\|^s_{p,\tilde\Gamma}\le C\sup|f|\frac{\|F\|^s_-}{\inf|\det(DF|_{(\mathbf{C}'_+)^\perp})|^{1/p}}\|\varphi\|^s_{p,F(\tilde\Gamma)}\,, \tag{56}$$

and, for any $\tilde n_s\ge 1$,

$$\|(\phi-\mathcal{R}_{\tilde n_s,\tilde\Gamma})\mathcal{M}_{c,\tilde\Gamma}(\varphi)\|^s_{p,\tilde\Gamma}\le C_{F,f,\delta}2^{-(r-1-\delta-|s|)\tilde n_s}\|\varphi\|^s_{p,F(\tilde\Gamma)}\,, \tag{57}$$

where, recalling $\psi^{Op(\tilde\Gamma)}_{n_s}$ from (16), we set, for $w\in\tilde\Gamma$,

$$\mathcal{R}_{\tilde n_s,\tilde\Gamma}(\varphi)(w)=\phi(w)\cdot\sum_{n_s\le\tilde n_s}\psi^{Op(\tilde\Gamma)}_{n_s}(\varphi)(w)\,.$$

To construct the decomposition and prove the claims above, set, for $w \in \tilde\Gamma$,

$$\mathcal{M}_{b,\tilde\Gamma}(\varphi)(w) = \sum_{n_s} \psi_{n_s}^{Op(\tilde\Gamma)} \sum_{\ell_s \hookrightarrow_s n_s} \mathcal{M}(\tilde\psi_{\ell_s}^{Op(F(\tilde\Gamma))}(\varphi))(w)\,,$$

where

$$\ell_s \hookrightarrow_s n_s \qquad \text{if } \|F\|_- 2^{\ell_s - 4} \le 2^{n_s}\,. \tag{58}$$

If $\|F\|_- 2^{\ell_s-4} > 2^{n_s}$ then we say $\ell_s \not\hookrightarrow_s n_s$. For $w \in \tilde\Gamma$, we put

$$\mathcal{M}_{c,\tilde\Gamma}(\varphi)(w) = \sum_{n_s} \psi_{n_s}^{Op(\tilde\Gamma)} \sum_{\ell_s \not\hookrightarrow_s n_s} \mathcal{M}(\tilde\psi_{\ell_s}^{Op(F(\tilde\Gamma))}(\varphi))(w)\,.$$

This gives (55). We next check (56) and (57).

First, since $s<0$, (56) follows from the definition of $\hookrightarrow_s$ combined with the fact that

$$\|\psi_{n_s}^{Op(\tilde\Gamma)}\tilde\varphi\|_{L_p(\mu_{\tilde\Gamma})} \le C\|\tilde\varphi\|_{L_p(\mu_{\tilde\Gamma})}\,,\ \|\psi_{\ell_s}^{Op(F(\tilde\Gamma))}\varphi\|_{L_p(\mu_{F(\tilde\Gamma)})} \le C\|\varphi\|_{L_p(\mu_{\tilde\Gamma})}\,,\ \text{and}$$
$$\|\mathcal{M}(\tilde\varphi)\|_{L_p(\mu_{\tilde\Gamma})} \le C\frac{\sup|f|}{\inf|\det DF|_{\tilde\Gamma}|^{1/p}}\|\tilde\varphi\|_{L_p(\mu_{F(\tilde\Gamma)})}\,,$$

simplifying the argument in [10, 11, 2] (see also the proof of the parallel statement on $\mathcal{M}_b$ in the proof of Lemma 4.2 above, in particular (48) and (49)).

Next, by definition of $\not\hookrightarrow_s$, there exists an integer $N(F,\mathcal{F})>0$ (depending on $F$ and the cones, but not $\tilde\Gamma$) such that if $\ell_s \not\hookrightarrow_s n_s$ then

$$\ell_s \ge n_s - N(F,\mathcal{F})$$

and

$$\inf_{w\in\mathbb{R}^{d_s}} d(\mathrm{supp}(\psi_{n_s}), D(\pi_{F(\tilde\Gamma)}\circ F\circ\pi_{\tilde\Gamma}^{-1})_w^{tr}(\mathrm{supp}(\tilde\psi_{\ell_s}))) \ge 2^{\max\{n_s,\ell_s\}-2N(F,\mathcal{F})}\,. \tag{59}$$

The proof of (57) is then obtained by $(r-1)$ integration by parts, in the sense of Appendix C, in the kernel $V^{\ell_s}_{n_s,\tilde\Gamma}(w,y)$, with $w,y\in\mathbb{R}^{d_s}$, for

$$\psi_{n_s}^{Op(\tilde\Gamma)}\mathcal{M}(\psi_{\ell_s}^{Op(F(\tilde\Gamma))}(\varphi))$$

when $\ell_s \not\hookrightarrow_s n_s$, using (59). Just like for the estimate (50) on $\mathcal{M}_c$ in the proof of Lemma 4.2 above, this is a simplification of the argument in [10, 11, 2], so we do not enter into details.

From now on, we fix $\Gamma$. To deduce (39) from (56–57), we shall need to couple wave packets in the cotangent spaces of $\mathbb{R}^d$ and $\tilde\Gamma = \Gamma + x$ for $x\in\mathbb{R}^d$. (For this, it is essential that we have $(n,-)$ in the left-hand side of (39).) Recalling the functions $b_m$ from (53), we claim that there exists a constant $C_0>1$ depending only on $C_\mathcal{F}$ and $\mathbf{C}_\pm$ so that, for any $\Gamma\in\mathcal{F}(\mathbf{C}_+)$ and all $n$, $n_s$, the kernels $V^{n,-}_{n_s,\Gamma+x}(w,y)$ defined for $w\in\Gamma$, $x\in\mathbb{R}^d$, and $y\in\Gamma$ by

$$\mathbb{F}^{-1}(\psi_{\Theta',n,-})(-x)(\phi\cdot\psi_{n_s}^{Op(\Gamma+x)}\tilde\varphi)(w+x) = \frac{1}{(2\pi)^{d+d_s}}\int_\Gamma V^{n,-}_{n_s,\Gamma+x}(w,y)\tilde\varphi(y+x)\,\mathrm{d}y\,,$$

satisfy[21]

$$|\int_{\mathbb{R}^d} V^{n,-}_{n_s,\Gamma+x}(w,y)\,\mathrm{d}x| \le C_0 2^{-(r-1)n} b_{n_s}(w-y) \text{ if } C_0 2^{n_s} \le 2^n \text{ or } 2^{n_s} \ge C_0 2^n\,. \tag{60}$$

To prove (60), recalling (16), notice that $\int V^{n,-}_{n_s,\Gamma+x}(w,y)\,\mathrm{d}x$ is just

$$\int_{x,\eta\in\mathbb{R}^d,\,\eta_s\in\mathbb{R}^{d_s}} \phi(w+x)|\det D\pi_{\Gamma+x}(y)|e^{-ix\eta}\cdot e^{i(\pi_{\Gamma+x}(w+x)-\pi_{\Gamma+x}(y+x))\eta_s} \times \psi^{(d_s)}_{n_s}(\eta_s)\psi_{\Theta',n,-}(\eta)\mathrm{d}\eta\mathrm{d}\eta_s\mathrm{d}x\,,$$

and integrate by parts (see Appendix C) $(r-1)$ times with respect to $x$ in the right-hand side, just like in [10, 11] (see also [2]) using the facts that $\pi_{\Gamma+x}(u+x)=\pi_\Gamma(u)+x_-$ if $x=(x_-,x_+)\in\mathbb{R}^{d_s}\times\mathbb{R}^{d_u}$, and that $\Gamma\in\mathcal{F}$.

We finally conclude the proof of (39). Let $n\ge m_0$ and $\varphi\in C^\infty$. Recall (5). For $w\in\Gamma$ and $x\in\mathbb{R}^d$, decomposing $\mathcal{M}\varphi=\phi\mathcal{M}\varphi$ via (55) for $\tilde\Gamma=\Gamma+x$, we get

$$(\mathbb{F}^{-1}\psi_{\Theta',n,-})(-x)\cdot(\mathcal{M}(\varphi))(w+x)=$$
$$(\mathbb{F}^{-1}\psi_{\Theta',n,-})(-x)\cdot(\phi\mathcal{M}_{b,\Gamma+x}(\varphi))(w+x) \tag{61}$$
$$+(\mathbb{F}^{-1}\psi_{\Theta',n,-})(-x)\cdot(\mathcal{R}_{\tilde n_s,\Gamma+x}\circ\mathcal{M}_{c,\Gamma+x})(\varphi)(w+x) \tag{62}$$
$$+(\mathbb{F}^{-1}\psi_{\Theta',n,-}(-x))\cdot(\phi-\mathcal{R}_{\tilde n_s,\Gamma+x})\mathcal{M}_{c,\Gamma+x}(\varphi)(w+x)\,. \tag{63}$$

We average over $x\in\mathbb{R}^d$. Then, recalling (38), the $\|\cdot\|^s_{p,\Gamma}$-norm of the contribution of (61) may be estimated by (56). Also, noting that if $\tilde n_s$ is large enough (depending on $f$, $F$, $s$, and $\mathcal{F}$, but not on $\Gamma$), then

$$C_{F,f,\delta}2^{-(r-1-\delta-|s|)\tilde n_s}\le C\sup|f|\frac{\|F\|^s_-}{\inf|\det(DF|_{(\mathbf{C}'_+)^\perp})|^{1/p}}\,,$$

the $\|\cdot\|^s_{p,\Gamma}$-norm of the contribution of the last term (63) may be controlled by (57).

It only remains to control the contribution of (62). Since we may combine (40) with (56) to show that there exists a constant $\tilde C_1$ depending only on $f$, $F$ and $\mathcal{F}$ (but not on $\Gamma$ or $x$)

$$\|\mathcal{M}_{c,\Gamma+x}\varphi\|^s_{p,\Gamma+x}=\|\mathcal{M}\varphi-\mathcal{M}_{b,\Gamma+x}\varphi\|^s_{p,\Gamma+x}\le\tilde C_1\|\varphi\|^s_{p,F(\Gamma+x)}\,,$$

it suffices to establish, setting $\tilde\varphi=\mathcal{M}_{c,\Gamma+x}(\varphi)$, that there exists a constant $C_1$ depending only on $F$ and $\mathcal{F}$ (but not on $\Gamma$) so that for any fixed $\tilde n_s$, if $m_0$ is large enough, then for any $n\ge m_0$

$$\|\int\mathbb{F}^{-1}(\psi_{\Theta',n,-})(-x)\cdot(\mathcal{R}_{\tilde n_s,\Gamma+x})(\tilde\varphi)(\cdot+x)\mathrm{d}x\|^s_{p,\Gamma+x}\le\sup_x C_1 2^{-(r-1)m_0}\|\tilde\varphi\|^s_{p,\Gamma+x}\,. \tag{64}$$

Taking $2^{m_0}>C_02^{\tilde n_s}$, the bound (60) gives (64). □

We end with a remark on the kneading operator approach:

[21]For the kernels $V^{n,+}_{n_s,\Gamma+x}(w,y)$ defined by replacing $\psi_{\Theta',n,-}$ with $\psi_{\Theta,n,+}$, we only get $C_0>1$ so that $|\int V^{n,+}_{n_s,\Gamma+x}(w,y)\,\mathrm{d}x|\le C_02^{-(r-1)n}b_{n_s}(w-y)$ if $C_02^{n_s}\ge 2^n$. In particular, $V^{n,+}_{n_s,\Gamma+x}$ need not be small if $n$ is big and $n_s$ small.

*Remark* 4.6 (Nuclear power decomposition). Using approximation numbers (47) as in [11] and [2], Lemma 3.5 should imply, not only compactness of $\mathcal{M}_c$, but also that there exists an integer $D \geq 2$ (depending only on $r$, $s$, $t$, and $d$) so that $\mathcal{M}_c^D$ is nuclear. (This is the desired "nuclear power decomposition.") Also, we expect that (adapting the arguments of [11, 2]) for any $0 < \kappa < 1$ there exists $C_\kappa > 1$ so that the flat trace [11] of the term $\mathcal{M}_b$ for the operator $\mathcal{M}$ associated to $T^{-n}$ and $g^{(-n)}$ is smaller than $C_\kappa \kappa^n$.

## Appendix A. Characteristic functions are not bounded multipliers on the "microlocal" anisotropic spaces from §2.3

For simplicity, we only consider the scale $W^{\Theta,t,0}_{2,\dagger}$ in dimension $d = 2$ for $t > 0$, and ignore the charts completely, but the argument extends to all spaces $W^{\Theta,t,s}_{p,\dagger}$, to the other spaces in [10] and [11] (if $s < 0$ or $t > 0$), and to the spaces introduced by Faure–Roy–Sjöstrand [19] and[22] their variants. We shall outline the proof[23] of the following claim:

**Proposition A.1** (Gouëzel [25]). *Let $1_E$ be the caracteristic function of a half-plane $E$ in $\mathbb{R}^2$. Let $F$ be a linear transformation of $\mathbb{R}^2$ fixing two lines $D_+$ and $D_-$. Let $\widetilde{\Theta}$ and $\Theta'$ be two cone systems so that the corresponding cones $\mathbf{C}'_-$, $\tilde{\mathbf{C}}_-$, and $\tilde{\mathbf{C}}_+$, $\mathbf{C}'_+$ in $\mathbb{R}^2$ are centered on $D_-$ and $D_+$, respectively. Then for any $t > 0$, the operator $\mathcal{E}(\varphi) = 1_E \cdot (\varphi \circ F)$ does not map $W^{\widetilde{\Theta},2t,0}_{2,\dagger}$ into $W^{\Theta',2t,0}_{2,\dagger}$.*

The basic idea is that, in Fourier transform, multiplication by a Heaviside function becomes (essentially) a convolution with $(i\xi)^{-1}$ , and such a convolution may transform a function with square integrable Fourier transform supported in $\tilde{\mathbf{C}}_-$, into a function with Fourier transform decaying slower than any $\xi^{-t}$ in $\mathbf{C}'_+$. (The main issue is that the support of the Fourier transform "leaks" from $\tilde{\mathbf{C}}_-$ into $\mathbf{C}'_+$, due to convolution with $(i\xi)^{-1}$. This creates similar problems for the spaces introduced by Faure–Roy–Sjöstrand [19].)

*Sketch of the proof of Proposition A.1.* We claim that it is enough to show that the operator of multiplication by $1_E$ does not map $W^{\Theta,2t,0}_{2,\dagger}$ into $W^{\Theta',2t,0}_{2,\dagger}$ for any quadruple of cones in $\mathbb{R}^2$, centered on $D_-$ and $D_+$. Indeed, denote by $\mathcal{M}_F$ the operator mapping $\varphi$ to $\varphi \circ F$. If $F$ maps $\tilde{\mathbf{C}}_-$ and $\tilde{\mathbf{C}}_+$ into cones respectively included in $\mathbf{C}_-$ and containing $\mathbf{C}_+$, then $\mathcal{M}_F$ maps $W^{\Theta,2t,0}_{2,\dagger}$ continuously into $W^{\widetilde{\Theta},2t,0}_{2,\dagger}$. Assume by contradiction that $\mathcal{E}$ maps $W^{\widetilde{\Theta},2t,0}_{2,\dagger}$ into $W^{\Theta',2t,0}_{2,\dagger}$. Then, precomposing with $\mathcal{M}_{F^{-1}}$ we would get that $\varphi \mapsto 1_E \cdot \varphi$ maps $W^{\Theta,2t,0}_{2,\dagger}$ into $W^{\Theta',2t,0}_{2,\dagger}$, contradicting our assumption and proving the claim.

From now on, we focus on the operator of multiplication by $1_E$. In order to compute the Fourier transform $\mathbb{F}(1_E\varphi)$, we compute the Fourier transform of $1_E$. As a starting point, let $\tilde{\chi} = \mathbf{1}_{[0,\infty)}$ in dimension 1. Then $\tilde{\chi}' = \delta_0$ the Dirac mass

[22] As we were finishing this paper, F. Faure and M. Tsujii [21] announced a new version of microlocal anisotropic spaces for which the wave front set is more narrowly constrained. The counter-example in this appendix may fail for these new spaces.

[23] We take $F$ linear and a domain given by a half-plane for simplicity, the general case is similar.

at 0. Thus, since the Fourier transform of the Dirac mass is the constant function equal to 1, we have, formally

$$\mathbb{F}(\tilde\chi)(\xi) = \frac{1}{i\xi}\mathbb{F}(\tilde\chi')(\xi) = \frac{1}{i\xi}\mathbb{F}(\delta_0) = \frac{1}{i\xi}\,. \tag{65}$$

(In fact, $\mathbb{F}(\tilde\chi)$ is the distribution obtained by summing a Dirac mass at 0 and the "principal value of $1/\xi$,", but it will be sufficient to work with the approximation above.)

Let now $1_E$ be the characteristic function of a half-plane $E$ bounded by a line through the origin (we can reduce to this case by translation) directed by a unit vector $v$. The function $1_E$ restricted to any line orthogonal to $v$ is just the characteristic function of a half-line. Since $\mathbb{F}(1_E\varphi) = \mathbb{F}(1_E) * \mathbb{F}(\varphi)$, we have for any $\xi \in \mathbb{R}^2$,

$$\mathbb{F}(1_E\varphi)(\xi) \sim \int_{\omega\in\mathbb{R}} \frac{(\mathbb{F}\varphi)(\xi+\omega w)}{\omega}\mathrm{d}\omega + (\mathbb{F}\varphi)(\xi)\,, \tag{66}$$

where $w$ is the unit vector orthogonal to $v$ pointing towards the interior of the half-plane. (The symbol $\sim$ above means that we neglect unimportant factors such as $i$.)

There are three main cases to consider, depending on the position of the boundary of the half-plane with respect to the cones: In the interior of $\mathbf{C}_+$, in the interior of $\mathbf{C}_-$, or in the complement of their union. (The remaining case when the boundary of the half-plane lies on the boundary of a cone is similar.) We discuss each case by considering concrete examples of lines $D_+$ and $D_-$. The general situation may be handled by analogous arguments.

For the first case, we take $\mathbf{C}_-$ around the vertical axis, $\mathbf{C}_+$ around the horizontal axis, and a left half-plane with vertical boundary through the origin, $w=(-1,0)$. (The boundary of the half-plane thus lies inside $\mathbf{C}_-$.) Let $\varphi \in L_2$ be so that $\hat\varphi := \mathbb{F}(\varphi) \in L_2$ is supported in $\mathbf{C}_-$. In view of (66), the Fourier transform of $\psi = 1_E\varphi$ is given by the following convolution (modulo trivial correcting factors and terms)

$$\hat\psi(\xi_1,\xi_2) = \int_{\omega\in\mathbb{R}} \frac{\hat\varphi(\xi_1-\omega,\xi_2)}{\omega}\mathrm{d}\omega\,. \tag{67}$$

We now construct $\varphi \in L_2$ (this implies $\varphi \in W^{\Theta,2t,0}_{2,\dagger}$) so that $\int_{\mathbf{C}'_+}|\hat\psi|^2(1+|\xi|^2)^t = +\infty$ for all $t>0$, implying that $1_E\varphi \notin W^{\Theta',2t,0}_{2,\dagger}$). For this, take $\varphi$ so that

$$\hat\varphi(\xi_1,\xi_2) = \mathbf{1}_{\mathbf{C}_-}\phi(\xi_2)\,,$$

with $\phi(\xi_2)>0$ if $\xi_2 \ge 2$, and $\phi(\xi_2)=0$ if $\xi_2<2$, assuming also

$$\int_{\mathbf{C}_-}|\hat\varphi|^2\mathrm{d}\xi = \int_{\xi_2\ge 2}\xi_2\phi(\xi_2)^2\mathrm{d}\xi_2 < \infty\,. \tag{68}$$

Then, it is easy to see that for $(\xi_1,\xi_2)$ in $\mathbf{C}'_+$,

$$\hat\psi(\xi_1,\xi_2) \sim \phi(\xi_2)\frac{|\xi_2|}{|\xi_1|}, \tag{69}$$

where $|\xi_2|$ corresponds to the width of $\mathbf{C}_-$ at height $\xi_2$, and $|\xi_1|^{-1}$ comes from the factor $1/\omega$ in the formula for $\hat{\psi}$. Therefore,

$$\begin{aligned}\int_{\mathbf{C}'_+} |\hat{\psi}(\xi)|^2(1+|\xi|^2)^t \mathrm{d}\xi &\sim \int_{\xi_1>2}\int_{2\le\xi_2\le c'\xi_1} \phi(\xi_2)^2\frac{\xi_2^2}{\xi_1^2}\xi_1^{2t}\mathrm{d}\xi_1\mathrm{d}\xi_2\\ &\sim \int_{\xi_2>2}\left(\int_{\xi_1\ge\xi_2}\xi_1^{2t-2}\mathrm{d}\xi_1\right)\xi_2^2\phi(\xi_2)^2\mathrm{d}\xi_2\\ &\sim \int_{\xi_2>2}\xi_2^{2t-1}\xi_2^2\phi(\xi_2)^2\mathrm{d}\xi_2 \sim \int_{\xi_2\ge2}\xi_2^{1+2t}\phi(\xi_2)^2\mathrm{d}\xi_2\,.\end{aligned}$$

If $t>0$, it is easy to find $\phi$ so that (68) holds but the integral above is infinite. (This cannot be achieved when $t=0$, reflecting the fact that multiplication by $1_E$ leaves $L_2$ invariant.)

For the second case, we keep the same cones, but now take the upper half-plane bounded by the horizontal axis through zero (i.e., $w=(0,1)$, and the boundary of the half-plane lies inside $\mathbf{C}_+$). Then, taking the same $\varphi$, we have for $(\xi_1,\xi_2)\in\mathbf{C}'_+$,

$$\hat{\psi}(\xi_1,\xi_2) = \int_{\omega\in\mathbb{R}}\frac{\phi(\xi_1,\xi_2+\omega)}{\omega}\mathrm{d}\omega \sim \int_{\omega\ge c\xi_1}\frac{\phi(\omega)}{\omega}\mathrm{d}\omega\,. \tag{70}$$

Then, for suitable $c>0$ and $c'>0$,

$$\begin{aligned}\int_{\mathbf{C}'_+} |\hat{\psi}(\xi)|^2(1+|\xi|^2)^t \mathrm{d}\xi &\sim \int_{\xi_1>2}\int_{|\xi_2|\le c'\xi_1}\left(\int_{\omega\ge c\xi_1}\frac{\phi(\omega)}{\omega}\mathrm{d}\omega\right)^2\xi_1^{2t}\mathrm{d}\xi_1\mathrm{d}\xi_2\\ &\sim \int_{\xi_1>2}\left(\int_{\omega\ge c\xi_1}\frac{\phi(\omega)}{\omega}\mathrm{d}\omega\right)^2\xi_1^{1+2t}\mathrm{d}\xi_1\,.\end{aligned}$$

Take $\phi(\xi_2)=1/(\xi_2\log\xi_2)$. Then (68) holds but

$$\begin{aligned}\int_{\mathbf{C}'_+} |\psi(\xi)|^2(1+|\xi|^2)^t \mathrm{d}\xi &\sim \int_{\xi_1>2}\left(\int_{\omega\ge c\xi_1}\frac{1}{\omega^2\log t}\mathrm{d}\omega\right)^2|\xi_1|^{1+2t}\mathrm{d}\xi_1\\ &\sim \int_{\xi_1>2}\left(\frac{1}{\xi_1\log\xi_1}\right)^2|\xi_1|^{1+2t}\mathrm{d}\xi_1 \sim \int_{\xi_1>2}\frac{|\xi_1|^{2t}}{|\xi_1|(\log\xi_1)^2}\mathrm{d}\xi_1\,.\end{aligned}$$

The above integral is infinite for $t>0$, as claimed. (Like in the first example, the integral converges for $t=0$.)

Finally, for the third case, we consider $\mathbf{C}_-=\{-\xi_2\le\xi_1\le-\xi_2/2\}$ and $\mathbf{C}'_+=\{\xi_2/2\le\xi_1\le\xi_2\}$, taking the left half-plane with vertical boundary through the origin ($w=(-1,0)$, like in the first case, but the boundary now lies in the complement of the union of the two cones). We take $\hat{\varphi}$ as above. Then $\hat{\varphi}\in L_2$ if and only if

$$\int_{\mathbf{C}_-}|\hat{\varphi}|^2\mathrm{d}\xi \sim \int_{\xi_2>2}\xi_2\phi(\xi_2)^2\mathrm{d}\xi_2<\infty\,. \tag{71}$$

(This condition is the same as (68) modulo a constant factor due to the new cone.) Using (66) again, the Fourier transform of $\psi=1_E\varphi$ on $\mathbf{C}'_+$ is given by (modulo trivial corrections)

$$\hat{\psi}(\xi_1,\xi_2)\sim\int_{\omega\in\mathbb{R}}\frac{\hat{\varphi}(\xi_1-\omega,\xi_2)}{\omega}\mathrm{d}\omega\sim\phi(\xi_2)\,, \tag{72}$$

where we used that $1 \le |\xi_2|/|\xi_1| \le 2$ on $\mathbf{C}'_+$. Therefore,

$$\int_{\mathbf{C}'_+} |\hat\psi(\xi)|^2(1+|\xi|^2)^t \mathrm{d}\xi \sim \int_{\xi_2>2} \phi(\xi_2)^2 \xi_2^{1+2t}\mathrm{d}\xi_2\,. \tag{73}$$

If $t>0$ it is easy to find $\phi$ satisfying (71) so that the integral above diverges. $\square$

## Appendix B. Heuristic comparison of $\mathcal{U}_1^{t,s}$ and the Gouëzel–Liverani spaces

In this appendix, we discuss informally the relation between $\mathcal{U}_p^{t,s}$ when $p=1$ and the geometric spaces of Gouëzel–Liverani [26]. (We do not claim that the norms are equivalent.)

For $s\in\mathbb{R}$ and $1\le p,q\le\infty$, let $B^s_{p,q}(\mathbb{R}^{d_s})$ be the classical Besov space [42] on $\mathbb{R}^{d_s}$. We introduce the local version of a new space $\widetilde{\mathcal{U}}^{t,s}_{p,q}$:

**Definition B.1** (The local space $\widetilde{\mathcal{U}}^{\mathbf{C}_\pm,t,s}_{p,q}(K)$)**.** Let $K\subset\mathbb{R}^d$ be a non-empty compact set. For a cone pair $\mathbf{C}_\pm=(\mathbf{C}_+,\mathbf{C}_-)$, so that $\mathbb{R}^{d_s}\times\{0\}$ is included in $\mathbf{C}_-$, real numbers $1\le p<\infty$, $1\le q\le\infty$, $s\le 0$, and integer $t\ge 1$, we set

$$\|\varphi\|_{\widetilde{\mathcal{U}}^{\mathbf{C}_\pm,t,s}_{p,q}} = \sup_{\Gamma\in\mathcal{F}(\mathbf{C}_+)}\Bigg(\sum_{|\vec t|\le t-1}\|[D^{\vec t}\varphi]\circ\pi_\Gamma^{-1}\|_{B^{s+|\vec t|}_{p,q}(\mathbb{R}^{d_s})} + \sup_{h\in\mathbb{R}^{d_s},h\ne 0}\Big\|\frac{((D^t\varphi)\circ\pi_\Gamma^{-1})(\cdot+h)-(\varphi\circ\pi_\Gamma^{-1})(\cdot)}{|h|}\Big\|_{B^s_{p,q}(\mathbb{R}^{d_s})}\Bigg)\,. \tag{74}$$

The space $\widetilde{\mathcal{U}}^{t,s}_{p,q}(T)$ is then defined using admissible charts (like in Definition 3.6).

We claim that if $s<-t$, the spaces $\widetilde{\mathcal{U}}^{t,s}_{1,1}$ are heuristically similar both to the spaces $\mathcal{B}^{t,|s+t|}$ of Gouëzel–Liverani [26, 27] and to our spaces $\mathcal{U}^{t,s}_1$. Indeed, as noticed above, the dual of the little Besov space $b^{|s+t|}_{p/(p-1),q/(q-1)}(\mathbb{R}^{d_s})$ is the Besov space $B^{s+t}_{p,q}(\mathbb{R}^{d_s})$ appearing in the definition of $\widetilde{\mathcal{U}}^{t,s}_{p,q}$ (see [42, 2.1.5 Remark 1]). Taking $p=1$ and $q=1$ we find the dual of the little Besov space $b^{|s+t|}_{\infty,\infty}(\mathbb{R}^{d_s})$, which is similar to the strong stable norm of Gouëzel and Liverani. So $\widetilde{\mathcal{U}}^{t,s}_{1,1}$ is related to the space $\mathcal{B}^{t,|s+t|}$ of Gouëzel–Liverani. (We abusively disregard here the fact that Gouëzel–Liverani take the sum over all $|\vec t|\le t$ while we use the Lipschitz quotient for the last derivative, recalling that we are taking the closure of $C^\infty(K)$, as well as Footnote 24.) Since $t\ge 1$ is an integer, in view of the Paley–Littlewood decomposition [42, Prop 2.1(vi)] (see also[24] [49, 2.3.5, 2.5.7]) of Besov-Lipschitz spaces $\Lambda^t_{p,q}=B^t_{p,q}$ for $p<\infty$ and $t>0$, the spaces $\mathcal{U}_1^{\mathbf{C}_\pm,t,s}$ and $\widetilde{\mathcal{U}}^{\mathbf{C}_\pm,t,s}_{1,1}$ are similar. (We explained in Remark 3.7 why we took $q=\infty$ instead of $q=1$ and why we expect our spaces would have the same qualitiative and quantitative features for $q=1$.)

*Remark* B.2 (The Demers–Liverani–Zhang spaces). It is more difficult to compare our spaces $\mathcal{U}^{t,s}_p$ to the spaces of Demers–Liverani [16] (even heuristically) for $p>1$ and $-1+1/p<s<-t<0<t<1/p$. The main problem is that their stable norm roughly involves the dual of the little Besov space $b^v_{1/\alpha,\infty}$ (abusively considering

[24]To make this rigorous we would need to replace $L_p(\mathbb{R}^d)$ in the arguments therein by mixed Lebesgue norms [13].

$|\Gamma|^\alpha \|\varphi\|_{b^v_{\infty,\infty}} \simeq \|\varphi\|_{b^v_{1/\alpha,\infty}}$) while the unstable norm involves[25] the dual of $b^1_{\infty,\infty}$. It follows that, although one should set $t = \beta$, one cannot assign a value to $s$ and $p$ depending on their parameters $\alpha$, $\beta$, $q$. (Note however that setting $p = 1/\alpha$ we recover the condition $\beta \le \alpha$ from [16] while their condition $\alpha \le 1 - q$ is reminiscent of $s > -1 + 1/p$ if in addition $s = -q$.) This also explains why one cannot immediately compare our Lasota–Yorke estimates (30) with [16, Prop. 2.7].

## Appendix C. Integration by parts and proof of Lemma 3.4

For the convenience of the reader, we recall what is meant by integration by parts in the present context (see e.g. [4]).

**Integration by parts.** Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be $C^2$ and let $f : \mathbb{R}^d \to \mathbb{R}$ be $C^1$ and compactly supported, with $\sum_{j=1}^d (\partial_j \Phi)^2 \neq 0$ in the support of $f$, and consider the average $\int_{\mathbb{R}^d} e^{i\Phi(w)} f(w) \mathrm{d}w$. By "integration by parts on $w$," we mean application, for a $C^2$ function $\Phi : \mathbb{R}^d \to \mathbb{R}$ and a compactly supported $C^1$ function $f : \mathbb{R}^d \to \mathbb{R}$ with $\sum_{j=1}^d (\partial_j \Phi)^2 \neq 0$ in the support of $f$, of the identity

$$\begin{aligned}\int e^{i\Phi(w)} f(w) \mathrm{d}w &= -\sum_{k=1}^d \int i(\partial_k \Phi(w)) e^{i\Phi(w)} \cdot \frac{i(\partial_k \Phi(w)) \cdot f(w)}{\sum_{j=1}^d (\partial_j \Phi(w))^2} \mathrm{d}w \\ &= i \cdot \int e^{i\Phi(w)} \cdot \sum_{k=1}^d \partial_k \left( \frac{\partial_k \Phi(w) \cdot f(w)}{\sum_{j=1}^d (\partial_j \Phi(w))^2} \right) \mathrm{d}w \,,\end{aligned}$$

where $w = (w_k)_{k=1}^d \in \mathbb{R}^d$, and $\partial_k$ denotes partial differentiation with respect to $w_k$.

**Regularised integration by parts.** If $\Phi$ is $C^r$ for some $r > 1$, we can only integrate by parts $[r] - 1$ times in the above sense, even if $f$ is $C^r$ and compactly supported. If $r$ is not an integer, then to integrate by parts $r - 1$ times, we proceed as follows: If $\Phi : \mathbb{R}^d \to \mathbb{R}$ is $C^{1+\delta}$ and $f : \mathbb{R}^d \to \mathbb{R}$ is compactly supported and $C^\delta$, for $\delta \in (0, 1)$, and $\sum_{j=1}^d (\partial_j \Phi)^2 \neq 0$ on $\mathrm{supp}(f)$, we set, for $k = 1, \ldots, d$

$$h_k := \frac{i(\partial_k \Phi(w)) \cdot f(w)}{\sum_{j=1}^d (\partial_j \Phi(w))^2} \,.$$

Each $h_k$ belongs to $C^\delta_0(\mathbb{R}^d)$. Let $h_{k,\epsilon}$, for small $\epsilon > 0$, be the convolution of $h_k$ with $\epsilon^{-d} \upsilon(x/\epsilon)$, where the $C^\infty$ function $\upsilon : \mathbb{R}^d \to \mathbb{R}_+$ is supported in the unit ball and satisfies $\int \upsilon(x) \mathrm{d}x = 1$. There is $C$, independent of $\Phi$ and $f$, so that for each small $\epsilon > 0$ and all $k$, $\|\partial_k h_{k,\epsilon}\|_{L_\infty} \le C \|h_k\|_{C^\delta} \epsilon^{\delta - 1}$ and $\|h_k - h_{k,\epsilon}\|_{L_\infty} \le C \|h_k\|_{C^\delta} \epsilon^\delta$. Finally, for every real number $\Lambda \ge 1$

$$\begin{aligned}&\int e^{i\Lambda\Phi(w)} f(w) \mathrm{d}w = -\sum_{k=1}^d \int i\partial_k \Phi(w) e^{i\Lambda\Phi(w)} \cdot h_k(w) \mathrm{d}w \\ &= \int \frac{e^{i\Lambda\Phi(w)}}{\Lambda} \cdot \sum_{k=1}^d \partial_k h_{k,\epsilon}(w) \mathrm{d}w - \sum_{k=1}^d \int i\partial_k \Phi(w) e^{i\Lambda\Phi(w)} \cdot (h_k(w) - h_{k,\epsilon}(w)) \mathrm{d}w \,.\end{aligned}$$

To conclude, we give the Proof of Lemma 3.4, which relies on the following standard result (see e.g. [10, Lemma 4.1] for a similar statement):

[25] In the spaces of Demers–Zhang [17] it is the dual of $b^w_{\infty,\infty}$ for $w \neq v$.

**Lemma C.1** (Paley–Littlewood proper support)**.** *Let $K \subset \mathbb{R}^d$ be compact, and let $1 < p \le \infty$. For any $P > 0$, $Q > 0$, and $\epsilon > 0$, there exists a constant $C > 0$ such that*

$$|\psi_n^{Op}\varphi(x)| \le C \cdot \frac{\sum_\ell 2^{-P\max\{n,\ell\}}\|\psi_\ell^{Op}\varphi\|_{L_p}}{d(x,\mathrm{supp}(\varphi))^Q} \tag{75}$$

*for any $n \in \mathbb{Z}_+$, $\varphi \in C^\infty(K)$, and all $x \in \mathbb{R}^d$ so that $d(x,\mathrm{supp}(\varphi)) > \epsilon$.*

*Proof of Lemma 3.4.* We may assume for both claims that $u > t$ is not an integer. Then (see e.g. [50, §1.3.4, Rk. 3, and §2.3.2]) the $C^u$ norm is equivalent to the norm $\|\varphi\|_{C^u_*} := \sup_{n\ge 0}\left(2^{un}\|\psi_n^{Op}\varphi\|_{L_\infty}\right)$.

Let $\tilde K$ be a compact neighbourhood of $K$. For the first claim, recalling (20), since

$$\|1_{\tilde K}\tilde\varphi\|^s_{p,q,\Gamma} \le C\|1_{\tilde K}\tilde\varphi\|_{L_p(\Gamma)} \le C_{\tilde K}\|\tilde\varphi\|_{L_\infty},$$

and $u > t$, we find, using Young's inequality in $L_p$ that

$$2^{tn}\|1_{\tilde K}\psi_n^{Op}\varphi\|^s_{p,q,\Gamma} \le C(u,\tilde K)\cdot\|\varphi\|_{C^u_*} \quad \text{for any } n \text{ and } \Gamma\,. \tag{76}$$

Using Lemma C.1 for large enough $P$ and $Q$, we estimate

$$2^{tn}\|1_{\mathbb{R}\setminus\tilde K}\psi_n^{Op}\varphi\|_{L_p(\Gamma)} \le \sum_\ell 2^{-(P-t)\ell}\|\psi_\ell^{Op}\varphi\|_{L_\infty}\,.$$

Since $u > t$, we obtain

$$2^{tn}\|1_{\mathbb{R}\setminus\tilde K}\psi_n^{Op}\varphi\|^s_{p,q,\Gamma} \le C(u,\tilde K)\cdot\|\varphi\|_{C^u_*} \quad \text{for any } n \text{ and } \Gamma\,. \tag{77}$$

Clearly (76) and (77) imply the first claim.

We move to the second claim. Decompose $\varphi \in C^\infty(K)$ and $v \in C^u(K)$ with $u > |s+t|$ as $\varphi = \sum_n \psi_n^{Op}\varphi$ and $v = \sum_{m\ge 0}\psi_m^{Op}v$. We get

$$\int \varphi\cdot v\,\mathrm{d}x = \sum_n \sum_{m:|m-n|\le 1}\int \psi_n^{Op}\varphi(x)\cdot\psi_m^{Op}v(x)\,\mathrm{d}x\,,$$

by Parseval's theorem. We decompose the integral above into the sum of $\int_{\tilde K}$ and $\int_{\mathbb{R}^d\setminus\tilde K}$. Up to changing coordinates, we can assume that $\Gamma = \mathbb{R}^{d_s}\times\{0\}$, and that every translated hyperplane $\mathbb{R}^{d_s}\times\{x_{d_u}\}$ lies in $\mathcal{F}(\mathbf{C}_+)$. Taking $Q_1 > d_s$ and $Q_2 > d_u$, Lemma C.1 gives a constant $C_{Q_1,Q_2,\tilde K}$ so that for all $x$ with $d(x,K) \ge d(K,\tilde K)$, in the new coordinates,

$$|\psi_m^{Op}v(x)| \le C_{Q_1,Q_2,\tilde K}\sum_\ell 2^{-P\ell}\frac{\|\psi_\ell^{Op}v\|_{L_\infty}}{(1+\|x_-\|)^{Q_1}(1+\|x_+\|)^{Q_2}}\,.$$

Therefore, if $|m-n|\le 1$ and $|m_s-n|\le 1$, recalling that $u>|s+t|=-s-t$,

$$
\begin{aligned}
&|\int_{\mathbb{R}^d\setminus\tilde K} \psi_n^{Op}\varphi(x)\cdot\psi_m^{Op}v(x)|\mathrm{d}x\\
&\quad\le \sum_{\ell\ge m} 2^{-P\ell}\int_{\mathbb{R}^{d_s}}\frac{1}{(1+\|x_-\|)^{Q_1}}|\int_{\mathbb{R}^{d_u}}\frac{1}{(1+\|x_-\|)^{Q_1}}\psi_n^{Op}\varphi(x)\cdot\psi_\ell^{Op}v(x)\mathrm{d}x_+|\mathrm{d}x_-\\
&\quad\le C\sum_{\ell\ge m} 2^{\ell(s+t-P)}\|v\|_{C^u}\int_{\mathbb{R}^{d_u}}\frac{1}{(1+\|x_+\|)^{Q_2}}\int_{\mathbb{R}^{d_s}}\frac{|\psi_n^{Op}\varphi(x)|}{(1+\|x_-\|)^{Q_1}}\mathrm{d}x_+\mathrm{d}x_-\\
&\quad\le C'2^{m(s+t)}\|v\|_{C^u}\int_{\mathbb{R}^{d_u}}\frac{1}{(1+\|x_+\|)^{Q_2}}\mathrm{d}x_+\cdot\sup_{x_+}\int_{\mathbb{R}^{d_s}}\frac{|\psi_n^{Op}\varphi(x)|}{(1+\|x_-\|)^{Q_1}}\mathrm{d}x_+\\
&\quad\le C'2^{nt}\|v\|_{C^u}\int_{\mathbb{R}^{d_u}}\frac{1}{(1+\|x_+\|)^{Q_2}}\mathrm{d}x_+\cdot 2^{ns}\sup_{x_+}\int_{\mathbb{R}^{d_s}}\frac{|\psi_n^{Op}\varphi(x)|}{(1+\|x_-\|)^{Q_1}}\mathrm{d}x_+\,.
\end{aligned}
$$

Since the foliation is trivial we have $\psi_n^{Op}\varphi=\sum_{n_s=n-2}^{n+2}\psi_n^{Op}(\psi_{n_s}^{(d_s)})^{Op}\varphi$, so that the right-hand side above is bounded by

$$
\le \tilde C 2^{nt}\|v\|_{C^u}\int_{\mathbb{R}^{d_u}}\frac{1}{(1+\|x_+\|)^{Q_2}}\mathrm{d}x_+\cdot\sup_\Gamma\|\psi_n^{Op}\varphi\|^s_{p,q,\Gamma}\le C\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}}\|v\|_{C^u}\,.
$$

The integral over $\tilde K$ is easier to estimate, and we obtain

$$
\left|\int\varphi\cdot v\mathrm{d}x\right|\le C\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}}\|v\|_{C^u}\,.
$$

$\square$

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