Spectra of analytic hyperbolic maps and flows: Correlation functions, Fredholm determinants and zeta-functions. *Lectures given at IHP, june 2005.*

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Abstract

In [13] we defined a class of so-called 'Hyperbolic Analytic Maps'. Given a map in this class one associates a Banach space and a family of transfer operators with analytic weights on the space. These operators are nuclear in the sense of Grothendieck. An elementary proof was given in the case of 1+1 dimensional maps [Fried has extended the proof to higher dimensional systems]. In this case such an operator admits a Fredholm determinant which is an entire function in the complex plane.

Applying a judicious choice of weights we may relate the zeroes of a determinant to resonances for certain ergodic measures on the underlying dynamical system. In particular, we consider real-analytic Anosov maps or Axiom A attractors (still in 1+1 dimensions) with their SRB (or natural) measure and show that the SRB-resonances form a discrete subset of the complex plane and are localized in the zero-set of a Fredholm determinant. Regarding an Axiom A flow as a suspension of a ditto map we prove similar results for the SRB-resonances in the flow case.

1 Introduction

In the theory of dynamical systems and in particular, in its physical applications, the so-called natural measures and their correlation functions play a particularly important role. Transfer operators acting on suitable function spaces provide a useful technical tool in this context. We will here limit our discussion to the case of real-analytic Axiom A attractors, 2 dimensional in the case of maps, 3 dimensional in the case of flows. The advantage is that function spaces are very explicit and easy to deal with, the disadvantage is that it restricts unnecessarily the systems we may consider. The present notes form a complement to reference [13] (and also [14]) in which we introduced function spaces associated to a collection of 2 dimensional hyperbolic analytic 'maps', a family of associated transfer operators and showed that these operators give rise to Fredholm determinants that extend to entire functions in the complex plane. Several extensions have followed in the wakes of the ideas introduced: Fried [6] removed the restrictions on the dimensions and Kitaev [7] relaxed the smoothness conditions. We refer to the excellent review in [2] for a more detailed account.

Here, we will focus upon certain questions regarding SRB measures which were partially discussed but left incomplete in [13]. Again we will content ourselves with the 2 (3) dimensional real-analytic case being conscient of the fact that the work of Kitaev [7] would allow us to go to higher dimensions and relax the smoothness conditions. Recent works of Liverani and Gouzel as well as Baladi and Tsujii seemingly superseed the results presented here but proofs are based on rather different techniques.

First, consider a map $f : M \to M$ of the real-analytic Riemann surface M. We assume that there is an open subset $U \subset M$ on which (1) f is real-analytic, (2) $f(U) \subset\subset U$ and (3) $\Lambda = \cap_{k\geq 0} f^k(U)$ is an Axiom A basic set for f. We say that (Λ, f) a real-analytic Axiom A attractor (embedded in M). We write $C^\omega(\Lambda)$ for the space of complex valued functions on Λ that admits a holomorphic extension to a complexified neighborhood of Λ (a space of analytic germs). Given an analytic weight, $g \in C^\omega(\Lambda)$, Theorem 1 in [14] shows that the generalized Fredholm determinant,

$$d_g(z) = \exp\left(- \sum_{n\in\mathbb{N}} \frac{z^n}{n} \sum_{x\in \mathrm{Fix} f^n|\Lambda} \frac{g(f^{n-1}x)\cdots g(x)}{|\det(Df^n(x) - \mathbf{1})|} \right), \tag{1.1}$$

extends to an entire function in the complex plane. We write simply $d_1 = d_{g\equiv 1}$ for the case when the weight equals one. Let $\mu_{\rm nat}$ denote the natural (SRB) measure associated with (Λ, f). Given continuous functions (observables) $A, B \in C^0(\Lambda)$ we consider the correlation function given by $C_{B,A}(n) = \mu_{\rm nat}(B \circ f^n\, A)$, $n \geq 0$ and its Fourier transform, $\widehat{C}_{B,A}(\omega) = \sum_{n\geq 0} C_{B,A}(n) e^{i\omega n}$, $\omega \in \mathbb{C}$. Our main goal is to show the following :

Theorem 1.1 Let $A, B \in C^\omega(\Lambda)$ be analytic observables. Then $\widehat{C}_{B,A}$ extends to a meromorphic function in the complex plane. Its poles is included in the zero-set of $\omega \in \mathbb{C} \mapsto d_1(e^{i\omega})$.

Second, let $\mathcal{M}$ be a 3 dimensional real-analytic manifold and ϕ^t a flow on $\mathcal{M}$. Again we assume that there is an open set $U \subset M$ on which (1) ϕ^t is real-analytic, (2) $\phi^t(U) \subset\subset U$, $t > 0$ and (3) $\Lambda = \bigcap_{t\geq 0} \phi^t U$ is an Axiom A attractor for the flow. Write $\mathcal{P}$ for the prime periodic orbits, $\lambda(\gamma)$ the return time for each $\gamma \in \mathcal{P}$ and finally let $P_* \phi_*^{\lambda(\gamma)}(\gamma)$ be the Jacobian of the return map with the flow-direction projected away and modulo conjugacy (we only consider the eigenvalues).

Then [14, Theorem 2 with $a \equiv i$] the (reciprocal) generalized Selberg zeta function

$$\zeta^{-1}(\omega) = \exp\left(- \sum_{m\in\mathbb{N}} \frac{1}{m} \sum_{\gamma\in\mathcal{P}} \frac{\exp(i\,\omega\, m\, \lambda(\gamma))}{|\det(P_* \phi_*^{\lambda(\gamma)}(\gamma) - \mathbf{1})|} \right), \tag{1.2}$$

extends to an entire function in the complex plane. As above we let $\mu_{\rm nat}$ denote the SRB measure associated to the flow, $C_{B,A}(\tau) = \mu_{\rm nat}(B \circ f^\tau A)$, $\tau \geq 0$ is a correlation function and $\widehat{C}_{B,A}(\omega) = \int_0^\infty C_{B,A}(\tau) e^{i\omega\tau}\, dt$ its Fourier transform. We show that :

Theorem 1.2 Let $A, B \in C^\omega(\Lambda)$ be analytic observables. Then $\widehat{C}_{B,A}$ extends to a meromorphic function in the complex plane. Its poles is included in the zero-set of $\omega \in \mathbb{C} \mapsto \zeta^{-1}(\omega)$.

Remarks 1.1

1. *It seems reasonable to conjecture that any zero of $d_1(e^{i\omega})$ or $\zeta^{-1}(\omega)$ may appear as a pole, i.e. for a suitable choice of A and B.*
2. *Our proofs of the above theorems are based on Thermodynamic Formalism, certain limiting properties of traces of transfer operators and analytic properties of zeta-functions and Fredholm determinants. It is likely that so-called sharp- or flat- traces would suffice to produce similar results in a less smooth category.*

2 Proof of Theorem 1

Recall that the natural measure may be characterized as the unique Gibbs measure for the potential $-\log Df_{|E^u}$. By thermodynamic formalism [12], we may therefore compute it as a weak-limit over periodic orbits. More precisely, for $A : \Lambda \to \mathbb{C}$ continuous we have

$$\int A\, d\mu_{\rm nat} = \lim_{n\to\infty} \sum_{x\in {\rm Fix} f^n|\Lambda} \frac{A(x)}{|Df^n_{|E^u}(x)|}. \tag{2.3}$$

Suppose now that $\{\Omega_i\}_{i=1..N}$ is a Markov partition for f with a transfer matrix t and the induced map $\widehat{f}$ between partition elements. Then also,

$$\int A\, d\mu_{\rm nat} = \lim_{n\to\infty} \sum_{x\in {\rm Fix} \widehat{f}^n|\Lambda} \frac{A(x)}{|D\widehat{f}^n_{|E^u}(x)|}. \tag{2.4}$$

The latter formula may count periodic orbits differently (when they belong to boundaries of rectangles) but the limits are the same. The central idea is now to replace unstable derivatives by determinants of full derivatives. This introduces exponentially small errors. More precisely, we rewrite the unstable derivative as follows :

$$|D\widehat{f}^n_{|E^u}(x)| = |\det(D\widehat{f}^n(x) - \mathbf{1})|(1 + \mathcal{O}(\theta^n)),$$

with $\theta < 1$ (use the exponential growth/decay of eigenvalues). Thus,

$$\int A \, d\mu_{\rm nat} = \lim_{n\longrightarrow\infty} \sum_{x \in {\rm Fix}\widehat{f}^n|\Lambda} \frac{A(x)}{|\det(D\widehat{f}^n(x) - \mathbf{1})|}, \tag{2.5}$$

the difference between the sums in (2.4) and (2.5) being exponential small in n.

Now, when A is an analytic observable the last expression may be written as a trace in a suitable function space, X. More precisely, we may introduce a multiplication operator $M_A \in L(X)$ and a nuclear transfer operator $\widehat{L} \in L(X)$ so that

$$\mathrm{tr}(\widehat{L}^n M_A) = \sum_{x \in {\rm Fix}\widehat{f}^n|\Lambda} \frac{A(x)}{|\det(D\widehat{f}^n(x) - \mathbf{1})|}. \tag{2.6}$$

More precisely, as in [13, section 4] we consider a function space of the form $X = A_\infty((\widehat{\mathbb{C}} \setminus \mathcal{D}_1) \times \mathcal{D}_2)$ where $\mathcal{D}_1$ and $\mathcal{D}_2$ are regular domains in $\mathbb{C}$. Given a function $A \in A_\infty(\mathcal{D}_1 \times \mathcal{D}_2)$ the mulitiplication operator M_A then acts upon $\eta \in X$ as follows :

$$(M_A\eta)(w_1, w_2) \equiv \oint_{\partial\mathcal{D}_1} \frac{A(\xi_1, w_2)}{w_1 - \xi_1} \, \eta(\xi_1, z_2) \, \frac{d\xi_1}{2\pi i}. \tag{2.7}$$

For example, let the 'function' $\eta(w_1, w_2) = \int_{I_1} \frac{\rho(x_1,w_2)}{w_1 - x_1} \, dx_1$ represent (for fixed $w_2 \in \mathcal{D}_2$) Lebesgue measure in $x_1 \in I_1 \subset \mathcal{D}_1$ with density ρ. Then $(M_A\eta)(w_1, w_2) = \int_{I_1} \frac{A(x_1,w_2)\rho(x_1,w_2)}{w_1 - x_1} \, dx_1$ is Lebegue measure with density given by $A\rho$.

It is known that the expression in (2.4) converges exponentially fast (for smooth observables). Comparing with (2.5) and (2.6) we see that the latter two also converge exponentially fast. It follows that 1 is a simle eigenvalue of $\widehat{L}$ and that all other eigenvalues are strictly smaller in absolute value, i.e. $\widehat{L}$ has a spectral gap. Let $P = h \otimes \nu$ be the associated one-dimensional projection in $L(X)$. As $\widehat{L}$ is nuclear we may take limits to obtain the following representation for the natural measure of an analytic observable :

$$\int A \, d\mu_{\rm nat} = \nu(M_A h). \tag{2.8}$$

For the correlation function we will use the following identity :

$$\mathrm{tr}(\widehat{L}^n M_B \widehat{L}^k M_A) = \sum_{x \in {\rm Fix}\widehat{f}^{(n+k)}|\Lambda} \frac{B \circ \widehat{f}^k(x) A(x)}{|\det(D\widehat{f}^{(n+k)}(x) - \mathbf{1})|}. \tag{2.9}$$

The proof of this necessitates the evaluation of a couple of Cauchy integrals as in [13, section 4, proof of equation (19)]. [Note that we here avoid the use of the more 'natural' expression $\mathrm{tr}(\widehat{L}^n M_{B\circ\widehat{f}^k A})$ because of the analyticity deterioration when composing B with $\widehat{f}$. In order to handle this expression we would have to refine the Markov partition as a function of k and this would bring havoc to the arguments]. Note now that for any fixed k,

$$C_{B,A}(k) = \lim_{n\longrightarrow\infty} \mathrm{tr}(\widehat{L}^n M_B \widehat{L}^k M_A) = \nu(M_B \widehat{L}^k M_A h), \tag{2.10}$$

so that for the Fourier transform and with $\mathrm{Im}\,\omega > 0$:

$$\widehat{C}_{B,A}(\omega) = \nu(M_B(1 - e^{i\omega}\widehat{L})^{-1} M_A h). \tag{2.11}$$

Again since $\widehat{L}$ is nuclear, the determinant $\widehat{d}_1(z) = \det(1 - z\widehat{L})$ extends to an entire function in z. Therefore, $\widehat{C}_{B,A}$ has indeed a meromorphic extension with its poles belonging to the ensemble of ω-values for which $\det(1 - e^{i\omega}\widehat{L})$ vanishes. Now, $\widehat{d}_1(e^{i\omega})$ is almost the same function as $d_1(e^{i\omega})$ except for the counting of periodic orbits within the determinant. As shown in [14] the ratio of the two expressions comes from the contribution of periodic orbits on the boundary of Markov partitions. But the result is correct for any choice of (sufficiently refined) Markov partition. By changing the Markov partition and proceeding as in [14, pp 817-819] we see that any 'spurious' zero coming from boundary points can not give rise to a pole in $\widehat{C}_{B,A}$ and Theorem 1 follows. □

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