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Decay of Correlations

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Abstract. This introduction to rates of mixing of discrete time dynamical systems contains an overview of known results (mostly for uniformly or nonuniformly hyperbolic maps and their SRB measures) and techniques (symbolic dynamics, towers with recurrence times, spectrum of Perron-Frobenius-Ruelle operator and dynamical zeta functions, projective metrics on Birkhoff cones, probabilistic coupling).

1. What are these notes about? Invariant measures, physical measures, and rates of mixing

We are interested in discrete-time dynamical systems represented by the iterates $f^n = f \circ f^{n-1}$ (where $n \in \mathbb{Z}_+$ represents time) of a transformation $f : M \to M$ which is "chaotic," in the sense that arbitrarily close distinct initial points become "separated" (assuming a metric structure or at least a topology on M) if one waits long enough. This separation (sometimes referred to as "sensitive dependence on initial conditions") often takes place at exponential speed, as in the elementary paradigm of the "angle-doubling" map $z \mapsto z^2$ on the circle $\{z \in \mathbb{C} \mid |z| = 1\}$, where the distance between $f^n(x)$ and $f^n(y)$ is 2^n times the distance between x and y if they are close enough.

Our aim is to describe the long-time behaviour of generic initial conditions. Generic is understood in a measure-theoretical sense, so that this task of *statistically describing* the asymptotics of "most" initial data is not rendered completely hopeless by the sensitive dependence property. We shall mostly discuss the case when M is a compact Riemann manifold, so that we have a natural a priori probability measure on M: the Lebesgue measure. We therefore seek to understand invariant probability measures μ (i.e., $\mu(f^{-1}(E)) = \mu(E)$ for every Borel set) which are somehow related to the a priori measure.

The first key concept is, of course, ergodicity (see [Wal] for this, and other, ergodic-theoretical notions). Recall that the celebrated *Birkhoff theorem* says that

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if μ is an f invariant Borel probability measure such that (f, μ) is *ergodic* (i.e., "indecomposable" in the sense that $E = f^{-1}E$ only if $\mu(E) = 0$ or 1), then for each continuous "test" function (also called "observable") $\varphi : M \to \mathbb{C}$ and μ-almost all $x \in M$ the "time average converges to the space average:"

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) = \int \varphi \, d\mu \, . \tag{1.1}$$

(In fact, convergence does not hold only for continuous observables, $L^1(d\mu)$ would suffice.) Of course, μ can be supported on a set of zero Lebesgue measure (the simplest example being a periodic orbit). In this case "μ-almost everywhere" is not very meaningful! Suppressing the observable, one may consider the set of points x (called the *ergodic basin* of μ) such that the convergence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x} = \mu \, . \tag{1.2}$$

holds (weakly, with δ_y denoting the Dirac mass at y). If the ergodic basin of x has positive Lebesgue measure, μ is called a *physical measure.* In recent literature, physical measures are sometimes called *SRB measures,* for Sinai-Ruelle-Bowen. We prefer to restrict the SRB terminology to its historical technical definition (absolute continuity with respect to Lebesgue of the measures induced along the unstable directions, see, e.g., [Yo2], see also [PeS, BoV, ABV] for generalisations) although it is often equivalent to the above concept of physical measure.

We shall discuss below classes of dynamical systems possessing a single, or finitely many, physical measure(s) the (union of the) ergodic basin(s) of which have total Lebesgue measure in M. (For saddles, or systems with "holes," the notion of physical measure is not appropriate and one should condition by the property of remaining in a neighbourhood. This gives rise to the notion of *quasi-SRB* or *generalised SRB* measure, see, e.g., Ruelle [Ru10] for the concept, Collet–Martínez–Schmitt [CMS] and Chernov–Markarian–Troubetzkoy [CMT] for examples.)

For an ergodic physical measure, one would like to quantify the spatial nature of the convergence (1.1–1.2). For a real-valued observable, assuming that $\int \varphi \, d\mu = 0$ (replacing if necessary φ by $\varphi - \int \varphi \, d\mu$), one says that the *central limit theorem* (CLT) holds for (f, μ) and φ if the "random variable"

$$x \mapsto \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ f^n(x)$$

(where x is distributed with law μ) converges in law to a normal distribution $\mathcal{N}(0, \sigma)$ for some $0 < \sigma < \infty$. Recall that in probability theory the central limit theorem holds, e.g., for independent identically distributed random variables [Dur], but also under weaker "independence" assumptions. Of course, a deterministic dynamical system does not behave like an i.i.d. random variable! Nevertheless we shall encounter below "weak independence" assumptions involving the *correlation functions* or *rates of mixing* which are the main topic of these notes, and which are sufficient for the CLT.

A statistical property which implies ergodicity is *mixing,* a kind of asymptotic independence: (f,μ) is mixing if for every pair of observables $\varphi,\psi \in L^2(d\mu)$

$$\lim_{n\to\infty}\int (\varphi\circ f^n)\,\psi\,d\mu = \int \varphi\,d\mu \int \psi\,d\mu\,. \tag{1.3}$$

For the special case when the observables are characteristic functions χ_A and χ_B of Borel sets $A, B \subset M$, this means

$$\lim_{n\to\infty}\mu(f^{-n}A\cap B) = \mu(A)\mu(B)\,, \tag{1.4}$$

which is indeed the classical notion of (asymptotic) independence. Often, (1.3) is equivalent to a more natural (and often more convenient) formulation of mixing (using dx for Lebesgue measure again):

$$\lim_{n\to\infty}\int (\varphi\circ f^n)\,\psi\,dx = \int \varphi\,d\mu \int \psi\,dx\,. \tag{1.5}$$

Indeed, the left hand side is now a well-defined expression even if μ is unknown. In particular, taking $\psi\equiv 1$ we see that (1.5) corresponds to *convergence to equilibrium:*

$$\lim_{n\to\infty}\int (\varphi\circ f^n)\,dx = \int \varphi\,d\mu\,. \tag{1.6}$$

The expression

$$\rho_{\varphi\psi}(n) = \int (\varphi\circ f^n)\,\psi\,d\mu - \int \varphi\,d\mu \int \psi\,d\mu\,, \tag{1.7}$$

viewed as a function of $n\in\mathbb{Z}_+$ is called the *correlation function* of (f,μ) for the observables φ, ψ. By a slight abuse of language the difference

$$\rho'_{\varphi,\psi}(n) = \int (\varphi\circ f^n)\,\psi\,dx - \int \varphi\,d\mu \int \psi\,dx\,, \tag{1.8}$$

may also be called a correlation function. We shall see that sometimes the correlation function $\rho_{\varphi\psi}(n)$ decays exponentially with n for suitable φ,ψ. It is thus natural to consider its (a priori formal) Fourier transform

$$\widehat{\rho}_{\varphi\psi}(\omega) = \sum_{n\in\mathbb{Z}} e^{in\omega}\rho_{\varphi\psi}(n) \tag{1.9}$$

(setting $\rho_{\varphi\psi}(n) = \rho_{\psi\varphi}(-n)$ for negative n if f is not invertible). Exponential decay (with a uniform rate in φ, ψ) would imply that $\widehat{\rho}_{\varphi\psi}(\omega)$ is analytic in a strip $|\Im\omega| < \eta$, with η independent of φ, ψ in some function class. In some cases $\widehat{\rho}_{\varphi\psi}(\omega)$ admits an extension meromorphic in a strip $\Im(\omega) < \eta'$ where the possible location of its poles (the *Ruelle resonances,* or the *correlation spectrum*) only depend on the dynamical system (f,μ) (and not on the observables). They are often described by the set of poles of a dynamical zeta function or the set of zeroes of a dynamical Fredholm determinant, as we shall explain in these notes.

Mixing alone is in general *not* enough to imply the CLT: we need to quantify this notion by studying the *speed* at which the convergence (1.3) (or (1.5)) takes place. This is one of the motivations to study the rate of decay of correlations functions. For this, it is in general necessary to restrict the test functions to a functional subspace $\mathcal{B}$ of $L^2(d\mu)$. It is often convenient to take for $\mathcal{B}$ a Banach space (suitable Hilbert spaces are usually not readily available). In many cases, $\rho'_{\varphi,\psi}(n) \sim \rho_{\varphi\psi}(n)$

when $n \to \infty$ if the observables belong to an appropriate function space. In Section 4 we shall give a catalogue of results for speeds of mixing. The rule of thumb is that if it is possible to find a space of observables $\mathcal{B}$ such that $|\rho_{\varphi\psi}(n)| \le C(n)$ for all $n \in \mathbb{Z}_+$ and all $\varphi, \psi \in \mathcal{B}$, then the central limit theorem holds for observables in $\mathcal{B}$ *if the speed of mixing* $C(n)$ is summable, i.e., if $\sum_n C(n) < \infty$. We refer to [Li2], and also [Yo2, Yo3], [V2] for precise statements, their proofs, and references (in particular to previous work of Gordin, and to a probabilistic Lindeberg-Levy central limit theorem for martingales [Bil], as well as an important martingale approximation theorem by Kipnis and Varadhan).

2. A pedestrian's view: Three elementary examples

Before quoting (mostly without proof) a series of known results, let us discuss three examples which can be treated "by hand." We hope this will ease the beginners' entrance into this topic, but we would like to point out that this "hands on" approach is *not* characteristic of the field. Most results are obtained by rather abstract and sometimes long proofs which require, besides the usual geometric and topological toolbox of dynamics, a not completely trivial amount of functional analysis (spectral theory of bounded operators on Banach spaces, projective metrics on convex cones). Probability theory also enters the picture in the more advanced results, e.g., via the implementation of large deviation ideas.

Example # 1: *Rotations on the circle.* (Ergodic isometry with no mixing.)

For $\alpha \in [0,1)$, consider the rotation $R_\alpha(z) = ze^{2i\pi\alpha}$ on the circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Lebesgue measure μ is clearly invariant under R_α. We shall see that (R_α, μ) is ergodic if and only if α is irrational (it is then an ergodic physical measure), but that it is *not* mixing.

If α is irrational then for any $\varphi \in L^2(d\mu)$ such that $\varphi = \varphi \circ R_\alpha$ (Lebesgue almost everywhere), we have, using the Fourier expansion $\varphi(z) = \sum_{\mathbb{Z}} b_n z^n$ of φ:

$$\varphi(e^{2i\pi\alpha} z) = \sum_{n\in\mathbb{Z}} b_n e^{2i\pi n\alpha} z^n \,, \tag{2.1}$$

so that $\varphi \circ R_\alpha = \varphi$ if and only if for all $n \in \mathbb{Z}$

$$b_n(e^{2i\pi n\alpha} - 1) = 0 \,. \tag{2.2}$$

This implies that $b_n = 0$ except perhaps for $n = 0$. Thus, φ is constant. (And if it is the characteristic function of a set, this set has either zero or full measure.) Now if $\alpha = p/q$ is rational then R_α^q is the identity map, which is clearly not ergodic.

We now check that (R_α, μ) is not mixing, assuming $0 < \alpha < 1/2$ (the case $\alpha \ge 1/2$ is similar). For this, take an arc of circle A of length $\alpha/10$ and let B be the arc of circle which is the complement of an arc of length $\alpha/2$ centered at A. Thus, the length of B is $> 2\alpha$. Clearly, there are arbitrarily large iterates n so that $R_\alpha^n(A)$ is contained in B but also arbitrarily large iterates m such that R_α^m is contained in the complement of B. (Draw a picture!) Thus, $\mu(R_\alpha^n A \cap B) = \mu(A) > 0$ but $\mu(R_\alpha^m A \cap B) = 0$, which is incompatible with mixing.

Example # 2: *Angle doubling on the circle.* (Linear expanding map with super-exponential mixing.)

We now move to an example of dynamics which is locally expanding but confined to a compact set: the angle-doubling map $f(z) = z^2$ on the circle $\{z \in \mathbb{C} \mid$

$|z| = 1\}$. Again, Lebesgue measure μ on the circle is preserved. (Draw the graph of the map $x \mapsto 2x$ modulo 1 on the interval.) Let us show that (f, μ) is mixing (and thus ergodic), again by a Fourier series argument. Write two elements φ, ψ of $L^2(d\mu)$ as

$$\varphi = \sum_{m \in \mathbb{Z}} a_m z^m \, , \quad \psi = \sum_{\ell \in \mathbb{Z}} b_\ell z^\ell \, , \tag{2.3}$$

and assume that $\int \psi \, d\mu = 0$, that is, $b_0 = 0$. Then, since $\varphi \circ f^n = \sum_m a_m (z^{2^n})^m$, we have

$$\int \varphi \circ f^n \, \psi \, d\mu = \sum_m \sum_\ell \int a_m (z^{2^n})^m b_\ell z^\ell \, d\mu \, . \tag{2.4}$$

The only nonvanishing terms in the double sum are those for which $\ell = -2^n m$, $\ell \neq 0$, and we are left with

$$\int \varphi \circ f^n \, \psi \, d\mu = \sum_{m \neq 0} a_m b_{-2^n m} \, , \tag{2.5}$$

which clearly goes to zero as $n \to \infty$. The reader is invited to prove that the convergence to zero is faster than any exponential if φ, ψ are analytic. (If they are only Lipschitz, exponential mixing with rate $1/2$ holds in general. The statement for C^r observables, with $0 < r \leq \infty$, is that mixing takes place at least with exponential rate $1/2^r$.)

The same property of superexponential mixing for analytic observables and exponential mixing for finitely differentiable observables occurs for linear uniformly hyperbolic maps, e.g., the Arnold "cat map" $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on the two-dimensional torus, which also preserves Lebesgue measure. For nonlinear but smooth (C^2, or at least $C^{1+\epsilon}$) uniformly expanding or uniformly hyperbolic dynamics, the physical measure first has to be constructed, and one can then prove that it mixes exponentially. (See Sections 3 and 4.)

Example # 3: *Full shift, finitely many symbols.* (Symbolic hyperbolic/expanding model.)

Subshifts of finite type will be discussed further later on. Let us restrict for the moment to the simplest case of *full shifts.* Let then $S = \{1, 2, \ldots, N\}$ and $\Sigma = S^{\mathbb{Z}}$ be the set of biinfinite sequences with coefficients in S. Define the (two-sided) shift σ to the left on Σ by $\sigma(x)_n = x_{n+1}$. We also consider the (nonivertible) one-sided shift σ_+ on $\Sigma_+ = S^{\mathbb{Z}_+}$. For each $0 < \theta < 1$ we may use the metric

$$d_\theta(x, y) = \sum_k \theta^{|k|} (1 - \delta(x_k, y_k)) \, , \tag{2.6}$$

where $\delta(\cdot, \cdot)$ is the Kronecker δ. (The shifts σ and σ_+ are continuous. The shift σ_+ is locally expanding, while σ is "hyperbolic" for this metric.)

If $N = 2$ the one-sided full shift is a symbolic model for the angle-doubling map f of Example 2. The reader who has never seen this is invited to construct a Lipschitz surjective map $\pi : \Sigma_+ \to S^1$ such that $f\pi = \pi\sigma_+$, i.e., π is a *semi-conjugacy,* by considering the partition of the circle into the two arcs $\{e^{2i\pi\alpha} \mid 0 < \alpha < 1/2\}$ and $\{e^{2i\pi\alpha} \mid 1/2 < \alpha < 1\}$.

The *Bernoulli measure* μ on Σ, respectively Σ_+ is an invariant probability measure for σ, respectively σ_+, constructed by assigning mass $(1/N)^k$ to each *k-cylinder* Z_k ($a_i \in S$, and ℓ in $\mathbb{Z}$ or $\mathbb{Z}_+$ respectively):

$$Z_k = Z_k(a_1, \ldots, a_k, \ell) = \{x \in \Sigma_{(+)} \mid x_{\ell+j-1} = a_j\, , j = 1, \ldots, k\}\, . \tag{2.7}$$

Standard results from probability theory guarantee that this defines a Borel probability measure on Σ or Σ_+, and it is not difficult to verify that this measure is shift-invariant. To see that the one-sided shift is mixing (and thus ergodic), express observables φ, $\psi \in L^2(d\mu)$ as weighted sums of characteristic functions of cylinders (at $\ell = 0$) of increasing length k, and use that

$$\int \chi_Z \circ (\sigma_+)^n \chi_{Z'}\, d\mu = \mu(Z)\, \mu(Z')\, ,$$

for two cylinders Z, Z' centered (at $\ell = 0$), if n is larger than the sum of the lengths of Z and Z'. The details are left to the reader.

The connection with physical measures is given by the semi-conjugacy π which lifts Lebesgue measure to the ergodic invariant measure μ.

By showing that the "transfer operator" $\mathcal{L}$ associated to this dynamical system has a "spectral gap" when acting on Lipschitz observables $\varphi : \Sigma_+ \to \mathbb{C}$ for d_θ, one proves that correlation functions associated to (μ, σ_+) and Lipschitz observables on Σ_+ (and similarly for Σ) decay exponentially fast with rate at least θ. (For Hölder observables of exponents $0 < \alpha < 1$ the rate is θ^α.) See Sections 3 and 4.

3. A road atlas: results for deterministic maps

Here are the main examples of possible speeds of mixing (we concentrate on the rate of decay of correlations (1.7)) for discrete-time dynamical systems $f : M \to M$ posessing a mixing *physical measure* μ:

Speed # 1: *Exponential.*

If there is a Banach space $\mathcal{B}$ of functions or distributions on M and a "rate" $0 < \tau < 1$ so that for each φ, $\psi \in \mathcal{B}$ there is $K(\varphi, \psi) > 0$ so that for all $n \in \mathbb{Z}_+$

$$\rho_{\varphi\psi}(n) \le K(\varphi, \psi)\, \tau^n\, , \tag{3.1}$$

the correlation function is said to enjoy *exponential decay.* (It is then summable so that our "rule of thumb" condition for the CLT holds.) The constant K usually depends on φ, ψ through their norms, a typical example is $K(\varphi, \psi) \le C \int |\varphi|\, dx\, \|\psi\|_{\mathcal{B}}$, where $C > 1$ only depends on f.

Exponential decay of correlations holds for the physical measure(s) of:

• *Smooth uniformly expanding dynamics.*

More precisely: assume that M is compact and connected, $f : M \to M$ is $C^{1+\epsilon}$ for some $\epsilon > 0$ and that there are $C > 0$ and $\lambda > 1$ with $\|Df^n v\| \ge C\lambda^n$ for all $v \in TM$. Then f possesses a unique (ergodic) absolutely continuous invariant measure μ, with a positive density with respect to Lebesgue. It is thus the unique physical measure, and also an SRB measure. This measure is mixing [KS]. The rate of mixing is exponential if the observables are at least Hölder (see, e.g., [Ru5] for a relatively recent reference). The result also holds for random compositions of smooth expanding maps [BKS].

• *One-dimensional piecewise smooth and piecewise (non)uniformly expanding dynamics.*

More precisely, $M = I$ is a compact interval, $f : I \to I$ is continuous and there is a finite partition of I into subintervals such that f restricted to each subinterval J is C^2 (or at least $C^{1+\epsilon}$), and $|f'|_J| \geq \lambda > 1$. In this context, Lasota and Yorke [LaY] showed existence of an ergodic absolutely continuous invariant measure μ, which is thus a physical measure. Later Li and Yorke [LiY] studied the number of ergodic absolutely continuous invariant measures for such systems. Rychlik [Ry] studied the case when there are countably many (maximal) intervals of monotonicity. Using a transfer operator approach, Hofbauer and Keller [HoK] obtained exponential decay of correlations for (f, μ), and for observables φ, ψ in the Banach space BV of functions of bounded variation, if f is topologically mixing. Keller [Ke] later refined the spectral analysis. One can reduce to the topological mixing case by taking a suitable iterate of f, using important work of Hofbauer (see [Ho] and references therein), in particular a "tower extension" of the dynamics which inspired much later work. See Buzzi [Bu1] for a proof of exponential decay of correlations for an absolutely continuous quasi-invariant measure for random compositions of piecewise monotone transformations of the interval. We refer to Section 5 for other recent results and connections with dynamical zeta functions.

The uniform expansion assumption has been weakened to include, e.g., the logistic map $f_a(x) = a - x^2$ for a positive measure set of parameters a. For the full parabola $a = 2$ existence and mixing of an absolutely continuous measure are not difficult to obtain: $2 - x^2$ may be explicitly conjugated to a "tent map" of slopes ± 2 (Ruelle [Ru2]), Misiurewicz extended this to logistic maps for which the postcritical orbit is bounded away from the critical point. Jakobson and Benedicks-Carleson showed that f_a admits an absolutely continuous invariant (physical) ergodic measure μ for a positive measure set of as (see, e.g., [dMvS]). Then, using a change of metric in a tower extension, Young [Yo1] and, independently, Keller and Nowicki [KeN] showed that if f is topologically mixing the correlation function of (f, μ) decays exponentially for observables of bounded variation. (Their results apply in fact to a larger class of nonuniformly expanding unimodal maps. See also Section 9 of [Yo2].) Exponential decay of correlations is stochastically stable [BV] for such nonuniformly hyperbolic unimodal maps.

Piecewise expanding maps in higher dimensions are discussed under the heading "expanding systems with singularities" below.

• *Smooth uniformly hyperbolic dynamics.*

More precisely, M is compact and f is a $C^{1+\epsilon}$ Anosov or Axiom A diffeomorphism. Existence of (finitely many) ergodic physical measures was obtained by Sinai (Anosov diffeomorphisms) and Ruelle (Axiom A attractors). If f is topologically mixing (one may reduce to this case via the Smale spectral decomposition) then (f, μ) is mixing and correlations decay exponentially fast for Hölder observables (see, e.g., [Bow], [Ru3]). Pollicott [Po1], Ruelle [Ru4] and Haydn [Ha1, Ha2] obtained important results relating the correlation function and the zeta function. The first proofs used Markov partitions and symbolic dynamics to reduce to a transfer operator defined from a subshift of finite type and a Hölder weight. Different proofs using Birkhoff cones [Li1], [V2], or the usual spectral gap approach but on towers with recurrence estimates [Yo2], were recently carried out.

• *Some smooth nonuniformly hyperbolic dynamical systems.*

The Hénon family is a two-parameter family $H_{a,b}$ of diffeomorphisms which is in some sense a two-dimensional version of the unimodal family f_a. Benedicks and Young [BeY3] recently showed that, for a positive measure set of parameters a, b, the physical measure μ that they had previously exhibited [BeY2] for $H_{a,b}$ enjoyed exponential decay of correlations for Hölder observables. (Using Young's tower construction [Yo2].) A recent work of Wang and Young [WY] extends this analysis to an axiomatic class of nonuniformly hyperbolic surface attractors.

• *Some partially hyperbolic maps.*

Some partially hyperbolic maps (such as time-one maps of hyperbolic geodesic flows or certain skew product constructions) naturally preserve volume, so that the physical measure is given a priori. (See under polynomial mixing for Dolgopyat's [Do5] rapid mixing results for some compact group extensions.) In more abstract, non necessarily volume-preserving situations, Bonatti-Viana [BoV] (assuming that the central direction is "mostly contracting") and more recently Alves-Bonatti-Viana [ABV] (if the central direction is "mostly expanding") recently showed that there are finitely many physical measures the union of the ergodic basins of which has full measure (building on previous work by Pesin and Sinai [PeS]). In the mostly contracting (topologically mixing) case, using methods in [Yo2], Castro [Cas] (under additional technical assumptions) and Dolgopyat [Do4] (under natural additional assumptions) showed that such physical measures enjoy exponential decay of correlations for Hölder observables. (Castro introduced a combinatorical variant of Young's tower construction.)

• *Some expanding/hyperbolic systems with "singularities."*

This class extends piecewise monotone interval maps to higher dimensions. Such dynamical systems appear naturally, e.g., when one considers billiards flows (which preserve volume by construction). In dimension two, Buzzi [Bu2] and Tsujii [Tsu1] independently proved that piecewise expanding and analytic maps admit an absolutely continuous invariant probability measure. An example of Tsujii ([Tsu2], see also [Bu3]) demonstrates that assuming only piecewise expansion and arbitrary high (but finite) differentiability is not sufficient to garantee existence of (a finite number of) such invariant measures. Exponential decay of correlations for a large class of piecewise smooth hyperbolic area-preserving systems was shown by Liverani [Li1]. Existence of a physical measure and exponential decay of its correlations was obtained for certain piecewise expanding or hyperbolic maps in higher dimensions by Young [Yo2] and then Chernov [Ch3], and for the Poincaré maps of certain billiards by Young [Yo2], Chernov [Ch4]. For recent work regarding weak sufficient conditions to obtain an absolutely continuous invariant measure for higher-dimensional piecewise expanding dynamical systems we mention Saussol [Sau]. See also Cowieson's generic transversality condition [Co]. Recently, Buzzi and Maume [BM] obtained weak sufficient conditions for exponential decay of the corresponding correlations.

Sometimes lower bounds $\rho_{\varphi\psi}(n) \geq K'\tau^n$ for the decay of correlations can be obtained by exhibiting eigenvalues of modulus τ with eigenfunctions in $\mathcal{B}$ for a suitable transfer operator. Then, mixing is "truly exponential."

Speed # 2: *Stretched exponential.*

If there is a Banach space $\mathcal{B}$ of functions or distributions on M, a "rate" $0 < \tau < 1$ and an "exponent" $0 < \xi < 1$ so that for each φ, $\psi \in \mathcal{B}$ there is $K(\varphi, \psi) > 0$ so that for all $n \in \mathbb{Z}_+$

$$\rho_{\varphi\psi}(n) \le K(\varphi, \psi)\, \tau^{n^\xi} , \tag{3.2}$$

the correlation function is said to enjoy *stretched exponential decay.* (It is then always summable so that the "rule of thumb" for the TCL is satisfied.) To the author's knowledge there are few concrete examples of systems with "true" stretched exponential decay of correlations (with lower bounds). Early estimates (by Bunimovich–Sinai–Chernov) for the decay of correlation of Poincaré maps of billiards gave stretched exponential upper bounds (Liouville volume is ergodic and invariant and thus a physical measure), but the more recent results [Yo2], [Ch4] mentioned above yield exponential asymptotics. Another example is given by the decay of correlation of certain hyperbolic flows: the stretched exponential bounds for correlations of Anosov flows obtained by Chernov [Ch2] and Liverani [Li3] were subsumed by the exponential estimates of Dolgopyat [Do1] *when the dynamical foliations are C^1*. (Note, however, that in the more generic case of Lipschitz foliations the stretched exponential bounds of Chernov [Ch2] are stronger than the rapid decay obtained by Dolgopyat [Do2, Do3]. Also, Liverani's results [Li3] apply to Hölder foliation, giving polynomial bounds.)

Stretched exponential upper bounds have recently been obtained for some random nonuniformly hyperbolic systems (Baladi-Benedicks-Maume [BBM], lower bounds are not available), and may be conjectured for the systems studied by Viana [V2] and Alves [A].

In Young's symbolic tower model, for systems with stretched exponential return asymptotics [Yo3] the rates of decay of correlations and the speed of approach to equilibrium are truly stretched exponential (see Theorem 4.2(3) below).

Speed # 3: *Polynomial.*

If there is a Banach space $\mathcal{B}$ of functions or distributions on M and an "exponent" $\gamma > 0$ so that for each φ, $\psi \in \mathcal{B}$ there is $K(\varphi, \psi) > 0$ so that for all $n \in \mathbb{Z}_+$

$$\rho_{\varphi\psi}(n) \le K(\varphi, \psi)\, \frac{1}{n^\gamma} , \tag{3.3}$$

the correlation function is said to enjoy *polynomial decay* of correlations. The correlation function is summable if $\gamma > 1$.

Polynomial decay of correlations has been proved for some "parabolic" or "intermittent" systems having an absolutely continuous invariant probability measure. They include (mixing) one-dimensional maps expanding everywhere except at a fixed point at the origin where the map has local expansion $f(x) \sim x + Cx^{1+\delta}$, and $0 < \delta < 1$. Then, $\gamma = 1 - 1/\delta$, so that the CLT holds if $0 < \delta < 1/2$ (we refer to Hu [Hu], who also obtained lower bounds, and Young [Yo3], as well as Isola [Is1, Is2] for strong results on a piecewise linear model). See also Liverani–Saussol–Vaienti [LSV2] for earlier estimates using a different approach. Hu has recently suggested the terminology *almost hyperbolic* for a class of dynamical systems including this model. Higher dimensional dynamical systems with neutral periodic orbits appear in connection with continued fraction algorithms from number theory. (There is in particular work of Yuri [Yu1, Yu2] regarding the existence of ergodic absolutely continuous measures and speed of approach to equilibrium.) We also mention a paper by Aaronson-Denker-Urbański [ADU]. Sometimes the presence of a nonhyperbolic periodic orbit is an obstruction to the existence of a good finite physical measure (see the two-dimensional example of Hu and Young [HuY]). Fayad [F] has very recently obtained polynomial rates of mixing for the a.c.i.m. of a class of smooth flows of the two-torus.

Dolgopyat [Do5] obtained rapid mixing (i.e., polynomial of arbitrary exponent) for C^∞ test functions and a class of Diophantine compact group extensions (here SRB is just volume), using asociated Brin groups.

In a recent paper, using the coupling technique on towers from [Yo3], Bruin, Luzzatto and van Strien [BLS] exhibit various possible rates of mixing (exponential to polynomial) within the quadratic family.

Other equilibrium states:
The physical measure is not the only invariant measure of natural interest. Other Gibbs states and equilibrium states (for Hölder and non Hölder weights) have been studied, together with their rates of mixing. We mention a few references into the vast literature dealing with thermodynamic formalism: Bowen [Bow], Ruelle [Ru5], Hofbauer-Keller [HK], Keller-Nowicki [KN], Maume [Mau1, Mau2], Mayer [Ma1], Liverani-Saussol-Vaienti [LSV1]. See also [Bal]. We shall not discuss this (interesting) problem further in these notes.

4. Choosing a vehicle and a highway: Two models and three approaches to study speed of mixing

Three different approaches have been applied successfully to estimate the rate of decay of correlation and the speed of approach to equilibrium. We shall describe them briefly, pointing out relations with dynamical zeta functions when relevant and giving a few references. Some of these methods (in particular cone methods) may be implemented on the dynamical system under study directly. Others require to construct first a "model" for the dynamics $f : M \to M$, usually in the form of an extension map $F : \Delta \to \Delta$ with a semi-conjugacy $\pi : \Delta \to M'$ with $M' \subset M$, such that $f \circ \pi = \pi \circ F$. Such a model should embody the main features of the system, and the projection (although neither necessarily surjective nor injective) should be such that not too much information is lost. Before picking a highway, we need to discuss cars types: Let us thus describe two such models: a very classical one (symbolic Markov dynamics) and a more recent and perhaps more flexible one (Markov towers with asymptotics of recurrence times).

4.A Two models: Symbolic dynamics and Towers with return times.

Model # 1: *Symbolic dynamics (SFT).*

Uniformly expanding or uniformly hyperbolic dynamical systems (such as Axiom A diffeomorphisms $f : M \to M$ restricted to an attracting transitive basic set) admit finite Markov partitions. This enables construction of a semi-conjugacy $\pi : \Sigma_A \to M$ with a subshift of finite type (SFT): the restriction of the full shift σ on N symbols to a compact invariant subset $\Sigma_A \subset \Sigma$ defined by a transition matrix A. It is known [Bow] that the unique "equilibrium state" of $\log |\det Df|_{E^u}|$ (where E^u denotes as usual the unstable bundle) is the physical measure of a transitive Axiom A diffeomorphisms.

Lifting the function $\log|\det Df|_{E^u}|$ to Σ_A, one gets a Lipschitz function (if θ is chosen in an appropriate way). Therefore, it suffices to understand equilibrium states of Lipschitz functions for a subshift of finite type. We refer, e.g., to the books of Bowen [Bow], Ruelle [Ru3], or Parry and Pollicott [PaPo]. Generalisations of this model consist in considering countably many symbols ([Br], [Mau2], [Sar]) or weakening ([Mau1], [Po2]) the Hölder/Lipschitz assumption.

Model # 2: *Markov towers with recurrence time asymptotics.*

The use of towers with return times to the basis is not new in dynamics (especially continuous-time dynamics): The grandfather of such constructions in ergodic theory is probably due to Rokhlin. More recently, Hofbauer [Ho] constructed Markov tower extensions of discrete-time interval maps. They were used, e.g., by Hofbauer-Keller [HoKe] and Keller–Nowicki [KN] to study the statistical properties of interval maps. Other tower constructions are due to Young [Yo1] and Baladi-Viana [BV]. We describe here Young's [Yo2, Yo3] recent axiomatised version :

Let $(\Delta_0, \mathcal{B}, m)$ be a probability space, which will be the "basis" of our tower. (The reader can imagine that Δ_0 is a topological space and $\mathcal{B}$ the Borel σ-algebra, but this is not necessary.) We view m as our reference measure, in applications it is often Lebesgue measure or a measure induced by Lebesgue. Assume we are given a countable partition $\Delta_0 = \cup_{i\in\mathbb{Z}}\Delta_{0,i}$, modulo m, together with a positive integer-valued function R (the return-time), defined m-almost everywhere on Δ_0 and constant on each $\Delta_{0,i}$. The *tower* is obtained by layering $R|_{\Delta_{0,i}}$ copies of $\Delta_{0,i}$ above itself:

$$\Delta = \{(x,\ell) \in \Delta_0 \times \mathbb{Z}_+ \, , \, \ell + 1 \le R(x)\} \, . \tag{4.1}$$

Set $\Delta_{\ell,i}$ for the subsets of $\Delta_\ell = \{x \in \Delta_0 \mid R(x) \ge \ell\}$ induced by the partition $\Delta_{0,i}$, and call $\mathcal{Z}$ the resulting partition of Δ. (We sometimes abuse slightly notation, writing $\Delta = \cup_{\ell\ge 0}\Delta_\ell$.)

The *tower* dynamics $F : \Delta \to \Delta$ takes the following form:

$$F(x,\ell) = \begin{cases} (x,\ell+1) & \ell+1 < R(x), \\ (y(x),0) & \ell+1 = R(x) \, , \end{cases} \tag{4.2}$$

where the map $x \mapsto y(x)$ is such that the restriction of $(x,0) \to F^{R(x)}(x,0)$ to $\Delta_{0,i}$ is a bijection $\Delta_{0,i} \to \Delta_0$, for all i.

In view of defining a "dynamical distance," introduce a separation time $s : \Delta \times \Delta \to \mathbb{Z}_+ \cup \{\infty\}$ by setting

$$s(x,y) = \min\{n \ge 0 \mid F^n(x) \text{ and } F^n(y) \text{ lie in distinct elements of } \mathcal{Z}\} \, . \tag{4.3}$$

The definition implies in particular that if (x,ℓ) and (y,ℓ) are both in some $\Delta_{\ell,j}$, then $s((x,0),(y,0)) \ge \ell$ and for any $(x,0),(y,0) \in \Delta_{0,j} \times \{0\}$

$$s((x,0),(y,0)) = R(x) + s(F^{R(x)}(x,0), F^{R(x)}(y,0)) \, . \tag{4.4}$$

The two main assumptions on this abstract model are:

• *Bounded distortion:* For each $\Delta_{0,i}$, the map $F^R|_{\Delta_{0,i}}$ and its inverse are nonsingular with respect to the measure m so that the Jacobian JF^R exists (and almost never vanishes). There are $C > 0$ and $0 < \beta < 1$ so that for all i and each $x, y \in \Delta_{0,i}$

$$\left| \frac{JF^R(x)}{JF^R(y)} - 1 \right| \le C\beta^{s(F^R(x),F^R(y))} \tag{4.5}$$

• *Integrability of return times:* $\int_{\Delta_0} R(x)dm(x) < \infty$.

Consider the "dynamical Lipschitz" function space on Δ:

$$\mathcal{F}_\beta(\Delta) = \{\varphi : \Delta \to \mathbb{C} \mid \exists C_\varphi\,, |\varphi(x) - \varphi(y)| \le C_\varphi \beta^{s(x,y)}\,, \forall x, y \in \Delta\}\,. \tag{4.6}$$

Young [Yo2], [Yo3] showed that these assumptions are enough to guarantee the existence of an F-invariant measure ν absolutely continuous with respect to m on Δ. The density of this measure belongs to $\mathcal{F}_\beta$ (see Theorem 4.2 (1) below). The interest of this method lies in its rather broad applicability to actual dynamical systems $f : M \to M$ possessing "some" hyperbolicity. The difficulty there is to find a subset $\Lambda \subset M$ and a return-time function R so that f^R has good properties on Λ, in order to construct a semi-conjugacy $\pi : \Delta \to \cup_{\ell \ge 0} f^R \Lambda$ between f and F, showing that F satisfies the desired axioms. (The semi-conjugacy map is neither surjective nor injective in general.) Then, if the return-times satisfy exponential asymptotic bounds (see Theorem 4.2 (4)) the transfer operator associated to F has a spectral gap on a suitable space [Yo2], and one may show exponential decay of correlations for (F, ν) and observables in (a modified version of) $\mathcal{F}_\beta$. The physical measure μ may be defined by projecting ν, and the result on correlations carries over to Lipschitz observables (Lipschitz functions are lifted to $\mathcal{F}_\beta$). See Theorem 4.2(5) (6) for the nonexponential case. Note that there are slight technical differences between the definitions of the tower models in [Yo2] and [Yo3].

4.B Three methods to estimate speeds of mixing.

Method # 1: *Spectrum of transfer operator.*

Spectral gap (spectral approach).

The spectrum of the Ruelle transfer operator $\mathcal{L}_g$, associated to a subshift of finite type and a Hölder or Lipschitz weight g, and acting on the Banach space of Lipschitz functions for a metric d_θ are discussed in the notes of M. Pollicott on *Zeta functions,* where its relationship with the zeta function

$$\zeta(g, z) = \exp \sum_{n \ge 1} \frac{z^n}{n} \sum_{x \in \mathrm{Fix} f^n} \prod_{k=0}^{n-1} g(f^k(x))$$

was presented. Let us sketch here how a spectral gap for a transfer operator is related to exponential decay of correlations, in the simplest possible case: smooth expanding dynamics on the circle S^1. Let then $f : S^1 \to S^1$ be C^r for some $r \ge 2$, and assume that f is uniformly expanding in the sense that there is $C > 0$ and $\lambda > 1$ so that $|(f^n)'| \ge C\lambda^n$ for all $n \in \mathbb{Z}_+$ (an example is given by considering a small C^r perturbation of the angle-doubling map from Section 2). The transfer operator $\mathcal{L}$, associated to f and the weight $g = 1/|f'|$, acting on the Banach space $\mathcal{B}$ of C^s functions on M (with $0 \le s \le r-1$, with fractal exponents denoting Hölder continuity) φ by

$$\mathcal{L}\varphi(x) = \sum_{fy=x} \varphi(y)g(y) = \sum_{fy=x} \frac{\varphi(y)}{|f'(y)|}\,, \tag{4.7}$$

is appropriate to study absolutely continuous invariant measures for f. Indeed, its dual $\mathcal{L}^*$ acting on $\mathcal{B}^*$ by $\mathcal{L}\nu(\varphi) = \nu(\mathcal{L}\varphi)$ for $\varphi \in \mathcal{B}$ preserves Lebesgue measure dx:

$$\int \mathcal{L}\varphi\, dx = \int \varphi\, dx\,.$$

(By the change of variable formula, the above equality holds in fact for all $\varphi \in L^1(dx)$, since $|f'|$ is the Jacobian of f with respect to Lebesgue measure.)

It can be shown (by arguments similar to those used to prove the Ruelle operator theorem in the notes of M. Pollicott on *Zeta functions*) that the spectral radius of $\mathcal{L}$ is equal to 1 and that its *essential spectral radius* is not larger than λ^{-s}. (This fact also holds for $s = 0$. In this case each point in the open disc of radius 1 is an eigenvalue of infinite multiplicity! Hence, the Banach space of continuous functions and a fortiori $L^1(dx)$ or $L^2(dx)$ are "too big"). See Section 5 for a proof of an analogous result. We just mention that the basic ingredient for the bound for the essential spectral radius in the case $s = 1$ is the "Lasota-Yorke" [LaY] inequality: there is $C > 0$ so that for all C^1 functions φ and all $n \in \mathbb{Z}^+$

$$\sup_M \left| \frac{d}{dx} (\mathcal{L}^n \varphi)(x) \right| \le C \lambda^{-n} \sup_M \left| \frac{d}{dx} \varphi(x) \right| + C \sup_M |\varphi(x)| \, . \tag{4.8}$$

The above estimate may be proved by using the Leibniz formula (derivation of a product), chain rule (derivation of a composition). The fact that the inverse branches of f^n contract by a factor at least λ^{-n} obviously enters the computation. (In other words: composing a C^1 function φ with a C^1 contraction *improves* the smoothness of φ in the sense that it decreases the C^1 norm.)

Assume from now on that $s > 0$. Since the spectrum of a bounded linear operator and its dual coincide outside of the disc of essential spectral radius, $\mathcal{L}$ has a fixed point $\varphi_0 \in \mathcal{B}$. In fact, using positivity of $\mathcal{L}$ and generalising the finite-dimensional *Perron-Frobenius* theorem (see, e.g., [Wal]), one shows that φ_0 is a positive function, which can be normalised to ensure $\int \varphi_0 \, dx = 1$. Easy algebra then shows that the absolutely continuous probability measure $\mu = \varphi_0 \, dx$ is f-invariant:

$$\int \varphi \circ f \varphi_0 \, dx = \int \mathcal{L}(\varphi \circ f \varphi_0) \, dx = \int \varphi (\mathcal{L} \varphi_0) \, dx = \int \varphi \, d\mu \, .$$

The eigenvalues of $\mathcal{L}$ on the unit circle consist in a finite set of roots of unity, they are simple because smooth expanding circle maps are topologically transitive. Using that smooth expanding circle maps are in fact topologically mixing, one shows that the eigenvalue 1 is simple and is the only eigenvalue of modulus 1 of $\mathcal{L}$ acting on $C^s(S^1)$ if $s > 0$. In other words, we have the following *spectral decomposition:*

$$\mathcal{L} \varphi = \varphi_0 \int \varphi \, dx \, + \, \mathcal{R} \varphi \, , \tag{4.9}$$

where $\mathcal{R}$ is a bounded operator of spectral radius $\tau < 1$ on $C^s(S^1)$. (Either τ corresponds to the modulus of an eigenfunction of $\mathcal{L}$ or it is just the essential spectral radius.) Thus, since $\mathcal{L}^n(\varphi \circ f^n \psi) = \varphi \mathcal{L}^n \psi$, for φ, $\psi \in C^s$,

$$\begin{aligned} | \int \varphi \circ f^n \psi \, dx - \int \varphi \, d\mu \int \psi \, dx | &= | \int \varphi (\mathcal{L}^n \psi - \varphi_0 \int \psi dx) dx | \\ &\le \sup |\mathcal{R}^n \psi| \int |\varphi| \, dx \end{aligned}$$

which proves exponential decay of correlations (for any rate $> \tau$) since $\sup |\varphi| \le \|\varphi\|$ for the Banach space $\mathcal{B} = C^s(S^1)$.

Beyond first gap: zeta function (spectral approach).

A more careful spectral decomposition than (4.9), writing out explicitly a sum over eigenvalues of modulus larger than $\lambda^{-s}+\epsilon$ for small ϵ, is the starting point to see that the Fourier transform

$$\widehat{\rho}(\omega)=\sum_{n\in\mathbb{Z}} e^{in\omega}\rho_{\varphi\psi}(n)$$

of the correlation function associated to φ, $\psi \in \mathcal{B}$ (setting $\rho_{\varphi\psi}(n)=\rho_{\psi\varphi}(-n)$ for negative n) is meromorphic in the strip $\Im(\omega)<\log\lambda^s$, where its poles (the *Ruelle resonances*) may only be located at ω with $\exp(i\omega)$ or $\exp(-i\omega)$ an eigenvalue of $\mathcal{L}$. Also, these eigenvalues of $\mathcal{L}$ on $C^s(S^1)$ are described (including multiplicity) by the poles of the zeta function $\zeta_g(z)$ for $g=1/|f'|$ (which is known to be meromorphic in the disc of radius λ [Ru5])

Further spectrum: Fredholm determinants (spectral approach).

Let us see what more can be said if f and g are of higher smoothness. In the case of analytic expanding maps on S^1, the transfer operator $\mathcal{L}$ acting on the Banach space of holomorphic bounded functions in a small annulus obtained by "thickening" the unit circle is a *compact* operator (in particular its essential spectral radius vanishes). This can be shown by using Montel's theorem and is due to the fact that composing a holomorphic function with a holomorphic contraction extends the domain of holomorphy ("improves analyticity"). In fact, as observed by Ruelle [Ru1] in the seventies, $\mathcal{L}$ is in this case a *nuclear operator in the sense of Grothendieck* [Gr1, Gr2]. This also holds for analytic Anosov diffeomorphisms with analytic invariant foliations (unfortunately a very strong assumption): see Ruelle [Ru1], Fried [Fr1], and Mayer's extensive study of the thermodynamic formalism of the Gauss map $x\mapsto\{1/x\}$ [Ma1] (and references therein). We especially recommend Mayer's very clear survey [Ma2]. Nuclear operators generalise classical Fredholm operators such as

$$\mathcal{K}:\varphi\to\int_0^1 K(\cdot,y)\varphi(y)\,dy\,,\qquad K(x,y)\text{ continuous on }[0,1]\times[0,1]\,, \tag{4.10}$$

acting on $C^0([0,1])$. In particular, nuclear operators possess a *trace* tr $\mathcal{K}$ which has many of the features of the trace (i.e., sum of eigenvalues) of a finite matrix. The most important of these features is perhaps the fact that the power series

$$\exp-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{tr}\ \mathcal{K}^n \tag{4.11}$$

is an entire function, called the Fredholm determinant of the operator $\mathcal{K}$, the zeroes of which are exactly the inverse of the eigenvalues of $\mathcal{K}$. (Clearly, this generalises the formula $\det(I-zA)=\exp-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{tr}\ A^n$ from linear algebra, for a finite matrix A.) In the case of the Fredholm operator $\mathcal{K}$ from (4.10), the trace is just given by the, intuitively obvious, "sum along the diagonal" $\int K(x,x)\,dx$. In the case of dynamically defined transfer operators (4.7) associated to analytic expanding dynamics f and analytic weights g (e.g., $1/|f'|$, the modulus being dealt with by a suitable argument), the trace can be computed by using Cauchy integrals:

$$\sum_{x\in\mathrm{Fix}\ f}\frac{g(x)}{1-(f_x^{-n})'(x)}=\sum_{x\in\mathrm{Fix}\ f}\frac{1}{\det(Df_x^n(x)-\mathrm{Id})}\,,$$

where the inverse branch f_x^{-n} is uniquely defined by the periodic orbit of x.

The expression in the denominator *morally* comes from the Jacobian of the change of variable associated to the map $\mathrm{Id} - f^{-1}$ arising from the operator with singular kernel:

$$\mathcal{L}\varphi(x) = \int \varphi(y)\delta(f(y)-x)g(y)|f'(y)|\,dy = \int \varphi(y)\delta(f(y)-x)g(y)|\mathrm{det}Df(y)|\,dy\,,$$

where δ is the Dirac delta.

Therefore, the Fredholm determinant whose zeroes describe the spectrum of $\mathcal{L}_g$ acting on holomorphic functions for $g = 1/|\mathrm{det}Df|$ (i.e., the correlation spectrum of the physical measure of the expanding analytic map f) is just

$$\begin{aligned}(4.12)\qquad &\exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \text{ Fix } f^n} \frac{1}{|\mathrm{det}Df^n(x)|} \frac{1}{\det(\mathrm{Id} - Df_x^{-n}(x))} \\ &\qquad = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \text{ Fix } f^n} \frac{1}{\det(Df^n(x) - \mathrm{Id})}\,.\end{aligned}$$

The denominator $\det(\mathrm{Id} - Df_x^{-n}(x))$, which is present here but not in the zeta function $\zeta_g(x)$, improves the convergence of the series and extends the domain in which its zeroes describe the inverse eigenvalues of $\mathcal{L}$. (The reader is invited to compute both the zeta function and the dynamical determinant of the angle doubling map for the weight $g = 1/2$.) See [Ru1], [Fr1].

Although the transfer operators (4.7) associated to finitely differentiable expanding maps and/or weights (say, C^r for $r \geq 1$, and assuming that the spectral radius is 1 for simplicity) are not compact, expressions of the type (4.12) have been shown to decribe their inverse spectrum in the disc of radius λ^r (Ruelle [Ru5, Ru6]), a larger radius than the value λ corresponding to the domain of "visibility" (meromorphy) of the zeta function.

Two-sided or hyperbolic dynamics (spectral approach).

The situation of two-sided shifts (or hyperbolic maps as opposed to expanding maps) is more delicate. Indeed, coexistence of contracting and expanding directions means that the transfer operator does not improve regularity of smooth functions. We first discuss the *"historical" approach.* It has been used in statistical mechanics, see Bowen's book [Bow], but also very recently in the axiomatic framework of Young [Yo2], e.g., for good Hénon maps [BeY3]. It consists in reducing to the one-sided dynamics by expressing the measure $\hat\mu$ under study as the natural extension of a one-sided measure μ (see below). The idea is thus to "quotient out" the stable directions and some information ("half of the smoothness") is lost in the process. On must also approximate a "two-sided" function (depending on both past and future) by sums of "one-sided" functions. Let us describe this in detail for an equilibrium measure $\hat\mu$ of a two-sided subshift of finite type $\sigma : \Sigma_A \to \Sigma_A$ and a positive Lipschitz weight $\hat g$. Let $\mathcal{C}_\theta$ and $\mathcal{C}_\theta^+$ be the Banach spaces of Lipschitz functions for d_θ on the two-sided and one-sided shift spaces Σ_A, Σ_A^+, writing $|\cdot|_\theta$ for the Lipschitz constant and $\|\cdot\|_\theta = \max(|\cdot|_\theta, \sup|\cdot|)$ for the norm. For $\varphi, \psi \in \mathcal{C}_\theta$, assuming that $\int_{\Sigma_A} \psi\, d\hat\mu = 0$, we wish to analyse

$$\rho_{\varphi\psi}(n) = \int \varphi \circ \sigma^n\, \psi\, d\hat\mu\,.$$

and the formal Fourier transform $\widehat{\rho}_{\varphi\psi}(\omega)$.

Our starting point is the fact that for each $\phi \in \mathcal{C}_\theta$, there is a sequence of functions $\phi_n \in \mathcal{C}^+_\theta$, $n \ge 0$ such that ϕ_n is constant on n-cylinders at 0, $\sup_{\Sigma_A^+} |\phi_n| \le \theta^n \|\phi\|_\theta$ (hence, $|\phi_n|_\theta \le 2\|\phi\|_\theta$) for all $n \ge 0$, and with

$$\phi = \sum_{n=0}^{\infty} \phi_n \circ \Pi \circ \sigma^{-n} \,, \tag{4.14}$$

where $\Pi : \Sigma_A \to \Sigma_A^+$ is the map which simply "forgets" x_i for $i < 0$. We may in fact construct the sequence ϕ_n explicitly: For each $1 \le j \le N$, fix $a^{(j)} \in \Sigma_A$ with $(a^{(j)})_0 = j$, and define a function $r : \Sigma_A \to \Sigma_A$ by setting

$$r(x)_k = \begin{cases} x_k & \text{if } k \ge 0 \\ (a^{(x_0)})_k & \text{if } k \le 0 \,. \end{cases} \tag{4.15}$$

Abusing notation, the function r can be viewed as $r : \Sigma_A^+ \to \Sigma_A$ with $\Pi \circ r = \mathrm{Id}$. Then, we may take $\phi_0 = \phi \circ r$, and for each $n \ge 1$, inductively define,

$$\phi_n = (\phi - \sum_{k=0}^{n-1} \phi_k \circ \Pi \circ \sigma^{-k}) \circ \sigma^n \circ r \,. \tag{4.16}$$

We finally associate a formal power series to $\phi \in \mathcal{C}_\theta$:

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} z^n \phi_n \ . \tag{4.17}$$

It is not difficult to see that $\tilde{\phi}(z)$ is an analytic $\mathcal{C}^+_\theta$-valued functions for $|z| < 1$, and is an analytic function with values in the space of bounded functions for $|z| < \theta^{-1}$. (See [DS, Definition VI.10.5] for the concept of analytic functions from $\mathbb{C}$ to an arbitrary Banach space.)

Now, it is known that there is $h \in \mathcal{C}_{\sqrt{\theta}}$ with $\hat{g} = g + h - h \circ \sigma$ where g only depends on the future and can thus be viewed as an element of $\mathcal{C}^+_{\sqrt{\theta}}$. (Note that one loses "one-half" of the differentiability: a Lipschitz function for $d_{\sqrt{\theta}}$ is 1/2-Hölder for d_θ.) Also, the one-sided shift invariant equilibrium state μ for g admits $\hat{\mu}$ as natural extension (see, e.g., [Bow]). The transfer operator $\mathcal{L}_g$ associated to σ^+ and g acts on $\mathcal{C}^+_{\sqrt{\theta}}$, with maximal eigenvalue $e^P > 0$ and essential spectral radius $< \sqrt{\theta} e^P$, and $\mu = \varphi_0 \, \nu_0$ with $\mathcal{L}_g \varphi_0 = e^P \varphi_0$ and $\mathcal{L}_g^* \nu_0 = e^P \nu_0$. There is a normalised operator $\tilde{\mathcal{L}}$ which satisfies $\tilde{\mathcal{L}}^* \mu = \mu$, $\tilde{\mathcal{L}} 1 = 1$, and which has the same spectrum as $e^{-P} \mathcal{L}_g$. (See, e.g., [PaPo].)

Finally, we get the following sequence of equalities between power series:

$$\begin{aligned} \hat{\rho}_{\varphi\psi}(\omega) &= \sum_{k=-\infty}^{\infty} e^{ki\omega} \int_{\Sigma_A} (\varphi \circ \sigma^{-k}) \, \psi \, d\hat{\mu} \\ &= \sum_{k=-\infty}^{\infty} \sum_{m,n=0}^{\infty} e^{ki\omega} \int_{\Sigma_A} (\varphi_m \circ \Pi \circ \sigma^{-k-m})(\psi_n \circ \Pi \circ \sigma^{-n}) \, d\hat{\mu} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m,n=0}^{\infty} e^{(n-m)i\omega} \Big[\sum_{k \le n-m} e^{(k+m-n)i\omega} \int_{\Sigma_A^+} (\varphi_m \circ (\sigma^+)^{n-m-k})\, \psi_n \, d\mu \\
&\quad + \sum_{k \ge n-m} e^{(k+m-n)i\omega} \int_{\Sigma_A^+} \varphi_m (\psi_n \circ (\sigma^+)^{k+m-n})\, d\mu - \int_{\Sigma_A^+} \varphi_m \, \psi_n \, d\mu \Big] \\
&= \sum_{\ell=0}^{\infty} e^{-\ell i\omega} \int (\tilde\varphi(e^{-i\omega}) \circ (\sigma^+)^\ell) \tilde\psi(e^{i\omega})\, d\mu \\
&\quad + \sum_{\ell=0}^{\infty} e^{\ell i\omega} \int \tilde\varphi(e^{-i\omega}) (\tilde\psi(e^{i\omega}) \circ (\sigma^+)^\ell)\, d\mu - \int \tilde\varphi(e^{-i\omega}) \tilde\psi(e^{i\omega})\, d\mu \\
&= \sum_{\ell=0}^{\infty} e^{-\ell i\omega} \int \tilde\varphi(e^{-i\omega}) \tilde{\mathcal{L}}^\ell (\tilde\psi(e^{i\omega}))\, d\mu \\
&\quad + \sum_{\ell=0}^{\infty} e^{\ell i\omega} \int \tilde\psi(e^{i\omega}) \tilde{\mathcal{L}}^\ell \tilde\varphi(e^{-i\omega})\, d\mu - \int \tilde\varphi(e^{-i\omega})\, \tilde\psi(e^{i\omega})\, d\mu \\
&= \int \tilde\varphi(e^{-i\omega})(1 - e^{-i\omega} \tilde{\mathcal{L}})^{-1} \tilde\psi(e^{i\omega})\, d\mu \\
&\quad + \int \tilde\psi(e^{i\omega})(1 - e^{i\omega} \tilde{\mathcal{L}})^{-1} \tilde\varphi(e^{-i\omega})\, d\mu - \int \tilde\varphi(e^{-i\omega})\, \tilde\psi(e^{i\omega})\, d\mu \,. \qquad (4.18)
\end{aligned}
$$

The spectral properties of $\mathcal{L}_g : \mathcal{C}^+_{\sqrt{\theta}} \to \mathcal{C}^+_{\sqrt{\theta}}$, together with the mentioned analyticity properties of the maps $z \mapsto \tilde\varphi(z)$, $\tilde\psi(z)$, imply that the right-hand side of (4.18) is an analytic function of ω in the strip $|\Im\omega| < \log(\tau^{-1})$ (τ as after (4.9)) which admits a meromorphic extension to the strip $|\Im\omega| < \log(\sqrt{\theta}^{-1})$. (For explicit formulas of the corresponding residues as well as an analysis of the order of the pole, see [Ru4], and also [Ha1].)

The above computations can be applied directly via symbolic dynamics and Markov partitions to show exponential decay for the physical measure (or other equilibrium measures) of Axiom A attractors ([Bow], [Po1], [Ru4]). We already proved that the corresponding ideas (quotienting out stable directions and approaching two sided functions by sums of one-sided functions) have been adapted by Young [Yo2] to towers with asymptotics of return times, in particular to prove exponential mixing for good Hénon maps [BeY3].

Another more *modern approach* to attack the *two-sided or hyperbolic* situation was initiated in the early nineties by Rugh [Rug1, Rug2] for analytic Axiom A diffeomorphisms (without assuming smoothness of the invariant foliations) and analytic weights. One of the main ideas was to introduce a Banach space of functions analytic in the unstable directions and "dual of analytic" in the stable directions. (In an analytic framework one can morally think of Laurent series.) The transfer operator associated to the full hyperbolic dynamics (*without* quotienting out stable directions) then has good spectral properties on this Banach space, it is even nuclear in the sense of Grothendieck. One thus obtains Fredholm determinants which are entire functions in this framework. The method was then extended further by Fried [Fr3].

Assuming only finite differentiability (here, partitions of unity are available), Kitaev [Kit] obtained a large radius of convergence (of the order of $\lambda^{r/2}$) for the dynamical determinant of C^r hyperbolic diffeomorphisms with C^r weights g. Fried has very recently (June 1999, IHES talk) announced a spectral interpretation of the zeroes of the corresponding Fredholm determinant.

Beyond uniform expansion (spectral approach).

In the list in Section 3 we mentioned that exponential decay holds for some nonuniformly expanding dynamical systems. The spectral approach has also been implemented there, by first constructing a *tower* model for the map. (Young [Yo1, Yo2], Benedicks-Young [BeY3], Keller-Nowicki [KN]). We refer to Section 5 for a decription of a form of regularisation, the kneading approach (Baladi-Ruelle [BR1, BR2]) inspired by Milnor-Thurston [MT].

Method # 2: *Projective (Hilbert) metrics on Birkhoff cones.*

One way to prove the finite-dimensional Perron-Frobenius theorem is by using the fact that any $n \times n$ matrix A with strictly positive coefficients maps the positive "quadrant" in $\mathbb{R}^n$ strictly into itself. Representing half-lines in the quadrant by their intersection with the simplex $\sum x_i = 1$, it is convenient to use the projective metric on the quadrant given by the cross ratio on the line joining x and y. One then proves that this projective metric is strictly decreased by the action of A (see, e.g., [Fur, § 15]). There exists an infinite dimensional version of this theory, where the positive quadrant is replaced by a convex cone Λ in a topological vector space (for example a space of functions or distributions), and A is replaced by a linear operator mapping Λ into itself. The cones are called Birkhoff cones in honor of Garrett Birkhoff (the main original reference by Birkhoff is [Bir], see also [Li1] for a reader-friendly exposition). These techniques were introduced into dynamical systems in particular by Ferrero-Schmitt in their pioneering paper [FS]. They were more recently used by Liverani [Li1] to prove exponential rates of mixing of some expanding and hyperbolic dynamical systems. This approach was generalised by Viana and is very well presented in his book [V2]. Castro [Cas] defined cones (in a tower setting) for some partially hyperbolic systems. In all these cases, a function cone is exhibited which is mapped strictly into itself with "finite diameter," allowing to prove simultaneously existence (and uniqueness) of the physical measure *and* exponential approach to equilibrium. Much more recently, Maume was able to use cone techniques to obtain other speeds of mixing (slower than exponential): first in a symbolic framework [Mau2] (with countably many symbols) and then in the axiomatic setting of towers with return times described in Section 4.A [Mau3]. In this recent work, the function cone is *not* mapped strictly into itself after a single iteration.

As there is no (apparent) direct relation between the invariant cone approach and zeta functions, we shall only give a very brief introduction to the main notions here, referring the interested reader to [Li1] or [V2].

Definition (Cones and partial order). Let $\mathcal{B}$ be a Banach space with the topology induced by its norm. A subset $\Lambda \subset \mathcal{B} \setminus \{0\}$ is called a *cone* if $\lambda\varphi \in \Lambda$ for all $\varphi \in \Lambda$ and all $\lambda \in \mathbb{R}^*_+$. The cone Λ is said to be *convex* if $\varphi + \psi \in \Lambda$ whenever both φ and ψ are in Λ. Any convex cone Λ defines a *partial ordering* compatible with the vector space structure $\preceq$ on $\mathcal{B}$ by setting

$$\varphi \preceq \psi \Leftrightarrow \psi - \varphi \in \Lambda \cup \{0\}\,.$$

DEFINITION(HILBERT PSEUDO-METRIC Θ_Λ). For φ, ψ in a cone Λ, set

$$\alpha(\varphi,\psi) = \sup\{\lambda \in \mathbb{R}_+ \mid \lambda\varphi \preceq \psi\}\,,\ \beta(\varphi,\psi) = \inf\{\lambda \in \mathbb{R}_+ \mid \psi \preceq \lambda\varphi\}\,,$$

(putting $\alpha = 0$ and $\beta = \infty$ when the corresponding sets are empty). Then let

$$\Theta_\Lambda(\varphi,\psi) = \log \frac{\beta(\varphi,\psi)}{\alpha(\varphi,\psi)}\,.$$

The Hilbert pseudo-metric Θ_Λ induces a distance on the quotient $\Lambda/\sim$ where $\varphi \sim \psi$ if $\psi = \lambda\varphi$ for some $\lambda \in \mathbb{R}_+^*$.

The key result is the fact that any linear operator $\mathcal{L}$ mapping a convex cone Λ inside itself does not increase the Hilbert pseudo-distance, with a strict contraction property whenever the Θ_Λ-diameter of $\mathcal{L}\Lambda$ is finite (write $\operatorname{diam}_{\Theta_\Lambda}(A) = \sup_{\varphi,\psi\in A} \Theta_\Lambda(\varphi,\psi)$):

THEOREM 4.1 (BIRKHOFF'S INEQUALITY). *Let Λ be a convex cone in a Banach space $\mathcal{B}$. If $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $\mathcal{L}(\Lambda) \subset \Lambda$, then for each $\varphi, \psi \in \Lambda$ we have*

$$\Theta_\Lambda(\mathcal{L}\varphi, \mathcal{L}\psi) \le \tanh\left(\frac{\operatorname{diam}_{\Theta_\Lambda}(\mathcal{L}\Lambda)}{4}\right) \cdot \Theta_\Lambda(\varphi,\psi)\,.$$

For example, to prove exponential mixing for C^2 expanding maps (see Speed #1 in Section 3), one may use a one-parameter family of Lipschitz cones Λ_L, indexed by $L > 0$:

$$\Lambda_L = \{\varphi \in C^0(M) \mid \varphi(x) > 0\ ,\ \frac{\varphi(x)}{\varphi(y)} \le e^{L d(x,y)}\,,\ \forall x,y \in M\}\,.$$

As mentioned above, in the hyperbolic case, more interesting cones were introduced by Liverani [Li1] and generalised by Viana [V2]. We refer to these sources for more, in particular discussions on how the pseudo-metric may be compared to the L^1 distance or some other metric on the function space.

Method # 3: *Probabilistic coupling.*

"Coupling" is a technique from probability theory [Lind] which is useful to compare sequences of probability measures in particular to estimate at which speed the distance between them decreases. This approach has been introduced into dynamical systems by Young [Yo3]. More precisely, she was able to apply it to the tower model with recurrence asymptotics described in Section 4.A. (See Bressaud-Fernandez-Galves [BFG], Bruin-Luzzatto-van Strien [BLS], and Baladi-Benedicks-Maume [BBM] for recent applications.) As there is no (apparent) direct relation between the invariant cone approach and zeta functions, we shall only state the main result in [Yo3] here.

Consider the tower model $F : \Delta \to \Delta$ with return time R and the function space $\mathcal{F}_\beta$ from (4.6). Let $\mathcal{F}_\beta^+$ denote those elements of $\mathcal{F}_\beta$ whose restriction to each $\Delta_{\ell,j}$ is either strictly positive or identically zero (see [Yo3] for a formal definition). Denote by $|\rho|$ the total mass of a signed measure ρ on Δ. (If ρ is absolutely continuous with respect to m then $|\rho| = \int |\frac{d\rho}{dm}|\, dm$.) Finally, define ($m$ almost everywhere) $\widehat{R} : \Delta \to \mathbb{Z}_+$ by setting $\widehat{R}(x,\ell) = R(x) - \ell$. Note that

$$m(\{x \in \Delta \mid \widehat{R}(x) > n\}) = \sum_{\ell > n} m(\Delta_\ell) = \sum_{\ell > n} m(\{x \in \Delta_0 \mid R(x) > \ell\})\,.$$

The main result is:

Theorem 4.2 (Recurrence times and rates of mixing [Yo3]).
(1) $F : \Delta \to \Delta$ admits an invariant probability measure ν which is absolutely continuous with respect to m. The density $d\nu/dm$ is an element of $\mathcal{F}^+_\beta(\Delta)$ and $\inf(d\nu/dm) > 0$.

(2) The pair (F,ν) is exact (and thus mixing) if and only if the greatest common divisor of the return times $\{R(x)\,, x \in \Delta_0\}$ is equal to one.

(3) Assume that (F,ν) is exact. Then there is a probability measure λ on Δ, absolutely continuous with respect to m, with $\frac{d\lambda}{dm} \in \mathcal{F}^+_\beta$ and $C > 0$ with

$$|F^n_* \lambda - \nu| \ge C\, m\{\widehat{R} > n\}\,, \forall n \in \mathbb{Z}_+\,.$$

(4) Assume that (F,ν) is exact and that there is $0 < \tilde\tau < 1$ with $m(\{\widehat{R} > n\}) = O(\tilde\tau^n)$. Then there is $\tilde\tau \le \tau < 1$ such that for each absolutely continuous probability measure λ on Δ with $\frac{d\lambda}{dm} \in \mathcal{F}^+_\beta$

$$|F^n_* \lambda - \nu| \le C\, \tau^n\,, \forall n \in \mathbb{Z}_+\,.$$

(5) Assume that (F,ν) is exact and that there are $0 < \tilde\tau < 1$ and $0 < \tilde\xi \le 1$ with $m(\{\widehat{R} > n\}) = O(\tilde\tau^{n^{\tilde\xi}})$. Then there are $\tilde\tau \le \tau < 1$ and $\xi \le \tilde\xi$ such that for each probability measure λ on Δ with $\frac{d\lambda}{dm} \in \mathcal{F}^+_\beta$

$$|F^n_* \lambda - \nu| \le C\, \tau^{n^\xi}\,, \forall n \in \mathbb{Z}_+\,.$$

(6) Assume that (F,ν) is exact, and that for some $\gamma > 0$ and all n, $m(\{\widehat{R} > n\}) = O(n^{-\gamma})$. Then for each probability measure λ on Δ with $\frac{d\lambda}{dm} \in \mathcal{F}^+_\beta$

$$|F^n_* \lambda - \nu| \le \frac{C}{n^\gamma}\,, \forall n \in \mathbb{Z}_+\,.$$

In the above theorem, the statements on rates of mixing are formulated in terms of bounds for the approach to equilibrium $|F^n_* \lambda - \nu|$. The corresponding upper bounds for decay of correlations are easily deduced.

Other methods:
We do not claim that the three approaches described are the only ones in existence! In particular "Markov approximation" techniques have been much investigated (see Chernov [Ch2, Ch1], Lecture 12 in Sinai's book [Sin] and also some work of Yuri [Yu1, Yu2]). See also the results of Jakobson and Newhouse [JN] for two-dimensional maps.

Also, there exist a few "tricks" to improve the spectral properties of the transfer operator. One of them involves "regularising" the operator (an instance of this method is the kneading operator approach presented in Section 5, see also the work of Rugh [Rug3] on analytic maps with a neutral fixed point). Finally, some of the results discussed involve constructing a transfer operator which is not directly associated to the dynamics to be studied ([Li3], [LSV2], [Po2]) but to a combination of a suitable iterate n of the dynamics and some "noise," letting the noise level ϵ go to zero and taking a suitable iterate $N(\epsilon)$. In some sense the method used by Chernov [Ch1] and Hu [Hu] also involves this idea.

5. Walking around a picturesque neighbourhood: One-dimensional dynamics and kneading operators

We discuss here a relatively recent strategy, inspired by the Milnor-Thurston kneading matrix, to implement the spectral approach (control of the essential spectral radius and link with dynamical zeta function) which has been studied in one-dimension. Let $I \subset \mathbb{R}$ be a compact interval, and Ω be a finite set of indices. For each $\omega \in \Omega$, let $I_\omega \subset I$ be an open interval, $\psi_\omega : I_\omega \to \psi_\omega(I_\omega)$ a homeomorphism (setting also $\epsilon_\omega = 1$ if ψ_ω preserves orientation, $\epsilon_\omega = -1$ otherwise). Finally, let $g_\omega : I \to \mathbb{C}$ be of bounded variation (in particular, dg_ω is a complex measure), continuous, and supported in I_ω. The assumption that each g_ω is both continuous and supported in I_ω is technical, to avoid boundary terms when integrating by parts. It is in general not satisfied if $g_\omega = |\psi'_\omega|$ with ψ_ω the inverse branches of a single map. Fortunately, it can basically be suppressed for Theorem 5.1 (see [Bal]) and greatly weakened for Theorem 5.2 [Rey].

We define a transfer operator acting on the space $\mathcal{B}$ of functions $\varphi : I \to \mathbb{C}$ of bounded variation modulo countable sets (or just on bounded functions) by

$$\mathcal{L}\varphi = \sum_{\omega\in\Omega} (\varphi \circ \psi_\omega)\, g_\omega \, . \tag{5.1}$$

We also introduce a formal dual of $\mathcal{L}$

$$\widehat{\mathcal{L}}\varphi = \sum_{\omega\in\Omega} \epsilon_\omega \,(\varphi \circ \psi_\omega^{-1})\,(g_\omega \circ \psi_\omega^{-1})\, . \tag{5.2}$$

The main example is when the ψ_ω are the local inverse branches of a piecewise monotone interval map f, i.e., there is a partition of a compact interval $I = [a_0, a_n]$, given by $a_0 < a_1 < \ldots < a_N$, and $f : I \to I$ so that $f|_{[a_i,a_{i+1}]}$ is continuous and monotone, $\Omega = \{1, \ldots, N\}$, $\psi_\omega = (f|_{[a_{\omega-1},a_\omega]})^{-1}$, and $g_\omega = g\circ\psi_\omega$. Then the formal dual is just a weighted composition operator $\widehat{\mathcal{L}}\varphi = (\varphi \circ f) g\epsilon_f$, where ϵ_f is the "sign of the slope of f."

Finally, writing $\|\cdot\|_\infty$ for the operator norm of $\mathcal{L}$ or $\widehat{\mathcal{L}}$ acting on bounded functions, we define

$$R := \lim_{n\to\infty} (\|\mathcal{L}^n\|_\infty)^{1/n}\, , \quad \widehat{R} := \lim_{n\to\infty} (\|\widehat{\mathcal{L}}^n\|_\infty)^{1/n}\, . \tag{5.3}$$

Up to exchanging $\mathcal{L}$ and $\widehat{\mathcal{L}}$, we can assume to fix ideas that $\widehat{R} \le R$. The inequality $\widehat{R} \le R$ is automatic in the setting of piecewise monotone and piecewise continuous interval maps: In this case $\widehat{R} = \lim_{n\to\infty} (\sup_x |g^{(n)}(x)|)^{1/n}$. (We write $g^{(n)}(x) = \prod_{k=0}^{n-1} g(f^k x)$.)

5.A Bounding the essential spectral radius.

THEOREM 5.1 (SPECTRUM OF $\mathcal{L}$ ON BV, RUELLE). *Assume $\widehat{R} \le R$.*
(1) The essential spectral radius of $\mathcal{L}$ on BV is not larger than $\widehat{R}$.
(2) The spectral radius $\rho(\mathcal{L})$ of $\mathcal{L}$ on BV satisfies $\rho(\mathcal{L}) \le R$.

(3) If all functions g_ω are real and nonnegative, then $\rho(\mathcal{L}) = R$. If, additionally, $\widehat{R} < R$ then R is an eigenvalue with nonnegative eigenfunction φ_g, and there is an eigenfunctional ν_g of $\mathcal{L}^*$ for R, with ν_g a Borel probability measure and $\nu_g(\varphi_g) > 0$. The measure $\varphi_g \nu_g$ is f-invariant if the ψ_ω are the inverse branches of a given f.

There is no "hyperbolicity" or "expansion" assumption in Theorem 5.1. Note however that if the ψ_ω are the inverse branches of f a piecewise differentiable map, and $g = 1/|f'|$ then $R = 1$, and the strict inequality $\widehat{R} < R$ is satisfied if $|f'| \geq \lambda > 1$. Under such assumptions, the result above had been proved previously by Hofbauer-Keller [HK] (see also Baladi-Keller [BK] and Ruelle [Ru7]).

Proof of Theorem 5.1. (1) We start by sketching Ruelle's elegant conceptual proof [Ru8] of the upper bound for the essential spectal radius, which emphasizes the role played by the Leibniz formula (also essential in the proof of (4.8)). See [Bal] for more details.

The key remark is that addition of a compact operator does not change the essential spectral radius of a bounded operator. Let $d : \mathcal{B} \to C(I)^*$ be the Banach space isomorphism between functions of bounded variation and signed (or complex) finite measures. We shall prove that the essential spectral radius of the conjugated operator $\mathcal{M} = d\mathcal{L}d^{-1}$ acting on $C(I)^*$ is at most $\widehat{R}$ by writing $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$, with $\mathcal{M}_2$ compact and $\mathcal{M}_1$ having spectral radius at most $\widehat{R}$.

By definition,

$$\begin{aligned} \mathcal{M}(\mu) &= d\mathcal{L}(d^{-1}\mu) \\ &= d \sum_\omega g_\omega \, (d^{-1}\mu) \circ \psi_\omega \, . \end{aligned} \tag{5.4}$$

We may assume that the functions appearing in the sum have at most regular discontinuities. Therefore, the Leibniz rule for functions of bounded variation, together with a change of variable formula, produce a decomposition $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$, where $\mathcal{M}_1$ has spectral radius on $C(I)^*$ bounded by $\widehat{R}$ and where

$$\mathcal{M}_2 \mu = \sum_\omega (d^{-1}\mu) \circ \psi_\omega \, dg_\omega \tag{5.5}$$

is compact. Compactness of $\mathcal{M}_2$ follows from finiteness of the sum and an intuitively obvious property of "compactness of integration." In the case when the ψ_ω are inverse branches of f, e.g., the bound on the spectral radius of $\mathcal{M}_1$ is an easy exercise left to the reader, since

$$\mathcal{M}_1 \mu = f_*(\epsilon_f \cdot g)\mu \, . \tag{5.6}$$

(2) Since we have just shown that $\rho_{\mathrm{ess}}(\mathcal{L}) \leq \widehat{R} \leq R$, to check that $\rho(\mathcal{L}) \leq R$ it suffices to see that whenever λ is an eigenvalue of $\mathcal{L}$ on $\mathcal{B}$ then $|\lambda| \leq R$. This is evident, since λ is then an eigenvalue for $\mathcal{L}$ acting on bounded functions modulo countable sets, and R is the spectral radius on that space.

(3) To construct $\varphi_g \in \mathcal{B}$ and a Borel probability measure ν_g with $\mathcal{L}(\varphi_g) = R\varphi_g$ and $\mathcal{L}^*(\nu_g) = R\nu_g$, one can use general remarks on positive operators, following Ruelle, Keller, and others (see, e.g., the proof of the Ruelle-Perron-Frobenius theorem in [Bal1]). □

5.B Kneading operators and sharp traces.

Up to this point, the dynamical zeta functions and generalised Fredholm determinants whose analytical properties were related to the spectral properties of transfer operators involved sums over all periodic points, requiring in particular the set of periodic points to be at most countable. In their pioneering paper on one-dimensional dynamics, Milnor and Thurston [MT] associated to any piecewise monotone interval map $f : I \to I$ (with finitely many, say N, monotonicity intervals) a *negative zeta function*

$$\zeta^-(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} 2\, \#\, \mathrm{Fix}^- f^n \,, \tag{5.7}$$

where the set of *negative fixed points* of f^n is

$$\mathrm{Fix}^- f^n = \{x \in I \mid f^n x = x \,, f^n \text{ monotone decreasing in a neighbourhood of } x\} \,.$$

(See also the notes on *Zeta functions* by M. Pollicott.) The important fact, of course, is that $\mathrm{Fix}^- f^n$ is a finite set. The principle "what goes down must go up" explains why it is natural to double the negative periodic points. If f is piecewise *expanding,* one can show that $\zeta^-(z)$ is just the usual unweighted dynamical zeta function (1.1) of f, up to a simple polynomial factor.

Milnor-Thurston [MT] proved (using a homotopy argument) the surprising equation

$$\zeta^-(z) \det(1 - D(z)) = 1 \,, \tag{5.8}$$

where $D(z)$ is the *kneading matrix,* a finite $(N+1) \times (N+1)$ matrix, with coefficients power series associated to the itineraries of the turning points. These power series (the *kneading invariants*) embody a rather complete description of the map f, and the one-dimensional kneading theory is by now well developed, see, e.g., de Melo-van Strien [MS]. (Milnor and Thurston's version of (5.8) involves a polynomial correction, due to the fact that they worked with an $N - 1 \times N - 1$ matrix.)

One can rewrite $\zeta^-(z)$ as a *Lefschetz zeta function,* $\zeta^L(z)$ where all periodic points x are counted, but with a weight $L(x) \in \{0, -1, 1\}$ (cancellations may occur, in particular in homtervals). This second formulation (Baladi-Ruelle [BR1]) makes it easier to define a weighted negative (Lefschetz) dynamical zeta function especially if the weight is locally constant. Formula (5.8) was extended to the weighted and "nonfunctional" case (ψ_ω, g_ω) situations in a series of papers (in particular Baladi-Ruelle [BR1], Ruelle [Ru8], Baladi-Ruelle [BR2]). The "usual" relationship between the poles of the zeta function and the inverse eigenvalues of a transfer operator acting on functions of bounded variation (or C^r with r-th derivative of bounded variation, Ruelle [Ru8]) was established in this framework. Again we limit ourselves to simplest statements. We shall use the sign function

$$\mathrm{sgn}\,(x) = -1 \text{ if } x < 0 \,, \ \mathrm{sgn}\,(x) = 0 \text{ if } x = 0 \,, \ \mathrm{sgn}\,(x) = 1 \text{ if } x > 0 \,, \tag{5.9}$$

which has the property that $\frac{1}{2} d(\,\mathrm{sgn})$ is δ_0 the Dirac mass at 0. Define also the *sharp trace* of the data $(\Omega, I_\omega, \psi_\omega, g_\omega)$ by

$$\mathrm{tr}^{\#} \mathcal{L} := \sum_{\omega \in \Omega} \int_{I_\omega} dg_\omega(x)\, \frac{1}{2} \mathrm{sgn}\,(\psi_\omega x - x) \,, \tag{5.10}$$

Definition (5.10) clearly extends to the iterates $\mathcal{L}^n$. The sharp trace has the trace property

$$\mathrm{tr}^{\#}(\mathcal{L}_1\mathcal{L}_2) = \mathrm{tr}^{\#}(\mathcal{L}_2\mathcal{L}_1) \tag{5.11}$$

for any transfer operators $\mathcal{L}_1$, $\mathcal{L}_2$ of the form (5.1).

THEOREM 5.2 (SHARP TRACES AND SHARP DETERMINANTS [BR2]). *Assume that $\widehat{R} \le R$. The sharp determinant*

$$det^{\#}(1 - z\mathcal{L}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} tr^{\#}\mathcal{L}^m \,. \tag{5.12}$$

defines a holomorphic function in the disc of radius $\widehat{R}^{-1}$ where its zeroes are exactly the inverses of the eigenvalues of $\mathcal{L}$ acting on $\mathcal{B}$ of modulus at least $\widehat{R}$. The order of λ^{-1} as a zero of $det^{\#}(1 - z\mathcal{L})$ coincides with the algebraic multiplicity of λ as an eigenvalue of $\mathcal{L}$.

An application of integration by parts and change of variables rule in an integral yields $\mathrm{tr}^{\#}\widehat{\mathcal{L}} = -\mathrm{tr}^{\#}\mathcal{L}$. We thus have the *functional equation*

$$\det{}^{\#}(1 - z\widehat{\mathcal{L}}) = \frac{1}{\det^{\#}(1 - z\mathcal{L})} \,. \tag{5.13}$$

Note also that one can rewrite $\det^{\#}(1 - z\mathcal{L})$ as a (weighted) Lefschetz zeta function whenever the set of "periodic orbits" is countable (using $d\,(\mathrm{sgn}) = \delta_0$). More precisely, if $\mathrm{Fix}\psi^{(n)}$ is finite for each relevant composition $\psi^{(n)} = \psi_{\omega_n} \circ \cdots \circ \psi_{\omega_1}$ then

$$\mathrm{tr}^{\#}\mathcal{L}^n = \sum_{\omega_1, \cdots, \omega_n} \sum_{x \in \mathrm{Fix}\psi^n} L(x, \psi^{(n)}) g^{(n)}(x) \,,$$

where the Lefschetz number $L(x, \psi)$, for x an isolated fixed point of a monotone function $\psi : (a, b) \to \mathbb{R}$ is defined by:

$$L(x, \psi) = \begin{cases} 0 & \text{if graph of } \psi \text{ doesn't cross the diagonal at } x, \\ \epsilon_\psi \lim_{y \to x} \frac{\mathrm{sgn}(\psi y - y)}{\mathrm{sgn}(x - y)} & \text{otherwise,} \end{cases}$$

with $\epsilon_\psi = 1$ if ψ preserves orientation and -1 otherwise.

SKETCH OF PROOF OF THEOREM 5.2. The proof is by regularisation. Specifically, it uses a family of almost trace class operators, the *kneading operators* $\mathcal{D}(z)$, which are analogues of Milnor and Thurston's kneading matrix. These operators act on $L^2(d\mu)$ for an auxiliary measure $\mu = \sum_\omega |dg_\omega| + \sum_\omega |d(g_\omega \circ \psi_\omega^{-1})|$, guaranteeing the existence of the Radon-Nikodym derivatives $dg_\omega / d\mu$. For any z not in the spectrum of $\mathcal{L}$, the operator $\mathcal{D}(z)$ is defined by

$$\mathcal{D}(z) = z\mathcal{N}(1 - z\mathcal{L})^{-1} S \,, \tag{5.14}$$

where the (smoothness improving, compact) integration operator S is defined by

$$\mathcal{S}\varphi(x) = \int_I \frac{1}{2}\mathrm{sgn}\,(x - y)\varphi(y)\, d\mu(y) \,, \tag{5.15}$$

and where the auxiliary operator $\mathcal{N}$ is given by

$$\mathcal{N}\varphi = \sum_{\omega\in\Omega} (\varphi \circ \psi_\omega) \frac{dg_\omega}{d\mu} . \tag{5.16}$$

We define $\widehat{\mathcal{D}}(z)$ analogously (replacing g_ω, ψ_ω and $\mathcal{M}$ by $\epsilon_\omega g_\omega \circ \psi_\omega^{-1}$, ψ_ω^{-1} and $\widehat{\mathcal{M}}$).

Since $\mathcal{D}(z)\varphi(x) = \int \mathcal{D}_{xy}(z)\varphi(y)\, d\mu(y)$ has a bounded kernel, whenever $1/z$ is not in the spectrum of $\mathcal{L}$, it is a Hilbert-Schmidt operator in $L^2(d\mu)$ so that the regularised determinant $\det_2(1+\mathcal{D}(z))$ of order 2 is well-defined (Simon [Sim]). We may thus set

$$\det{}^*(1+\mathcal{D}(z)) := e^{\int \mathcal{D}_{xx}(z)\, d\mu(x)} \det{}_2(1+\mathcal{D}(z)) .$$

The key identity in the proof of Theorem 5.2 is the following analogue of (5.8):

$$\det{}^\#(1 - z\mathcal{L}) \det{}^*(1+\mathcal{D}(z)) = 1 . \tag{5.17}$$

The first proof of (5.17) in Baladi-Ruelle [BR2] was by a series resummation argument (involving repeated use of integration by parts and change of variables). A more conceptual proof can be extracted from the method used in a one-dimensional complex partial analogue of Theorem 5.2 (Baladi et al. [BKRS]). We do not state the complex result for lack of space, mentioning only that $\mathrm{sgn}(x)/2$ is replaced by the function $\sigma(z) = 1/(\pi z)$, and derivation is replaced by $\bar\partial$ (using in particular $\bar\partial\sigma = \delta_0$), and that in the simplest cases we have the formula $\mathrm{tr}^\# \mathcal{L} = \sum_{\omega\in\Omega} \sum_{x\in \mathrm{Fix}\psi_\omega(x)} g_\omega(x)/(1-\partial\psi_\omega(x))$ (note the absence of absolute value).

We sketch the conceptual proof of (5.17) now: after unifying the sharp trace (5.10) of transfer operators with the "trace" of kernel operators defined by

$$\mathrm{tr}^*\mathcal{D}(z) := \int \mathcal{D}_{xx}(z)\, d\mu(x) ,$$

we check that this unified trace, noted tr^*, satisfies (5.11). Hence, the usual determinant formulas hold for $\det^*(1-\mathcal{M}) = \exp - \sum_n \frac{1}{n} \mathrm{tr}^* \mathcal{M}^n$. The argument is then quite literally a three-line proof:

$$\begin{aligned}
\det{}^*(1+\mathcal{D}(z)) \det{}^\#(1-z\mathcal{L}) &= \det{}^*(1 + z\mathcal{N}(1-z\mathcal{L})^{-1}S) \det{}^*(1-z\mathcal{L}) \\
&= \det{}^*(1 + zS\mathcal{N}(1-z\mathcal{L})^{-1}) \det{}^*(1-z\mathcal{L}) \\
&= \det{}^*(1 + z(S\mathcal{N} - \mathcal{L})) = 1 ,
\end{aligned}$$

where we used $\mathrm{tr}^\#\mathcal{L} = \mathrm{tr}^*(S\mathcal{N})$, and more generally the less trivial identities $\mathrm{tr}^*(S\mathcal{N} - \mathcal{L})^n = 0$ for $n \geq 1$, to get the last equality. $\square$

Notational warning: We use g instead of $\exp(g)$ for the weight function in these notes, because we want to admit the possibility that $g(x) = 0$. We have reserved the letter $\mathcal{F}$ for the function spaces in [Yo2, Yo3] and use $\mathcal{C}$ for the Lipschitz functions in the subshift of finite type model.

Final Comment: These notes and in particular the bibliography do not have any pretension to being exhaustive. We are grateful to the referee for pointing out some glaring omissions and apologize to authors for any remaining ones. The book [Ba1] contains a more detailed account of some topics.

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