SMOOTH DEFORMATIONS OF PIECEWISE EXPANDING UNIMODAL MAPS

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Abstract. In the space of C^k piecewise expanding unimodal maps, $k \geq 1$, we characterize the C^1 smooth families of maps where the topological dynamics does not change (the "smooth deformations") as the families tangent to a continuous distribution of codimension-one subspaces (the "horizontal" directions) in that space. Furthermore such codimension-one subspaces are defined as the kernels of an explicit class of linear functionals. As a consequence we show the existence of $C^{k-1+Lip}$ deformations tangent to every given C^k horizontal direction, for $k \geq 2$.

1. Introduction

The topological class of a dynamical system f is the set of all maps topologically conjugate with f. A *smooth deformation* of a dynamical system f_0 is a smooth family of dynamical systems $t \mapsto f_t$ inside the topological class of f_0. We also say that a smooth deformation f_t is a family with "no bifurcations." Deciding whether or not there are bifurcations in a family is one of the primary problems concerning dynamical systems.

In the theory of complex dynamical systems, specially for rational functions, this type of study was very sucessful. One of the most powerful tools in complex dynamics are the quasiconformal methods: quasiconformal maps, quasiconformal vector fields, and holomorphic motions. For example, they allow us to easily find a holomorphic deformation between two holomorphic dynamical systems which are conjugate by a quasiconformal map, using the so-called Beltrami path. Beltrami paths are examples of *holomorphic motions,* whose importance in complex dynamics can not be overstated since the time they were introduced in the seminal work by Mañé, Sad and Sullivan [9]. Holomorphic motions are a key tool in the characterization of structurally stable rational maps and families of rational maps with no bifurcations [9] (see also [10]). The study of the regularity of hybrid classes of quadratic-like maps [7] and topological classes of analytic unimodal maps [2] also depends heavily on quasiconformal methods.

Unfortunately quasiconformal methods do not seem to be applicable for real one-dimensional maps which do not have a holomorphic extension to the complex plane, like piecewise expanding C^2 unimodal maps (see Section 2 for formal definitions).

In our first main result, Theorem 1 (Section 4), we characterize all smooth families in the space of piecewise expanding unimodal maps which are smooth deformations: they are precisely the families tangent to a continuous distribution of

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codimension-one subspaces in that space. Following the notation in [7], these subspaces will be called "horizontal directions." See Section 3 for the definition of the linear functional $J(f, \cdot)$ whose kernel defines the horizontal directions.

We observe that for families of smooth unimodal maps, the condition

$$J(f_{t_0}, \partial_t f_t|_{t=t_0}) \neq 0$$

is the "nondegeneracy condition" for a family f_t at a Collet-Eckmann parameter f_{t_0} that appeared in a generalization of Jakobson's Theorem by Tsujii [11]. On the other hand, the condition $J(f, v) = 0$ for v to be horizontal is the same which is well-known for smooth unimodal maps satisfying certain summability condition (see e.g. [1], [2]). This condition first appeared in the context of piecewise expanding maps in [3].

One can wonder if there *exist* deformations of a given piecewise expanding unimodal map which are non trivial, i.e., so that the f_t are not smoothly conjugate to f_0 . (Not only because this is an intrisically natural question, but also because it recently became clear that this is crucial to understand some dynamically defined quantities, see below.) We answer this question in Theorem 2 (Section 5). In particular, for each "good" piecewise expanding unimodal map f_0 and each horizontal direction v, we construct a smooth deformation of f_0 tangent to v at f_0, i.e., $\partial_t f_t|_{t=0} = v$. In other words, Theorem 2 shows that the theory of smooth deformations is very rich, since there are plenty of deformations of a piecewise expanding unimodal map in "horizontal" directions.

In both theorems we heavily use "smooth motions," that is, we exploit the fact that the conjugacies h_t depend smoothy on t. In Theorem 1, $B \Rightarrow C$ (see [4]), we use them in the phase space, and in Theorem 2, in the parameter space.

Given a smooth family of dynamical systems, one can ask how dynamically defined quantities, such as the average of a given observable with respect to the SRB measures, the Lyapunov exponents, and the Hausdorff dimension of invariant sets, change along this family. Studying smoothness of these quantities can be a tricky issue. For example, SRB measures are often described as eigenvectors of Ruelle-Perron-Frobenius operators acting on infinite-dimensional spaces with a complicated structure.

In the case of piecewise expanding maps, Hölder continuity of SRB measures (for all exponents < 1) has been known for a long time [5]. However any hope of higher regularity for families transversal to the topological class was annihilated by the examples in [3] (see also [8]). In order to have a satisfactory theory about smooth variation of dynamically defined quantities, at least in the case of the SRB measure of piecewise expanding unimodal maps, it was recently discovered that we need to restrict ourselves to families tangent to topological classes [4]. Theorem 2 and its corollaries imply the result announced as Theorem 2.8 in [4]. This result was not used to obtain the other claims in [4], but it shows that there are plenty of families satisfying the restriction of tangency to the topological class needed there.

2. Preliminaries

Denote $I = [-1, 1]$ and $\mathbb{N} = \mathbb{Z}_+$.

For $k \geq 0$, we define the set $\mathcal{B}^k(I)$ of *piecewise C^k functions* to be the linear space of continuous functions $f \colon I \to \mathbb{R}$ such that f is C^k on the intervals $[-1, 0]$

and $[0,1]$, with $f(1) = f(-1)$. Then $\mathcal{B}^k(I)$ is a Banach space for the norm

$$|f|_k = \max\{|f|_{C^k[-1,0]}, |f|_{C^k[0,1]}\}, \text{ where } |f|_{C^k(Q)} = \max_{0\le i\le k}\{|D^i f|_{L^\infty(Q)}\}.$$

For $k \ge 1$, we define the set $\mathcal{U}^k$ of *piecewise expanding C^k unimodal* maps to be the set of maps $f \in \mathcal{B}^k(I)$ such that

I. *(Invariance of ∂I)* $f(-1) = f(1) = -1$.
II. *(Expanding condition)* $\inf_{x\in[-1,0]} Df(x) > 1$ and $\sup_{x\in[0,1]} Df(x) < -1$.
III. *(Invariance of I)* $f(0) \le 1$ (by I.–II. this implies $f(I) \subset I$).

The set $\mathcal{U}^k$ is a convex subset of the affine subspace

$$\{f \in \mathcal{B}^k(I) \text{ s.t. } f(-1) = f(1) = -1\}$$

and $\mathcal{U}^k \cap \{f \in \mathcal{B}^k(I)\colon\ f(0) < 1\}$ is a convex and open set of the same affine subspace. We call elements of $\mathcal{U}^1$ simply *piecewise expanding unimodal maps.* The point $c = 0$ is called the *critical point* of a piecewise expanding unimodal map. Set

$$\lambda_f = \min_x |Df(x)| > 1.$$

The itinerary of $x \in I$ for a piecewise expanding unimodal map f is the sequence

$$(\sigma_0(x), \sigma_1(x), \sigma_2(x), \dots) \in \{L, C, R\}^{\mathbb{N}}$$

such that $\sigma_i(x) = L$ if $f^i(x) < c$, $\sigma_i(x) = C$ if $f^i(x) = c$, and $\sigma_i(x) = R$ if $f^i(x) > c$. We write $\sigma_i = \sigma_i(c)$.

Let $1 \le j \le k$, with k an integer, and j either an integer or $j = k - 1 + Lip$. A *C^j family* f_t of piecewise expanding C^k unimodal maps is a C^j map

$$t \mapsto f_t \text{ from } [-\delta, \delta] \text{ to } \mathcal{U}^k. \tag{1}$$

(In particular, for such a family, the map $(t, x) \mapsto f_t(x)$ is continuous on $[-\delta, \delta] \times I$, and it is C^j on the sets $[-\delta, \delta] \times [-1, 0]$ and $[-\delta, \delta] \times [0, 1]$.)

If f_t is a family of piecewise expanding unimodal maps, we consider

$$R_t := \{(i, j) \in \mathbb{N} \times \mathbb{N}\colon\ f_t^i(c) = f_t^j(c) \text{ and } i < j\},$$

the set of *critical relations* of f_t. Note that if the forward orbit of c (also called *postcritical orbit*) is infinite then R_t is empty.

We say that a piecewise expanding unimodal map f is *good* if either c is not periodic or, writing $p \ge 2$ for the prime period of c, if

$$|Df^{p-1}(f(c))| \min\{|Df^+(c)|, |Df^-(c)|\} > 2. \tag{2}$$

A map $f\colon I \to I$ is *ϵ-expansive* if for every interval $L \subset I$ there is $i \ge 1$ so that

$$|f^i(L)| > \epsilon.$$

A piecewise expanding unimodal map f_0 is *stably ϵ-expansive* if every piecewise expanding unimodal map f close enough to f_0 (for $|\cdot|_1$) is ϵ-expansive. We give the easy proof of the following useful result for completeness:

Proposition 2.1. *Let f be a piecewise expanding unimodal map. Then there exists $\epsilon > 0$ so that f is ϵ-expansive. If we assume furthermore that f is good, then there exists $\epsilon > 0$ so that f is stably ϵ-expansive.*

Proof. Choose N_0 such that $\frac{1}{2}\lambda_f^{N_0-1} > \lambda_f$, and ϵ such that

$$c \notin \mathrm{int}\,(f^i[-\epsilon,\epsilon]) \text{ for } i = 1,\ldots,N_0.$$

Then for every interval $Q \subset [-\epsilon,\epsilon]$ we have

$$|f^{N_0}(Q)| > \lambda_f |Q|. \tag{3}$$

If the turning point is not periodic it is easy to see that (3) remains true for any small enough perturbation of f.

Consider an interval $Q \subset I$ and suppose that $|f^i(Q)| < \epsilon$ for every $i \in \mathbb{N}$. Define $n_0 = 0$ and $n_1, n_2, n_3, \ldots$ in the following way: If

$$c \notin Q_s := f^{n_0+n_1+\cdots+n_s}(Q)$$

define $n_{s+1} = 1$. In this case

$$|Q_{s+1}| = |f(Q_s)| \geq \lambda_f |Q_s|.$$

Otherwise set $n_{s+1} = N_0$. Note that $Q_s \subset [-\epsilon,\epsilon]$ and that (3) implies

$$|Q_{s+1}| = |f^{N_0}(Q_s)| \geq \lambda_f |Q_s|,$$

so $|Q_s| \geq \lambda_f^s |Q|$, which implies that $|Q| = 0$, proving that f is ϵ-expansive.

If c is periodic but (2) holds, the argument above can be easily modified to show stable ϵ-expansiveness. □

3. The linear functional $J(f,\cdot)$

3.1. **Definition and relation with the twisted cohomological equation.** We shall associate a bounded linear functional $J(f,\cdot) \in (L^\infty(I))^*$ to each piecewise expanding unimodal map f. This functional will play a main role in this work. Let $v\colon I \to \mathbb{R}$ be a bounded function. If the critical point c is not periodic, we define

$$J(f,v) = \sum_{i=0}^{\infty} \frac{v(f^i(c))}{Df^i(f(c))}.$$

The above expression is not well defined if the critical point c is periodic, since the derivative at the critical point does not exist. If c has prime period p we set

$$J(f,v) = \sum_{i=0}^{p-1} \frac{v(f^i(c))}{Df^i(f(c))}.$$

Note that in both cases (non periodic and periodic critical points) we have

$$|J(f,v)| \leq \frac{|v|_{L^\infty}}{1-\lambda_f^{-1}}.$$

It is easy to see that $v \mapsto J(f,v)$ is not the zero functional on $C(I)$, so for every $k \in \mathbb{N}$, by the density of $C^k(I)$ in $C(I)$, there exists $v \in C^k(I)$ with $J(f,v) \neq 0$.

The meaning of the expression for $J(f,v)$ can be clarified by the following comments. Let f_t be a C^1 family of piecewise expanding C^1 unimodal maps such that $\partial_t f_t|_{t=0} = v$, $f_0 = f$. (We shall sometimes call the argument v of $J(f,v)$ a *vector field*.) Then, for any $k \geq 1$, if $f^j(x) \neq c$ for $1 \leq j \leq k-1$

$$\partial_t f_t^k(x)|_{t=0} = \sum_{i=0}^{k-1} Df^{k-1-i}(f^{i+1}(x)) \cdot v(f^i(x)),$$

so if $f^j(x) \neq c$ for $1 \leq j \leq k-1$, then

$$\frac{\partial_t f_t^k(x)|_{t=0}}{Df^{k-1}(f(x))} = \sum_{i=0}^{k-1} \frac{v(f^i(x)))}{Df^i(f(x))}. \tag{4}$$

So, if c has prime period k, then

$$J(f,v) = \frac{\partial_t f_t^k(c)|_{t=0}}{Df^{k-1}(f(c))}, \quad (v = \partial_t f_t|_{t=0}, f = f_0), \tag{5}$$

and if c is not periodic for f, then

$$J(f,v) = \lim_{k\to\infty} \frac{\partial_t f_t^k(c)|_{t=0}}{Df^{k-1}(f(c))}, \quad (v = \partial_t f_t|_{t=0}, f = f_0). \tag{6}$$

In other words, the derivatives in the phase and parameter spaces along the critical orbit are related by $J(f,v)$.

We also mention that if $v = X \circ f$ then $J(f,v) = s_1^{-1}\mathcal{J}(f,X)$, where $s_1 < 0$ is the jump $s_1 = -\lim_{x\to 1}\rho(x)$ at 1 of the invariant density ρ of f, and where $\mathcal{J}(f,X)$ was introduced in [3] and used in [4]. It was observed in [4] (see also Proposition 3.1 below) that elements v of the kernel of $\mathcal{J}(f,\cdot)$ satisfy $\sum_{i=0}^{\infty} \frac{v(f^i(c))}{Df^i(f(c))} = 0$ if c is not periodic and $\sum_{i=0}^{p-1} \frac{v(f^i(c))}{Df^i(f(c))} = 0$ if c has prime period p. Such $v \in \operatorname{Ker}((\mathcal{J}(f,\cdot))$ deserve to be called *horizontal vector fields,* by analogy with the theory for smooth unimodal maps ([7], [2]) and in view of the results in [4] (in particular Corollary 2.6 and Remark 2.7 there). Our Theorems 1 and 2 also justify this terminology.

We next recall the relation between $J(f,v)$ and the *twisted cohomological equation (7)* from [4, Lemma 2.2].

Proposition 3.1. *For every piecewise expanding unimodal map f and $v \in L^\infty(I)$ the following holds: Let $\mathcal{D}$ be the set of $x \in I$ with a forward orbit that does not contain c. There exists a unique bounded function $\alpha\colon \mathcal{D} \to \mathbb{R}$ such that*

$$v(x) = \alpha \circ f(x) - Df(x)\cdot \alpha(x), \tag{7}$$

for every $x \in \mathcal{D}$. There exists a unique bounded function $\alpha\colon I \to \mathbb{R}$ such that $\alpha(c) = 0$ and (7) holds for every $x \neq c$.

Furthermore $J(f,v) = 0$ if and only if $v(c) = \alpha(f(c))$.

Proof. We refer to [4, Lemma 2.2]. We just recall that for each $x \in \mathcal{D}$

$$\alpha(x) = -\sum_{i=0}^{\infty} \frac{v\circ f^i(x)}{Df^{i+1}(x)},$$

and for $x \in I \setminus \mathcal{D}$, with $x \neq c$, setting $k = \min\{i > 0 : \ f^i(x) = c\}$,

$$\alpha(x) = -\sum_{i=0}^{k-1} \frac{v\circ f^i(x)}{Df^{i+1}(x)}.$$

□

3.2. **Continuity of** $\operatorname{Ker}(J(f,\cdot))$**.** Observe that $f \mapsto J(f,v)$ is continuous at piecewise expanding unimodal maps with non periodic critical point:

Proposition 3.2. *Let $f_0 \in \mathcal{U}^1$ be a piecewise expanding unimodal map. If the critical point of f_0 is not periodic then*

A. *For every $\eta > 0$ there exists a neighborhood W of f_0 in $\mathcal{U}^1$ such that $|J(f,v) - J(f_0,v)| \le \eta |v|_1$ for every $v \in \mathcal{B}^1(I)$ and $f \in W$.*
B. *For every $v_0 \in \mathcal{B}^0(I)$ the function $f \mapsto J(f,v_0)$ is continuous at $f = f_0$, considering the $\mathcal{U}^1$ norm.*

Proof of Claim A. Taking W small enough, we have $\theta = \inf_{f\in W} \inf_x |Df(x)| > 1$. Let N be such that $\frac{\theta^N}{1-\theta} < \frac{\eta}{8}$. Reducing W, if necessary, we can assume that $f^i(f(c)) \neq c$ for every $f \in W$ and $i \le N$, and that

$$\sum_{i\le N} \Big| \frac{1}{Df^i(f(c))} - \frac{1}{Df_0^i(f_0(c))} \Big| < \frac{\eta}{4} \text{ and } \sum_{i\le N} |f^i(c) - f_0^i(c)| \le \frac{\eta}{4}. \tag{8}$$

Then

$$|J(f,v) - J(f_0,v)| \le \sum_{i<N} \Big| \frac{v(f^{i+1}(c))}{Df^{i+1}(f(c))} - \frac{v(f_0^{i+1}(c))}{Df_0^{i+1}(f_0(c))} \Big| + \frac{\eta}{4} |v|_0.$$

Estimating $\sum_{i<N} \Big| \frac{v(f^{i+1}(c))}{Df^{i+1}(f(c))} - \frac{v(f_0^{i+1}(c))}{Df_0^{i+1}(f_0(c))} \Big|$ by

$$\sum_{i<N} |v(f^{i+1}(c))| \Big| \frac{1}{Df^{i+1}(f(c))} - \frac{1}{Df_0^{i+1}(f_0(c))} \Big| + \sum_{i<N} \frac{|v(f^{i+1}(c)) - v(f_0^{i+1}(c))|}{|Df_0^{i+1}(f_0(c))|},$$

we get the claim from (8) and our choice of N. □

Proof of Claim B. We can assume that $|v_0| \le 1$. Fix η, and let W and N be like in the proof of Claim A. Reducing W if necessary, we have

$$\sum_{i\le N} |v_0(f^i(c)) - v_0(f_0^i(c))| \le \frac{\eta}{4}, \forall f \in W.$$

The calculations in the proof of Claim A imply that $|J(f,v) - J(f_0,v)| < \eta$. □

Continuity of $f \mapsto J(f,v)$ fails at maps f_0 with periodic critical points. However, to prove Theorem 2, the next result (which, losely speaking, implies that when $f_t \to f_0$ then $J(f_t,v) \to 0$ if and only if $J(f_0,v) = 0$) will suffice:

Proposition 3.3. *Let f_0 be a good piecewise expanding C^1 unimodal map with periodic critical point of prime period p_0. There exist $C_+, C_- > 0$, such that:*

A. *For every $\eta > 0$, there exists a neighborhood W of f_0 in $\mathcal{U}^1$ such that, setting*

$$\mathcal{M} = \{f \in W\colon\ f^{p_0}(c) = c,\ f^i(c) \neq c \text{ for } i < p_0\}, \tag{9}$$

the set $W \setminus \mathcal{M}$ has two connected components, W_+ and W_-, so that, for any $v \in \mathcal{B}^1(I)$, if $f \in \mathcal{M}$ then $|J(f,v) - J(f_0,v)| \le \eta |v|_1$, if $f \in W_+$ then $|J(f,v) - C_+ J(f_0,v)| \le \eta |v|_1$, if $f \in W_-$ then $|J(f,v) - C_- J(f_0,v)| \le \eta |v|_1$.
B. *For every $v \in \mathcal{B}^0(I)$ and $\eta > 0$, there exists a neighborhood W of f_0 in $\mathcal{U}^1$ such that $W \setminus \mathcal{M}$ (with $\mathcal{M}$ defined by (9)) has two connected components, W_+ and W_-, so that if $f \in \mathcal{M}$ then $|J(f,v) - J(f_0,v)| \le \eta$, if $f \in W_+$ then $|J(f,v) - C_+ J(f_0,v)| \le \eta$, if $f \in W_-$ then $|J(f,v) - C_- J(f_0,v)| \le \eta$.*

A consequence of Propositions 3.2 and 3.3 (B.) is that Ker $(J(f,\cdot))$ is a continuous distribution of codimension-one subspaces for *any* good f:

Corollary 3.1 (Continuity of Ker $(J(f,\cdot))$). *Let f be a good piecewise expanding C^1 unimodal map. Suppose that f_n is a sequence of piecewise expanding C^1 unimodal maps with $|f_n - f|_1 \to 0$, and that $v_n \in \mathcal{B}^0(I)$ and $v \in \mathcal{B}^0(I)$ are such that*

$$|v_n - v|_0 \to 0 \text{ and } J(f_n, v_n) = 0\,, \forall n$$

then $J(f,v) = 0$.

Proof of Proposition 3.3. Since

$$J(f_0,\cdot) : \{u \in \mathcal{B}^1(I), u(-1) = u(1) = 0\} \to \mathbb{R}$$

is a non trivial linear functional, there is $w \in \mathcal{B}^1(I)$, with $w(-1) = w(1) = 0$, so that $J(f_0, w) > 0$. Define the subspace

$$K = \{u \in \mathcal{B}^1(I),\ u(-1) = u(1) = 0,\ J(f_0, u) = 0\}.$$

We can identify a neighborhood of f_0 in $\mathcal{U}^1$ with a neighborhood $\widetilde{W}$ of $(0,0)$ in $K \times \mathbb{R}$ via $(u,a) \to f_0 + u + aw$. Consider the functional $F\colon \widetilde{W} \to \mathbb{R}$ defined by

$$F(u,a) = (f_0 + u + aw)^{p_0}(c) - c,$$

where p_0 denotes the prime period of c for f_0.

Note that F is a C^1 Fréchet differentiable function and (recall (4))

$$\partial_a F|_{(u,a)=(0,0)} = Df_0^{p_0-1}(f_0(c))J(f_0,w) \neq 0. \tag{10}$$

So by the Implicit Function Theorem for C^1 Fréchet differentiable functions on Banach spaces (see [6, page 17]), there exists a neighborhood $W \subset \mathcal{U}^1$ of f_0 in which $\mathcal{M} \subset W$, defined by (9), is a C^1 Banach submanifold and so that $W \setminus \mathcal{M}$ has two connected components.

Let W_+ be the connected component containing the maps f satisfying

$$f^{p_0}(c) > c,$$

and let W_- be the other component.

We claim that our assumption that f_0 is stably ϵ-expansive implies that for every n there exist neighborhoods $W_n \subset W_{n-1}$ of f_0 with the following properties: the sets $W_n \cap W_+$ and $W_n \cap W_+$ are connected and the critical point c of every map

$$f \in W_n \setminus \mathcal{M}$$

is either non periodic or periodic with prime period $\geq n$. Indeed, consider open intervals $I_0, I_1, \ldots, I_{p_0-1}$, with pairwise disjoint closures, such that

$$|I_i| < \epsilon \text{ and } f_0^i(c) \in I_i \text{ for every } 0 \leq i < p_0.$$

Let W_n be a small enough neighbourhood of f_0 so that every $f \in W_n$ satisfies

$$f^i(c) \subset I_{i \ mod \ p_0}, \tag{11}$$

for every $0 \leq i \leq n$. In particular, if f has a critical point with prime period $p < n$ then (11) holds for *every* i. We claim that $f^{p_0}(c) = c$. It is enough to show that

$$\ell = \#\{f^i(c) :\ f^i(c) \in I_1\} = 1.$$

Define

$$y = \inf\{f^i(c) :\ f^i(c) \in I_1\}.$$

Then

$$f^i[y, f(c)] \subset I_{i+1 \ mod \ p_0} \text{ for every } i.$$

Since f is ϵ-expansive, this implies that $y = f(c)$, so that $\ell = 1$, as desired.

Consequently, the itinerary of the critical point up to the n-th iteration is the same for all maps in $W_n \cap W_+$. The same statement holds for $W_n \cap W_-$. In particular there exist sequences

$$\sigma_+ = (\sigma_0^+ = C, \sigma_1^+, \sigma_2^+, \dots) \text{ and } \sigma_- = (\sigma_0^- = C, \sigma_1^-, \sigma_2^-, \dots), \quad \sigma_i^+, \sigma_i^- \in \{R, L\},$$

such that the itinerary of the critical point of a map $f \in W_+$ converges to σ_+ (in the product topology of $\{C, R, L\}^{\mathbb{N}}$) when the map converges to f_0, and an analogous statement holds for σ_- and W_-. It is not difficult to see that if $\sigma = (C, \sigma_1, \sigma_2, \dots)$ is the itinerary of the critical point of f_0, then $\sigma_i^+ = \sigma_i^- = \sigma_i$ if $p_0 \neq i$.

Define

$$C_+ := \sum_{i=0}^{\infty} \frac{1}{[Df_0^{p_0-1}(f_0(c))]^i} \prod_{j=0}^{i} \frac{1}{Df_{0,\sigma^+_{jp_0}}(c)},$$

where we put $Df_{0,R}(c) = \lim_{x\to c, x>c} Df_0(x)$, $Df_{0,L}(c) = \lim_{x\to c, x<c} Df_0(x)$, and $Df_{0,C}(c) = 1$. Since f_0 is good there is $\beta > 1$ so that

$$|Df_{0,s}(c)\ Df_0^{p_0-1}(f_0(c))| > 2\beta$$

for $s \in \{L, R\}$, so

$$\frac{1}{2\beta}\frac{2\beta-2}{2\beta-1} \le C_+ \le \frac{2\beta}{2\beta-1}.$$

Set $\lambda := \inf_{f\in W} \inf_x |Df(x)| > 1$. For each $f \in W_{p_0 m} \setminus \mathcal{M}$ we have

$$\Big|\prod_{j=0}^{i-1} \frac{1}{[Df^{p_0-1}(f^{jp_0+1}(c))]} \frac{1}{Df(f^{(j+1)p_0}(c))} - \frac{1}{[Df_0^{p_0-1}(f_0(c))]^i} \prod_{j=0}^{i} \frac{1}{Df_{0,\sigma^+_{jp_0}}(c)}\Big|$$

$$\le 2\lambda^{-p_0 m}, \quad \forall 1 \le i \le m, \tag{12}$$

and

$$\Big|J(f,v) - \sum_{i=0}^{m}\Big[\prod_{j=0}^{i-1} \frac{1}{[Df^{p_0-1}(f^{jp_0+1}(c))]} \frac{1}{Df(f^{(j+1)p_0}(c))}\Big] \sum_{\ell=0}^{p_0-1} \frac{v(f^{p_0 i+\ell}(c))}{Df^{\ell}(f^{p_0 i+1}(c))}\Big|$$

$$\le \frac{\lambda^{-p_0 m}|v|_0}{1-\lambda^{-1}}.$$

Also, we have

$$|C_+ J(f_0, v) - \sum_{i=0}^{m} \frac{1}{[Df_0^{p_0-1}(f_0(c))]^i} \prod_{j=0}^{i} \frac{1}{Df_{0,\sigma^+_{jp_0}}(c)} J(f_0,v)| \le \frac{\lambda^{-p_0 m}|v|_0}{1-\lambda^{-1}}.$$

Fix $\delta > 0$ and let $m \ge 1$ be such $2m\lambda^{-p_0 m} < \delta$. If we assume, as in Claim A, that $v \in \mathcal{B}^1(I)$, it is not difficult to see that there is a neighborhood $\widetilde{W}_{\delta,p_0 m}$ of f_0 such that if $f \in \widetilde{W}_{\delta,p_0 m} \cap W_+ \cap W_{p_0 m}$ then

$$|\sum_{\ell=0}^{p_0-1} \frac{v(f^{p_0 i+\ell}(c))}{Df^{\ell}(f^{p_0 i+1}(c))} - J(f_0, v)| \le \delta |v|_1, \text{ for every } 0 \le i \le m. \tag{13}$$

Consequently if $f \in \widetilde{W}_{\delta,p_0 m} \cap W_+ \cap W_{p_0 m}$ then

$$|J(f,v) - C_+ J(f_0,v)| \le \frac{3\delta}{1-\lambda^{-1}}|v|_0 + \frac{\delta}{1-\lambda^{-p_0}}|v|_1.$$

This proves Claim A for W_+.

To show Claim B for W_+, consider, without loss of generality, $v \in \mathcal{B}^0(I)$ with $|v|_0 \le 1$. Then we can find $\widetilde{W}_{\delta,p_0 m}$ such that

$$|\sum_{\ell=0}^{p_0-1} \frac{v(f^{p_0 i+\ell}(c))}{Df^\ell(f^{p_0 i+1}(c))} - J(f_0,v)| \le \delta, \quad \forall\ 0 \le i \le m,$$

holds for every $f \in \widetilde{W}_{\delta,p_0 m} \cap W_+ \cap W_{p_0 m}$. Then

$$|J(f,v) - C_+ J(f_0,v)| \le \frac{3\delta}{1-\lambda^{-1}} + \frac{\delta}{1-\lambda^{-p_0}},$$

completing the proof of Claim B for W_+.

We can apply a similar argument to $f \in W_{p_0 m} \cap W_-$ and

$$C_- := \sum_{i=0}^{\infty} \frac{1}{[Df_0^{p_0-1}(f_0(c))]^i} \prod_{j=0}^{i} \frac{1}{Df_{0,\sigma^-_{jp_0}}(c)},$$

with $\frac{1}{2\beta}\frac{2\beta-2}{2\beta-1} \le C_- \le \frac{2\beta}{2\beta-1}$.

The proof of the claims for $f \in \mathcal{M}$ is easier. □

4. Bifurcations in families of expanding unimodal maps

We are going to see in this section that if a C^1 family f_t of good piecewise expanding C^1 unimodal maps is tangent to the distribution of codimension-one subspaces

$$f \mapsto \mathrm{Ker}\,(J(f,\cdot))$$

then there are no bifurcations in this family, that is, there are homeomorphisms h_t such that $h_t \circ f_0 = f_t \circ h_t$ for every t. In other words, the family is a smooth deformation of f_0. The reverse statement also holds: If f_t is a family such that $J(f_0, \partial_t f_t|_{t=0}) \ne 0$ and if f_0 is good, then there are bifurcations in this family.

Theorem 1 (Characterization of smooth deformation)**.** *Let f_t, $t \in (-\delta,\delta)$, be a C^k family of piecewise expanding C^k unimodal maps, with $k \ge 1$. Then the following properties are equivalent:*

A. *For small t, the set of critical relations R_t is constant.*
B. *For small t, there exists a family $h_t\colon I \to I$ of homeomorphisms so that h_t is a conjugacy between f_0 and f_t,*
$$h_t \circ f_0 = f_t \circ h_t.$$
C. *For small t, there are conjugacies h_t, as in B, and we have that*
$$(x,t) \mapsto h_t(x)$$
is continuous and for each $x \in I$ the function $t \mapsto h_t(x)$ is $C^{k-1+Lip}$. Furthermore if we restrict t to a compact interval $Q \subset (-\epsilon,\epsilon)$ we have that this family is a bounded subset in $C^{k-1+Lip}(Q)$. (In fact, there is a universal constant C so that the diameter of this subset is $\le C \sup_{t\in Q} \frac{|f_t|_k}{1-\lambda_{f_t}^{-1}}$.)

Furthermore A, B and C imply

D. *For small t we have that $J(f_t, \partial_s f_s|_{s=t}) = 0$.*

Note that for these implications we do not assume that f_0 is good. But if we assume in addition that f_0 is stably ϵ-expansive, then D is equivalent to A, B, and C.

Proof. Note that C trivially implies B and A. □

Proof of A implies B. This implication is a consequence of Milnor-Thurston theory of kneading invariants, but we will give a self-contained argument.

Let $\delta > 0$ be so that R_t is constant for $t \in (-\delta, \delta)$. Note that the itinerary σ^t of the critical point of f_t is constant for $t \in (-\delta, \delta)$. Indeed, if $f_t^i(c) = c$ for some i and some t, then by definition $(0, i) \in R_t$, and assumption A implies $f_s^i(c) = c$ for every small s. By the continuity of the family f_t, this implies that the itinerary of c is constant (also if $R_t = \emptyset$).

Let $\mathcal{P}_t$ be the set of points which are either periodic or eventually periodic points of f_t, and whose forward orbit does not contain the critical point. It is easy to see that $\mathcal{P}_t$ is dense in I. We claim that up to taking a smaller $\delta > 0$, each point $p \in \mathcal{P}_0$ has an analytic continuation $h_t(p)$, defined for every $|t| < \delta$. Moreover

$$h_t \colon \mathcal{P}_0 \to \mathcal{P}_t$$

is a bijection. In fact, since the forward orbit of p does not contain the critical point, we can find a maximal open interval Q, where the analytic continuation $h_t(p)$ is (uniquely) defined. If there exists $t_\infty \in \partial Q \cap (-\delta, \delta)$, choose $t_n \in Q$, with $lim_{n\to\infty} t_n = t_\infty$.

Since Q is maximal, every accumulation point q of the sequence $h_{t_n}(p)$ has a priori the itinerary of p, replacing at least one of its symbols by C. But note that since the f_t are piecewise expanding, and since we proved that the itinerary of the critical point under f_t is constant, every itinerary obtained by replacing C by either R or L symbols in the itinerary of the critical point is forbidden for p. So $\partial Q \cap (-\delta, \delta) = \emptyset$ and $h_t(p)$ is defined for every t. Of course $h_t(p) \in \mathcal{P}_t$. Furthermore h_t is injective, since $h_t(p)$ has the same itinerary as p and distinct points in $\mathcal{P}_0$ have distinct itineraries.

It remains to prove that $h_{t_0}(\mathcal{P}_0) = \mathcal{P}_{t_0}$, for every t_0. This can be achieved by considering a smooth re-parametrization g_u of the family f_t such that $g_0 = f_{t_0}$ and and applying the argument above to construct $h_{t_0}^{-1}$.

Due the uniqueness of the analytic continuation

$$h_t \circ f = f_t \circ h_t \tag{14}$$

on $\mathcal{P}_0$. Moreover $p < q$ implies $h_t(p) < h_t(q)$ for every t. By the density of $\mathcal{P}_t$, for every t, we can extend h_t to a homeomorphism $h_t \colon I \to I$. The continuity of h_t and Eq. (14) imply that f_0 is conjugate to f_t by h_t. □

Proof of B implies C. See [4, Proposition 2.4] (the proof there works for $k \geq 1$). □

Proof of A, B, C implies D. It is enough to show that A. implies D. First, suppose that $R_t \neq \emptyset$. Then f_t has a periodic critical point with prime period p, for all small t, that is, $f_t^{p-1}(f_t(c)) = c$ for small t. Differentiating with respect to t, we obtain

$$\partial_t f_t^{p-1}(f_t(c)) + Df_t^{p-1}(f_t(c))\, \partial_t f_t(c) = 0.$$

So (using (5) for f_t and $\partial_t f_t$)

$$J(f_t, \partial_t f_t) = \frac{\partial_t (f_t^{p-1} \circ f_t)(c)}{Df_t^{p-1}(f_t(c))} = \frac{\partial_t f_t^{p-1}(f_t(c))}{Df_t^{p-1}(f_t(c))} + \partial_t f_t(c) = 0.$$

Now assume that $R_t = \emptyset$ for small t and suppose for a contradiction that $J(f_{t_0}, \partial_t f_t|_{t=t_0}) \neq 0$ for some small t_0. By Proposition 3.2, Claim B., either $J(f_t, \partial_t f_t) \geq \xi > 0$ for every t close to t_0, or $J(f_t, \partial_t f_t) \leq \xi < 0$ for every t

close to t_0. Without loss of generality, assume the first case. Using (6) for f_t and $\partial_t f_t$, and the fact that $\theta = \inf_{t,x} |Df_t(x)| > 1$, we find $\delta > 0$ and $k_0 \geq 1$ so that

$$\frac{\partial_t f_t^k(c)}{Df_t^{k-1}(f_t(c))} \geq \frac{\xi}{2}, \quad \forall |t - t_0| \leq \delta, \forall k \geq k_0.$$

So,

$$2 \geq |f_{t_0+\delta}^k(c) - f_{t_0}^k(c)| \geq \frac{\xi}{2}\theta^{k-1}\delta,$$

for every $k \geq k_0$, which is absurd since $\theta > 1$. □

Proof of D implies A. We assume stable ϵ-expansivity of f_0. Consider the set of uniformly bounded functions

$$c_n\colon \{t\colon\ |t| < \delta\} \to I,$$

with $c_n(t) = f_t^n(c)$. We claim that this family is equicontinuous.

Write $v_t = \partial_s f_s|_{s=t}$. By Proposition 3.1, there exists for each t a unique bounded function $\alpha_t\colon I \to \mathbb{R}$ satisfying $\alpha_t(c) = 0$ and

$$v_t(x) = \alpha_t(f_t(x)) - Df_t(x)\alpha_t(x) \tag{15}$$

for every $x \neq c$. In addition, since we assumed $J(f_t, v_t) = 0$, we have

$$v_t(c) = \alpha_t(f_t(c))\,, \forall t. \tag{16}$$

Consider a solution $g\colon \{t\colon\ |t| < \delta\} \to I$ of the differential equation

$$\frac{dg}{dt}(s) = \alpha_s(g(s)). \tag{17}$$

We claim that the function $G\colon \{t\colon\ |t| < \delta\} \to I$ defined by

$$G(t) = f_t(g(t))$$

is also a solution of (17). Indeed, as a consequence of (15), if $g(t_0) \neq c$, we have

$$\begin{aligned}\frac{dG}{dt}(t_0) &= v_{t_0}(g(t_0)) + Df_{t_0}(g(t_0))\frac{dg}{dt}(t_0) \\ &= v_{t_0}(g(t_0)) + Df_{t_0}(g(t_0))\alpha_{t_0}(g(t_0)) = \alpha_{t_0}(f_{t_0}(g(t_0))) = \alpha_{t_0}(G(t_0)).\end{aligned}$$

If $g(t_0) = c$ then by (17) we have $\frac{dg}{dt}(t_0) = 0$. Therefore, using that the f_t are piecewise uniformly Lipschitz and the family is C^1, we get

$$\begin{aligned}\frac{G(t_0+\eta) - G(t_0)}{\eta} &= \frac{f_{t_0+\eta}(g(t_0+\eta)) - f_{t_0+\eta}(g(t_0))}{\eta} + \frac{f_{t_0+\eta}(g(t_0)) - f_{t_0}(g(t_0))}{\eta} \\ &= \frac{O(|g(t_0+\eta) - g(t_0)|)}{\eta} + v_{t_0}(g(t_0)) + o(h) = v_{t_0}(g(t_0)) + o(\eta).\end{aligned}$$

So

$$\frac{dG}{dt}(t_0) = v_{t_0}(g(t_0)) = v_{t_0}(c) = \alpha_{t_0}(f_{t_0}(c)) = \alpha_{t_0}(f_{t_0}(gt_0))).$$

Consequently $g_n(t) = f_t^n(g(t)))$ is a solution of (17), for every n.

Of course the constant function $c_0(t) = c$ is a solution of (17). Since the functions α_t can be uniformly bounded by a constant which is independent of t, we conclude by (17) that the set of functions $c_n(\cdot)$ is equicontinuous.

Suppose now that there is t_0 so that c is a periodic point of f_{t_0}. If the prime period of c is p, choose open intervals $I_0, I_1, \ldots I_{p-1}$, with pairwise disjoint closures, $|I_i| < \epsilon$, and such that

$$f_{t_0}^i(c) \in I_{i \mod p} \quad \forall i.$$

Since $\{c_n(\cdot)\}$ is an equicontinuous set of functions, there exists $\delta_0 > 0$ such that

$$f_t^i(c) \in I_i \mod p \tag{18}$$

for every i and every t such that $|t - t_0| < \delta_0$. We claim that if $|t - t_0| < \delta_0$ then the map f_t has a periodic critical point with the same itinerary as that of c for f_{t_0}. By (18), it is enough to show that

$$N = \#\{f_t^i(c)\colon i \mod p = 1\} = 1.$$

Define

$$y = \inf\{f_t^i(c)\colon i \mod p = 1\}.$$

The definition of y implies

$$f_t^p[y, f_t(c)] \subset [y, f_t(c)],$$

and

$$f_t^i[y, f_t(c)] \subset I_{i+1 \mod p}$$

for every p. This implies $|f_t^i([y, f_t(c)])| < \epsilon$ for every i. By the stable ϵ-expansivity of f_0 we must have $y = f_t(c)$ which implies $N = 1$.

So we conclude that for every itinerary σ of length p, the set of parameters $\mathcal{O}$ such that f_t has a p-periodic critical point with itinerary σ is an open set. Of course for all parameters in the closure of $\mathcal{O}$, f_t has a p-periodic critical point, but, *a priori,* not with prime period p. But if we apply the same argument to this boundary parameter, we conclude that its critical point has the same itinerary as points in $\mathcal{O}$. This implies that either $\mathcal{O} = \emptyset$ or $\mathcal{O} = \{t\colon\ |t| < \delta\}$. If $\mathcal{O} = \emptyset$ for each finite orbit, then each f_t has an infinite postcritical orbit and an empty R_t. So the set of critical relations R_t does not depend on t. □

We mention an easy consequence of Theorem 1 which will be useful in the proof of Theorem 2:

Corollary 4.1 (Unstable families). *Let f_t be a C^1 family of piecewise expanding C^1 unimodal maps such that f_0 is good and*

$$J(f_0, \partial_t f_t|_{t=0}) \neq 0.$$

A. *If f_0 has a periodic critical point then there exists a sequence of parameters $t_n \to 0$ such that the critical point of f_{t_n} is not periodic.*
B. *If f_0 has a non periodic critical point then there exists a sequence of parameters $t_n \to 0$ such that the critical point of f_{t_n} is periodic.*

Proof of Claim A. By Corollary 3.1 and the continuity of $t \mapsto \partial_t f_t$ in the $\mathcal{B}^0(I)$ norm, there exists $\tilde{\delta} > 0$ such that

$$J(f_t, \partial_s f_s|_{s=t}) \neq 0, \quad \forall |t| \leq \delta_0. \tag{19}$$

Suppose by contradiction that for all parameters $|t| \leq \delta_1 < \delta_0$ the critical point of f_t is periodic. Define

$$P_n := \{t\colon\ f_t^n(c) = c \text{ and } |t| \leq \delta_1\}.$$

Of course P_n is closed. By the Baire Theorem there exists $n_0 \geq 1$ so that P_{n_0} contains a nonempty connected open set $Q \subset P_{n_0}$. For each $1 \leq i \leq n_0$, let $P'_i \subset P_{n_0}$ be set of parameters for which f_t has a critical point whose prime period is equal or larger than i. Of course each P'_i is an open subset of P_{n_0}. Let

$$p = \max\{i\colon\ P'_i \neq \emptyset\}.$$

Then there exists an open set U such that the critical point of f_t has prime period p if $t \in U$. In particular the set of critical relations R_t is constant on U. By the implication $A \Rightarrow D$ in Theorem 1, $J(f_t, \partial_t f_t) = 0$ for every $t \in U$, which contradicts (19). □

Proof of Claim B. The proof in this case is even easier. By Proposition 3.2 B., we have (19) for some $\delta_0 > 0$. If there are non periodic critical points for f_t for all small enough t, then the set of critical relations R_t is empty for those t. By $A \Rightarrow D$ in Theorem 1, $J(f_t, \partial_t f_t) = 0$ for all small enough t, which contradicts (19). □

5. Finding or approximating families tangent to a given horizontal direction

We can now state and prove our second main result:

Theorem 2. *Let $k \geq 2$, let f be a good piecewise expanding C^k unimodal map, and let $v, w \in \mathcal{B}^k(I)$ satisfy $v(-1) = v(1) = w(-1) = w(1) = 0$, $J(f, v) = 0$ and $J(f, w) \neq 0$. Then for every C^k family f_t of piecewise expanding C^k unimodal maps such that $f_0 = f$ and $\partial_t f_t|_{t=0} = v$, there exists $\delta > 0$ and a unique continuous function $b : (-\delta, \delta) \to \mathbb{R}$, such that $b(0) = 0$ and that*

$$\tilde{f}_t = f_t + b(t)w$$

is topologically conjugate with f for all $|t| < \delta$.

Furthermore this unique function b is in fact $C^{k-1+Lip}$ and satisfies $b'(0) = 0$ (in particular $\partial_t \tilde{f}_t|_{t=0} = v$), and the family $\tilde{f}_t$ is a $C^{k-1+Lip}$-family of piecewise expanding C^k unimodal maps.

In addition, there exists a sequence of C^k families of piecewise expanding C^k unimodal maps $t \mapsto g_{t,n}$ ($t \in (-\delta, \delta)$) such that

- *the map $g_{t,n}$ is topologically conjugate with $g_{0,n}$, for each t and n,*
- *the critical point of $g_{0,n}$ is periodic for each n,*
- *For each t the map $g_{t,n}$ converges to the map $\tilde{f}_t$ in the $\mathcal{B}^{k-1}(I)$ topology.*

Proof of the existence of $\tilde{f}_t$. Note that if $f_0(c) = +1$, then either $f_t(c) = +1$ for all small enough t (in which case we may take $b(t) \equiv 0$, so that existence of $\tilde{f}_t$ is proved) or $f_t(c) < 1$ for all nonzero small enough t. Denote $v_t = \partial_s f_s|_{s=t}$. For small $\eta > 0$ set $M_\eta = \{|t| < \eta,\ |\theta| < \eta\}$ and consider

$$f_{(t,\theta)} = f_t + \theta w, \quad (t, \theta) \in M_\eta. \tag{20}$$

(In fact, if $f_0(c) = +1$ but $f_t(c) < 1$ for all small nonzero t we must take $M_\eta = \{|t| < \eta, |\theta| < \Theta(t)\}$ with Θ a C^1 function so that $\Theta(0) = 0$, $\Theta(t) > 0$ for $t \neq 0$.) Since $J(f_0, v_0) = 0$ but $J(f_0, w) \neq 0$, Propositions 3.3 and 3.2 and Corollary 3.1 imply that if η is small enough then for all $(t, \theta) \in M_\eta$ the linear space

$$\mathbb{L}(t, \theta) = \{(\alpha, \beta) \in \mathbb{R}^2 : J(f_{(t,\theta)}, \alpha v_t + \beta w) = 0\}$$

is a one-dimensional subspace of $\mathbb{R}^2$, which depends continuously on (t, θ) and never coincides with the vertical line $\{0\} \times \mathbb{R}$. In other words: There exists a uniquely defined function $d : M_\eta \to \mathbb{R}$ so that

$$v_t + d(t, \theta)w \in \operatorname{Ker}(J(f_{(t,\theta)}, \cdot)). \tag{21}$$

In addition, $d(0,0) = 0$ and d is continuous. Consider the C^1-integral curve b of the ordinary differential equation

$$\frac{db}{dt} = d(t, b(t)), \quad b(0) = 0.$$

Since $d(0,0) = 0$, if η is small, then the solution b is defined for $|t| < \eta$. As a consequence, the family $\tilde f_t = f_{(t,b(t))} = f_t + b(t)w$ satisfies

$$J(\tilde f_t, \partial_s \tilde f_s|_{s=t}) = 0, \quad \forall |t| < \eta.$$

By $D \Rightarrow B$ in Theorem 1, $\tilde f_t$ is topologically conjugate with f_0 for small t. □

Proof of the uniqueness of $\tilde f_t$. Suppose that $\underline{b}$, $\bar b$ are two continuous functions with $\underline{b}(0) = \bar b(0) = 0$ and such that both maps

$$f_t + \underline{b}(t)w, \quad f_t + \bar b(t)w, \tag{22}$$

are topologically conjugate to f for each small $|t| < \delta$. Using the map $d : M_\eta \to \mathbb{R}$ from (21) in the proof of the existence of $\tilde f_t$, choose $0 < \hat\eta < \tilde\eta < \eta$ such that if $(t_0, \theta_0) \in M_{\hat\eta}$ then the ordinary differential equation

$$\frac{db(t)}{dt} = d(t, b(t)) \tag{23}$$

with initial condition $b(t_0) = \theta_0$, has a C^1-solution defined for every $|t| < \tilde\eta$, and, moreover, we have $|b(t)| < \tilde\eta$ for $|t| < \tilde\eta$. Such $\hat\eta$, $\tilde\eta$ exist, since $d(0,0) = 0$.

Suppose there is $|t_0| < \hat\eta$ such that $\underline{b}(t_0) \neq \bar b(t_0)$. Since $\bar b$ and $\underline{b}$ are continuous, up to taking a smaller t_0 we may assume that $\max(|\underline{b}(t_0)|, |\bar b(t_0)|) < \hat\eta$. To fix ideas, assume $0 \le \underline{b}(t_0) < \bar b(t_0)$ (the other cases are similar). Then for every $\theta_0 \in (\underline{b}(t_0), \bar b(t_0))$ we can find a solution

$$b \colon (-\tilde\eta, \tilde\eta) \to \mathbb{R}$$

for (23) such that $b(t_0) = \theta_0$. By the Intermediate Value Theorem, there exists $t_1 \in [0, t_0)$ such that $b(t_1) = \underline{b}(t_1)$.

Note that by (23) and the definition of $d(\cdot,\cdot)$

$$J(f_t + b(t)w, v_t + b'(t)w) = 0. \tag{24}$$

Thus, by $D \Rightarrow B$ in Theorem 1 for b, and by assumption for $\underline{b}$, $\bar b$, all maps

$$f_t + b(t)w, \quad f_t + \underline{b}(t)w, \quad f_t + \bar b(t)w, \quad |t| < \hat\eta$$

are in the topological class of f. As a consequence, for every $\theta \in (\underline{b}(t_0), \bar b(t_0))$, the map $f_{t_0} + \theta w$ is topologically conjugate with f. Consequently there is no change of combinatorics in the C^∞ family of piecewise expanding C^k unimodal maps

$$s \mapsto f_{t_0} + (\theta_0 + s)w, \quad |s| < \eta(\theta_0),$$

and $B \Rightarrow D$ in Theorem 1 gives $J(f_{t_0} + \theta_0 w, w) = 0$.

Taking a sequence $t_n \to 0$ such that $\underline{b}(t_n) \neq \bar b(t_n)$, the argument above gives a sequence $\theta_n \to 0$ so that $J(f_{t_n} + \theta_n w, w) = 0$, and Corollary 3.1 implies $J(f_0, w) = 0$, contradicting the assumption on w. □

Proof of the $C^{k-1+Lip}$ regularity, construction of $g_{t,n}$. Recall (20) and the characterization (21) of $d(\cdot,\cdot)$. By Corollary 4.1.B, there exists a sequence $\theta_n \to 0$ such that $f_{(0,\theta_n)}$ has a periodic critical point (if f_0 has a periodic critical point, define

$\theta_n = 0$, for every n). Consider the C^1-integral curves b_n of the ordinary differential equation

$$\frac{db_n}{dt} = d(t, b_n(t)), \quad b_n(0) = \theta_n. \tag{25}$$

Note that since $d(0,0) = 0$, if η is small, then the solution b_n is defined for $|t| < \eta$, provided n is large enough. As a consequence, for all large enough n,

$$g_{t,n} = f_{(t,b_n(t))} = f_t + b_n(t)w \tag{26}$$

satisfies

$$J(g_{t,n}, \partial_s g_{s,n}|_{s=t}) = 0, \quad \forall |t| < \eta. \tag{27}$$

Let p_n be the prime period of the turning point of $g_{n,0} = f_{(0,\theta_n)}$. By (27) and $D \Rightarrow B$ in Theorem 1 we have that $g_{t,n}$ is topologically conjugate with $g_{n,0}$, so $g_{t,n}$ has a critical point with the same prime period p_n.

We shall first prove that each $g_{t,n} \in \mathcal{U}^k(I)$, by showing that each function b_n is C^k. Indeed consider the non-linear functional

$$F_n(t, \theta) = f^{p_n}_{(t,\theta)}(c) - c.$$

Then F_n is C^k on M_η, and if $f_{(t,\theta)}$ has a periodic point with prime period p_n our assumption on w gives (recalling (4), as for (10))

$$\partial_\theta F_n(t, \theta) = Df^{p_n - 1}_{(t,\theta)}(f_{(t,\theta)}(c)) J(f_{(t,\theta)}, w) \neq 0.$$

Since $F_n(t, b_n(t)) = 0$, the Implicit Function Theorem implies that $t \mapsto b_n(t)$ is C^k. For further use, note also that

$$\partial_t F_n(t, b_n(t)) + \partial_\theta F(t, b_n(t)) b_n'(t) = 0,$$

and since $\partial_t F_n(t, \theta) = Df^{p_n - 1}_{(t,\theta)}(f_{(t,\theta)}(c)) J(g_{t,n}, v_t)$, we obtain

$$b_n'(t) = -\frac{J(g_{t,n}, v_t)}{J(g_{t,n}, w)}. \tag{28}$$

We shall next show that the families $\{(t, x) \mapsto g_{t,n}(x), \quad n \in \mathbb{N}\}$ form a bounded subset of C^2 (in the sense of families (1)) as the first step in the inductive proof that this set is bounded for C^k. Let $\lambda = \inf_{t,n} \lambda_{g_{t,n}}$. We have $\lambda > 1$.

Since f_t and w are in $\mathcal{B}^2(I)$, and b_n, b_n' are uniformly bounded in n (use (25)), there exist by the definition (26) uniform upper bounds for the derivatives

$$\partial_t \, g_{t,n}, \; \partial_x \, g_{t,n}, \; \partial^2_x \, g_{t,n}, \; \partial^2_{xt} \, g_{t,n}, \; \partial^2_{tx} \, g_{t,n}.$$

So we must only estimate $\partial^2_{tt} \, g_{t,n}$. By (26), it is enough to show that $\{b_n\}_n$ is a bounded subset in C^2. Since we already know that each b_n is C^2, it is enough to get an n-uniform bound on the Lipschitz constant of b_n'. In view of (28), we first show that for every vector field $u \in \mathcal{B}^1(I)$, the map $t \mapsto J(g_{t,n}, u)$ is Lipschitz, and its Lipschitz norm does not depend on n. By $B \Rightarrow C$ in Theorem 1, the conjugacies $h_{t,n} \circ g_{0,n} = g_{t,n} \circ h_{t,n}$ are such that (the $C^{k-1+Lip}$ map) $t \mapsto h_{t,n}(x)$ is K-Lipschitz, with

$$K \le \tilde{C} \, \frac{\sup_{n,|t|<\eta} |v_t + b_n'(t) w|_0}{1 - \lambda^{-1}}.$$

This implies in particular that $t \mapsto u(g^i_{t,n}(c))$ is Lipschitz uniformly in t, i and n.

So (we omit n in $g_{t,n}$ and $h_{t,n}$to avoid a cumbersome notation)

$$
\begin{aligned}
(29)\qquad |J(g_{t+\delta}, u) - J(g_t, u)| &\le \sum_{i=0}^{p_n-1} \Big| \frac{u(g^i_{t+\delta}(c))}{Dg^i_{t+\delta}(g_{t+\delta}(c))} - \frac{u(g^i_t(c))}{Dg^i_t(g_t(c))}\Big| \\
&\le \sum_{i=0}^{p_n-1} \frac{|u(h_{t+\delta}(g^i_0(c)) - u(h_t(g^i_0(c))|}{|Dg^i_{t+\delta}(g_t(c))|} \\
&\quad + \sum_{i=0}^{p_n-1} |u(g^i_t(c))| \Big| \frac{1}{Dg^i_{t+\delta}(g_{t+\delta}(c))} - \frac{1}{Dg^i_t(g_t(c))}\Big| .
\end{aligned}
$$

Note that

$$
\begin{aligned}
(30)\qquad &\Big|\frac{1}{Dg^i_{t+\delta}(g_t(c))} - \frac{1}{Dg^i_t(g_t(c))}\Big| \\
&\le \sum_{j=0}^{i-1} \frac{1}{\lambda^{i-1}} \Big| \frac{1}{Dg_{t+\delta}(h_{t+\delta}(g_0^{j+1}(c)))} - \frac{1}{Dg_t(h_t(g_0^{j+1}(c)))}\Big| \\
&\le \frac{1}{\lambda^{i-1}} \Big(\sum_{j=0}^{i-1} \Big| \frac{1}{Dg_{t+\delta}(h_{t+\delta}(g_0^{j+1}(c)))} - \frac{1}{Dg_t(h_{t+\delta}(g_0^{j+1}(c)))}\Big| \\
&\qquad + \Big| \frac{1}{Dg_t(h_{t+\delta}(g_0^{j+1}(c)))} - \frac{1}{Dg_t(h_t(g_0^{j+1}(c)))}\Big| \Big) \le C\frac{i}{\lambda^i}\delta.
\end{aligned}
$$

In the last step we used that $\partial^2_{tx}\, g_{t,n}$ is bounded uniformly in n. So (29) gives

$$
|J(g_{t+\delta,n}, u) - J(g_{t,n}, u)| \le \sum_{i=0}^{p_n-1} \Big[\frac{C|u|_1\delta}{\lambda^i} + \frac{|u|_0 i\delta}{\lambda^i}\Big] \le C|u|_1\delta,
$$

which proves that $t \mapsto J(g_{t,n}, u)$ is $C|u|_1$-Lipschitz, uniformly in n.

Next, we obtain from (28) that

$$
\begin{aligned}
|b'_n(t+\delta) - b'_n(t)| &\le \frac{|J(g_{t+\delta}, v_{t+\delta}) - J(g_{t+\delta}, v_t)| + |J(g_{t+\delta}, v_t) - J(g_t, v_t)|}{|J(g_{t+\delta}, w)|} \\
&\quad + |J(g_t, v_t)| \Big| \frac{1}{J(g_{t+\delta}, w)} - \frac{1}{J(g_t, w)}\Big| \le K\delta \max(\sup_t |v_t|_1, |w|_1).
\end{aligned}
$$

We used that $J(g, \cdot)$ is linear and that $J(g_{t,n}, w)$ is bounded away from zero and infinity, uniformly in n and $|t| < \eta$ (by Propositions 3.2 and 3.3 since $g_{t,n}$ and f_0 are $\mathcal{B}^2(I)$-close, using that $\sup_{n,t} |b_n(t)| < \infty$).

This proves our claim that b_n is C^{1+Lip}, and thus C^2, uniformly in n, and thus the uniform C^2 claim on $g_{t,n}$. Note that by $B \Rightarrow C$ in Theorem 1 the maps $t \mapsto h_{t,n}(x)$ are C^{1+Lip} uniformly in n and x. If $k = 2$, we are done. If $k \ge 3$, we have concluded the first step in the induction.

Before we perform the inductive step, we introduce some notation and terminology. Let X be a set and let f_λ, $\lambda \in \Lambda$, be an indexed family $\mathcal{F}$ of functions on X. For each formal *monomial* $\lambda_1\lambda_2 \dots \lambda_n$, we can associate the function $f_{\lambda_1} f_{\lambda_2} \dots f_{\lambda_n}$. This function is called a $\mathcal{F}$-monomial combination of degree n. A *$\mathcal{F}$-polynomial* combination of degree n is a finite sum of $\mathcal{F}$-monomials whose maximal degree is n. Note that if Λ_1 is a finite subset of Λ and $P_n \in \mathbb{N}_n[\Lambda_1]$ is a polynomial with non negative integer coefficients, we can associate to it a $\mathcal{F}$-polynomial combination of degree n. We will call this combination the P-combination of the family $\mathcal{F}$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families indexed by Λ_1 and Λ_2, we will denote by $\mathcal{F}_1 \cup \mathcal{F}_2$ the disjoint union of these families, indexed by the disjoint union Λ_1 and Λ_2.

If X is an open interval, all functions in $\mathcal{F}$ are differentiable and $\mathcal{F}'$ is the indexed family of derivatives of functions in $\mathcal{F}$, then the derivative of a sum of m S-monomials of degree n is a sum of $m \cdot n$ $\mathcal{F} \cup \mathcal{F}'$-monomials of degree n.

Suppose by induction that $\{b_n\}_n$ is a bounded family in C^q, with $2 \leq q < k$. Then $g_{t,n}$ is a C^q family. By $B \Rightarrow C$ in Theorem 1, the conjugacies $h_{t,n} \circ g_{0,n} = g_{t,n} \circ h_{t,n}$ are such that $t \mapsto h_{t,n}(x) \in C^{q-1+Lip}$, uniformly in n and x, and, setting

$$\mathcal{C}^{q-1}_{n,i} = \{\partial^j_t h_{t,n}(g^\ell_{0,n}(c)) \colon\ 1 \leq j \leq q-1, \ell \leq i\},$$

then

$$\cup_n \mathcal{C}^{q-1}_{n,p_n-1} \tag{31}$$

is a bounded family in C^{Lip}. Note also that for every $x \neq c$

$$\partial^a_x \partial^b_t g_{t,n}(x) = \partial^b_t \partial^a_x g_{t,n}(x) = \partial^a_x f_t(x) + \partial^b_t b_n(t)\ \partial^a_x w(x)$$

for every $a \leq k$ and $b \leq q$. In particular for each $i < p_n$, if we define the indexed families of functions

$$\mathcal{G}^q_{n,i} := \{\partial^a_x \partial^b_t g_{t,n}(h_{t,n}(g^\ell_{0,n}(c))) \colon\ b < q, a \leq q, 1 \leq \ell \leq i\}$$

then

$$\cup_n \mathcal{G}^q_{n,p_n-1} \tag{32}$$

is also a bounded subset in C^{Lip}. It is easy to show by induction on q that there exist C_q and d_q such that for each $1 \leq \ell < p_n$, there exists a $P_{q,\ell}$-combination $\psi_{n,\ell,q}$ of the family $\mathcal{C}^{q-1}_{n,\ell} \cup \mathcal{G}^q_{n,\ell}$ such that

$$\partial^{q-1}_t \frac{1}{Dg_{t,n}(h_t(g^\ell_{0,n}(c)))} = \frac{\psi_{n,\ell,q}}{(Dg_{t,n}(h_t(g^\ell_{0,n}(c)))^{2^{q-1}}},$$

and $P_{q,\ell}$ is a sum of at most C_q monomials of maximal degree d_q. In particular if we define the indexed families

$$\mathcal{F}^{q-1}_{n,i} := \{\partial^r_t \frac{1}{Dg_{t,n}(h_t(g^\ell_{0,n}(c)))} \colon\ 0 \leq r \leq q-1, 1 \leq \ell < p_n\}$$

then

$$\cup_n \mathcal{F}^{q-1}_{n,p_n-1} \tag{33}$$

is a bounded set in C^{Lip}.

Let $u \in \mathcal{B}^k(I)$. We claim that $\{J(g_{t,n}, u)\}_n$ is a bounded subset of functions in $C^{q-1+Lip}$. Indeed, for each n and $i < p_n$, define the indexed families of functions

$$\mathcal{O}^{q-1}_{n,i} = \{D^j_x u(h_{t,n}(g^\ell_{0,n}(c))) \colon\ 0 \leq j \leq q-1, 1 \leq \ell \leq i\},$$

Of course

$$\cup_n \mathcal{O}^{q-1}_{n,p_n-1} \tag{34}$$

is a bounded subset of functions in C^{Lip}.

Note that for each $i < p_n$

$$u_{0,i,n} := \frac{u(g^i_{t,n}(c))}{Dg^i_{t,n}(g_{t,n}(c))} = u(h_{t,n}(g^i_{0,n}(c))) \prod_{\ell=1}^{i} \frac{1}{Dg_{t,n}(h_{t,n}(g^\ell_{0,n}(c)))},$$

in particular there exists a monomial $P_{0,i}$ of degree $i+1$ such that $u_{0,i,n}$ is a $P_{0,i}$-combination of the family $\mathcal{C}^0_{n,i} \cup \mathcal{F}^0_{n,i} \cup \mathcal{O}^0_{n,i}$. It can be easily proven by induction on q that for every $1 \le i < p_n$ there exists a polynomial $P_{q,i}$ such that

$$\partial_t^{q-1} \frac{u(g^i_{t,n}(c))}{Dg^i_{t,n}(g_{t,n}(c))}$$

is a $P_{q,i}$-combination of the family $\mathcal{C}^{q-1}_{n,i} \cup \mathcal{F}^{q-1}_{n,i} \cup \mathcal{O}^{q-1}_{n,i}$. Furthermore $P_{q,i}$ is the sum of at most $(i+q-1)!/i!$ monomials with maximal degree $i+q$, and each monomial contains at least $\max\{0, i-q+1\}$ indexes in $\mathcal{F}^0_{n,i}$. So if

$$\bigcup_n \mathcal{C}^{q-1}_{n,p_n-1} \cup \mathcal{F}^{q-1}_{n,p_n-1} \cup \mathcal{O}^{q-1}_{n,p_n-1}.$$

belongs to the ball of radius R in C^{Lip}, it is easy to see that

$$|\partial_t^{q-1} J(g_{t,n}, u)|_{Lip} \le \sum_{j=0}^{q-1} \frac{(j+q)!}{j!} R^{j+1+q} + \sum_{i \ge q}^{\infty} \frac{(i+q)!}{i!} \frac{R^q}{\lambda^{i-q}}. \tag{35}$$

Thus, by (28), b_n belongs to a bounded subset in C^{q+Lip}, and thus in C^{q+1}. That concludes the induction step. In particular $\{g_{t,n}(x), n\}$ belongs to a bounded set of C^k families (in the sense of families (1)).

We can thus choose a convergent subsequence $\lim_{j\to\infty} g_{n_j,t} = g_{\infty,t}$ in the $\mathcal{B}^{k-1}(I)$-topology, where the family $g_{\infty,t}$ is a $C^{k-1+Lip}$ family of piecewise expanding C^k unimodal maps. Note that $g_{\infty,0} = f$. Since $J(g_{t,n}, \partial_t g_{t,n}) = 0$ and

$$\lim_{j\to\infty} \partial_t g_{n_j,t}|_{t=t_0} = \partial_t g_{\infty,t}|_{t=t_0},$$

in the $\mathcal{B}^0(I)$ topology, we conclude, by Corollary 3.1, that $J(g_{\infty,t}, \partial_s g_{\infty,s}|_{s=t}) = 0$. By $D \Rightarrow B$ in Theorem 1, the map $g_{\infty,t}$ is topologically conjugate with f. By uniqueness, $g_{\infty,t} = \tilde{f}_t$ and $\lim_{n\to\infty} b_n = b$. □

For $k \ge 2$, let $f \in \mathcal{U}^k(I)$ be a good map, and let $w \in \mathcal{B}^k(I)$ be such that $w(-1) = w(1) = 0$ and $J(f, w) \ne 0$. Using a technique similar to the proof of Theorem 2, one can prove that there exist a neighborhood $\mathcal{V}_1$ of f and a neighborhood $\mathcal{V}_2$ of 0 in $\operatorname{Ker} J(f,\cdot)$ (we consider $J(f,\cdot) : \{v \in \mathcal{B}^k, v(-1) = v(1) = 0\} \to \mathbb{R}$) such that each topological class in $\mathcal{V}_1$ is of the form $\{f + v + \psi(v)w\colon\ v \in \mathcal{V}_2\}$, where ψ is a $C^{k-1+Lip}$ real-valued function defined in $\mathcal{V}_2$. Moreover if $f_n \in \mathcal{V}_1$ for all integers $n \ge 0$, with $\lim_{n\to\infty} f_n = f_\infty \in \mathcal{V}_1$, and if $\psi_n\colon \mathcal{V}_2 \to \mathbb{R}$ defines the topological class of f_n in $\mathcal{V}_1$, then $\lim_{n\to\infty} \psi_n = \psi_\infty$ in C^{k-1} defines the topological class of f_∞.

We end by mentioning two immediate corollaries of Theorem 2.

Corollary 5.1. *Let f be a piecewise expanding C^k unimodal map and $v \in \mathcal{B}^k(I)$, with $k \ge 2$, such that $v(-1) = v(1) = 0$ and $J(f, v) = 0$. Then there exists a $C^{k-1+Lip}$ family of piecewise expanding C^k unimodal maps $\tilde{f}_t$ such that $\tilde{f}_0 = f$, $\partial_t \tilde{f}_t|_{t=0} = v$ and $\tilde{f}_t$ is topologically conjugate with f, for every t.*

Proof. Choose $w \in \mathcal{B}^k(I)$ with $w(-1) = w(1) = 0$ such that $J(f, w) \ne 0$. Consider the family $f_t = f + tv$ and apply Theorem 2. □

Corollary 5.2. *Let f_t be a C^k family of piecewise expanding C^k unimodal maps, $k \ge 2$, such that f_t is topologically conjugate with f_0, for every t. Then there exists a sequence of C^k-families $t \in (-\delta, \delta) \mapsto g_{t,n}$ such that*

- *The map $g_{t,n}$ is topologically conjugate with $g_{0,n}$, for every t.*
- *The critical point of $g_{0,n}$ is periodic.*
- *The families $g_{t,n}$ converge to the family f_t in the $\mathcal{B}^{k-1}$ topology.*

Proof. Apply Theorem 2 to the family f_t, noting that Theorem 1 implies that $J(f_0, \partial_t f_t|_{t=0}) = 0$. So there exists a unique family $\tilde{f}_t$ such that $\tilde{f}_t$ is topologically conjugate with f_0, for every small t. We conclude that $f_t = \tilde{f}_t$. To finish, apply the last claim in Theorem 2. □

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