Determinant of Sturm-Liouville operators and semi-classical trace formula for Fourier integral operators

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The semi-classical trace formula (SCTF), also called "Gutzwiller formula", has been proved in a rigorous way by several people in the seventies. This formula is a kind of approximate quantization rule for the eigenvalues of a Schrödinger operator $\hat{H} = -h^2\Delta + V$ in the semi-classical limit $h \to 0$. More precisely, it relates the $h \to 0$ asymptotics of the regularized density of eigenvalues

$$D_\rho(E) := \sum_j \frac{1}{h}\rho(\frac{E - E_j}{h})$$

to the closed orbits of the Newton flow $x'' = -\nabla V$. The SCTF is a generalization of the Poisson sommation formula and of the Selberg trace formula.

We will give the precise statement and a proof for the trace of a Fourier integral operator associated to a twist symplectic map χ with a generating function $S(x, y)$ given by $\chi(y, -\partial_y S) = (x, \partial_x S)$. The proof uses the stationnary phase approximation and a formula for the determinant of a discrete Sturm-Liouville operator going back to a formula of Levit and Smilansky. The Gutzwiller formula can be deduced easily from the previous one at least in the case of a regular Hamiltonian.

References

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