The Magnet and the Butterfly:
Thermodynamic formalism and the ergodic theory of chaotic dynamics

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Abstract. Some of the formal tools first developed in the sixties to make rigorous the theory of phase transitions in statistical mechanics have turned out to be "unreasonably effective" in the study of chaotic dynamical systems. The evolution over three decades of this fruitful, ongoing, reciprocal influence between theoretical physics and mathematics is briefly evoked, while a selection of ideas at the hinges of the connection is presented.

1. Introduction

The modern theory of dynamical systems is one of the most mediatized mathematical advances of the second half of this century. Its roots reach back to Poincaré in the nineteen-hundreds, but, as many cultivated laymen know, interest in the theory was revived among other things by the famous article of Lorenz [1963] describing the "chaotic" properties of a deterministic model for convection in the atmosphere. Chaos here means essentially sensitive dependence on initial conditions, and the average educated man in the street is familiar with the metaphor of the Brazilian butterfly whose fluttering wings may induce a Texan tornado days, weeks, or maybe even years later.

It turns out that such chaotic dynamical systems, although by essence unpredictable, admit a very good statistical description, in the sense that it is often possible to describe the average *asymptotic* – that is, long-term – behaviour of *almost all initial conditions.* We state this now in more precise terms, leaving details to the discussion below, and restricting our attention to the simplest case of a discrete-time attractor in a finite dimensional manifold M, where Lebesgue measure (the measure associated to Riemannian volume) is a natural reference mass. This means that we consider a diffeomorphism $f : M \to M$, and assume that $\Omega \subset M$ is a closed f-invariant set admitting an open neighbourhood U with $\cap_{n\ge 0} f^n(U) = \Omega$. (We write f^n for the n^{th} iterate $\overbrace{f \circ \cdots \circ f}^{n \text{ times}}$ of the map f, viewing $f^n(x)$ as the state at time $n \in \mathbb{Z}^+$ of our system, starting from an initial condition

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$x \in M$.) Under suitable hyperbolicity assumptions on f (which imply sensitive dependence on initial conditions), one may prove the existence of a unique probability measure μ on Ω, invariant under the dynamics (i.e., $\mu(f^{-1}E) = \mu(E)$ for all Borel sets E), and which has the property that for any reasonable observable $\varphi : M \to \mathbb{C}$, the time average *for Lebesgue almost any starting point* x in the neighbourhood U of the attractor Ω converges to the space average with respect to the measure μ:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int_\Omega \varphi(y)\, d\mu(y)\,. \tag{1.1}$$

(Going back to our metaphor: The average tropical butterfly has no influence on the number of tornados that a Texas ranch will weather during the lifetimes of seven generations of owners.) A measure μ satisfying (1.1) for a set of x of positive Lebesgue measure is in some sense the *natural* or *physical measure.* In Section 3 we shall discuss a condition of "absolute continuity along unstable directions" which implies that a measure is natural in this sense: the *SRB property,* after the names of Sinai, Ruelle, and Bowen, part of whose work will be sketched in this text. When several attractors Ω_i for a given f coexist, one may often study independently their physical measures μ_i.

Figure 1 A butterfly: Points on the Lorenz attractor

Many readers will have recognized the flavour of *ergodicity* in the characterization (1.1) of the physical measure (note however that the measure μ is often singular with respect to Lebesgue). In many cases, the convergence of the time average to the space average can be quantified by a *central limit theorem.* Often, one can show additionally that the system has exponential loss of memory (fast mixing) in the sense that the time *correlation functions* (see Sections 2 and 3) associated to the physical measure and smooth observables decay exponentially rapidly. Cases where the decay is subexponential (e.g. a power law) seem to appear naturally in physical problems, but have been the object of mathematical attention only more recently.

Many of the techniques which were used to obtain the results sketched in the previous paragraphs have their origin in theoretical physics, specifically *statistical mechanics.* Statistical mechanics is concerned with the description of the macroscopic thermodynamic properties (temperature, pressure, volume, energy, etc.) of physical or chemical systems composed of a *very large number* of identical subunits (such as atoms or molecules). Its aim is to explain, or deduce, macroscopic phenomena from assumptions at the microscopic level. Statistical mechanics is here used in the sense of *classical* statistical mechanics of *lattice systems* for *translation invariant* interactions.

Equilibrium statistical mechanics deals with systems already in equilibrium, and one of its key problems is that of phase transitions. An iron magnet can be described in first approximation as being composed of elementary magnetic moments, the spins, which are viewed as vectors in space placed on the vertices of a crystal lattice. Since iron is a ferromagnetic metal, the interactions between the spins tend to favor alignment of nearest neighbours, but this tendency is limited by thermal motion at high temperature. These competing forces are encoded in a Hamiltonian function. Starting from this simple mathematical model based on the microscopic features of the magnet, one task of equilibrium statistical mechanics is to deduce a theory describing in particular spontaneous magnetization, i.e., the fact that below a certain threshold temperature value the spins align and induce a macroscopic magnetic field even after the external field is turned off.

Figure 2 A magnet: Spin configurations on a one-dimensional lattice

The equilibrium state of the magnet for a given temperature can be described by one or several translation invariant (the system is homogeneous and very large) probability measures on the space of spin configurations which are called *Gibbs measures* or *equilibrium states.* Nonuniqueness corresponds to the existence of several directions of magnetization for the same material and experimental settings, as is the case if the spontaneous magnetization depends on the initial value of the external magnetic field (which has been turned off at the end of the experiment).

Translating from the language of statistical mechanics on one-dimensional lattices to that of dynamics mainly involves formally exchanging the *large volume* parameter of the former with the *large time* parameter of the latter. The analogy between the equilibrium states of the magnet and the invariant measures μ_i of the chaotic attractors mentioned above can be pursued into a very deep level of detail: The aim of this text is to explain how many of the mathematical tools and structures which underlie thermodynamic formalism (the formalism of statistical mechanics) have been applied to study the fine statistical properties of chaotic dynamical systems, concentrating mainly on the theory of SRB measures. We have nurtured no ambition to write a systematic historical account, but only to convey the "flavour" and the beauty of the subject to non-specialists.

The text is organized as follows. In Section 2 we give an elementary description of some basic ideas in thermodynamic formalism, starting from the notion of Gibbs ensemble on a finite set, moving then to the one-dimensional Ising model, and ending by results obtained in the sixties on infinite range, exponentially decaying, interactions, introducing the crucial notion of a transfer operator along the way. In Section 3 we explain how symbolic dynamics was used in the seventies to link a class of chaotic dynamical systems to the statistical mechanical models of Section 2, yielding in particular proofs of existence of SRB measures for uniformly hyperbolic attractors. We discuss in Section 4 some topics developed in the eighties and nineties, for which transfer operators play a key role: The construction of

SRB measures for systems enjoying only some (non uniform) hyperbolicity, the delicate question of the decay of correlations for continuous-time dynamics (i.e., flows), and the theory of functional determinants for transfer operators (including dynamical zeta functions). In the last section, we mention a few recently emerging topics: Application of Birkhoff cones techniques, infinite dimensional dynamics, slow decay of correlations, and nonequilibrium statistical mechanics.

We end this introduction with a quote from Ruelle's [1978] book: "the relation [between magnets and butterflies] is at an abstract mathematical level, and fairly inobvious at first sight. It is evident that the study of the physical world is a powerful source of inspiration for mathematics. That this inspiration can act in such a detailed manner is a more remarkable fact, which the reader will interpret according to his own philosophy."

2. Thermodynamic formalism for pedestrians

This section contains a brief description of a few ideas from thermodynamic formalism which are relevant to the mathematical study of dynamical systems. For detailed overviews, and more physical approaches, we refer e.g. to the monographs by Ruelle [1969, 1978], Israel [1978], Sinai [1982], Georgii [1988], and Simon [1993].

2.A Gibbs distributions.

In order to motivate the definition of a Gibbs ensemble and to prepare for the variational principle relating Gibbs states with equilibrium states, we start with an exquisitely simple mathematical lemma:

Variational principle on a finite set. *Let $n \ge 1$ be a fixed integer and consider n real numbers $a_1, \dots, a_n$. Then the maximum over all probability vectors $(p_1, \dots, p_n)$ (with $p_i \ge 0$ and $\sum_i p_i = 1$) of the expression*

$$-\sum_{i=1}^{n} p_i \log p_i + \sum_{i=1}^{n} p_i a_i \tag{2.1}$$

is equal to $\log \sum_{i=1}^{n} e^{a_i}$ and is attained if and only if

$$p_i = \frac{e^{a_i}}{\sum_{j=1}^{n} e^{a_j}}\,, \quad i = 1, \dots, n\,. \tag{2.2}$$

For the short proof (which follows from the strict convexity of the function $x \mapsto x \log x$ on $[0,1]$) we refer the reader e.g. to Walters [1982, Lemma 9.9]. The nonnegative number $-\sum_{i=1}^{n} p_i \log p_i$ is the *entropy* of the probability measure p_i on the finite set $S_n = \{1, \dots, n\}$.

Consider now a large finite set S_n of cardinality n (e.g. the possible spin configurations of a finite sets of atoms). Assume that the energy of each element of S_n, or configuration, at a fixed temperature T is given by the number $-\beta E_j$, with $\beta = 1/kT$, for $k > 0$ the Boltzmann constant, and where $\{E_j \mid 1 \le j \le n\}$ is a set of real numbers. The probability measure

$$p_i = \frac{e^{-\beta E_i}}{\sum_{j=1}^{n} e^{-\beta E_j}}\,, \quad i = 1, \ldots, n\,, \tag{2.3}$$

on S_n is called the *Gibbs distribution* (or Gibbs ensemble, or Gibbs state) associated to the Hamiltonian function $H(j) = -\beta E_j$, in the honour of the physicist G.W. Gibbs (1839-1903). The variational principle mentioned above shows that the Gibbs distribution maximizes entropy when the Hamiltonian vanishes, and otherwise (multiplying by -1 and changing the inequality sign) minimizes the difference (called free energy) between the average of the energy and the entropy. The link between this mathematical observation and a full understanding of why the Gibbs Ansatz (2.3) describes equilibrium of physical systems (for very large n) is to our knowledge not complete. (The traditional wisdom relates the Gibbs "canonical ensemble" to the "microcanonical ensemble," and uses the ergodic hypothesis, see Ruelle [1969].)

The object of thermodynamic formalism is the study of Gibbs measures generalizing (2.3), and equilibrium states maximizing a generalization of (2.1), under a *thermodynamic limit* $n \to \infty$, assuming that the set S_n has some extra structure. In the present text, since we have in view applications to dynamics, we shall restrict to the one-dimensional lattice $\mathbb{Z}$, i.e., situations where the possible configurations are a subset of $\mathcal{S}^{\mathbb{Z}}$, where the spin set $\mathcal{S}$ is a finite set (sometimes a countable or compact set).

2.B The one-dimensional Ising model.

Consider the simple following mathematical model, first studied by Ising in 1924, for an infinite one-dimensional ferromagnetic lattice. The set of spins is $\mathcal{S} = \{+1, -1\}$, and the configuration space is the set of two-sided sequences $\mathcal{S}^{\mathbb{Z}} = \{x_j \mid j \in \mathbb{Z}; x_j \in \{\pm 1\}\}$ (this set, endowed with the product topology of the discrete topology on $\mathcal{S}$, is a Cantor set). We assume that the contribution of site $j \in \mathbb{Z}$ to the total energy is the sum of two terms: An *interaction* with its right nearest neighbour (favorable to spin alignment), and the effect of an external magnetic field on the spin. (Note that the spin j is the right neighbour of its left neighbour.) The contribution of the spin at site 0 to the total energy of the lattice is thus encoded in the Hamiltonian function:

$$H : \mathcal{S}^{\mathbb{Z}} \to \mathbb{R}\,, \quad H(x) = J x_0 x_1 + h x_0\,, \tag{2.4}$$

for the two parameters $J > 0$ (a negative J would correspond to an antiferromagnetic lattice) and $h \in \mathbb{R}$. The contribution of the jth site is given by $H(\sigma^j(x))$, where $\sigma : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$ is the two-sided shift to the left: $(\sigma(x))_i = x_{i+1}$.

We wish to generalize the statements of Section 2.A, i.e., we seek a Borel probability measure on the compact space $\mathcal{S}^{\mathbb{Z}}$ which should be translation invariant (since the Hamiltonian is), and should satisfy a generalization of either (2.1) or (2.2–2.3). We shall now briefly describe four equivalent ways of attaining this goal, referring to Ruelle [1978] or Georgii [1988] for proofs (of much more general results) and references.

Thermodynamic limit

The most natural way of generalizing Section 2.A is to perform an infinite volume limit, using the special structure of the configuration space. Concretely, we may define for each fixed n a probability measure μ_n on $\mathcal{S}^{[-n,n]\cap\mathbb{Z}}$ by setting for any configuration $x^{(n)} = (x_{-n}, \ldots, x_n)$

$$\tag{2.5} \begin{aligned} \mu^{(n)}(x^{(n)}) &= \frac{e^{hx_n + \sum_{j=-n}^{n-1} H(\sigma^j(x))}}{\sum_{z\in\mathcal{S}^{[-n,n]\cap\mathbb{Z}}} e^{hz_n+\sum_{j=-n}^{n-1} H(\sigma^j(z))}} \\ &= \frac{e^{hx_n+\sum_{j=-n}^{n-1} Jx_jx_{j+1}+hx_j}}{\sum_{z\in\mathcal{S}^{[-n,n]\cap\mathbb{Z}}} e^{hz_n+\sum_{j=-n}^{n-1} Jz_jz_{j+1}+hz_j}}, \end{aligned}$$

and by defining a *Gibbs ensemble* $\mu = \mu_{h,J}$ for the Ising model to be any local weak accumulation point of $\mu^{(n)}$ as $n \to \infty$. There are other ways to perform the thermodynamic limit, in particular by introducing boundary terms, or considering more general subsets $\Lambda \subset \mathbb{Z}$ with $|\Lambda| \to \infty$. In the Ising model, all reasonable choices lead to the same result.

Conditional probabilities: The DLR equations

A probability measure μ on $\Omega = \mathcal{S}^{\mathbb{Z}}$ is called a *Gibbs state* for the Ising model if for every finite subset $\Lambda = [-n, n] \cap \mathbb{Z}$, and any fixed configuration $y^{\mathbb{Z}\setminus\Lambda} \in \mathcal{S}^{\mathbb{Z}\setminus\Lambda}$ the conditional probability that $x^\Lambda = w^\Lambda$, knowing that $x^{\mathbb{Z}\setminus\Lambda} = y^{\mathbb{Z}\setminus\Lambda}$, is given by

$$\tag{2.6} \frac{e^{Jy_{-n-1}w_{-n}+Jw_ny_{n+1}+hw_n+\sum_{j=-n}^{n-1} H(\sigma^j w)}}{\sum_{z\in\mathcal{S}^\Lambda} e^{Jy_{-n-1}z_{-n}+Jz_ny_{n+1}+hz_n+\sum_{j=-n}^{n-1} H(\sigma^j z)}}.$$

(Note the presence of the term $Jy_{-n-1}w_{-n} + Jw_ny_{n+1}$, taking into account the interaction between Λ and $\mathbb{Z} \setminus \Lambda$, which in this special nearest neighbour case only occurs on the boundary of Λ.)

Using the definition of conditional probabilities, (2.6) can be rewritten as a family of equations for μ, the so-called DLR equations. The terminology comes from Dobrushin [1968a, 1968b] who introduced Gibbs states (in a more general setting), and Lanford-Ruelle [1969] who independently discovered this characterization.

Using the trivial fact that the contribution $H(\sigma^k(x))$ of a single site k to the total energy is uniformly bounded (by $|h| + J$), it is not so difficult to show that any local accumulation point of (2.5) satisfies the DLR condition (2.6). Therefore, for the Ising model, a Gibbs ensemble obtained from performing a thermodynamic limit is also a Gibbs state.

Transfer matrix

Let us restrict to the one-sided lattice $\Omega_+ = \mathcal{S}^{\mathbb{Z}_+}$ (this reduction is trivial in the Ising model, but can be very tricky in more general settings, see also Section 4.C). Consider the one-sided shift to the left $\sigma_+ : \Omega_+ \to \Omega_+$, and define a *transfer* operator $\mathcal{L}$ acting linearly on bounded or continuous functions $\varphi : \Omega_+ \to \mathbb{C}$ by

$$\mathcal{L}\varphi(x) = \sum_{\substack{y\in\Omega_+ \\ \sigma_+(y)=x}} \varphi(y)e^{H(y)} = \sum_{y_0\in\{\pm 1\}} \varphi(y_0 \wedge x)e^{hy_0+Jy_0x_0}\,, \tag{2.7}$$

where $y_0 \wedge x = (y_0x_0x_1x_2\ldots)$. When restricted to the two-dimensional vector space of functions depending only on the value of the spin x_0, the operator $\mathcal{L}$ in the canonical basis is represented by the 2×2 transfer matrix

$$\begin{pmatrix} \exp(h+J) & \exp(+h-J) \\ \exp(-h-J) & \exp(-h+J) \end{pmatrix}\,. \tag{2.8}$$

The Perron-Frobenius theorem (see e.g. Walters [1982]) guarantees that this positive matrix has a simple eigenvalue $\lambda > 0$ equal to its spectral radius, and strictly positive corresponding left and right eigenfunctions (they can be computed explicitly in this case). The other eigenvalue has absolute value strictly smaller than λ. The left eigenfunction of λ can be viewed as an eigenfunctional ν of the dual $\mathcal{L}^*$ of $\mathcal{L}$. It is in fact a finite positive measure on Ω_+, and it is not difficult to check that it satisfies the (one-sided version of the) DLR condition. Furthermore, noting ψ for the right eigenvector of $\mathcal{L}$, one checks that the (normalized) positive measure $d\mu_+ = \psi\, d\nu$, when extended to the two-sided space Ω by the standard natural extension construction, is a (shift invariant) probability measure which satisfies the (two-sided) DLR condition (2.6). One can also show that *any* Gibbs state can be expressed as a combination of a left and right eigenvector of $\mathcal{L}$. This yields *uniqueness* of the Gibbs state $\mu_{h,J}$ (and thus of the Gibbs ensemble) for the one-dimensional Ising model. Ruelle [1968] used an analogous transfer operator (see below) to show uniqueness in a more general setting.

Variational principle

By using arguments essentially based on the variational principle on a finite set from Section 2.A, one shows that the Gibbs state of the Ising model is the unique shift invariant probability measure on $\mathcal{S}^{\mathbb{Z}}$ which attains the maximum

$$P(H) = \sup\{h_\rho(\sigma) + \int_\Omega H(x)\, d\rho(x) \mid \rho \text{ a } \sigma\text{-invariant probability measure on } \mathcal{S}^{\mathbb{Z}}\}\,, \tag{2.9}$$

where $h_\rho(\sigma)$ denotes the Kolmogorov (or metric) entropy of the σ-invariant measure ρ (a generalization of the entropy $-\sum_i pi \log p_i$ of a finite probability vector, which measures the rate of creation of information under the shift, see e.g. Walters [1982]). A measure realizing the maximum (2.9) is called an *equilibrium state* for the Hamiltonian or weight function $H = H_{h,J}$. Thus, for the one-dimensional Ising model there is a unique Gibbs state, which coincides with the unique equilibrium state (and is an ergodic measure for the shift). One proves also that the maximum $P(H)$, which is called the *topological pressure,* or the free energy, is equal to the logarithm $\log(\lambda)$ of the spectral radius of $\mathcal{L}$. Since λ is a simple isolated eigenvalue, perturbation theory gives that the pressure $P(H)$ is an analytic function of the parameters h and J (this spectral approach to prove analyticity was used by Ruelle in the setting of Section 2.C, based on an idea of Araki [1969]). The variational principle was first obtained in a statistical mechanical lattice setting (see also Section 2.C) by Ruelle [1967] and then extended by Walters [1976] to a more general dynamical situation.

2.C Infinite range interactions for one-dimensional lattices.

To obtain a more realistic model on the configuration space $\mathcal{S}^{\mathbb{Z}}$ ($\mathcal{S}$ a finite set), an obvious generalization consists in admitting translation invariant interactions $\phi_{k-j}(x_j, x_k)$ between spins which are arbitrary far apart, assuming some decay property with respect to the distance, or at least a summability property. (The contribution $\sum_{j\in\mathbb{Z}_+} |\phi_j(x_0, x_j)|$ of the spin at 0 to the total energy should indeed be finite.)

The strongest possible assumption is that of *exponential decay:* $|\phi_j| \le \theta^j$ for some $0 < \theta < 1$ and all large j. It turns out that for weight functions H constructed from exponentially decaying couplings, most results valid for the Ising model still hold (Ruelle [1967, 1968]), in particular the Gibbs ensemble obtained from a thermodynamic limit is the Gibbs state satisfying DLR equations. There is one slight difference: The transfer operator $\mathcal{L}$ defined by the first equality in (2.7) does not preserve a finite dimensional subspace in general, and thus cannot be represented by a finite matrix. However, the exponential decay of interactions implies that H is Lipschitz, and therefore that $\mathcal{L}$ preserves the Banach space of Lipschitz functions on Ω_+ (for the metric $d_\theta(x, y) = \sum_{k\ge 0} \theta^k d(x_k, y_k)$ on Ω_+ associated to a metric on $\mathcal{S}$). Although this operator is not compact, it has the following quasi-compactness property: There is a simple eigenvalue equal to the spectral radius λ, and the rest of the spectrum is contained in a disc of strictly smaller radius $\tau\lambda$ ($\tau < 1$). This result is often referred to as the *Ruelle-Perron-Frobenius theorem,* one also says that the operator has a *spectral gap.* The Gibbs state μ can be obtained again from the maximal eigenfunctions ψ, ν of $\mathcal{L}$ and its dual, and it is the unique measure realizing the maximum (2.9).

For φ_1, φ_2 two complex-valued test functions in $L^1(\Omega_+, d\mu_+)$, we may define the *correlation function* $C_{\varphi_1,\varphi_2} : \mathbb{Z}_+ \to \mathbb{C}$ of the one-sided Gibbs state μ_+ by

$$C_{\varphi_1,\varphi_2}(k) = \int (\varphi_1 \circ \sigma_+^k)\, \varphi_2 \, d\mu_+ - \int \varphi_1 \, d\mu_+ \int \varphi_2 \, d\mu_+ \,. \tag{2.9}$$

Since $\mu_+ = \psi \, d\nu$, one easily obtains that

$$C_{\varphi_1,\varphi_2}(k) = \int \varphi_1 [\mathcal{L}^k(\varphi_2\psi) - \psi \int \varphi_2 \psi \, d\nu] \, d\nu \,. \tag{2.10}$$

Because of the spectral gap, it is not difficult to see that whenever φ_2 is Lipschitz, the correlation function decays exponentially with a rate τ. (In statistical mechanics, this is called the *exponential cluster* property.)

The pressure is again the logarithm of the simple maximal eigenvalue, and thus depends analytically on smooth weight functions. Note that (also in more general situations) one may relate the equilibrium states to "tangents to the graph" of the pressure, so that phase transitions (nonuniqueness of equilibrium states) correspond to discontinuities in the derivative.

We have seen that for one-dimensional lattices, the case of exponentially decaying interactions is relatively simple (physicists refer to it as "short range interactions"), in particular there is no phase transition and we even have exponentially decaying correlations. More interesting phenomena appear when the decay is slower (there may be more than one Gibbs state, as was shown by Dyson – see also the example in Ruelle [1978, Chapter 5, exercise 8] inspired from Fisher [1967] – and the correlations may decay only like power laws). In the dynamical model, this corresponds either to weakening the smoothness assumption of the dynamics, or (which is perhaps more relevant) to weakening the hyperbolicity assumption, allowing e.g. periodic points with zero Lyapunov exponents (see Section 5).

3. SRB measures for uniformly hyperbolic dynamics

We now turn to discrete-time dynamical systems, i.e., iterations of maps $f : M \to M$, and we sketch the first proofs, obtained in the seventies, of the existence of physical measures for uniformly hyperbolic attractors. Useful references for this section include the monographs of Bowen [1975], Ruelle [1978], Walters [1982], Shub [1987], Parry-Pollicott [1990], and Katok-Hasselblatt [1995]. A review by Eckmann-Ruelle [1985] contains a stimulating account of the results available at the time, with many references into the physical literature.

3.A Markov partitions for expanding maps.

We start with one of the simplest examples, to introduce symbolic dynamics in an easy setting. (We stress that the techniques described here can mostly be bypassed for expanding maps: we refer in particular to Krzyżewski-Szlenk [1969] and Shub [1969] for original results.) Let $f : S^1 \to S^1$ be a C^2 transformation of the circle which is *expanding,* i.e., there are $\gamma > 1$ and $\epsilon > 0$ such that any pair of points x, y in the circle with $d(x, y) < \epsilon$ satisfies $d(f(x), f(y)) > \gamma\, d(x, y)$. The prototype of such a map is the angle doubling map $z \mapsto z^2$. Sensitive dependence on initial conditions here follows from the definition: The distance between two arbitrarily close initial points (closer than $\epsilon\gamma^{-n}$) is multiplied by γ^n after n steps.

Because of the expanding condition, it is possible to construct a finite partition of the circle into d arcs I_j with the property that whenever $f(I_j)$ intersects the interior of I_k then $I_k \subset \overline{f(I_j)}$. A partition with this property is called a *Markov* partition. The Markov property implies that the dynamics of f can be coded by a

$d \times d$ transition matrix:

$$A_{jk} = \begin{cases} 1 & \text{if } I_k \subset \overline{f(I_j)} \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

in the following sense. Consider the restriction of the one-sided shift σ_+ to the space of sequences $\mathcal{S}_A \subset \mathcal{S}^{\mathbb{Z}_+}$, where $\mathcal{S} = \{1, \dots, d\}$, and the elements x of $\mathcal{S}_A$ are characterized by the admissibility condition $A_{x_i x_{i+1}} = 1$ for all $i \geq 0$. This is a *subshift of finite type.* One can prove that the matrix A is irreducible and aperiodic (actually, for circle maps one can arrange that $A_{ij} = 1$ for all i, j, the reader may check this by drawing a picture). Define $\pi : \mathcal{S}_A \to S^1$ by setting $\pi(x) = \cap_{k \geq 0} f^{-k} I_{x_k}$. One verifies that π is a surjection and that the following diagram commutes

$$\begin{array}{ccc} \mathcal{S}_A & \xrightarrow{\sigma_+} & \mathcal{S}_A \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ S^1 & \xrightarrow{f} & S^1 . \end{array} \tag{3.2}$$

The projection π is not injective but the set of points in the circle with more than one preimage is countable. Moreover, if we endow $\mathcal{S}_A$ with a metric d_θ as in Section 2.C, for $\theta > 1/\gamma$, then the map π is Lipschitz.

To summarize, we have replaced our dynamical system by a subshift of finite type, the simplest instance of *symbolic dynamics.* Recall now that our aim was to find a good invariant probability measure for f. If we find a measure μ which is f-invariant, ergodic, and has a strictly positive *density* φ_0 with respect to Lebesgue measure on the circle, i.e., $d\mu = \varphi_0(x)dx$ (in other words, the measure μ is equivalent with Lebesgue measure), then Birkhoff's ergodic theorem (see e.g. Walters [1982]) will imply the "physical" condition (1.1) (for Lebesgue almost all $x \in S^1$). For an *expanding map,* we will thus say that a measure is *SRB* if it is *absolutely continuous* with respect to Lebesgue.

Consider now a transfer operator, acting on continuous functions $\varphi : S^1 \to \mathbb{C}$ by

$$\mathcal{L}\varphi(x) = \sum_{f(y)=x} \frac{\varphi(y)}{|f'(x)|} . \tag{3.3}$$

Since $|f'(x)|$ is the Jacobian of f with respect to Lebesgue mass, the change of variable formula in an integral implies that Lebesgue measure is a fixed point of the dual operator $\mathcal{L}^*$. We can lift the weight function $1/|f'|$ via π to the symbolic space, obtaining a positive Lipschitz function $e^{H(x)}$ on $\mathcal{S}_A$. We may then apply a version of the Ruelle-Perron-Frobenius theorem in Section 2.C to the transfer operator associated with $e^{H(x)}$ and the subshift of finite type on $\mathcal{S}_A$. Carrying down to S^1 the results from Section 2.C, we find a positive Lipschitz fixed point φ_0 for $\mathcal{L}$, and get that the unique SRB measure $\varphi_0\,dx$, obtained by combining the right and left eigenvalues of $\mathcal{L}$ is an equilibrium state for the function $-\log|f'(x)|$, and has correlation functions which decay exponentially fast for Lipschitz test functions. We have mentioned the transfer operator and variational principle viewpoints: The SRB measure for expanding circle maps can also be obtained from a "thermodynamic limit" by taking a limit point of f^{*n}(Lebesgue) as $n \to \infty$, and may be characterized by conditional probabilities analogous to the DLR equations (see e.g. Sinai [1994, Lecture 12]).

The symbolic dynamics technique just sketched works for expanding maps on compact manifolds in any dimension. (In fact, the C^2 condition can be weakened to $C^{1+\epsilon}$ for any $\epsilon > 0$, but not to C^1.) One may consider Gibbs states for other (Hölder) weight functions than $-\log|\det Df|$ (for other weights, property (1.1) does not hold). The technique is also useful to study repellors, for example maps $f_a(x) = a - x^2$ with $a > 2$, which have a Cantor invariant Ω_a set in the interval $[-2, 2]$: The Hausdorff dimension $0 < \beta < 1$ of Ω_a is defined by the condition $P(-\beta \log|f'|) = 0$, and can thus be shown to depend analytically on a (Ruelle [1982]).

3.B Markov partitions for Anosov diffeomorphisms.

We now extend the symbolic dynamics techniques to hyperbolic differentiable maps for which the uniform expansion discussed in the previous paragraph coexists with uniform contraction. The simplest possible case is a map f on the two-torus defined by a unimodular matrix in $GL(2,\mathbb{Z})$, for example Anosov's famous cat map associated to the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This matrix is *hyperbolic,* meaning that both its eigenvalues have modulus different from unity, therefore both are real with $0 < |\lambda_s| < 1 < |\lambda_u|$. There is thus an expanding eigendirection and a contracting eigendirection. When folded back from $\mathbb{R}^2$ to the two-torus, the two directions give the so-called stable and unstable manifolds (see below) of the fixed point at the origin, these two one-dimensional manifolds are densely immersed in the torus.

Sensitive dependence on the initial conditions holds (and is clearly visible in Figure 3). If instead of folding back to the torus the whole stable and unstable directions in $\mathbb{R}^2$, one stops when they meet as indicated on Figure 3, a partition of the torus into finitely many rectangles R_j is obtained. (Such partitions for toral automorphisms were studied by Adler and Weiss [1970] in the sixties.) This partition satisfies a hyperbolic version of the Markov property mentioned in Section 3.A: Whenever $f(R_j)$ meets the interior of R_k, it cuts across R_k completely in the unstable direction, and whenever $f^{-1}(R_k)$ meets the interior of R_j it cuts across R_j along the stable direction. Using this property, and defining a transition matrix A according to the intersections $f(R_j)$ and R_k as in (3.1), one constructs a semiconjugacy π between f and a subshift σ_+ on $\mathcal{S}_A$ as in (3.2).

Figure 3 Anosov's cat map, and a Markov partition for the map

In the cat map case, finding the physical measure is easy: Lebesgue measure itself is preserved since the map is linear, and proving ergodicity and even exponential decay of correlations for smooth observables can be done by hand. For *nonlinear* uniformly hyperbolic maps, for example small C^2 perturbations of the torus cat map, the situation changes drastically as we explain now. A splitting of the tangent space into stable and unstable directions still exists, in the sense that the tangent space $T_x M$ over each point in the torus decomposes as $E^u_x \oplus E^s_x$, where the unstable and stable bundles E^u_x, E^s_x are preserved and uniformly exponentially contracted by $T_x f^{-n}$ respectively $T_x f^n$: There is a constant $\gamma > 1$ so that for all $n \geq 0$

$$(3.4) \qquad \|T_x f^n v\| \leq \gamma^{-n} \|v\| \, , \forall v \in E^s_x \, , \quad \|T_x f^{-n} w\| \leq \gamma^{-n} \|w\| \, , \forall w \in E^u_x \, .$$

At each point x there is an unstable manifold W^u_x (those points in the torus which converge to x under backward iteration) and stable manifold W^s_x (defined analogously), tangent to the respective bundles. Note that even if f is analytic, the decomposition $TM = E^u \oplus E^s$ is in general nondifferentiable and only satisfies a Hölder condition. By definition an Anosov diffeomorphism, (a notion due to Anosov [1967]) is a diffeomorphism $f : M \to M$ on a compact manifold such that (3.4) holds for all x. Important results on foliations by stable and unstable manifolds, in particular for Anosov maps, were obtained by Hirsch-Pugh-Shub [1977]. Sinai [1968] constructed Markov partitions into finitely many "rectangles" for general Anosov diffeomorphisms $f : M \to M$ on a compact manifold, so that symbolic dynamics can be implemented. Using Smale's [1967] *spectral decomposition,* one can reduce to the case where the corresponding matrix A is irreducible, and even aperiodic (equivalently, where f is topologically transitive or even topologically mixing, see e.g. Walters [1982] for definitions).

Figure 4 SRB measures and absolute continuity along unstable foliations

However, Sinai [1972] proved that for a generic Anosov diffeomorphism there is *no* f-invariant measure absolutely continuous with respect to Lebesgue. This does not mean that we must give up our hope to construct a physical measure in the sense of (1.1): In fact, such a measure need only be ergodic and *absolutely continuous* with respect to Riemannian volume along the *unstable directions,* as we explain now (in Section 3.C we shall see how to find a measure with this property). If μ is ergodic and has absolutely continuous induced measures $\mu_{W^u_x}$ on unstable manifolds (Rokhlin's theory of measurable partitions is essential here, we do not go into details), then there exists by ergodicity a piece of unstable manifold $E \subset W^u_x$ with positive W^u_x-Lebesgue mass (and $\mu_{W^u_x}(E) > 0$), and the property that $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k y}$ (with δ_z the Dirac mass at $z \in M$) converges weakly to μ as $n \to \infty$, for *all* $y \in E$. By definition of stable manifolds this weak convergence holds on the union F of the manifolds W^s_y through points $y \in E$ (the set F has positive μ-mass by definition of induced measures, see also Figure 4). We now use an important property of the *stable* foliation of an Anosov diffeomorphism: It is *absolutely continuous.* Roughly said, this means that the projection from one piece of unstable manifold to another one along the stable foliation has a Jacobian (with respect to Lebesgue measure) which is bounded below in a controlled way. But this is precisely what we need to see that the *Lebesgue* mass of F is strictly positive, as desired (to get (1.1) for a set of positive Lebesgue mass). By definition, an invariant probability measure is called *SRB* if it is *absolutely continuous along unstable foliations* (this is consistent with the definition in Section 3.A).

3.C Construction of SRB measures for Axiom A attractors.

Generalizing Anosov maps, Smale [1967] introduced the following notion of *Axiom A* diffeomorphisms. Consider a C^1 diffeomorphism $f : M \to M$ of a compact manifold which admits a hyperbolic splitting of the tangent space $T_x M = E^u_x \oplus E^s_x$ over each point x of the compact, invariant nonwandering set Ω (a point is wandering if it admits a neighbourhood V with $f^n V \cap V = \emptyset$ for all $n \geq 1$), and assume also that periodic points (i.e., points $x \in M$ with $f^n(x) = x$ for some $n \geq 1$) are dense in Ω. (The bundles E^s and E^u should be uniformly exponentially contracted by Df, respectively Df^{-1}, as in (3.4).) The stable and unstable manifold theory, including the absolute continuity of stable foliations, also works in the Axiom A case (Hirsch-Pugh-Shub [1977]). Smale's spectral decomposition still holds and allows us to reduce to the topologically transitive or mixing case $f : \Omega_i \to \Omega_i$.

Let f be a transitive C^2 Anosov diffeomorphism, or a C^2 Axiom A map on an attracting transitive invariant set Ω_i (i.e. the basin of attraction of Ω_i contains a neighbourhood of Ω_i). Sinai, Ruelle, and Bowen were inspired by the various equivalent characterizations of Gibbs states (see Section 2.B) to construct SRB measures for f. To reduce to the lattice setting of Section 2.C, they applied Markov partitions and symbolic dynamics, and used in some way the fact that the *unstable Jacobian* of f, i.e., the Jacobian of the restriction of the derivative Df along the unstable bundle E^u, although generically not as smooth as f, satisfies a Hölder condition (Hirsch-Pugh-Shub [1977]). We now explain this in more detail.

Figure 5 Two Axiom A diffeomorphisms: a horseshoe and the solenoid attractor

One may construct μ by performing a *thermodynamic limit,* i.e., by letting the iterated dynamics act on Lebesgue measure, via a family of expanding maps associated to f by projecting along stable manifolds. This was the approach of Sinai [1972] (for Anosov maps), who showed the convergence of this procedure and that it produced a (Gibbs) measure with appropriate *conditional probabilities* satisfying constraints equivalent to the DLR equations (2.6). Then he proved that the DLR equations implied the desired absolute continuity property along unstable manifolds (see the end of Section 3.B). Uniqueness follows from topologically transitivity of f.

Ruelle [1976a] generalized Sinai's result to Axiom A attractors, using the Markov partitions for Axiom A systems constructed by Bowen [1970, 1975]. He used a *transfer operator* associated to an expanding map corresponding to f, in a more explicit way. His main point was that the dual of this transfer operator preserved Lebesgue mass on pieces of unstable manifolds (similarly as in Section 3.A), so that the f-invariant measure constructed with its maximal eigenfunctions automatically had the required absolute continuity property. In the same paper, Ruelle applied transfer operators to show exponential decay of correlations for Hölder functions when f is topologically mixing on Ω_i. He also proved, using the symbolic model, that the measure so constructed was an equilibrium state for the *variational principle*

(3.5)
$$P(-\log|\det Df|_{E^u}|) = \sup\{h_\rho(f) - \int \log|\det Df|_{E^u}(x)|\,d\rho(x) \mid \rho \ f\text{-invariant probability on } \Omega\}\,.$$

Note that $P(-\log|\det Df|_{E^u}|) = 0$ in the case of an Axiom A attractor.

The methods sketched also allow to study Gibbs states for Axiom A diffeomorphisms and arbitrary Hölder real weights (in particular the zero weight, which gives rise to the measure of maximal entropy), getting central limit theorems, and exponential decay of correlations in the mixing case.

Finally, the approach in Bowen-Ruelle [1975] and Bowen [1975] also works for Axiom A *flows* on attractors and does not use directly the fact that the measure is absolutely continuous along unstable foliations. The point of view is to apply symbolic dynamics to construct an ergodic equilibrium state for the lift of (3.5) on $\mathcal{S}_A$, and then recall that such an equilibrium state is also a Gibbs state. Bowen has yet another equivalent characterization of Gibbs states: Writing H for the lift of $-\log|\det Df|_{E^u}|$ on $\mathcal{S}_A$, he shows that μ is a Gibbs state if and only if there is a constant $C > 0$ so that the μ-measure of any n-cyclinder satisfies

$$\frac{\mu(\{x \in \mathcal{S}_A \mid x_i = X_i\,, i = 0, \ldots, n\})}{\exp(-nP(H) + \sum_{k=0}^{n-1} H(\sigma_+^k(X)))} \in [C, 1/C]\,. \tag{3.6}$$

Property (1.1) for a positive Lebesgue measure set of initial points is then obtained from Birkhoff's ergodic theorem, by combining the Bowen characterization (3.6) with an important *volume lemma,* which is also true for non-attracting Axiom A diffeomorphisms, and is useful to study escape rates. Fix some small number ϵ, choose a point $x \in \Omega_i$ and define the time-n dynamical ball $B_n^\epsilon(x)$ of radius ϵ at x to be the set of points y in M such that $d(f^k(x), f^k(y)) < \epsilon$ for all times $0 \le k < n$. The volume lemma says that for the Lebesgue mass of $B_n^\epsilon(x)$ is (up to a constant factor in $[C, 1/C]$, as in (3.6)) the same as $|\det Df^n|_{E^u}(x)|$ for all x and n. This result agrees with our intuition since we only have to remove points which leave along the unstable manifolds, when refining the dynamical balls.

4. More applications of transfer operators to dynamics

The ergodic theory of dynamical systems has flourished in many directions in the eighties and nineties, we have selected three topics which all involve transfer operators.

4.A Obtaining SRB measures with less hyperbolicity.

The strong uniform expansion or hyperbolicity assumptions that we have discussed seem unrealistic in view of dynamical systems originating from the physical world (e.g. the Lorenz model, or billiards such as the Lorentz gas with infinite horizon, see e.g. Cvitanović [1996] for examples and references). We now discuss some possible weakenings of these assumptions.

Lyapunov exponents and the Pesin equality

Oseledec's [1968] multiplicative ergodic theorem asserts that any diffeomorphism $f : M \to M$ on a finite dimensional manifold admits growth exponents (the *Lyapunov exponents*) at most points, i.e., for any f-invariant probability measure μ and μ-almost all $x \in M$, there is a splitting $T_x M = \oplus_{i=1}^{d(x)} E_i$ and real numbers $\lambda^1(x) < \lambda^2(x) < \cdots < \lambda^{d(x)}(x)$, such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n v\| = \lambda^i(x) \tag{4.1}$$

for all $0 \neq v \in E_i$. If μ is ergodic, its Lyapunov exponents are almost everywhere constant. Clearly, strong hyperbolicity is related to the absence of zero exponents, and the presence of at least one nonvanishing exponent is an indication of chaotic behaviour. In the second half of the seventies, Pesin [1976] started to develop a beautiful theory around Oseledec's theorem, constructing in particular stable and unstable manifolds associated to nonzero Lyapunov exponents. The famous Ruelle inequality relates the entropy and the Lyapunov exponents of any f-invariant probability measure ρ by:

$$h_\rho(f) - \int \sum_{\lambda^i(x)>0} \lambda^i(x)\, d\rho(x) \le 0\,. \tag{4.2}$$

(Using the unstable manifolds, (4.2) can be rewritten as $P(-\log|\det Df|_{E^u}|) \le 0$, where the pressure is defined by the variational principle (3.5).) A striking theorem of Ledrappier-Young [1985] shows that for C^2 diffeomorphisms equality holds in (4.2) if and only if the measure ρ is absolutely continuous along the unstable manifolds. A result of absolute continuity of stable foliations due to Pugh-Shub [1989] then yields that for an ergodic measure with no zero Lyapunov exponents, the absolute continuity along unstable foliations implies property (1.1) for a positive Lebesgue measure set of initial conditions (see the discussion at the end of Section 3.B). Of course, these characterizations of SRB measures do not solve the problem of the existence (or uniqueness) of the SRB measure for a general attracting diffeomorphism without vanishing exponents.

Logistic and Hénon families

We discuss now a class of non uniformly hyperbolic systems for which the SRB measure has been shown to exist and to possess good statistical properties. They consist in low-dimensional maps which were explored by Jakobson and Collet-Eckmann, and further studied by Benedicks-Carleson and Benedicks-Young. An axiomatization of this theory (applicable in principle in any finite dimension) has recently been obtained by Young.

Figure 6 The Ulam logistic map ($a = 2$) and a Hénon attractor ($a = 1.4$, $b = 0.3$)

The so-called *logistic* one-parameter family of interval maps

$$f_a : x \mapsto a - x^2 , \quad a \in [0, 2] ,$$

(in which the famous Feigenbaum period-doubling was first exhibited, and renormalization theory was first developed, see de Melo-Van Strien [1993] for references) clearly fails to satisfy any expansion condition at the critical point 0. However, for the Ulam parameter $f_2(x) = 2 - x^2$ one can construct "by hand" an absolutely continuous invariant probability measure (with an unbounded density, however) and study its ergodic properties. An important fact for $a = 2$ is that the critical orbit $0 \mapsto 2 \mapsto -2$ stabilizes on a expanding fixed point $f_2(-2) = -2$ with $|f_2'(-2)| > 1$. It turns out that for a positive measure set of values of parameter a, the so-called Collet-Eckmann condition that the postcritical orbit has a positive Lyapunov exponent is satisfied, and the map admits an SRB measure. This important theorem was first proved by Jakobson [1981] and then extended by Benedicks and Carleson (for a more general class of unimodal interval maps). Benedicks-Carleson [1991] later obtained fundamental results on a two-dimensional version of the logistic family, the *Hénon family,* which is a two-parameter family of non uniformly hyperbolic diffeomorphisms of the plane $H_{a,b}(x, y) = (1 - ax^2 + y, bx)$ ($0 < a < 2$, $b > 0$). Very roughly, the one-dimensional critical point unfolds into homoclinic tangencies between stable and unstable manifolds. The Hénon map was shown to admit an SRB measure for a positive measure set of parameters a, b by Benedicks and Young [1993].

Regarding rates of mixing for the SRB measure, Young [1992] and Nowicki-Keller [1992] independently showed that good unimodal maps (for Collet-Eckmann or Benedicks-Carleson parameters) enjoy exponential decay of correlations for test functions of bounded variation (there is also a central limit theorem). This result was recently obtained for a positive measure set of parameters of the Hénon map by Benedicks-Young [1996], using an abstract scheme of Young [1996] which is based on constructing a countable Markov extension (or tower) over the dynamical system. Young's approach works for many systems (including scattering billiards) which are hyperbolic, except at some points, where there exist exponentially controlled recovery times. It seems that it is best suited to the study of statistical properties of SRB measures, and that new ideas would be needed to understand other Gibbs states.

Countable Markov extension of dynamical systems had been implemented already in the eighties by Hofbauer [1986], for general piecewise monotone interval maps. Hofbauer and Keller [1982] used Hofbauer's tower to construct a quasi-compact transfer operator for such maps and obtain exponential decay of correlations for Gibbs measures corresponding to weights of bounded variation (this of course does not include $-\log|f'|$ for the logistic family). It is in fact this strategy which was later extended by Keller and Nowicki [1992] to Collet-Eckmann unimodal maps.

See Jakobson-Newhouse [1996] for another approach to the problem of existence of SRB measures for nonuniformy hyperbolic maps. We refer to the book of Viana [1997] for references (in particular by Alves [1997], and Bonatti and Viana [1997]) about recent existence and uniqueness results of SRB measures.

4.B Decay of correlations for continuous-time dynamics.

Up to this point, we have discussed iterated maps, i.e., discrete-time dynamics. The case of continuous-time dynamics (flows) is perhaps physically more natural since it arises when solving differential equations (such as the Lorenz equations in $\mathbb{R}^3$). A flow on a manifold M is a one-parameter family of invertible maps $\Phi^s : M \to M$, with $\Phi^{s+t}(x) = \Phi^s(\Phi^t(x))$. One defines invariant measures, correlations, etc. similarly as in the discrete-time case. Often, the study of flows may be reduced to the study of maps by performing a so-called *Poincaré section:* A family S of codimension one sets (with boundary) transverse to the flow direction is selected so that each orbit crosses at least one set. The Poincaré (or first return) map $f : S \to S$ is then defined by $f(x) = \Phi^r(x)$, where $r = r(x)$ is the smallest strictly positive time such that $\Phi^r(x) \in S$. The converse construction is called a *suspension,* and produces a flow from an invertible map $f : S \to S$ and a return time $r : S \to \mathbb{R}^+$. The function r is an essential ingredient of the construction: Suspending a mixing diffeomorphism under a *constant* return time, gives a "periodic" flow which is not mixing at all. By using good sections (Ratner [1969] was the first to construct Markov sections for flows) SRB measures and other Gibbs states were obtained and studied for Axiom A flows (Bowen-Ruelle [1975]). Note that the Lorenz flow mentioned in the introduction does not fit in the uniformly hyperbolic setting, and other methods are needed to study it (as shown by Luzzatto and Viana).

We explained in Section 3 how exponential speed of mixing for smooth observables and Gibbs measures was obtained for uniformly hyperbolic discrete-time dynamics using symbolic dynamics. In good cases, the exact rate of mixing can be interpreted as the "second eigenvalue" of a transfer operator and (see Section 4.C) as the "second pole" of a dynamical zeta function. For continuous-time dynamics however, the situation is quite different. The definition of Anosov, or Axiom A, flows requires the existence of an invariant decomposition into three bundles: $T_x M = E^u_x \oplus E^s_x \oplus E^0_x$, where E^0_x is one-dimensional in the direction of the flow. There is of course no expansion or contraction along the flow direction, which corresponds to a zero Lyapunov exponent, a heuristic indication of why rates of mixing for flows are more difficult to obtain. This difficulty is not only apparent: In 1983 Ruelle and, independently, Pollicott exhibited examples of Axiom A flows (*not* on attractors) with correlation functions (for Gibbs measures) decaying slower than exponentially. These counter-examples were obtained by suspending a subshift of finite type under a return time taking two rationally independent (but sufficiently well approximable by rationals) values α and β. The slow mixing is due to the quantifiable fact that linear combinations of α and β can be arbitrarily close to one.

Geodesic flows on (the unit tangent bundle of) compact surfaces (or higher dimensional manifolds) of strictly negative curvature were among the motivating examples for Anosov flows. In this case Liouville measure is the SRB measure. When the curvature is *constant,* it had been proved by using the underlying algebraic structure, that correlation functions for smooth observables decay exponentially (Ratner [1987], Moore [1987]). This algebraic approach fails when the curvature is strictly negative, but non-constant, and a more dynamical line of attack seems necessary. A technical difficulty is that the method used to construct the SRB measure gives rise not to a single transfer operator, the spectrum of which encodes the ergodic properties of the flow, but to a one-parameter family of transfer operators $\mathcal{L}_s$, weighted by $\exp(-sr(x))$, where $r(x)$ is a return time and $s \in \mathbb{C}$. It turns out that to get exponential decay of correlations, Paley-Wiener type theorems require the existence of an appropriate half-plane where there are no values of s such that $\mathcal{L}_s$ has the real number 1 in its spectrum, a rather difficult property to check or exclude.

The question whether topologically mixing Anosov flows (or Axiom A flows on attractors) were always exponentially mixing for smooth observables remained basically open for more than a decade after the discovery of the counter-examples mentioned above. Very recently, two important breakthroughs were obtained by different methods. First Chernov [1995] proved for a class of three-dimensional Anosov flows (with foliations satisfying a uniform nonintegrability condition) that correlation functions $C_{\psi_1,\psi_2}(t)$ of smooth observables decay at least as fast as $e^{-\sqrt{t}}$. He used Markov approximations based on a time-discretization of the suspended flow constructed from a Markov section. The method has then been conceptualized and generalized to higher-dimensional Anosov flows admitting Lipschitz stable and unstable foliations by Liverani [1996], who observed that one of the key ideas behind Chernov's approach was to introduce a suitably chosen small random perturbation (the size of which is ultimately allowed to go to zero in a controlled way) after each step of the discretized flow (thus destroying the zero exponent in the flow direction). Although we will see in the next paragraph that the subexponential decay $e^{-\sqrt{t}}$ can be improved in some cases, the Chernov-Liverani approach, besides its intrinsic elegance, enjoys a flexibility which will perhaps be useful to study systems with non uniform hyperbolicity.

A second breakthrough was obtained by Dolgopyat [1996a], who was inspired by Chernov's uniform nonintegrability condition, and went back to the symbolic dynamics transfer operator approach of Axiom A flows of Bowen and Ruelle. Pushing the analytical study into a great degree of sophistication, Dolgopyat proved that the correlation functions of Hölder continuous observables for topologically mixing Anosov flows admitting C^1 stable and unstable foliations decay *exponentially* (this applies to geodesic flows in variable negative curvature). To do this, he controls the operator norms, and thus spectra, of the transfer operators $\mathcal{L}_s$, in order to obtain the Paley-Wiener half-plane mentioned above. This very fine study produces information about the correlation spectra of uniformly hyperbolic flows, which can be translated into knowledge about periodic orbits, via dynamical zeta functions (see subsection 4.C). Dolgopyat [1996b] also shows that for more general Axiom A flows, the decay of correlations for Gibbs measures is often rapid in the sense of Schwartz.

4.C Dynamical zeta functions and generalized Fredholm determinants.

We hope to have transmitted the principle that the spectra of the various transfer operators arising in dynamical systems are useful to understand their statistical properties. We shall now briefly describe a theory relating analytic properties of dynamical zeta functions to the spectra of transfer operators.

Dynamical zeta functions were introduced in the sixties by Artin and Mazur [1965] for maps and Smale [1967] for flows (there were precursors, such as the Selberg zeta function which can be expressed in terms of dynamical zeta functions of geodesic flows, and the number-theoretic zeta functions studied by Deligne which can be viewed as dynamical zeta functions for the Frobenius morphism, see e.g. Ruelle [1994, Chapter I]). Given a transformation $f : M \to M$ such that f^n has finitely many fixed points (i.e., points x so that $f^n(x) = x$) for all n, the dynamical zeta function is the formal power series

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \#\{x \in M \mid f^n x = x\}\,. \tag{4.3}$$

For flows with countably many closed orbits ξ, denoting the orbit-length by $\ell(\xi)$, one considers the formal product

$$\zeta^*(s) = \prod_{\xi} \frac{1}{1 - e^{-s\ell(\xi)}}\,. \tag{4.4}$$

In particular because of the formal analogy with the *partition functions* of statistical mechanics (roughly said: the denominators of the expressions (2.5) leading to Gibbs ensembles for one-dimensional lattices), it was observed by Ruelle that the extension to *weighted* dynamical zeta functions could be fruitful. A weighted dynamical zeta function is obtained by choosing a (say, bounded) function $g : M \to \mathbb{C}$ and writing

$$\zeta_g(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in M, f^n x = x} \prod_{k=0}^{n-1} g(f^k(x))\,. \tag{4.5}$$

(In the statistical mechanics framework, the function g is usually real and positive, and can be written $g = \exp H$.) One recovers $\zeta(z)$ when setting $g = 1$. Examples such as $g(x) = |\det Df(x)|$ for differentiable dynamics, or $g(x) = \exp(sr(x))$ when constructing a flow by suspension show that the formula (4.5) is natural from a dynamical point of view.

We present once more a rather trivial but intuitive example. Consider a subshift of finite type defined by a transition matrix A as in Subsection 3.A. Observe that the number of fixed points of this subshift is exactly the number of 1's in the diagonal of A, i.e., the trace of this matrix. Similarly, the number of fixed points of σ_+^n in $\mathcal{S}_A$ is the trace of the matrix A^n. Therefore, using the formula $\exp \operatorname{Tr} \log B = \det B$ for finite matrices, the unweighted dynamical zeta function of the subshift is just

(4.6)
$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \#\{x \in M \mid f^n x = x\} = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{Tr} A^n = \frac{1}{\det(1 - zA)},$$

proving that the zeta function of any dynamics conjugated to a subshift of finite type is rational. (By studying the semi-conjugacy between Axiom A diffeomorphisms and subshifts of finite type, Manning [1971] proved that the unweighted zeta function of any Axiom A diffeomorphism is rational.) By (4.6), the poles of $\zeta(z)$ are exactly the inverses of the nonzero eigenvalues of A. Since this matrix represents the unweighted ($H = 0$) transfer operator of σ_+ acting on a finite dimensional vector space, we have established the link between the zeta function and the spectrum of a transfer operator.

Introducing a locally constant weight g in the zeta function corresponds to weighing the matrix A and transfer operator accordingly. Things become interesting for arbitrary Hölder weights, since a finite matrix representation of the weighted operators $\mathcal{L}$ defined by the first equality in (2.7) is unavailable, and formula (4.6) does not make sense literally. One of the fundamental results in the theory of dynamical zeta functions, obtained by Haydn [1990] (who built on previous key results by Pollicott and Ruelle, based on symbolic dynamics), is that any Hölder-weighted dynamical zeta function of an Axiom A diffeomorphism is a meromorphic function in a disc, where its poles are in bijection with the inverses of the eigenvalues of a transfer operator (acting on a Banach space of Hölder functions) associated to the corresponding weight and an expanding map obtained by projecting along stable manifolds. (One first obtains an upper bound for the essential spectral radius of the operator.) We refer to the review book of Parry-Pollicott [1990] for more details and for their beautiful analogue, for Axiom A flows, of the prime number theorem, obtained via a dynamical zeta function (the objects being asymptotically counted, instead of the prime numbers, are the periodic orbits).

The lower bound for the radius of meromorphy of the zeta function in Haydn's theorem is in some sense optimal when the weight is truly only Hölder. In many cases however, the hyperbolic dynamics satisfies higher smoothness properties, and this information is lost through symbolic dynamics, or just by going to the expanding system (the invariant foliations are usually only Hölder). This has been a source of frustration for many years, since on the one hand numerical experiments (see e.g. Cvitanović [1996] and references therein) indicate that the correlation spectrum of (hyperbolic) smooth dynamics and weights extends much farther, and on the other hand it had been observed by Ruelle [1976b] that for analytic expanding maps and analytic weights, transfer operators acting on holomorphic functions are compact (in fact, nuclear in the sense of Grothendieck), and that the dynamical zeta function could be replaced by a *generalized Fredholm determinant.* This determinant is an entire function, whose zeroes describe exactly the inverses of the eigenvalues of the operator, it is given by:

$$d_g(z) = \exp - \sum_{n\ge 1} \frac{z^n}{n} \sum_{x\in M\,, f^n(x)=x} \frac{\prod_{k=0}^{n-1} g(f^k(x))}{|\det(Df^n(x) - 1)|}\,. \tag{4.7}$$

The breakthrough came in the early nineties, when Rugh [1992] constructed a Banach space of functions smooth along unstable foliations and rough along stable ones, on which a transfer operator associated to the *full hyperbolic dynamics* (as opposed to an expanding map obtained by projecting) is nuclear in the sense of Grothendieck (for hyperbolic and analytic surface transformations and analytic weights). This yields a proof that the determinant $d_g(z)$ is an entire function in the whole complex plane (and the corresponding zeta function meromorphic in $\mathbb{C}$) for two-dimensional transformations. Fried [1995] later proved meromorphy of the zeta function in any finite dimension. Kitaev [1995] obtained more recently analogous results for systems of uniformly hyperbolic diffeomorphisms of finite differentiablity, a technically challenging situation. (There are previous important results of Tangerman and Ruelle in the finite differentiability case.)

The study of dynamical zeta functions (or generalized Fredholm determinants) for suitable algebras of transfer operators can also be viewed as a part of functional analysis. In this direction, we mention that the transfer operator theory of Hofbauer and Keller for (non Markov) piecewise monotone interval maps and function spaces of bounded variation led to a result on the corresponding weighted zeta functions (Baladi-Keller [1990]), under an assumption guaranteeing that the number of periodic orbits of each fixed period is finite. In the seventies Milnor and Thurston [1988] started to develop a rich *kneading theory* for one-dimensional maps. They associated an unweighted dynamical zeta function to *any* piecewise monotone map, by counting (twice) only the (transverse) intersections of the diagonal with the graph of f^n where this graph has negative slope. One of their most striking theorems is a formula expressing this zeta function in terms of the determinant of a finite kneading matrix. This formula was generalized recently to the weighted situation by Baladi-Ruelle [1996], who introduce sharp determinants, and use regularization techniques, involving a kneading operator related to Milnor and Thurston's kneading matrix, to show that the zeroes of the sharp determinant in a disc are indeed the inverse discrete eigenvalues of a transfer operator. The formulas for the *sharp trace* and sharp determinant of a transfer operator

$\mathcal{L}\varphi = \sum_i g_i \cdot (\varphi \circ \psi_i)$ associated to a family of interval maps ψ_i and continuous weights of bounded variation are given by

$$\text{(4.8)} \qquad \mathrm{Tr}^{\#}\mathcal{L} = \sum_i \int dg_i(x)\,\frac{1}{2}\,\mathrm{sgn}\,(\psi_i x - x)\,, \quad d^{\#}(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}^{\#}\mathcal{L}^n\,,$$

where sgn is the sign function $x/|x|$. If the g_i vanish on the boundary, and the maps ψ_i have finitely many fixed points, integration by parts of $\mathrm{Tr}^{\#}\mathcal{L}$ yields a weighted sum over fixed points, as in (4.5), but with Lefschetz signs introduced.

5. Current topics

Before we end by commenting on a few recent trends, we would like to emphasize that many areas involving applications of thermodynamic formalism to dynamics have not even been mentioned in this text, for example regarding complex dynamics (see e.g. Zinsmeister [1996]), applications to the Riemann zeta function (see in particular Knauf [1994]), quantum chaos (see Cvitanović [1996] and references therein) etc. Another important aspect that we have neglected is the study of random dynamical systems (see e.g. Kifer [1988]), and in particular stability of the SRB or other Gibbs measures, or of the spectrum of transfer operators, under small stochastic perturbations of chaotic dynamical systems (there are important results by Kifer and Young, and more recently by Benedicks–Viana)).

We have seen that symbolic dynamics was a crucial ingredient in the early development of the statistical theory of chaotic dynamics. The method has several drawbacks: For example there is in general no canonical choice for the Markov partition, and it is rather sensitive to perturbations of the system. On the one hand, we believe that it is too early to pronounce the death of the Markov partition: Its avatars (often with countably many pieces) are still being applied today successfully to various problems (e.g. by Benedicks-Young [1996], Dolgopyat [1996a, 1996b]) see also the application of Markov sieves to billiards by Chernov, Sinai, and others, and also the work of Bricmont-Kupiainen [1996] on infinite dimensional dynamical systems described below). On the other hand, however, powerful techniques to construct SRB measures without Markov partitions are emerging now, namely the application of fixed point theorems involving Hilbert-type metrics in *Birkhoff cones* (Garrett Birkhoff [1967]). Contraction properties of such metrics have been applied to dynamics in the past (we mention in particular Ferrero-Schmitt [1988]), but interest has been renewed recently by Liverani [1995] who obtained a simple proof of exponential decay of correlations for area-preserving Anosov maps. He did this by constructing a suitable invariant cone of functions, and proving that this cone is mapped strictly into itself under the action of a transfer operator: The procedure is straightforward for expanding maps, but for hyperbolic diffeomorphisms the coexistence of contracting and expanding directions once more requires some care (see the results of Rugh and Kitaev in Section 4.C). Viana [1997] recently extended Liverani's approach to (non area-preserving) discrete-time Axiom A attractors.

Although the Birkhoff cone approach is very elegant, and suited to the analysis of small deterministic or random perturbations, we mention that it seems for the moment limited to the understanding of the "first gap" of the transfer operator (assuming that there is a single maximal eigenvalue). We believe that the wealth of information contained in the full correlation spectrum (which is interesting in view of the Ruelle resonances, see e.g. Eckmann [1989], but also in other cases such as the application to Feigenbaum period-doubling, see e.g. Artuso et al. [1990], or quantum chaos), which is available through dynamical zeta functions or generalized Fredholm determinants in many settings, should not be discarded.

We have discussed some dynamical systems on finite dimensional manifolds. *Infinite dimensional* settings occur naturally, e.g. coming from partial differential equations. A class of infinite dimensional models which has received a lot of attention from physicists, but where rigorous results are still scarce, is given by so-called *coupled map lattices.* A coupled map lattice is a transformation $\mathcal{F}$ on a space $M^{\mathbb{Z}^d}$ where M is, say, a compact manifold, and $d \in \mathbb{Z}_+$, which is obtained e.g. by perturbing an uncoupled map $f^{\mathbb{Z}^d}$ for $f : M \to M$ (Bunimovich-Sinai [1988]). If f has some hyperbolic structure, and if the coupling is not too strong, one can attempt to transport the statistical knowledge on f to information on $\mathcal{F}$. Recently, Bricmont and Kupiainen [1996] were able to show existence of an SRB measure (the definition in the infinite dimensional case is slightly different) and exponential decay of spatio-temporal correlations under suitable assumptions. Computer experiments seem to indicate that a rich variety of behaviour is possible when the coupling strength is increased (see e.g. Bunimovich [1995] and references therein).

Statistical mechanical examples with decay of correlations slower than exponential, or with nonunique Gibbs measures may be constructed by weakening the assumptions on the Hamiltonian from Section 2.C. (In some sense, this weaker setting allowing phase transitions is the nontrivial case.) In dynamics, slow decay of correlations seems to occur e.g. in the presence of certain types of (so-called neutral) periodic orbits with a zero Lyapunov exponent (sometimes called *intermittent* orbits in the physics literature), but has not been much investigated rigorously up to now (see however Prellberg [1991]). Understanding such systems seems to be one of the next important challenges for mathematicians. One of the difficulties lies in the more complicated relation between the spectrum of the – gapless – transfer operator (or the nature of the first singularity of the dynamical zeta function), and the decay of correlations. Recent results including polynomial decay of correlations for certain one-dimensional maps have been obtained by Isola [1996], using inducing techniques. Applying methods from complex dynamics, Rugh [1996] very recently showed, for certain piecewise analytic one-dimensional maps with a neutral fixed point, that the spectrum of the transfer operator is the union of a discrete set and the interval $[0, 1]$, a situation quite different from the quasicompact transfer operators that we have encountered above. A thorough analysis of the tail behaviour of the recovery times (as was done by Young [1996] for systems enjoying some hyperbolicity) of orbits which come close to the neutral periodic has very recently been realized by Young [1997].

Finally, several recent results are concerned with situations where only a *quasi-attractor* is present, i.e., an f-invariant set $\Omega \subset M$ which is attracting neither for f nor for f^{-1}. There is then no SRB measure in the sense of Section 3.B or in the sense of (1.1), but (if it exists) the *generalized SRB measure* which realizes the variational principle for the weight $-\log|\det Df|_{E^u}|$ may contain useful information, in particular about escape rates from the quasi-attractor. Alternatively, one can introduce *generalized SRB quasi-invariant* measures, using a definition with the same ergodic flavour as (1.1), but conditioning by keeping at each step only those points which are still in a fixed neighbourhood of the quasi-attractor. Existence and properties of these two measures have been known for a long time in Axiom A situations (Bowen-Ruelle [1975], see also Ruelle [1996a]), but they are now being studied also for other systems (see e.g. Chernov-Markarian-Troubetzkoy [1996] and references therein, Collet-Martinez-Schmitt [1996]; see also the physical studies of Kaplan, Yorke, Kantz, Grassberger, Gaspard, and Nicolis, as well as the recent counter-example by Baladi, Bonatti, and Schmitt to the conjecture of Eckmann-Ruelle). These mathematical results have direct applications to the theory of *nonequilibrium statistical mechanics,* which has recently attracted a revived interest (see e.g. Gallavotti [1996], Ruelle [1996], Gallavotti-Ruelle [1996] and references therein). It is perhaps not impertinent to hope that this is the beginning of another very productive episode in the stimulating relationship between statistical mechanics and chaotic dynamics.

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