

Θ HAND-OUT*

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1. SCHRÖDINGER AND SHALE-WEIL REPRESENTATION

1.1. Let ω be the standard symplectic form on \mathbb{R}^{2k} , i.e.,

$$\omega(\xi, \xi') = \mathbf{x} \cdot \mathbf{y}' - \mathbf{y} \cdot \mathbf{x}',$$

where

$$\xi = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, \quad \xi' = \begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix}, \quad \mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathbb{R}^k.$$

The *Heisenberg group* $\mathbb{H}(\mathbb{R}^k)$ is then defined as the set $\mathbb{R}^{2k} \times \mathbb{R}$ with multiplication law [2]

$$(\xi, t)(\xi', t') = (\xi + \xi', t + t' + \frac{1}{2}\omega(\xi, \xi')).$$

Note that we have the decomposition

$$\left(\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, t \right) = \left(\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}, 0 \right) \left(\begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}, 0 \right) \left(\mathbf{0}, t - \frac{1}{2}\mathbf{x} \cdot \mathbf{y} \right).$$

1.2. The *Schrödinger representation* of $\mathbb{H}(\mathbb{R}^k)$ on $f \in L^2(\mathbb{R}^k)$ is given by (cf. [2], p. 15)

$$[W\left(\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}, 0\right)f](\mathbf{w}) = e(\mathbf{x} \cdot \mathbf{w}) f(\mathbf{w}), \quad \text{with } \mathbf{x}, \mathbf{w} \in \mathbb{R}^k,$$

$$[W\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}, 0\right)f](\mathbf{w}) = f(\mathbf{w} - \mathbf{y}), \quad \text{with } \mathbf{y}, \mathbf{w} \in \mathbb{R}^k,$$

$$W(\mathbf{0}, t) = e(t) \text{id}, \quad \text{with } t \in \mathbb{R}.$$

We have therefore for a general element (ξ, t) in $\mathbb{H}(\mathbb{R}^k)$

$$[W\left(\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, t\right)f](\mathbf{w}) = e\left(t - \frac{1}{2}\mathbf{x} \cdot \mathbf{y}\right) e(\mathbf{x} \cdot \mathbf{w}) f(\mathbf{w} - \mathbf{y}).$$

1.3. For every element M in the symplectic group $\text{Sp}(k, \mathbb{R})$ of \mathbb{R}^{2k} , we can define a new representation W_M of $\mathbb{H}(\mathbb{R}^k)$ by

$$W_M(\xi, t) = W(M\xi, t).$$

All such representations are irreducible and, by the Stone-von Neumann Theorem, unitarily equivalent (see [2] for details). That is, for each $M \in \text{Sp}(k, \mathbb{R})$ there exists a unitary operator $R(M)$ such that

$$R(M) W(\xi, t) R(M)^{-1} = W(M\xi, t).$$

The $R(M)$ is determined up to a unitary phase factor and defines the projective *Shale-Weil representation* of the symplectic group. *Projective* means that

$$R(MM') = c(M, M')R(M)R(M')$$

with cocycle $c(M, M') \in \mathbb{C}$, $|c(M, M')| = 1$, but $c(M, M') \neq 1$ in general.

1.4. For our present purpose it suffices to consider the group $\text{SL}(2, \mathbb{R})$ which is embedded in $\text{Sp}(k, \mathbb{R})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a 1_k & b 1_k \\ c 1_k & d 1_k \end{pmatrix}$$

where 1_k is the $k \times k$ unit matrix.

The action of $M \in \text{SL}(2, \mathbb{R})$ on $\xi \in \mathbb{R}^{2k}$ is then given by

$$M\xi = \begin{pmatrix} a\mathbf{x} + b\mathbf{y} \\ c\mathbf{x} + d\mathbf{y} \end{pmatrix}, \quad \text{with } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \xi = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}.$$

1.5. For $M \in \text{SL}(2, \mathbb{R}) \hookrightarrow \text{Sp}(k, \mathbb{R})$ we have the explicit representations (see [2], p. 61f.)

$$[R(M)f](\mathbf{w}) = \begin{cases} |a|^{k/2} & e(\frac{1}{2}\|\mathbf{w}\|^2 ab) f(a\mathbf{w}) \\ & (c = 0) \\ |c|^{-k/2} & \int_{\mathbb{R}^k} e\left[\frac{\frac{1}{2}(a\|\mathbf{w}\|^2 + d\|\mathbf{w}'\|^2) - \mathbf{w} \cdot \mathbf{w}'}{c}\right] f(\mathbf{w}') d\mathbf{w}' \\ & (c \neq 0). \end{cases}$$

Here $\|\cdot\|$ denotes the euclidean norm in \mathbb{R}^k ,

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_k^2}.$$

1.6. If

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix},$$

$\in \text{SL}(2, \mathbb{R})$ with $M_1 M_2 = M_3$, the corresponding cocycle is

$$c(M_1, M_2) = e^{-i\pi k \text{sign}(c_1 c_2 c_3)/4},$$

where

$$\text{sign}(x) = \begin{cases} -1 & (x < 0) \\ 0 & (x = 0) \\ 1 & (x > 0). \end{cases}$$

1.7. In the special case when

$$M_1 = \begin{pmatrix} \cos \phi_1 & -\sin \phi_1 \\ \sin \phi_1 & \cos \phi_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} \cos \phi_2 & -\sin \phi_2 \\ \sin \phi_2 & \cos \phi_2 \end{pmatrix},$$

we find

$$c(M_1, M_2) = e^{-i\pi k(\sigma_{\phi_1} + \sigma_{\phi_2} - \sigma_{\phi_1 + \phi_2})/4}$$

where

$$\sigma_\phi = \begin{cases} 2\nu & \text{if } \phi = \nu\pi, \\ 2\nu + 1 & \text{if } \nu\pi < \phi < (\nu + 1)\pi. \end{cases}$$

1.8. Every $M \in \text{SL}(2, \mathbb{R})$ admits the unique Iwasawa decomposition

$$M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} = (\tau, \phi),$$

where $\tau = u + iv \in \mathfrak{H}$, $\phi \in [0, 2\pi)$. This parametrization leads to the well known action of $\text{SL}(2, \mathbb{R})$ on $\mathfrak{H} \times [0, 2\pi)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, \phi) = \left(\frac{a\tau + b}{c\tau + d}, \phi + \arg(c\tau + d) \bmod 2\pi \right).$$

We will sometimes use the convenient notation $(M\tau, \phi_M) := M(\tau, \phi)$ and $u_M := \text{Re}(M\tau)$, $v_M := \text{Im}(M\tau)$.

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1.9. The (projective) Shale-Weil representation of $\mathrm{SL}(2, \mathbb{R})$ reads in these coordinates

$$\begin{aligned} [R(\tau, \phi)f](\mathbf{w}) &= [R(\tau, 0)R(i, \phi)f](\mathbf{w}) \\ &= v^{k/4} e^{(\frac{1}{2}\|\mathbf{w}\|^2 u)} [R(i, \phi)f](v^{1/2}\mathbf{w}) \end{aligned}$$

and

$$[R(i, \phi)f](\mathbf{w}) = \begin{cases} f(\mathbf{w}) & (\phi = 0 \bmod 2\pi) \\ f(-\mathbf{w}) & (\phi = \pi \bmod 2\pi) \\ |\sin \phi|^{-k/2} \int_{\mathbb{R}^k} e^{\left[\frac{\frac{1}{2}(\|\mathbf{w}\|^2 + \|\mathbf{w}'\|^2) \cos \phi - \mathbf{w} \cdot \mathbf{w}'}{\sin \phi} \right]} f(\mathbf{w}') d\mathbf{w}' & (\phi \neq 0 \bmod \pi). \end{cases}$$

Note that $R(i, \pi/2) = \mathcal{F}$ is the Fourier transform.

2. THETA SUMS

2.1. The *Jacobi group* is defined as the semidirect product [?]

$$\mathrm{Sp}(k, \mathbb{R}) \ltimes \mathbb{H}(\mathbb{R}^k)$$

with multiplication law

$$(M; \boldsymbol{\xi}, t)(M'; \boldsymbol{\xi}', t') = (MM'; \boldsymbol{\xi} + M\boldsymbol{\xi}', t + t' + \frac{1}{2}\omega(\boldsymbol{\xi}, M\boldsymbol{\xi}')).$$

This definition is motivated by the fact that, since

$$R(M)W(\boldsymbol{\xi}', t') = W(M\boldsymbol{\xi}', t')R(M),$$

(recall 1.3) we have

$$\begin{aligned} W(\boldsymbol{\xi}, t)R(M)W(\boldsymbol{\xi}', t')R(M') &= W(\boldsymbol{\xi}, t)W(M\boldsymbol{\xi}', t')R(M)R(M') \\ &= c(M, M')^{-1}W(\boldsymbol{\xi} + M\boldsymbol{\xi}', t + t' + \frac{1}{2}\omega(\boldsymbol{\xi}, M\boldsymbol{\xi}'))R(MM'). \end{aligned}$$

Hence

$$R(M; \boldsymbol{\xi}, t) = W(\boldsymbol{\xi}, t)R(M)$$

defines a projective representation of the Jacobi group, with cocycle $c(M, M')$ as above, the so-called *Schrödinger-Weil representation*.

Let us also put

$$\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t) = W(\boldsymbol{\xi}, t)\tilde{R}(\tau, \phi).$$

2.2. Jacobi's theta sum. We define Jacobi's theta sum for $f \in \mathcal{S}(\mathbb{R}^k)$ by

$$\Theta_f(\tau, \phi; \boldsymbol{\xi}, t) = \sum_{\mathbf{m} \in \mathbb{Z}^k} [\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f](\mathbf{m}).$$

More explicitly, for $\tau = u + iv$, $\boldsymbol{\xi} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$,

$$\begin{aligned} \Theta_f(\tau, \phi; \boldsymbol{\xi}, t) &= v^{k/4} e^{(t - \frac{1}{2}\mathbf{x} \cdot \mathbf{y})} \\ &\sum_{\mathbf{m} \in \mathbb{Z}^k} f_\phi((\mathbf{m} - \mathbf{y})v^{1/2}) e^{(\frac{1}{2}\|\mathbf{m} - \mathbf{y}\|^2 u + \mathbf{m} \cdot \mathbf{x})}, \end{aligned}$$

where

$$f_\phi = \tilde{R}(i, \phi)f.$$

It is easily seen that if $f \in \mathcal{S}(\mathbb{R}^k)$ then $f_\phi \in \mathcal{S}(\mathbb{R}^k)$ for ϕ fixed, and thus also $\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f \in \mathcal{S}(\mathbb{R}^k)$ for fixed $(\tau, \phi; \boldsymbol{\xi}, t)$. This guarantees rapid convergence of the above series. We have the following uniform bound.

2.3. Lemma. Let $f_\phi = \tilde{R}(i, \phi)f$, with $f \in \mathcal{S}(\mathbb{R}^k)$. Then, for any $R > 1$, there is a constant c_R such that for all $\mathbf{w} \in \mathbb{R}^k$, $\phi \in \mathbb{R}$, we have

$$|f_\phi(\mathbf{w})| \leq c_R(1 + \|\mathbf{w}\|)^{-R}.$$

2.4. The following transformation formulas are crucial for our further investigations:

Jacobi 1.

$$\Theta_f\left(-\frac{1}{\tau}, \phi + \arg \tau; \begin{pmatrix} -\mathbf{y} \\ \mathbf{x} \end{pmatrix}, t\right) = e^{-i\pi k/4} \Theta_f(\tau, \phi; \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, t).$$

Proof. The Poisson summation formula states that for any $f \in \mathcal{S}(\mathbb{R}^k)$ we have

$$\sum_{\mathbf{m} \in \mathbb{Z}^k} [\mathcal{F}f](\mathbf{m}) = \sum_{\mathbf{m} \in \mathbb{Z}^k} f(\mathbf{m})$$

where \mathcal{F} is the Fourier transform. Because

$$\mathcal{F} = R(i, \pi/2) = R(S), \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and secondly $\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f \in \mathcal{S}(\mathbb{R}^k)$ for fixed $(\tau, \phi; \boldsymbol{\xi}, t)$, the Poisson summation formula yields

$$\sum_{\mathbf{m} \in \mathbb{Z}^k} [R(S)\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f](\mathbf{m}) = \sum_{\mathbf{m} \in \mathbb{Z}^k} [\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f](\mathbf{m}).$$

We have

$$\begin{aligned} R(S)\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t) &= R(S)W(\boldsymbol{\xi}, t)\tilde{R}(\tau, 0)\tilde{R}(i, \phi) \\ &= W(S\boldsymbol{\xi}, t)R(S)R(\tau, 0)\tilde{R}(i, \phi); \end{aligned}$$

furthermore

$$R(S)R(\tau, 0) = R\left(-\frac{1}{\tau}, \arg \tau\right) = R\left(-\frac{1}{\tau}, 0\right)R(i, \arg \tau),$$

since $(\tau, 0)$ and $(-\frac{1}{\tau}, 0)$ are upper triangular matrices, and hence the corresponding cocycles are trivial, i.e., equal to 1 (recall 1.6). Finally, since $0 < \arg \tau < \pi$ for $\tau \in \mathfrak{H}$,

$$R(i, \arg \tau)\tilde{R}(i, \phi) = e^{i\pi k/4}\tilde{R}(i, \arg \tau)\tilde{R}(i, \phi) = e^{i\pi k/4}\tilde{R}(i, \phi + \arg \tau).$$

Collecting all terms, we find

$$R(S)\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t) = e^{i\pi k/4}\tilde{R}\left(-\frac{1}{\tau}, \phi + \arg \tau; S\boldsymbol{\xi}, t\right),$$

and hence

$$\begin{aligned} \sum_{\mathbf{m} \in \mathbb{Z}^k} [\tilde{R}\left(-\frac{1}{\tau}, \phi + \arg \tau; S\boldsymbol{\xi}, t\right)f](\mathbf{m}) \\ = e^{-i\pi k/4} \sum_{\mathbf{m} \in \mathbb{Z}^k} [\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f](\mathbf{m}). \end{aligned}$$

which proves the claim. \square

Jacobi 2.

$$\Theta_f\left(\tau + 1, \phi; \begin{pmatrix} \mathbf{s} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, t + \frac{1}{2}\mathbf{s} \cdot \mathbf{y}\right) = \Theta_f(\tau, \phi; \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, t),$$

with

$$\mathbf{s} = {}^t\left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right) \in \mathbb{R}^k.$$

Proof. Clearly for any $f \in \mathcal{S}(\mathbb{R}^k)$

$$\sum_{\mathbf{m} \in \mathbb{Z}^k} [\tilde{R}(i+1, 0; \begin{pmatrix} \mathbf{s} \\ \mathbf{0} \end{pmatrix}, 0)f](\mathbf{m}) = \sum_{\mathbf{m} \in \mathbb{Z}^k} f(\mathbf{m}),$$

and hence also (replace f with $\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f$)

$$\sum_{\mathbf{m} \in \mathbb{Z}^k} [\tilde{R}(i+1, 0; \begin{pmatrix} \mathbf{s} \\ \mathbf{0} \end{pmatrix}, 0)\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f](\mathbf{m}) = \sum_{\mathbf{m} \in \mathbb{Z}^k} [\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f](\mathbf{m}). \quad \text{SL}(2, \mathbb{Z}) \rightarrow N \backslash \Gamma^k, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \begin{pmatrix} abs \\ cds \end{pmatrix} + \mathbb{Z}^{2k} \right)$$

We conclude by noticing

$$\begin{aligned} & \tilde{R}(i+1, 0; \begin{pmatrix} \mathbf{s} \\ \mathbf{0} \end{pmatrix}, 0)\tilde{R}(\tau, \phi; \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, t) \\ &= \tilde{R}(\tau+1, \phi; \begin{pmatrix} \mathbf{s} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, t + \frac{1}{2}\mathbf{s} \cdot \mathbf{y}), \end{aligned}$$

where we have used that $c(i, 0), (\tau, \phi) = 1$ since $(i, 0)$ is an upper triangular matrix, cf. 1.6. \square

Jacobi 3.

$$\Theta_f(\tau, \phi; \begin{pmatrix} \mathbf{k} \\ \mathbf{l} \end{pmatrix} + \boldsymbol{\xi}, r + t + \frac{1}{2}\omega(\begin{pmatrix} \mathbf{k} \\ \mathbf{l} \end{pmatrix}, \boldsymbol{\xi})) = (-1)^{\mathbf{k} \cdot \mathbf{l}} \Theta_f(\tau, \phi; \boldsymbol{\xi}, t)$$

for any $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^k, r \in \mathbb{Z}$.

Proof. By virtue of 1.2 we have for all f

$$\sum_{\mathbf{m} \in \mathbb{Z}^k} [W(\begin{pmatrix} \mathbf{k} \\ \mathbf{l} \end{pmatrix}, r)f](\mathbf{m}) = e(-\frac{1}{2}\mathbf{k} \cdot \mathbf{l}) \sum_{\mathbf{m} \in \mathbb{Z}^k} f(\mathbf{m}),$$

and therefore, replacing f with $W(\boldsymbol{\xi}, t)\tilde{R}(\tau, \phi)f$,

$$\begin{aligned} & \sum_{\mathbf{m} \in \mathbb{Z}^k} [W(\begin{pmatrix} \mathbf{k} \\ \mathbf{l} \end{pmatrix}, r)W(\boldsymbol{\xi}, t)\tilde{R}(\tau, \phi)f](\mathbf{m}) \\ &= e(-\frac{1}{2}\mathbf{k} \cdot \mathbf{l}) \sum_{\mathbf{m} \in \mathbb{Z}^k} [W(\boldsymbol{\xi}, t)\tilde{R}(\tau, \phi)f](\mathbf{m}), \end{aligned}$$

which gives the desired result. \square

2.5. In what follows, we shall only need to consider products of theta sums of the form

$$\Theta_f(\tau, \phi; \boldsymbol{\xi}, t)\overline{\Theta_g(\tau, \phi; \boldsymbol{\xi}, t)},$$

where $f, g \in \mathcal{S}(\mathbb{R}^k)$. Clearly such combinations do not depend on the t -variable. Let us therefore define the semi-direct product group

$$G^k = \text{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^{2k}$$

with multiplication law

$$(M; \boldsymbol{\xi})(M'; \boldsymbol{\xi}') = (MM'; \boldsymbol{\xi} + M\boldsymbol{\xi}'),$$

and put

$$\Theta_f(\tau, \phi; \boldsymbol{\xi}) = v^{k/4} \sum_{\mathbf{m} \in \mathbb{Z}^k} f_\phi((\mathbf{m} - \mathbf{y})v^{1/2})e(\frac{1}{2}\|\mathbf{m} - \mathbf{y}\|^2 u + \mathbf{m} \cdot \mathbf{x}).$$

By virtue of Lemma 2.3 and the Iwasawa parametrization 1.8, $\Theta_f \overline{\Theta_g}$ is a continuous \mathbb{C} -valued function on G^k .

2.6. A short calculation yields that the set

$$\Gamma^k = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \begin{pmatrix} abs \\ cds \end{pmatrix} + \mathbf{m} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \mathbf{m} \in \mathbb{Z}^{2k} \right\},$$

with $\mathbf{s} = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{R}^k$, is closed under multiplication and inversion, and therefore forms a subgroup of G^k . Note also that the subgroup

$$N = \{1\} \ltimes \mathbb{Z}^{2k}$$

is normal in Γ^k .

2.7. Lemma. Γ^k is generated by the elements

$$\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \mathbf{0} \right), \quad \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} \mathbf{s} \\ \mathbf{0} \end{pmatrix} \right), \quad \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \mathbf{m} \right), \quad \mathbf{m} \in \mathbb{Z}^{2k}.$$

Proof. The map

$$\text{SL}(2, \mathbb{Z}) \rightarrow N \backslash \Gamma^k, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \begin{pmatrix} abs \\ cds \end{pmatrix} + \mathbb{Z}^{2k} \right)$$

defines a group isomorphism. The matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $\text{SL}(2, \mathbb{Z})$, hence the lemma. \square

2.8. Proposition. The left action of the group Γ^k on G^k is properly discontinuous. A fundamental domain of Γ^k in G^k is given by

$$\mathcal{F}_{\Gamma^k} = \mathcal{F}_{\text{SL}(2, \mathbb{Z})} \times \{\phi \in [0, \pi)\} \times \{\boldsymbol{\xi} \in [-\frac{1}{2}, \frac{1}{2})^{2k}\}.$$

where $\mathcal{F}_{\text{SL}(2, \mathbb{Z})}$ is the fundamental domain in \mathfrak{H} of the modular group $\text{SL}(2, \mathbb{Z})$, given by $\{\tau \in \mathfrak{H} : u \in [-\frac{1}{2}, \frac{1}{2}), |\tau| > 1\}$.

Proof. As mentioned before, the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $\text{SL}(2, \mathbb{Z})$, which explains $\mathcal{F}_{\text{SL}(2, \mathbb{Z})}$. Note furthermore that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ generates the shift $\phi \mapsto \phi + \pi$. \square

2.9. Proposition. For $f, g \in \mathcal{S}(\mathbb{R}^k)$, $\Theta_f(\tau, \phi; \boldsymbol{\xi})\overline{\Theta_g(\tau, \phi; \boldsymbol{\xi})}$ is invariant under the left action of Γ^k .

Proof. This follows directly from Jacobi 1–3, since the left action of the generators from 2.7 is

$$(\tau, \phi; \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}) \mapsto \left(-\frac{1}{\tau}, \phi + \arg \tau; \begin{pmatrix} -\mathbf{y} \\ \mathbf{x} \end{pmatrix}\right),$$

$$(\tau, \phi; \boldsymbol{\xi}) \mapsto \left(\tau + 1, \phi; \begin{pmatrix} \mathbf{s} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}\right),$$

and

$$(\tau, \phi; \boldsymbol{\xi}) \mapsto (\tau, \phi; \boldsymbol{\xi} + \mathbf{m}),$$

respectively. \square

We find the following uniform estimate.

2.10. Proposition. Let $f, g \in \mathcal{S}(\mathbb{R}^k)$. For any $R > 1$, we have

$$\begin{aligned} & \Theta_f(\tau, \phi; \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix})\overline{\Theta_g(\tau, \phi; \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix})} \\ &= v^{k/2} \sum_{\mathbf{m} \in \mathbb{Z}^k} f_\phi((\mathbf{m} - \mathbf{y})v^{1/2})\overline{g_\phi((\mathbf{m} - \mathbf{y})v^{1/2})} + O_R(v^{-R}) \end{aligned}$$

uniformly for all $(\tau, \phi; \boldsymbol{\xi}) \in G^k$ with $v > \frac{1}{2}$.

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