DECAY OF RANDOM CORRELATION FUNCTIONS FOR UNIMODAL MAPS

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Since the pioneering results of Jakobson and subsequent work by Benedicks-Carleson and others, it is known that quadratic maps $f_a(x) = a - x^2$ admit a unique absolutely continuous invariant measure for a positive measure set of parameters a. For topologically mixing f_a, Young and Keller-Nowicki independently proved exponential decay of correlation functions for this a.c.i.m. and smooth observables. We consider random compositions of small perturbations $f + \omega_t$, with $f = f_a$ or another unimodal map satisfying certain nonuniform hyperbolicity axioms, and ω_t chosen independently and identically in $[-\epsilon, \epsilon]$. Baladi-Viana showed exponential mixing of the associated Markov chain, i.e., averaging over all random itineraries. We obtain stretched exponential bounds for the random correlation functions of Lipschitz observables for the sample measure μ_ω of almost every itinerary.

1. Introduction

1.1. Mixing and rates of mixing

A discrete-time deterministic dynamical system given by a transformation f on a Riemann manifold M is said to admit a physical measure if there is an ergodic f-invariant

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measure μ so that, for each integrable observable $\varphi : M \to \mathbf{C}$, Lebesgue almost every initial condition x_0 (or at least a set of initial conditions of positive Lebesgue measure) has an orbital average converging to the space average:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k x_0) = \int \varphi \, d\mu \, . \tag{1}$$

If μ is not only ergodic, but also mixing, we have *convergence to equilibrium:*

$$\lim_{n\to\infty} \int \varphi \circ f^n \, d\mathrm{Leb} = \int \varphi \, d\mu \, , \tag{2}$$

and more generally, for any pair of square integrable observables, *decay of (operational) correlations:*

$$\lim_{n\to\infty} \left(\int \varphi_1 \circ f^n \, \varphi_2 \, d\mathrm{Leb} - \int \varphi_1 \, d\mu \int \varphi_2 \, d\mathrm{Leb} \right) = 0 \, . \tag{3}$$

We are being slightly sloppy here: strictly speaking, decay of the correlations is defined by the relation

$$\lim_{n\to\infty} \left(\int \varphi_1 \circ f^n \, \varphi_2 \, d\mu - \int \varphi_1 \, d\mu \int \varphi_2 \, d\mu \right) = 0 \, . \tag{4}$$

In the framework of the present paper we obtain both (3) and (4). Since $\int \varphi_1 \circ f^n \, \varphi_2 \, d\mathrm{Leb}$ is more accessible experimentally, we concentrate on (3) for the sake of simplicity.

When (2–3–4) hold, a natural question is: "how fast does convergence take place?" Such asymptotic information on *rates of mixing* may sometimes be obtained for smooth enough observables, and often yields a central limit theorem for them. The article [15] (see also references therein) contains a discussion of this class of problems as well as some specific examples of uniformly and nonuniformly hyperbolic dynamical systems where the rate of mixing is exponential. One of these examples is given by the quadratic family $x \mapsto a - x^2$ on the interval for "good" (Collet–Eckmann or Benedicks–Carleson) values of the parameter a, or more generally unimodal maps satisfying certain axioms (see (H1)-(H4) below).

We study random perturbations of dynamical systems, more precisely *independent identically distributed (i.i.d.) small perturbations of interval maps:* Let $f : I \to I$ be a smooth transformation such that $f(I)$ is contained in the interior of I, and for small $\epsilon > 0$, let ν_ϵ be a probability on $[-\epsilon, \epsilon]$. In the following subsections we present two models for the random compositions of $f + \omega_0$, with ω_0 selected in $[-\epsilon, \epsilon]$ following the law ν_ϵ.

1.2. Markov chain.

Here, we average over all possible random realisations. Because of the i.i.d. assumption, this can be done by averaging at each time-step. More formally, this means considering the *Markov chain* $\{X_n\}_{n=1}^{\infty}$ with transition probabilities ($x \in I$, and $E \subset I$ with characteristic function χ_E):

$$\mathrm{Prob}\,(X_{n+1} \in E \mid X_n = x) = \int_{-\epsilon}^{\epsilon} \chi_E(f(x) + \omega_0)\, d\nu_\epsilon(\omega_0)\,. \tag{5}$$

Under rather weak assumptions, it is possible to show that this Markov chain admits a *unique* invariant probability measure. Recall that a measure μ_ϵ on I is invariant for the Markov chain (5) if for each Borel set E

$$\mu_\epsilon(E) = \int_{-\epsilon}^{\epsilon} \int_I \chi_E(f(x) + \omega_0)\, d\mu_\epsilon(x)\, d\nu_\epsilon(\omega_0)\,. \tag{6}$$

Let σ be the shift $(\sigma\omega)_k = \omega_{k+1}$ on $\Omega_\epsilon = [-\epsilon, \epsilon]^{\mathbf{Z}}$. Writing $f_\omega(x) = f(x) + \omega_0$, and by induction $f_\omega^n(x) = f_{\sigma\omega}^{n-1} f_\omega(x)$ for $\omega \in \Omega_\epsilon$, one defines *operational correlation functions*

$$\int_{-\epsilon}^{\epsilon} \int_I \varphi_1 \circ f_\omega^n\, \varphi_2\, d\mathrm{Leb} \prod_{i=0}^{n-1} d\nu_\epsilon(\omega_i) - \int_I \varphi_1 d\mu_\epsilon \int_I \varphi_2 d\mathrm{Leb}\,. \tag{7}$$

for the Markov chain. (The equivalent of the classical correlations (4) may be defined similarly.) Besides obtaining upper bounds for the difference (7) as $n \to \infty$, it is of obvious interest to study *stochastic stability,* i.e., whether $\mu_\epsilon \to \mu$ (at which speed? in which topology?) and whether the rate of decay of correlations (7) is stable as $\epsilon \to 0$.

For smooth expanding (also in higher dimension) or piecewise smooth and piecewise expanding interval maps, the Markov chain model was studied by Baladi–Young [5] who proved exponential decay of correlations and strong stochastic stability. Baladi–Viana [4] then extended these Markov chain results to a positive measure set of nonuniformly expanding unimodal maps, for which Benedicks–Young [6] had previously obtained a weaker form of stochastic stability. (We also refer to results of Katok-Kifer [11] for more general perturbations of unimodal maps, but under a Misiurewicz assumption, as well as to unpublished work of Collet [10].)

1.3. Random skew product.

Alternatively, we may consider the *random skew product* $T : I \times \Omega_\epsilon \to I \times \Omega_\epsilon$, defined by

$$T(x, \omega) = (f_\omega(x), \sigma(\omega))\,. \tag{8}$$

The objects of study are the T-invariant probability measures $\mu_\omega(d\mathrm{Leb})\, P_\epsilon(d\omega)$ with

$$P_\epsilon = \nu_\epsilon^{\mathbf{Z}}\,, \tag{9}$$

in particular those for which almost each μ_ω is absolutely continuous with respect to Lebesgue. In this case, we have a family of *quasi-invariant measures* μ_ω, so called because $(f_\omega)_*\mu_\omega = \mu_{\sigma\omega}$ for almost every ω. It is natural in this context to consider both the *future* ("aiming at a moving target"), and the *past rates* of *convergence to equilibrium:*

$$R_\omega^{(f)}(n) = |f_\omega^n(d\mathrm{Leb}) - \mu_{\sigma^n\omega}| \quad \text{and} \quad R_\omega^{(p)}(n) = |f_{\sigma^{-n}\omega}^n(d\mathrm{Leb}) - \mu_\omega|\,. \tag{10}$$

We may also consider the "future" and "past" *operational random correlations:*

$$C^{(f)}_{\omega,\varphi_1,\varphi_2}(n) = \int (\varphi_1 \circ f^n_\omega)\varphi_2 \, d\mathrm{Leb} - \int \varphi_1 \, d\mu_{\sigma^n\omega} \int \varphi_2 \, d\mathrm{Leb}$$
$$C^{(p)}_{\omega,\varphi_1,\varphi_2}(n) = \int (\varphi_1 \circ f^n_{\sigma^{-n}\omega})\varphi_2 \, d\mathrm{Leb} - \int \varphi_1 \, d\mu_\omega \int \varphi_2 \, d\mathrm{Leb} \qquad (11)$$

(Note that it is not immediate in general to obtain bounds on the future random correlation functions from estimates on the past random correlation functions, and vice versa.) Again, one could also consider the classical correlations.

We wish to obtain *for almost every random itinerary* ω, bounds of the type $C_\omega \cdot \psi(n)$ on the moduli of expressions in (10–11), where $\psi(n) \to 0$ at a certain rate, independently of ω. Probabilistic bounds on

$$P_\epsilon(\{\omega \mid C_\omega > m\}) \qquad (12)$$

are also desirable. The stochastic stability questions mentioned in the framework of the Markov chain may also be asked here.

A large body of literature is available (in particular by Kifer, and the school of L. Arnold in Bremen) regarding these randow skew products, and we only mention results connected to the present article. Bogenschütz [7] and Baladi et al. [3] studied random correlations for smooth expanding dynamics, proving exponential decay of future and past correlations together with a strong form of stochastic stability. The work of Khanin–Kifer [12] concerns more general equilibrium states for random compositions of maps expanding in average (they studied neither stability nor rates of mixing). More recently, Buzzi [8, 9] considered random compositions of piecewise monotone interval maps having some expansion property (but not necessarily close to a fixed map) and showed existence of absolutely continuous quasi-invariant measures and exponential decay of both future and past correlations.

Relating the two approaches

Obviously, bounding (7) (or the classical correlation) is not enough to estimate (11). In the other direction, estimates (11) yield corresponding bounds for (7) whenever the control in (12) is enough to guarantee that $C_\omega \in L^1(P_\epsilon)$. (In fact, some additional information is needed – and available – to estimate expressions of the type $\int \phi_1(\sigma^n\omega)\phi_2(\omega)\, dP_\epsilon(\omega) - \int \phi_1(\omega)\, dP_\epsilon(\omega) \int \phi_2(\omega)\, dP_\epsilon(\omega)$.) Also, it may be argued that a control of "almost all random itineraries" with information of the type (12) is more relevant to actual physical experiments than the averaged bounds (11): After all, only finitely many experiments may be actually realised!

1.4. Informal statement of results

Starting from a "good" unimodal map f (our assumptions are stated axiomatically as (H1)–(H4) in Section 2, they apply to a positive measure set of parameters of the quadratic family) and an atomless probability ν_ϵ on $[-\epsilon, \epsilon]$ (the precise assumption is (14)) for small enough ϵ, we consider i.i.d. compositions of $f + \omega_0$. We show that for small enough ϵ and P_ϵ-almost every $\omega \in \Omega_\epsilon$:

1. There is a unique family of absolutely continuous quasi-invariant probability measures $\mu_\omega = h_\omega \, d\mathrm{Leb}$ (we call the h_ω *quasi-invariant densities*);
2. We have *stretched exponential decay* for the rates of mixing. More precisely, there is $0 < u < 1$, independent of ϵ, and for each small ϵ a random variable C_ω on Ω_ϵ with the following properties. First, there is $v > 1$ so that

$$P_\epsilon(\{\omega \mid C_\omega > m\}) \le \frac{1}{m^v} \tag{13}$$

for large enough m. Second, for all Lipschitz test functions φ_i, $i = 1, 2$, there is $C(\varphi_1, \varphi_2)$, depending only on their Lipschitz constants and on ϵ so that, with $R^{(f)}_\omega(n)$ $R^{(p)}_\omega(n)$, $C^{(f)}_{\omega,\varphi_1,\varphi_2}(n)$, $C^{(p)}_{\omega,\varphi_1,\varphi_2}(n)$ as in (10), (11), we have for all $n \in \mathbf{Z}_+$

$$\max\left(R^{(f)}_\omega(n), R^{(p)}_\omega(n), C^{(f)}_{\omega,\varphi_1,\varphi_2}(n), C^{(p)}_{\omega,\varphi_1,\varphi_2}(n)\right) \le C(\epsilon) C_\omega \, C(\varphi_1, \varphi_2) \; e^{-n^u} .$$

We believe that this is the first time that estimates have been obtained for the random rates of mixing in a concrete nonuniformly hyperbolic dynamical setting.

The bound (13) on C_ω is integrable, and averaging our results on the random correlations gives that the Markov chain correlation decays faster than $\exp(-n^u)$ for some $0 < u < 1$, a result not as good as the exponential decay obtained in [4]. In view also of the fact that the transition from exponential to stretched exponential bounds occurs rather late in our proof (it is a consequence of the waiting times interfering with some combinatorical bounds similar to those appearing in Section 7.6 of [15]), it is not clear whether the subexponentiality is an artefact of our method.

We would like to emphasize that our constants C_ω and $C(\varphi_1, \varphi_2)$ blow up as $\epsilon \to 0$. In particular, we do *not* address in the present paper the question of stochastic stability.

One of the advantages of this work as contrasted to the previous studies ([4, 5, 6], etc.) of the Markov chain approach is that it is naturally suited to extensions to the non-autonomous case. More precisely, instead of assuming full i.i.d., that is $P_\epsilon = \nu_\epsilon^{\mathbf{Z}}$, we could suppose that P_ϵ is a "rapidly" mixing stationary measure for σ.

Our result was announced for the first time during the 31st Symposium in Mathematical Physics in May 1999 in Toruń. Its detailed proof is too long for the 12-page limit of the present proceedings and is available in the article *Almost sure rates of mixing for i.i.d. unimodal maps* by the same authors [2].

2. Statement of results and elements of proof

2.1. Formal statement of results

Let $I = [-1, 1]$ and $f : I \to I$ be a C^2 unimodal map (i.e., f is increasing on $[-1, 0]$, decreasing on $[0, 1]$) satisfying $f''(0) \neq 0$, and,

(H1) There are $\alpha > 0$ and $1 < \lambda \le 4$ with $200\alpha < (\log \lambda)^2$, and $\sup_I |f'| < 8$ so that
(i) $|(f^n)'(f0)| \ge \lambda^n$ for all $n \in \mathbf{Z}$;
(ii) $|f^n(0)| \ge e^{-\alpha n}$, for all $n \ge 1$.

(H2) For each small enough $\delta > 0$, there is $M = M(\delta) \in \mathbf{Z}_+$ for which
(i) If $x, \dots, f^{M-1}(x) \notin (-\delta, \delta)$ then $|(f^M)'(x)| \geq \lambda^M$;
(ii) For each n, if $x, \dots, f^{n-1}(x) \notin (-\delta, \delta)$ and $f^n(x) \in (-\delta, \delta)$, then $|(f^n)'(x)| \geq \lambda^n$.
(H3) $f(I)$ is a subset of the interior of I.
(H4) f is topologically mixing on $[f^2(0), f(0)]$.

Examples of unimodal maps satisfying (H1)–(H4) are quadratic maps $a - x^2$ for a positive measure set of parameters a (see e.g. [4] for notations similar to those of the present paper). Condition (H1)(i) is often replaced by $|(f^n)'(f(0))| \geq \lambda^n / C$, for some $C > 0$ and all $n \geq 1$, but our formulation also works. Condition (H2) is in fact implied by the hypothesis that there exist $\delta > 0$ and $M \in \mathbf{Z}_+$ such that (H2)(i)–(ii) hold. Also, (H1)(i) implies (H1)(ii) for *some* α (not necessarily such that $200\alpha < (\log \lambda)^2$), e.g. if $f'(-1) > 1$, see Corollary 4.10 in [13] (based on previous results of Nowicki).

Fixing $\epsilon_0 > 0$ small enough to guarantee $f(x) \pm \epsilon_0 \in I$ for all $x \in I$, we assume that we are given a constant $C > 0$ and for each $0 < \epsilon < \epsilon_0$ a probability measure ν_ϵ on $[-\epsilon, \epsilon]$ such that for any subinterval $J \subset [-\epsilon, \epsilon]$,

$$\nu_\epsilon(J) \leq \frac{C|J|}{\epsilon} . \tag{14}$$

The bound (14) holds e.g. if ν_ϵ has a bounded density with respect to Lebesgue. We do not need to assume that 0 belongs to the support of ν_ϵ.

We study random compositions of maps $f_\omega(x) = f(x) + \omega_0$ with $\omega \in \Omega_\epsilon$ following the i.i.d. law P_ϵ from (9). For $n \geq 1$, write

$$f^n_\omega(x) = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x) ,$$

and let $\mathrm{Lip}\, \varphi$ denote the Lipschitz norm of a complex valued function. Denoting Lebesgue measure on I by $d\mathrm{Leb}$, and $|\mu|$ for the total mass of a signed measure, our main result is:

Theorem 1. **(Stretched exponential mixing for i.i.d. unimodals)** *If ϵ is small enough, then for P_ϵ-almost each $\omega \in \Omega_\epsilon$ there is a quasi-invariant density $h_\omega \in L^1(d\mathrm{Leb})$. For some $0 < u < 1$, independent of ϵ, and P_ϵ-almost all $\omega \in \Omega_\epsilon$, there are constants $C^{(1)}_\omega > 0$ (depending on ϵ and u) such that for each Lipschitz function $\varphi : I \to \mathbf{C}$, and all $n \geq 1$,*

$$\left|(f^n_{\sigma^{-n}\omega})_*(\varphi\, d\mathrm{Leb}) - (h_\omega\, d\mathrm{Leb})\right| \leq C^{(1)}_\omega \,\mathrm{Lip}\, \varphi\, e^{-n^u} . \tag{15}$$

Additionally, for almost every $\omega \in \Omega_\epsilon$, there are constants $C^{(2)}_\omega > 0$, $C^{(3)}_\omega > 0$ such that for all Lipschitz functions $\varphi_1, \varphi_2 : I \to \mathbf{C}$, the "past" and "future" random operational correlation functions satisfy for all $n \geq 1$

$$\left| \int \varphi_1 \circ f^n_{\sigma^{-n}\omega}\, \varphi_2\, d\mathrm{Leb} - \int \varphi_1 h_\omega\, d\mathrm{Leb} \int \varphi_2\, d\mathrm{Leb} \right| \leq C^{(2)}_\omega \sup|\varphi_1|\, \mathrm{Lip}\, \varphi_2\, e^{-n^u} , \tag{16}$$

and

$$\left| \int \varphi_1 \circ f_\omega^n \, \varphi_2 \, d\mathrm{Leb} - \int \varphi_1 h_{\sigma^n \omega} \, d\mathrm{Leb} \int \varphi_2 \, d\mathrm{Leb} \right| \le C_\omega^{(3)} \sup |\varphi_1| \, \mathrm{Lip} \, \varphi_2 \, e^{-n^u} . \quad (17)$$

Finally, there are $v > 1$ *and* $C(\epsilon) > 0$ *so that* $C_\omega = \max(C_\omega^{(1)}, C_\omega^{(2)}, C_\omega^{(3)})$ *satisfies*

$$P_\epsilon(\{\omega \in \Omega_\epsilon \mid C_\omega > m\}) \le C(\epsilon) \frac{1}{m^v} . \quad (18)$$

Before commenting the proof in the next subsection, we list few remarks. We refer to [3] for the operational significance of, and experimental access to, the rates in (15–17).

In the present i.i.d. setting, the almost everywhere existence of the quasi-invariant measures (but not their absolute continuity) could be obtained by disintegrating the skew-product absolutely continuous invariant measure which is just a natural extension of $\mu_\epsilon \times \nu_\epsilon^{\mathbf{Z}_+}$ where μ_ϵ is the Markov chain invariant measure in [4] or [6].

Our argument also yields a statement on random countable Markov chains with estimates (after waiting times) on the recurrence times. (See Main Corollary in [2].)

2.2. Idea of proof

The key ingredient is to construct a random version of the towers used by Young [16], showing that the coupling method there can be randomised. The first difficulty is to modify the standard partition of the interval associated to good unimodal maps (see e.g. [15]) and obtain good estimates on the mass of points with large return times. Here, a beautiful idea due to Alves [1] was instrumental. Alves studied (maps close to) a deterministic skew product $T(x, \theta) = (a - x^2 + \epsilon\theta, D\theta \bmod 1)$ where $D \gg 1$. The map $\theta \mapsto D\theta$ is a "rapidly mixing" deterministic dynamical system on the circle. In order to construct absolutely continuous invariant measure for T on the cylinder, Alves introduced good partitions into rectangles, involving a crucial notion of "hyperbolic time" (an abstraction of the escape time relevant in unimodal or Hénon maps, which was later applied by Alves-Bonatti-Viana and Castro to analyze partially hyperbolic systems). Alves exploited important bounds on "exceptional sets" of points (x, θ) previously obtained by Viana [14], who was the first to study this skew product model and proved that it possesses two positive Lyapunov exponents. Although we consider a slightly more general framework than the Misiurewicz case in [1] and [14], many properties become easier to prove in our i.i.d. setting (the basic large deviations argument used to estimate exceptional sets is much simpler). The key observation is that the bounds on the mass of those ω such that a given x "behaves well" after n iterates, which are obtained by adapting [1, 14], are uniform in x. Thus, a careful application of Fubini's theorem allows us to exchange x and ω (up to a zero-measure set of bad ω:s which may be excluded). On the other hand, we are forced to introduce "waiting times" when performing the random parameter exclusion. These waiting times make the construction of the tower and coupling argument more intricate. We content ourselves with describing in the next subsection the axioms for the random towers that we construct, referring to [2] for details.

Once the towers were established, we noticed that an estimate of Young (see the "choice of n_0" in Lemma 1 of [16]) which was a trivial consequence of the mixing property of the measure in the deterministic case, becomes much more troublesome in the random case. To deal with this, we bootstraped from the mixing property of the Markov chain on the tower (which follows from mixing of the random skew product) applied in (yet) another large deviation argument within the coupling estimates. The corresponding statement is presented and proved below as Lemma 3.

2.3. Random towers with waiting times

In [2] we explain how to choose a suitable small subinterval $\Lambda \subset [-1,1]$, independently of ϵ (and ω), close to $+\delta$ or $-\delta$, and how to construct for almost all ω a countable partition of Λ into $\Lambda_{\omega,j}$, as well as a return-time function $R_\omega : \Lambda \to \mathbf{Z}_+ \cup \{\infty\}$.

Starting from this data, we now define tower extensions $F_\omega : \Delta_\omega \to \Delta_{\sigma\omega}$. Set

$$\Delta_\omega = \left\{ (x,\ell) \mid x \in \bigcup_j \Lambda_{\sigma^{-\ell}\omega,j}\,, 0 \le \ell \in \mathbf{Z}_+\,, \ell \le R_{\sigma^{-\ell}\omega}(x) - 1 \right\}.$$

In other words, layer $R_\omega(x) - 1$ disjoint copies of $\Lambda_{\omega,j}$ in "Pisa – or Toruń – tower fashion." Denote by $\Delta_{\omega,\ell}$ the ℓth level of the tower $\{(x,\ell) \in \Delta_\omega\}$.

The dynamics $F_\omega : \Delta_\omega \to \Delta_{\sigma\omega}$ consists in hopping from one tower to the next above $(x,0)$, stopping at level $R_\omega(x) - 1$ if $R_\omega(x) < \infty$, and falling down to the zeroth level of $\Delta_{\sigma^{R_\omega(x)}\omega}$ using the return map $f_\omega^{R_\omega} : \bigcup_j \Lambda_{\omega,j} \to \Lambda$ defined by $f_\omega^R(x) = f_\omega^{R_\omega(x)}(x)$. In other words,

$$F_\omega(x,\ell) = \begin{cases} (x, \ell+1), & \text{if } \ell + 1 < R_{\sigma^{-\ell}\omega}(x), \\ (f_{\sigma^{-\ell}\omega}^{R_{\sigma^{-\ell}\omega}}(x), 0), & \text{if } \ell + 1 = R_{\sigma^{-\ell}\omega}(x). \end{cases}$$

In particular, $F_\omega^{R_\omega}|_{\Delta_{\omega,0}} = f_\omega^R|_\Lambda$. Clearly, the projection $\pi_\omega : \Delta_\omega \to [-1,1]$ defined by $\pi_\omega(x,\ell) = f_{\sigma^{-\ell}\omega}^\ell x$ satisfies $f_\omega \circ \pi_\omega = \pi_{\sigma\omega} \circ F_\omega$ and $\pi_\omega(\Delta_\omega) = \bigcup_{\ell \ge 0} f_{\sigma^{-\ell}\omega}^\ell (\bigcup_j \Lambda_{\sigma^{-\ell}\omega,j})$. It is through this projection that we relate the ergodic properties of f and those of F.

For each ℓ, we consider the countable partition $\mathcal{Z}_{\omega,\ell}$ of $\Delta_{\omega,\ell}$ induced by $\bigcup_j \Lambda_{\sigma^{-\ell}\omega,j}$

$$\Delta_{\omega,\ell} = \bigcup_{j \text{ s.t. } R_\omega|_{\Lambda_{\sigma^{-\ell}\omega,j}} \ge \ell+1} \Lambda_{\sigma^{-\ell}\omega,j}\,.$$

We also let $\mathcal{Z}_\omega$, $\mathcal{Z}$ be the corresponding partitions of Δ_ω, respectively $\bigcup_\omega \Delta_\omega$.

Finally, for almost each ω, we introduce a *separation time* $s_\omega : \Delta_\omega \times \Delta_\omega \to \mathbf{Z}_+ \cup \{\infty\}$

$$s_\omega(x,y) = \min\left\{ n \ge 0 \mid F_\omega^n(x) \text{ and } F_\omega^n(y) \text{ lie in distinct elements of } \mathcal{Z} \right\}.$$

The separation time is used in (A.IV) below to express the "contraction" of inverse branches of F. It is also useful to define dynamical Lipschitz function spaces $\mathcal{F}$ and $\mathcal{F}^+$, for which correlations of F are studied. We refer to Section 5 in [2] for details.

We list the properties of the tower, which allow us to implement the approach in [16] to construct the quasi-invariant measure and study its rate of mixing:

(A.I) **[Return and separation times]** There is $p(\epsilon)$ so that $R_\omega : \Delta_\omega \to \mathbf{Z}_+$ is constant on each interval of the partition $\mathcal{Z}_\omega$; with $R_\omega \ge p(\epsilon)$. If (x,ℓ) and (y,ℓ) are both in the same interval of $\mathcal{Z}_\omega$, then $s_\omega((x,0),(y,0)) \ge \ell$. For any $(x,0),(y,0)$ in the same interval of $\mathcal{Z}_\omega$.

$$s_\omega(x,y) = R_\omega(x) + s_{\sigma^{R_\omega}}\left(f^{R_\omega(x)}(x), f^{R_\omega(y)}(y)\right) .$$

(A.II) **[Markov property]** For each element $\Lambda_{\omega,j}$ of $\mathcal{Z}_{\omega,0}$, the map $F_\omega^{R_\omega}|_{\Lambda_{\omega,j}} : \Lambda_{\omega,j} \to \Lambda$ is a bijection.

(A.III) **[Weak forward expansion]** The partition $\mathcal{Z}_\omega$ is generating for F_ω: the diameters of the partitions $\bigvee_{j=0}^n F_{\sigma^{-j}\omega}^{-j} \mathcal{Z}_\omega$ tend to zero as $n \to \infty$.

(A.IV) **[Bounded distortion]** There are $C(\epsilon) > 1$ and $0 < \beta < 1$ (β is independent of ϵ) such that for all ω and each element $\Lambda_{\omega,j}$ in $\mathcal{Z}_\omega$, the map $F^{R_\omega}|_{\Lambda_{\omega,j}}$ and its inverse are nonsingular with respect to the measure Leb. Also, writing $JF_\omega^{R_\omega} > 0$ for the Jacobian of $F_\omega^{R_\omega}$, we have for each $x, y \in \Lambda_{\omega,j}$, writing s for $s_{\sigma^{R_\omega(x)}\omega}$

$$\left| \frac{JF_\omega^{R_\omega}(x)}{JF_\omega^{R_\omega}(y)} - 1 \right| \le C(\epsilon)\, \beta^{s(F_\omega^{R_\omega}(x,0), F_\omega^{R_\omega}(y,0))}$$

(A.V) **[Return times asymptotics after waiting time]** There exist an exponent $0 < u < 1$, and, for each small enough $\epsilon > 0$, a random variable $n_1(\omega)$ (the *waiting time*) with

$$P_\epsilon(\{n_1(\omega) > n\}) \le C(\epsilon) e^{-n^u} ,$$

and such that for almost all ω and all $\ell \ge n_1(\omega)$:

$$\mathrm{Leb}(\{x \in \Lambda \mid R_\omega(x) \ge \ell\}) \le C(\epsilon) e^{-\ell^u} .$$

Furthermore there is an almost everywhere defined random variable $n_2 \ge n_1$ so that

$$\begin{cases} n_1(\sigma^{-\ell}\omega) \le \ell \text{ and } n_1(\sigma^\ell \omega) \le \ell , & \forall \ell \ge n_2(\omega), \\ P_\epsilon\left(\{\omega \mid n_2(\omega) > n\}\right) \le C(\epsilon)\, e^{-n^u} , \forall n . \end{cases}$$

In particular, for almost all ω we have the *summability* property:

$$\begin{aligned} \mathrm{Leb}_\omega(\Delta_\omega) &= \sum_{\ell \in \mathbf{Z}_+} \mathrm{Leb}(\{x \in \Lambda \mid R_{\sigma^{-\ell}\omega} \ge \ell + 1\}) \\ &\le n_2(\omega) + C(\epsilon) \sum_{\ell > n_2(\omega)} e^{-\ell^u} < \infty . \end{aligned} \tag{19}$$

(A.VI) **[Mixing: Gcd(Return times)=1]** There are $N \ge 1$ and $\{t_i \in \mathbf{Z}_+ , i = 1, \dots N\}$ with gcd $\{t_i\} = 1$ such that for P_ϵ-almost all ω, all $n \in \mathbf{Z}$ and all $1 \le i \le N$, we have $\mathrm{Leb}(\{x \in \Lambda \mid R_\omega(x) = t_i\}) > 0$. In fact, we have the following stronger property: there is $M(\epsilon) \ge 1$ so that for almost all ω and each $r \ge M$ the set of $x \in \Lambda$ with $R_\omega(x) = r$ has positive Lebesgue measure.

The mixing assumption (H4) is only used to get (A.VI). Properties (A.I)–(A.IV) together with (19) are enough to prove:

THEOREM 2. **(Quasi-invariant densities)** *There is for almost each $\omega \in \Omega$ a quasi-invariant absolutely continuous probability measure $\tilde{\mu}_\omega = h_\omega \, dLeb$ on Δ_ω for F_ω.*

Additionally there is $0 < u < 1$, independent of ϵ and, for almost all ω, a constant $K_\omega > 0$ so that for each $x \in \Delta_\omega$

$$K_\omega^{-1} \le h_\omega(x) \le K_\omega \, ,$$

and $P_\epsilon(\{\omega \mid K_\omega > K\}) \le e^{-K^u}$ for all large enough K.

We obtain μ_ω from $\tilde{\mu}_\omega$ by using the semi-conjugacy π_ω.

2.4. Large deviations and returns to the basis

In this last subsection we explain one of the additional lemmas needed to go from the coupling approach for deterministic towers in [16] to the argument for the random towers described in Subsection 2.3.

For $\ell \in \mathbf{Z}_+$, consider the random variable:

$$V_\omega^\ell = \mathrm{Leb}(\Delta_{\omega,0} \cap F_\omega^{-\ell}(\Delta_{\sigma^\ell \omega,0})) = \int (\chi_{\Delta_{\sigma^\ell \omega,0}} \circ F_\omega^\ell) \cdot \chi_{\Delta_{\omega,0}} \, d\mathrm{Leb}.$$

Recall that μ_ϵ is the invariant measure for the Markov chain and $\Delta_{\omega,0}$ is Λ for all ω. For small $\gamma > 0$ to be chosen later, the mixing property of the skew product F together with the fact that the function $\chi_{\Delta_{\omega,0}}/h_\omega$ belongs to $L^2(\mu_\omega)$ (see Theorem 2. and [2]) give that there exists ℓ_0 such that for all $\ell \ge \ell_0$, the expectation of V_ω^ℓ satisfies

$$\left| \int_{\Omega_\epsilon} V_\omega^\ell \, dP_\epsilon(\omega) - \mathrm{Leb}(\Lambda) \cdot \mu_\epsilon(\Lambda) \right| < \gamma \, . \tag{20}$$

For any $m \in \mathbf{Z}_+$ and each fixed sequence of integers ("stopping times") $0 = \tau_0 < \tau_1 < \cdots < \tau_m$ such that $\tau_{i+1} - \tau_i \ge \ell_0$, define:

$$S_m^{\{\tau_i\}}(\omega) = \sum_{i=1}^m V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i - \tau_{i-1}} \, .$$

LEMMA 3. **(Large deviations for returns to basis)** *There exist $\rho > 0$ and $0 < \kappa < 1$ such that for each $m \in \mathbf{Z}_+$ and every sequence of integers $0 = \tau_0 < \tau_1 < \cdots < \tau_m$ with $\tau_{i+1} - \tau_i \ge \ell_0$, we have*

$$P_\epsilon(\{\omega \mid S_m^{\{\tau_i\}}(\omega) < m\rho\}) \le \kappa^m \, . \tag{21}$$

Proof of Lemma 3. The random variable V_ω^ℓ depends only on $\omega_0, \ldots, \omega_{\ell-1}$, so V_ω^ℓ and $V_{\sigma^j \omega}^k$ are independent provided $j \ge \ell$. In particular, $S_m^{\{\tau_i\}}(\omega)$ is a sum of independent random variables. For any $v > 0$ and $t > 0$,

$$
\begin{aligned}
P_\epsilon(\{S_m^{\{\tau_i\}}(\omega) < t\}) &\le \int \exp[\upsilon(t - S_m^{\{\tau_i\}}(\omega))]\, dP_\epsilon(\omega) \\
&\le e^{\upsilon \cdot t} \int \exp[-\upsilon S_m^{\{\tau_i\}}(\omega)]\, dP_\epsilon(\omega) \\
&\le e^{\upsilon \cdot t} \prod_{i=0}^{m-1} \int \exp[-\upsilon V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i - \tau_{i-1}}]\, dP_\epsilon(\omega) \text{ (by independence)}.
\end{aligned}
$$

We have $0 \le V_\omega^\ell \le \mathrm{Leb}(\Lambda)$ and, by (20),

$$
\int V_\omega^\ell\, dP_\epsilon(\omega) \ge \mathrm{Leb}(\Lambda) \cdot \mu_\epsilon(\Lambda) - \gamma ,
$$

provided $\ell \ge \ell_0$. Now, since $0 \le \upsilon V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i - \tau_{i-1}} \ll 1$,

$$
\int \exp[-\upsilon V_{\sigma^{\tau_{i-1}}\omega}^{\tau_i - \tau_{i-1}}]\, dP_\epsilon(\omega) \le 1 - \upsilon \Big[\mathrm{Leb}(\Lambda)\mu_\epsilon(\Lambda) - \gamma - \mathrm{Leb}(\Lambda)^2 \frac{\upsilon}{2}\Big] =: a(\upsilon, \gamma) .
$$

Take $0 < \upsilon < 2\mu_\epsilon(\Lambda)$ and then $\gamma > 0$ small enough so that $0 < a(\upsilon,\gamma) < 1$. We find

$$
P_\epsilon(\{S_m^{\{\tau_i\}}(\omega) < m\rho\}) \le (e^{\upsilon\rho} \cdot a(\upsilon,\gamma))^m \le \kappa^m
$$

for some $0 < \kappa < 1$, by choosing $0 < \rho < \upsilon^{-1} \log(1/a(\upsilon,\gamma))$. This proves the lemma.

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