# WHITNEY–HÖLDER CONTINUITY OF THE SRB MEASURE FOR TRANSVERSAL FAMILIES OF SMOOTH UNIMODAL MAPS

VIVIANE BALADI, MICHAEL BENEDICKS, AND DANIEL SCHNELLMANN

ABSTRACT. We consider $C^2$ families $t \mapsto f_t$ of $C^4$ nondegenerate unimodal maps. We study the absolutely continuous invariant probability (SRB) measure $\mu_t$ of $f_t$, as a function of $t$ on the set of Collet–Eckmann (CE) parameters:

*Upper bounds:* Assuming existence of a transversal CE parameter, we find a positive measure set of CE parameters $\Delta$, and, for each $t_0 \in \Delta$, a set $\Delta_0 \subset \Delta$ of polynomially recurrent parameters containing $t_0$ as a Lebesgue density point, and constants $C \geq 1$, $\Gamma > 4$, so that, for every 1/2-Hölder function $A$,

$$\left| \int A \, d\mu_t - \int A \, d\mu_{t_0} \right| \leq C\|A\|_{C^{1/2}} |t - t_0|^{1/2} |\log|t - t_0||^{\Gamma} \,, \ \forall t \in \Delta_0 \,.$$

In addition, for all $t \in \Delta_0$, the renormalisation period $P_t$ of $f_t$ satisfies $P_t \leq P_{t_0}$, and there are uniform bounds on the rates of mixing of $f_t^{P_t}$ for all $t$ with $P_t = P_{t_0}$. If $f_t(x) = tx(1-x)$, the set $\Delta$ contains almost all CE parameters.

*Lower bounds:* Assuming existence of a transversal mixing Misiurewicz–Thurston parameter $t_0$, we find a set of CE parameters $\Delta'_{MT}$ accumulating at $t_0$, a constant $C \geq 1$, and a $C^\infty$ function $A_0$, so that

$$C|t - t_0|^{1/2} \geq \left| \int A_0 \, d\mu_t - \int A_0 \, d\mu_{t_0} \right| \geq C^{-1}|t - t_0|^{1/2} \,, \ \forall t \in \Delta'_{MT} \,.$$

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let $f : X \to X$ preserve an ergodic invariant probability measure $\mu$ which is absolutely continuous with respect to Lebesgue. Then Birkhoff's theorem implies that there is a positive Lebesgue measure set of points $x$ for which the time averages of iterated Dirac masses $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}$ converge (in the weak-$*$ topology) to $\mu$. A measure $\mu$ with this last property is also called an SRB measure. All SRB measures studied in the present paper are absolutely continuous, but there exist SRB measures which are not absolutely continuous, in particular those [46] constructed by Sinai, Ruelle, and Bowen for smooth hyperbolic systems such as Anosov diffeomorphisms.

We are interested in differentiable one-parameter families $t \mapsto f_t$, $t \in \mathbb{R}$ of differentiable dynamical systems where $f_t$ admits a unique SRB measure for a positive measure set of parameters $t$. In the case where each $f_t$ is a smooth transitive Anosov diffeomorphism, Ruelle [32, 34, 21] (see also [23]) showed that $t \mapsto R_A(t) := \int A d\mu_t$ is differentiable if $A$ is a smooth enough observable, and he obtained a formula (the *linear response formula*) for $\partial_t R_A(t)$. Since $\mu_t$ can be obtained as a fixed point for a Ruelle–Perron–Frobenius transfer operator with a spectral gap, this formula can be proved via perturbation theory on a suitable Banach space. This result led to the hope that linear response would hold in other dynamical situations where the SRB measure is related to a transfer operator with good spectral properties [33, 35, 36]. This hope was shattered when it was discovered ([6, 9], see also [29]) that linear response does not always hold in the simple situation of unimodal piecewise expanding interval maps. (Contrarily to Anosov maps, piecewise expanding maps are not structurally stable.) More precisely, $t \mapsto R_A(t)$ is differentiable at $t = 0$ if and only ([9, Thm 7.1] see also the remarks after Theorem 1.7 below) if the family $f_t$ is *horizontal,* that is, tangent to the topological class of $f_0$ at $t = 0$. Horizontality is an explicit codimension-one condition on the vector field $\partial_t f_t$ [11]. In the *transversal* (non-horizontal) case, Theorem 7.1 of [9] (see also corrigendum) shows that the $|t \log |t||$ modulus of continuity which had been discovered long before by Keller [24] is in fact optimal. (See also the discussion about parameters just after Theorem 1.7.)

Piecewise expanding maps can be viewed as a toy model for the more difficult case of smooth unimodal maps: While all unimodal piecewise expanding maps admit a unique absolutely continuous invariant probability measure, this does not hold in the smooth case, where vanishing of the derivative at the critical point means that hyperbolicity is not guaranteed (and occurs at best nonuniformly). The celebrated Collet–Eckmann condition (see (3) below) implies existence of a (unique) absolutely continuous invariant probability measure for smooth unimodal maps, with exponential decay of correlations in the mixing case. Indeed, one can then construct the SRB measure as a fixed point of a transfer operator, via a suitable tower construction (there are several such constructions; see, e.g., [45], [26], [13]). A generic family of smooth unimodal maps $f_t$ is transversal (or non-horizontal; see (4) below for a definition of transversality and [42], [3], [27], [12], [19] for previous occurrences in the literature). In a transversal family, the set of Collet–Eckmann parameters has positive measure, but does not contain any intervals. One natural question for families of smooth unimodal maps is then to study the regularity of $t \mapsto R_A(t)$, restricting $t$ to subsets of the Collet–Eckmann parameters. We consider only the nondegenerate case, where $f_t'(c) = 0$ and $f_t''(c) < 0$ at the critical point $c$. This includes the famous logistic (or quadratic) family $f_t = tx(1 - x)$. Continuity of $t \to R_A(t)$ was proved on a subset of "good" Collet–Eckmann parameters $t$ by Tsujii [43] and Rychlik–Sorets [39] in the 90's. Parameters in this subset enjoy not only qualitative slow recurrence ensuring the Collet–Eckmann property, but also quantitative control on the various relevant constants. (See Definition 2.2 for the notion of goodness used in the present paper.)

Quantified goodness is indeed necessary to ensure continuity, as we explain next: A parameter $t$ is called superstable if the critical point is periodic. For the quadratic family, e.g., Thunberg proved [41, Theorem C] that there are superstable parameters $s_n$ of periods $p_n$, with $s_n \to t$, where $t$ is a good Collet–Eckmann parameter, so

that $\nu_{s_n} \to \nu$, where $\nu_{s_n} = \frac{1}{p_n} \sum_{k=0}^{p_n-1} \delta_{f_{s_n}^k(c)}$, and $\nu$ is the sum of atoms on a repelling periodic orbit of $f_t$. Other sequences $t_n \to t$ of superstable parameters have the property that $\nu_{t_n} \to \mu_t$, the absolutely continuous invariant measure of $f_t$. Dobbs and Todd [17] have pointed out to us that it is not very difficult to construct, starting from Thunberg's result, sequences of *renormalisable* Collet–Eckmann maps (with nonuniform "goodness," in the terminology introduced below) converging to a Collet–Eckmann map, but such that the SRB measures do not converge. Dobbs and Todd [17] have recently generalised this result, finding *non-renormalisable* Collet–Eckmann maps $f_{t'_n}$ (with nonuniform "goodness") converging to a Collet–Eckmann map $f_t$, but such that the SRB measures do not converge. Such counter-examples can be constructed while requiring that $f_t$ and all maps $f_{t'_n}$ are Misiurewicz–Thurston. (Misiurewicz–Thurston maps are the smooth unimodal maps enjoying the most expansion, see below (5) for a definition.) These examples show that continuity of $R_A(t)$ cannot hold on the set of *all* Collet–Eckmann (or even Misiurewicz–Thurston) parameters: Some uniformity in the constants is needed.

Existence of the SRB measure holds under conditions much weaker than Collet–Eckmann (see [31] and references therein). Continuity of the SRB measure can be studied on suitable sets of "good" parameters enjoying this weaker property. (We would like also to draw attention to the exciting new approach of Shen [40] to stochastic stability.) Our aim here however is to study *moduli* of continuity of $t \mapsto R_A(t)$ for families of smooth unimodal maps — in order to go beyond mere continuity, it seems wise (and perhaps necessary) to restrict to subsets of good Collet–Eckmann parameters.

Until the present work, the only results going beyond continuity concerned fully horizontal families, that is, when all $f_t$ are topologically conjugated to $f_0$. Even in this "trivial" setting, where linear response can indeed be obtained ([10],[37], [12]), proofs were technically involved, in particular in [12], where analyticity was not assumed and the slow recurrence assumption was relatively weak.

We address here for the first time the modulus of continuity of the SRB measures in *transversal* families of (nondegenerate) smooth unimodal maps. We conjectured ([7, (3) in §3.2], making more precise [6, Conj. B]) that for $C^1$ observables $A$ the function $R_A(t)$ is $\eta$-Hölder for all $\eta < 1/2$. Our first main result, Theorem 1.2, gives a strengthening of this conjecture: We show there is a set $\Delta$ of Collet–Eckmann parameters, with $\Delta$ of positive measure, and, for each $t_0 \in \Delta$, a set $\Delta_0 \subset \Delta$ of polynomially recurrent parameters containing $t_0$ as a Lebesgue density point, and constants $C \geq 1$, $\Gamma > 4$, so that, for every $1/2$-Hölder function $A$,

$$\left| \int A \, d\mu_t - \int A \, d\mu_{t_0} \right| \leq C \|A\|_{C^{1/2}} |t - t_0|^{1/2} |\log |t - t_0||^{\Gamma}, \ \forall t \in \Delta_0.$$

This immediately implies a more precise result in the analytic case (Corollary 1.6). In particular, for the logistic family $f_t(x) = tx(1-x)$, the set $\Delta$ contains almost all Collet–Eckmann parameters.

Our proof implies that the renormalisation period $P_t$ is $\leq P_{t_0}$ for $t \in \Delta_0$, as well as uniform bounds on the exponential mixing for the ergodic components of $f^{P_t}$ for $t \in \Delta_0$ so that $P_t = P_{t_0}$ (Theorem 1.3).

We expected Conjecture B of [6] to be "essentially optimal." Making this more precise, we asked in [12] whether one can "construct a (non-horizontal) smooth family $f_t$ of quadratic unimodal maps, with $f_0$ a good map, so that $t \to \mu_t$, as a distribution of any order, is not differentiable (even in the sense of Whitney, for

reasonably large subsets) at $t = 0[$, or$]$ so that it is not Hölder for any exponent $> 1/2$." Our second main result, Theorem 1.7, answers this question positively: Assuming that the family is transversal at a mixing Misiurewicz–Thurston map $f_{t_0}$, we find a set of Collet–Eckmann parameters $\Delta'_{MT}$, accumulating at $t_0$, a constant $C \geq 1$, and a $C^\infty$ function $A_0$, so that

$$C|t - t_0|^{1/2} \geq \left| \int A_0 \, d\mu_t - \int A_0 \, d\mu_{t_0} \right| \geq C^{-1}|t - t_0|^{1/2} \, , \ \forall t \in \Delta'_{MT} \, .$$

We would like to point out that, in the piecewise expanding setting, the first counterexamples to differentiability of the SRB (see [6], [29]) had been obtained for sequences of maps having pre-periodic critical points converging to a map $f_{t_0}$ with a pre-periodic critical point. They were only later generalised to essentially all $f_{t_0}$ [9], and (except when the postcritical orbit of $f_{t_0}$ is dense) any $t \to t_0$.

Our results lead to several challenging questions for families of smooth unimodal maps, in particular regarding the size of the largest possible set $\Delta'_{MT}$, and what can be done if the Misiurewicz–Thurston assumption on $f_{t_0}$ is relaxed. (See the comments after the statements of Theorems 1.2 and 1.7 and Corollary 1.6 below.) We would like to note here a quantitative difference with respect to the piecewise expanding case [9] where the modulus of continuity in the transversal case was $|\log |t - t_0|||t - t_0|$, so that violation of linear response arose from the logarithmic factor alone.

More open questions are listed in [7] and [12]. In particular, the results in the present paper also give hope that analogous problems (see [7] and [38]) can be studied for (the two-dimensional) Hénon family, which is transversal, and where continuity of the SRB measure in the sense of Whitney in the weak-$*$ topology was proved by Alves et al. [2], [1].

We would like also to suggest here a weakening of the linear response problem: Consider a one-parameter family $f_t$ of (say, smooth unimodal maps) through $f_{t_0}$ and, for each $\epsilon > 0$, a random perturbation of $f_t$ with unique invariant measure $\mu_t^\epsilon$, e.g., like in [40]. Then for each positive $\epsilon$, it should not be very difficult to see that the map $t \to \mu_t^\epsilon$ is differentiable at $t_0$ (for essentially any topology in the image). Can we say something (existence? dependence on the perturbation? relation with the susceptibility function or some of its "extensions" [8]?) about the limit as $\epsilon \to 0$ of this derivative? (For a weak topology in the image, like Radon measures, or distributions of positive order.)

Before sketching the contents of the paper, we would like to highlight here some of the difficulties we had to face, and what are the new ideas and techniques with respect to the construction in [12]: We wish to compare the SRB measure of $f_0$ (assume $t_0 = 0$) to that of $f_t$ for suitable small $t$. Let us start with the similarities with [12]: Just like in [12], we use transfer operators $\widehat{\mathcal{L}}_t$ acting on towers, with a projection $\Pi_t$ from the tower to $L^1(I)$ so that $\Pi_t \widehat{\mathcal{L}}_t = \mathcal{L}_t \Pi_t$, where $\mathcal{L}_t$ is the usual transfer operator, and $\Pi_t \hat{\phi}_t = \phi_t$ with $\mu_t = \phi_t \, dx$ (here, $\hat{\phi}_t$ is the fixed point of $\widehat{\mathcal{L}}_t$, and $\phi_t$ is the invariant density of $f_t$). In [12], we adapted the tower construction in [13], allowing in particular the use of Banach spaces of continuous functions via the introduction of smooth cutoff functions (see the definition after (59)). We start from this adaptation. Another idea we import from [12] is the use of truncated operators $\widehat{\mathcal{L}}_{t,M}$ acting on truncated towers, where the truncation level $M$ must be chosen carefully depending on $t$. Roughly speaking, the idea is that

$f_t$ is comparable to $f_0$ for $M$ iterates (corresponding to the $M$ lowest levels of the respective towers), this is the notion of an admissible pair $(M, t)$. Denoting by $\hat{\phi}_{t,M}$ the maximal eigenvector of $\widehat{\mathcal{L}}_{t,M}$, the starting point for both our upper and lower bounds is (like in [12]) the decomposition (see (90))

(1)
$$\phi_t - \phi_0 = \left[ \Pi_t(\hat{\phi}_t - \hat{\phi}_{t,M}) + \Pi_0(\hat{\phi}_{0,M} - \hat{\phi}_0) \right] + \left[ \Pi_t(\hat{\phi}_{t,M} - \hat{\phi}_{0,M}) \right] + \left[ (\Pi_t - \Pi_0)(\hat{\phi}_{0,M}) \right],$$

for admissible pairs. The idea is then to get upper bounds on the first two terms by using perturbation theory à la Keller–Liverani [25], and to control the last (dominant) term by explicit computations on $\Pi_t - \Pi$ (which represents the "spike displacement," i.e., the effect of the replacement of $1/\sqrt{|x - f_0^k(c)|}$ by $1/\sqrt{|x - f_t^k(c)|}$ in the invariant density).

We now move to the differences: Using a tower with exponentially decaying levels as in [13] or [12] would limit us at best to an upper modulus of continuity $|t|^\eta$ for $\eta < 1/2$, and would not yield any lower bound. For this reason, we use instead tower levels with *polynomially decaying* sizes, working with polynomially recurrent maps ("fat towers"). In order to construct the corresponding parameter set, we need to *make use of very recent results of Gao and Shen [19]*. (See [5] for previous parameter exclusion results in the setting of the quadratic family.)

It turns out that applying directly the results of Keller–Liverani [25] would only give that the contributions of the first and second terms of (1) are bounded by $|t|^\eta$ for $\eta < 1/2$. In order to estimate the second term, we prove that $\widehat{\mathcal{L}}_{t,M} - \widehat{\mathcal{L}}_{0,M}$ *acting on the maximal eigenvector* is $O(|\log|t||^\Gamma |t|^{1/2})$ *in the strong norm* (see Lemma 4.5, which is used in Proposition 4.2; in the Misiurewicz–Thurston case we get get a better $O(|t|^{1/2})$ control). It is usually not possible to obtain strong norm bounds when bifurcations are present [14, 25] (see [18] for an exception), and this remarkable feature here is due to our choice of admissible pairs (combined with the fact that the towers for $f_t$ and $f$ are identical up to level $M$, just like in [12], see Lemma 3.8). In order to estimate the first term, we enhance the Keller–Liverani argument (Proposition 4.1), using again that it suffices to estimate the perturbation for the operators *acting on the maximal eigenvector*.

The changes just described are already needed to obtain the exponent $1/2$ in Theorem 1.2. In order to get lower bounds of Theorem 1.7, we use that *the tower associated to a Misiurewicz–Thurston map $f_0$ can be required to have levels with sizes bounded from below,* and that the truncation level can be chosen to be slightly larger ($2M$ instead of $M$). The final change, that we explain next, is also only needed to obtain the lower bound in Theorem 1.7. Working with Banach norms based on $L^1$ as in [12] would give that the first two terms in (1) are $\leq C|t|^{1/2}$, while the third is $\geq C^{-1}|t|^{1/2}$ for some large constant $C > 1$. In other words, the estimates are too tight. However, *introducing Banach–Sobolev norms based on $L^p$ for $p > 1$ instead*, we are able to control the constants and make sure that the third term dominates the other two, as needed (see Section 5).

The paper is organised as follows: In the remainder of this section, we furnish precise definitions, as well as formal statements of our main results. In Section 2.1, we construct the good parameter sets $\Delta_0 \subset \Delta$ (Proposition 2.1), and we define the corresponding (polynomially recurrent) good maps. In Section 2.2, we construct the tower, and we collect the needed expansion and distortion bounds. Section 2.3 contains Definition 2.7 of admissible pairs $(M, t)$. In Section 3.1, we introduce the

strong and weak Banach norms $(\mathcal{B}_t^{W_1^1}, \mathcal{B}_t^{L^1}, \mathcal{B}_t^{L^p})$ on the tower, define the transfer operator $\widehat{\mathcal{L}}_t$ associated to $f_t$ and acting on these spaces, and list its main spectral properties. In Section 3.2, we introduce the truncated transfer operators $\widehat{\mathcal{L}}_{t,M}$ which play a key role in our analysis. Section 3.3 contains the construction of the parameter set $\Delta_{MT}$ and a brief description of the modifications which can be used to take advantage of the Misiurewicz–Thurston setting. Then, we prove Theorem 1.2 in Section 4 and Theorem 1.7 in Section 5. The two appendices contain necessary but straightforward adaptations of bounds in [12].

## 1.1. **Setting.**

**Definition 1.1.** The *smooth one-parameter families of smooth nondegenerate unimodal maps* $f_t$ studied in the present paper are defined as follows: Let $I = [0, 1]$, and fix $c$ in the interior of $I$. We consider $C^2$ maps $t \mapsto f_t$, from a nontrivial closed interval $\mathcal{E}$ of $\mathbb{R}$ to $C^3$ endomorphisms $f_t$ of $I$. We *assume* that each $f_t$ is a $C^4$ unimodal map with negative Schwarzian derivative and critical point $c$ (independent on $t$), and that the $C^4$ norm of $f_t$ is bounded uniformly in $t$. We *suppose* further that $f_t''(c) < 0$ (this is the nondegeneracy, or quadratic-like property), and that $f_t(0) = f_t(1) = 0$. Put $c_{k,t} = f_t^k(c)$, for $k \geq 0$, and set

$$v_t = \partial_s f_s|_{s=t}.$$

The function $v_t : I \to \mathbb{R}$ is $C^1$ (with a bound on the norm independent on $t$) by assumption. Finally, we *assume* that there exist uniformly $C^1$ functions $X_t : I \to \mathbb{R}$ so that

$$v_t = X_t \circ f_t.$$

The archetypal example is the logistic family

$$(2) \qquad\qquad f_t(x) = tx(1-x), \quad t \in \mathcal{E} \subset (0, 4],$$

where $c = 1/2$, and $X_t(x) \equiv 1/t$. The map $f_4$ (for which $c_{1,4} = f_4(c) = 1$ and $f_4(c_{2,4}) = c_{2,4} = 0$) is called the Ulam–von Neumann map.

A map $f_t$ (or the corresponding parameter $t$) is called $(\lambda_c, H_0)$-*Collet–Eckmann* for some $\lambda_c > 1$ and $H_0 \geq 1$ (or simply Collet–Eckmann, if the meaning is clear) if

$$(3) \qquad\qquad |(f_t^k)'(c_{1,t})| \geq \lambda_c^k, \quad \forall k \geq H_0.$$

Recall [16] that any Collet–Eckmann unimodal map $f_t$ admits a unique absolutely continuous invariant probability measure $\mu_t = \phi_t\, dx$, also called the SRB measure. This measure is ergodic and supported inside $[c_{2,t}, c_{1,t}]$. A map $f_t$ (or the corresponding parameter $t$) is called *mixing* if $f_t$ is topologically mixing on $[c_{2,t}, c_{1,t}]$. The support of the SRB measure $\mu_t$ of a mixing map is equal to $[c_{2,t}, c_{1,t}]$. A unimodal map $f_t$ is *renormalisable* if there exists an interval neighbourhood $\mathcal{R}_{c,t}$ of $c$ so that the first return map to this interval is again a unimodal map, and the smallest return time $P_t$ is at least two. The largest such $P_t$ is called the *renormalisation period*. The map $f_t$ is mixing if and only if $f_t$ is not renormalisable — we say that the renormalisation period $P_t$ of $f_t$ is equal to 1 in this case.

The family $f_t$ is called *transversal* at a Collet–Eckmann parameter $t_1$, if $t_1$ lies in the interior of $\mathcal{E}$, and

$$(4) \qquad\qquad \mathcal{J}_{t_1} := \sum_{j=0}^{\infty} \frac{\partial_t f_t(c_{j,t_1})|_{t=t_1}}{(f_{t_1}^j)'(c_{1,t_1})} \neq 0.$$

Slightly abusing language, we say that $t$ is a transversal Collet–Eckmann parameter if $t$ is a Collet–Eckmann parameter in the interior of $\mathcal{E}$ and (4) holds.

## 1.2. Whitney–Hölder regularity for smooth families of nondegenerate smooth unimodal maps.
Our first result settles the upper bound conjecture in [6, Conj. B] (see also [7, §3.2]).

**Theorem 1.2** (Whitney–Hölder regularity for transversal families). *Let $f_t$ be a smooth one-parameter family of smooth nondegenerate unimodal maps. If there exists a transversal Collet–Eckmann parameter $t_1$, then there exists a positive Lebesgue measure set $\Delta \subset \mathcal{E}$ of Collet–Eckmann parameters such that for all $t_0 \in \Delta$ and all $\Gamma > 4$ there is a set $\Delta_0 \subset \Delta$, which has $t_0$ as a Lebesgue density point, and a constant $C$ such that for every $t \in \Delta_0$ and each $1/2$-Hölder function $A$, we have*

$$\left| \int_I A(x) d\mu_t - \int_I A(x) d\mu_{t_0} \right| \le C|t - t_0|^{1/2} |\log |t - t_0||^{\Gamma} \|A\|_{C^{1/2}},$$

*where*

$$\|A\|_{C^{1/2}} = \|A\|_{L^\infty} + \sup_{x \ne y} \frac{|A(x) - A(y)|}{|x - y|^{1/2}}.$$

As the referee pointed out, the conclusion of Theorem 1.2 implies the following statement, which is closer to the classical Egorov or Lusin theorems: There exists a positive Lebesgue measure set $\Delta \subset \mathcal{E}$ of Collet–Eckmann parameters such that for each $\epsilon > 0$ and every $\Gamma > 4$ there exists a subset $\Delta_\epsilon \subset \Delta$ with $|\Delta \setminus \Delta_\epsilon| < \epsilon$ and a constant $C = C(\epsilon)$ such that the inequality of the theorem holds for all $\{t, t_0\} \subset \Delta_\epsilon$.

Restricting to $C^1$ functions $A$ (or functions of higher smoothness) should not improve the upper bound (see (95)).

As a byproduct of our proof, we obtain the following result :

**Theorem 1.3** (Uniform bounds on renormalisation periods and rates of mixing). *In the setting of Theorem 1.2, the renormalisation period $P_t$ of $f_t$ is not larger than $P_{t_0}$ for all $t \in \Delta_0$. In addition, for any any $\zeta > 0$ there exists $\Theta_1 > 1$ so that for all $t \in \Delta_0$ for which $P_t = P_{t_0}$, each ergodic component $\mu_{j,t}$, $j = 1, \ldots P_t$, of $(f_t^{P_t}, \mu_t)$, and all $C^\zeta$ functions $\psi$ and $\varphi$, there exists $C_{\varphi,\psi}$ so that*

$$\left| \int (\varphi \circ f_t^{kP_t}) \psi \, d\mu_{j,t} - \int \varphi \, d\mu_{j,t} \int \psi \, d\mu_{j,t} \right| \le C_{\varphi,\psi} \Theta_1^{-k}.$$

(Theorem 1.3 is an immediate corollary of Lemma 4.6.)

We next discuss the sets $\Delta$ and $\Delta_0$.

A map $f_t$ is called *polynomially recurrent of exponent* $\alpha > 0$, if there is $H_0 \ge 1$ so that

$$(5) \qquad\qquad |c_{k,t} - c| > k^{-\alpha}, \quad \text{for all } k \ge H_0.$$

A map $f_t$ is called polynomially recurrent of exponent 0 if there is $C \ge 1$ so that $|c_{k,t} - c| > 1/C$ for all $k \ge 1$. A map $f_t$, or a parameter $t$, is called *Misiurewicz–Thurston* if the critical point of $f_t$ is pre-periodic, but not periodic (the postcritical periodic orbit is then necessarily a strictly expanding orbit). Misiurewicz–Thurston maps are Collet–Eckmann and polynomially recurrent of exponent 0. Misiurewicz–Thurston maps are not generic.

All parameters in the set $\Delta$ constructed in Theorem 1.2 are polynomially recurrent for some exponent $\alpha > 1$. *Understanding the largest possible sets $\Delta$ and*

$\Delta_0$ *for which Theorem 1.2 holds, and whether the logarithmic factor can be suppressed is a challenging question.* We conjecture that Theorem 1.2 holds for $\Delta$ the set of all "sufficiently slowly recurrent" parameters, where sufficiently slowly recurrent should include polynomial recurrence of exponent $\alpha = 0$ (the so-called Misiurewicz case). See Corollary 1.6 for analytic families (where $\Delta$ contains almost all Collet–Eckmann parameters), and the upper bound in Theorem 1.7 when $t_0$ is Misiurewicz–Thurston (without the Lebesgue density point property for the analogue of $\Delta_0$).

*Remark* 1.4. If the transversality condition (4) holds for almost all Collet–Eckmann parameters $t_1 \in \mathcal{E}$, then the set $\Delta$ in Theorem 1.2 can be taken equal to the set of Collet–Eckmann parameters. This follows from Proposition 2.1. For example, a non-trivial analytic family of nondegenerate unimodal maps has this property, in particular this holds for the logistic family $f_t(x) = tx(1 - x)$. (See Section 1.3.)

*Remark* 1.5 (Mixing). In [12, Beginning of §5.2] it is claimed incorrectly that 1 is always the only eigenvalue of the transfer operator on the unit circle. Since we did not assume mixing in [12], there could be in fact finitely many other simple eigenvalues of modulus one in general (they are roots of unity — see the proof of Proposition 3.6 in Appendix B below and the reference [22] to Karlin there). So, when constructing the contour integrals in [12, (112), Step 1 in §6], we should avoid not only a neighbourhood of the disc of radius $\theta_t$ there (see also (82)), but also neighbourhoods of these other eigenvalues of modulus 1 (see the circle $\gamma$ in the proofs of Propositions 3.7 and 4.2 below). Note also that exponential decay of correlations is not needed (up to replacing $\widehat{\mathcal{L}}^n$ by $k^{-1} \sum_{n=0}^{k-1} \widehat{\mathcal{L}}^n$ in the proof of the last claim of [12, Proposition 4.11], see (144)).

Note finally that we cannot apply the exactness argument from [13, Corollary 2] to show that 1 is a simple eigenvalue for a nonnegative eigenvector of the transfer operator (contrarily to what was stated in the proof of Proposition 4.11 of [12]), because the transfer operator $\widehat{\mathcal{L}}_t$ is associated to a probabilistic and not a deterministic tower map. However, we may apply classical results on positive operators [22] (details are given in Appendix B below).

1.3. **A stronger result in the analytic case.** In the case of the logistic family $f_t(x) = tx(1 - x)$, $t \in (0, 4]$, Benedicks and Carleson [15] showed that the set of parameters $t$ for which $f_t$ is Collet–Eckmann has positive Lebesgue measure. A parameter $t$ is called *regular* if the critical point $c$ of $f_t$ belongs to the basin of a hyperbolic periodic attractor. The parameter $t$ is called *stochastic* if $f_t$ has an absolutely continuous invariant measure. By Lyubich [28], Lebesgue almost every parameter is either regular or stochastic. Avila and Moreira [5] proved that for almost every stochastic parameter $t$, the map $f_t$ is Collet–Eckmann. Further, in [3] and [4] the results in [28] and [5] are extended to non-trivial analytic families of nondegenerate unimodal maps. (*Analytic* means that each $f_t$ is analytic and $t \mapsto f_t$ is analytic, *non-trivial* means that the family is not contained in a topological class.) Since all but countably many Collet–Eckmann parameters of a non-trivial analytic family of nondegenerate maps $f_t$ are transversal (see [27, 3, 4]), Theorem 1.2, Theorem 1.3, and Remark 1.4, give the following result.

**Corollary 1.6** (Application to analytic families of nondegenerate maps)**.** *Let* $f_t$, *be a non-trivial analytic family of nondegenerate (analytic) unimodal maps. For almost every Collet–Eckmann parameter* $t_0 \in \mathcal{E}$ *and all* $\Gamma > 4$ *there is a set* $\Delta_0 \subset \mathcal{E}$

*of Collet–Eckmann parameters which has $t_0$ as a Lebesgue density point and a constant $C$ such that, for all $t \in \Delta_0$ and $A \in C^{1/2}([0,1])$,*

$$\Big| \int_I A(x) d\mu_t - \int_I A(x) d\mu_{t_0} \Big| \le C|t - t_0|^{1/2} |\log|t - t_0||^{\Gamma} \|A\|_{C^{1/2}} \,.$$

*In addition, the renormalisation period $P_t$ of $f_t$ is not larger than $P_{t_0}$ for all $t \in \Delta_0$, and for any $\zeta > 0$ there exists $\Theta_1 > 1$ so that for all $t \in \Delta_0$ for which $P_t = P_{t_0}$, each ergodic component $\mu_{j,t}$, $j = 1, \ldots, P_t$, of $(f_t^P, \mu_t)$ and all $C^\zeta$ functions $\psi$, $\varphi$ there exists $C_{\varphi,\psi}$ so that*

$$\Big| \int (\varphi \circ f_t^{kP_t}) \psi \, d\mu_{j,t} - \int \varphi \, d\mu_{j,t} \int \psi \, d\mu_{j,t} \Big| \le C_{\varphi,\psi} \Theta_1^{-k} \,.$$

Again, understanding *the largest possible set of parameters $t_0$ and $\Delta_0$ for which Corollary 1.6 holds, and whether the logarithmic factor can be suppressed is a challenging question.* We conjecture that Corollary 1.6 holds for all Collet–Eckmann parameters $t_0$ with "sufficiently slow" recurrence.

1.4. **Hölder upper and lower bounds for Misiurewicz–Thurston parameters.** Our second main result addresses the lower bound in Conjecture B in [6] (see also [7, §3.2]).

**Theorem 1.7** (Hölder upper and lower bounds)**.** *Let $f_t$ be a smooth one-parameter family of smooth unimodal maps. Let $t_0$ be a mixing transversal Misiurewicz–Thurston parameter. Then there exist an observable $A \in C^\infty$, a constant $C \ge 1$, and a sequence of Collet–Eckmann parameters $t_{(n)}$, $n \ge 1$, with $t_{(n)} \to t_0$ as $n \to \infty$, such that*

$$(6) \quad \frac{|t_{(n)} - t_0|^{1/2}}{C} \le \Big| \int_I A(x) d\mu_{t_{(n)}} - \int_I A(x) d\mu_{t_0} \Big| \le C|t_{(n)} - t_0|^{1/2}, \quad \forall\, n \ge 1 \,.$$

The mixing assumption is for simplicity (the proof shows that it suffices to suppose that the deepest renormalisation of — the finitely renormalisable map – $f_{t_0}$ is not conjugated to the Ulam–von Neumann map).

The proof of Theorem 1.7 produces a set $\Delta_{MT}$ of parameters $t_{(n)}$ which are also Misiurewicz–Thurston (see Lemma 3.9 and its proof), and either all $> t_0$ or all $< t_0$ (see Remark 3.10). Using continuity of the absolutely continuous invariant measures in the sense of Whitney (see the works of Tsujii [43] and Rychlik–Sorets [39], or more recently Alves et al. [2], [1]), we can then easily construct sequences of Collet–Eckmann parameters $\tilde{t}_{(n)}$ which are not Misiurewicz–Thurston, and for which the lower bound in Theorem 1.7 hold. However, we do not know if $t_0$ is a Lebesgue density point of the set of all such $\tilde{t}_{(n)}$. *Understanding the set of sequences for which Theorem 1.7 holds is a challenging question. In fact, the toy model analogue of this question is also open (see [9, Theorem 7.1] and its corrigendum, even in the case of a pre-periodic critical point).*

Last, but not least, the lower bounds in the piecewise expanding toy model of [9, Theorem 7.1] had been first obtained only in the case when the critical point is preperiodic [6, Theorem 6.1]. *We conjecture that the conclusion (6) of Theorem 1.7 should also hold when $t_0$ enjoys a more generic slow recurrence condition (such as polynomial recurrence), perhaps up to introducing a power of $|\log|t_{(n)} - t_0||$ in both sides of (6).*

## 2. Preliminaries – Towers – Transfer operators

In Section 2.1, we show that transversality at a Collet–Eckmann parameter ensures that polynomial recurrence holds for uniform constants $\lambda_c > 1$, $\alpha > 1$, $H_0 \geq 1$, for a positive measure set of parameters $t$, with Collet–Eckmann parameters as Lebesgue density points. In Section 2.2 we adapt the tower map construction in [12] to our polynomially recurrent setting. Section 2.3 contains Lemma 2.6 giving estimates on iterated unimodal maps for all $f_s$ close enough to a good parameter $f_t$ and all iterates not too big compared with $|t - s|$, as well as the definition of admissible pairs $(M, t)$.

### 2.1. **Uniformity of constants.**

**Proposition 2.1** (Parameter set of good maps with uniform constants). *Assume that $t_1 \in \mathcal{E}$ is a transversal Collet–Eckmann parameter. Then, there exists a set $\Delta \subset \mathcal{E}$ of transversal Collet–Eckmann parameters, for which $t_1$ is a Lebesgue density point [1] (in particular, $\Delta$ has positive Lebesgue measure), so that for all $t_0 \in \Delta$ and all $\alpha > 1$ there exist $H_0 \geq 1$ and a set $\Delta_0 \subset \Delta$ containing $t_0$ as a Lebesgue density point such that, for all $t \in \Delta_0$, the map $f_t$ is $(\lambda_c, H_0)$-Collet–Eckmann and the polynomial recurrence condition* (5) *of exponent $\alpha$ holds for all $k \geq H_0$.*

*Furthermore, for each $t_0 \in \Delta$, there exist constants $\rho > 1$ (we may assume $\rho < \sqrt{\lambda_c}$) and $C_0 > 0$ such that, for all $\delta > 0$ sufficiently small, there is a constant $\epsilon(\delta) > 0$ such that, for all $t \in (t_0 - \epsilon(\delta), t_0 + \epsilon(\delta))$,*

$$(7) \qquad |(f_t^n)'(x)| \geq C_0 \delta \rho^n \,, \forall x \text{ so that } |f_t^j(x) - c| \geq \delta \,, \forall 0 \leq j < n \,,$$

*and*

$$(8)$$
$$|(f_t^n)'(x)| \geq C_0 \rho^n \,, \forall x \text{ so that } |f_t^j(x) - c| \geq \delta \,, \forall 0 \leq j < n \,, \text{ and } |f_t^n(x) - c| < \delta \,.$$

The following definition summarises the properties of parameters in the positive measure set $\Delta_0$ constructed in the previous proposition:

**Definition 2.2** (Good maps). A map $f_t$ (or the corresponding parameter $t$) is called good for the constants

$$\lambda_c > 1 \,, H_0 \geq 1 \,, \alpha \geq 0 \,, \rho > 1 \,, C_0 > 0 \,,$$

(called its "goodness constants", or simply "goodness") if $f_t$ is $(\lambda_c, H_0)$-Collet–Eckmann (3), if it satisfies the polynomial recurrence condition (5) for $\alpha$ and $H_0$ (when $\alpha = 0$ we require that $|c_{k,t} - c| \geq C_0$, for all $k \geq 1$), and if the expansion conditions (7) and (8) hold for $\rho$, $C_0$, and any small $\delta > 0$.

*Proof of Proposition 2.1.* We apply a recent work by Gao and Shen [19]. Since the map $f_{t_1}$ is Collet–Eckmann, all periodic orbits are repelling. Since $f_{t_1}$ is also transversal, we can apply [19, Main Theorem] which implies that there exists a set $\mathcal{E}_1$ for which $t_1$ is a Lebesgue density point such that, for all parameters $t \in \mathcal{E}_1$, the map $f_t$ satisfies the polynomial recurrence condition (5) for any $\alpha > 1$, $f_t$ is transversal at $t$, and $f_t$ is Collet–Eckmann. For $j, H, \ell \geq 1$, set

$$\Omega_{j,H,\ell} = \left\{ t \in \mathcal{E}_1 \mid |c_{k,t} - c| \geq k^{-(1+j^{-1})}, \text{ and } |(f_t^k)'(c_{1,t})| \geq e^{k/\ell}, \ \forall k \geq H \right\} .$$

---

[1]We do not claim that $t_1 \in \Delta$.

For a set $\Omega$, let $\Omega^L \subset \Omega$ denote the set of Lebesgue density points within $\Omega$. We set $\Delta = \cap_{j \geq 1}(\cup_{\ell \geq 1} \cup_{H \geq 1} \Omega^L_{j,H,\ell})$. By Lebesgue's density theorem, $\Delta$ is equal to the set $\mathcal{E}_1$ up to some zero Lebesgue measure set. Now, by construction, for all $t_0 \in \Delta$ and all $\alpha > 1$, choosing $j_0 \geq$ such that $(1 + j_0^{-1}) \leq \alpha$ we find $H_0 \geq 1$ and $\ell_0 \geq 1$ such that $t_0 \in \Omega^L_{j_0,H_0,\ell_0}$. Setting $\Delta_0 = \Omega^L_{j_0,H_0,\ell_0} \cap \Delta$ this concludes the first part of the proof of Proposition 2.1.

Regarding the second part we will apply a lemma by Tsujii [42]. The conditions (ND), (CE)(i), (Hyp), (W), and (NV) in [42] are satisfied for $f_{t_0}$. The only point we have to check is the "backward Collet–Eckmann condition" (CE)(ii), i.e., the existence of constants $C > 0$ and $\hat{\rho} > 1$ such that

$$|(f_{t_0}^k)'(b)| \geq C\hat{\rho}^k, \quad \text{if } k \geq 1 \text{ and } f_{t_0}^k(b) = c.$$

Since $f_{t_0}$ is Collet–Eckmann, unimodal, and has negative Schwarzian derivative, [30, Theorem A] guarantees that the backward Collet–Eckmann condition holds. By [42, Lemma 5.1 (2)], there exist $\rho > 1$, $\delta_0 > 0$, and $C_0 > 0$ such that, for all $0 < \delta \leq \delta_0$, there is $\epsilon(\delta) > 0$ such that (7) and (8) hold, for all $f_t$ with $|t - t_0| \leq \epsilon(\delta)$. (The fact that the constant in front of $\rho^n$ in (7) depends linearly on $\delta$ follows from the last line in the proof of [42, Lemma 5.1].) $\qquad\square$

2.2. **The tower map for good $f_t$ — Distortion estimates.** Assume that $f_t$ is good (recall Definition 2.2) for $\lambda_c$, $\alpha > 1$, $H_0$, $\rho$, and $C_0$. (The case when $f_t$ is Misiurewicz–Thurston with $\alpha = 0$ is treated in Section 3.3.) Let $\delta > 0$ be small, to be determined later as a function essentially of the goodness parameters and of the $C^2$ norm of $t \mapsto f_t$ (see, e.g., condition (11), condition just above (12), inequality (39) in Lemma 2.6, and Proposition 2.4, in which $\delta$ is so small such that (135) is satisfied; in (11) and in (135), $\delta$ depends on $L$, which is chosen in Lemma 3.8 via Lemma 2.6 again).

We introduce a tower map $\hat{f}_t : \hat{I}_t \to \hat{I}_t$ similar to the one constructed in [12, Section 3] (see also [13]). The tower in the present paper is "fatter" since our polynomial recurrence assumption allows us to choose the size of the levels polynomially small, instead of exponentially small. (To get the lower bound of Theorem 1.7, we shall later use levels of constant size, under a Misiurewicz–Thurston assumption.) Fix a constant

$$(9) \qquad\qquad\qquad\qquad \beta > \alpha + 1.$$

The *tower* $\hat{I}_t$ associated to $f_t$ is the union $\hat{I}_t = \cup_{k \geq 0} E_{k,t}$ of levels $E_{k,t} = B_{k,t} \times \{k\}$ satisfying the following properties: The ground floor interval $B_{0,t} = B_0$ is the interval $I$. Fix a constant $L > 1$ (the value of $L$ will be chosen in Lemma 3.8). For $k \geq 1$, the interval $B_{k,t}$ is centered at $c_{k,t}$ and such that

$$(10) \qquad \left[c_{k,t} - \frac{k^{-\beta}}{L^3}, c_{k,t} + \frac{k^{-\beta}}{L^3}\right] \subset B_{k,t} \subset \left[c_{k,t} - \frac{k^{-\beta}}{L}, c_{k,t} + \frac{k^{-\beta}}{L}\right].$$

(Observe that since $\beta > \alpha$ and $L > 1$, we have $c \notin B_{k,t}$ for all $k \geq H_0$.) Note that for given $\lambda_c > 1$ and $H_0 \geq 1$, there exists a constant $C > 1$ such that $\inf_{1 \leq k \leq H_0} |c_{k,t} - c| \geq 2C^{-1}$, for all $(\lambda_c, H_0)$-Collet–Eckmann maps $f_t$, $t \in \mathcal{E}$. (This follows directly from $\sup_{t \in \mathcal{E}} |f_t''(c)| < \infty$ combined with the expansion (3).) Henceforth, (for given $\lambda_c > 1$ and $H_0 \geq 1$) we assume that $\delta > 0$ is so small that there exists a uniform $c_u > 0$ so that for all $x$ with $|x - c| \leq \delta$ and all $(\lambda_c, H_0)$-Collet–Eckmann maps $f_t$,

$t \in \mathcal{E}$, we have

(11) $$|f_t^j(x) - c| \geq c_u, \qquad \text{for all } 1 \leq j \leq H_0.$$

For $(x, k) \in E_{k,t}$ we set

$$\hat{f}_t(x, k) = \begin{cases} (f_t(x), k+1) & \text{if } k \geq 1 \text{ and } f_t(x) \in B_{k+1,t}, \\ (f_t(x), k+1) & \text{if } k = 0 \text{ and } |x - c| \leq \delta, \\ (f_t(x), 0) & \text{otherwise.} \end{cases}$$

Denoting $\pi : \hat{I}_t \to I$ the projection to the first factor, we have $f_t \circ \pi = \pi \circ \hat{f}_t$ on $\hat{I}_t$. Define $H(\delta)$ to be the minimal $k \geq 1$ such that there exist some $x \in [c - \delta, c + \delta]$ such that $\hat{f}_t^{k+1}(x, 0) \in E_0$. (In other words if a point starts to climb the tower, it climbs the tower at least until level $H(\delta)$.) For fixed goodness parameters $(\lambda_c, H_0, \alpha, \rho, C_0)$, fixed $\beta > 1 + \alpha$, and fixed $L$, observe that for each $H \geq 1$, we can choose $\delta > 0$ so small such that $H(\delta) \geq H$ for all good maps and their corresponding towers with these fixed parameters. (This follows immediately from the fact that $\sup_{t \in \mathcal{E}} \|f_t\|_{C^1} < \infty$ and $|B_{k,t}| \geq 2k^{-\beta}/L^3$.) We assume throughout that $\delta > 0$ is so small that $H(\delta) \geq \max(2, H_0)$.

We decompose $[c - \delta, c + \delta] \setminus \{0\}$ as a disjoint union of intervals

$$[c - \delta, c + \delta] \setminus \{0\} = \cup_{j \geq H(\delta)} I_{j,t}, \quad I_{j,t} := I_{j,t}^+ \cup I_{j,t}^-,$$

(12) $$I_{j,t}^\pm := \left\{ |x| < \delta, \pm x > 0, \hat{f}_t^\ell(x, 0) \in E_{\ell,t}, 0 \leq \ell < j, \hat{f}_t^j(x, 0) \in E_0 \right\}.$$

In other words, $I_{j,t}$ is the set of points which climb up the tower $j - 1$ levels and fall back to the level $E_0$ at the $j$-th iteration. Note that $I_{j,t}$ can be empty for some $j$. (In particular $I_{j,t} = \emptyset$ for $1 \leq j < H(\delta)$.) For $k \geq 0$, let $J_{k,t}$ denote the set of points in $[c - \delta, c + \delta]$ which climb up the tower at least $k$ levels, i.e.,

(13) $$J_{k,t} := \{c\} \cup \bigcup_{j \geq k+1} I_{j,t}.$$

Obviously, we have $J_{k,t} = [c - \delta, c + \delta]$, for all $0 \leq k \leq H(\delta)$. Later, when considering a fixed good map $f_0$, we will write $B_k$, $I_k$ and $J_k$ for $B_{k,0}$, $I_{k,0}$ and $J_{k,0}$, respectively.

The following lemma is the adaptation of the distortion estimates from [12, Lemma 3.3] (which was an avatar of [13], [44, Lemma 5.3(1)]) to our fat towers:

**Lemma 2.3** (Bounded distortion in the bound period). *Let $f_t$, $t \in \mathcal{E}$, be good and assume that condition (11) is satisfied. Then, there exists $C > 1$ depending only on the goodness constants $(\lambda_c, H_0, \alpha)$, and on $\beta$ (in particular $C$ does not depend on $L$), such that for every $j \geq 1$, and every $k \leq j$,*

(14) $$C^{-1} \leq \frac{|(f_t^k)'(x)|}{|(f_t^k)'(y)|} \leq C, \qquad \forall x, y \in f_t(J_{j,t}).$$

*Proof.* Recall the intervals $B_{\ell,t}$ from (10). For $1 \leq \ell \leq k \leq j$, pick $x_\ell$ and $y_\ell$ in $f_t^\ell(J_{j,t}) \subset B_{\ell,t}$. Observe that there exists a constant $C$ such that $|f_t'(y)| \geq |y - c|/C$, for all $y \in I$ and $t \in \mathcal{E}$. We have

$$\prod_{\ell=1}^k \frac{|f_t'(x_\ell)|}{|f_t'(y_\ell)|} \leq \prod_{\ell=1}^k \left( 1 + \frac{\sup |f_t''|}{|f_t'(y_\ell)|} |x_\ell - y_\ell| \right) \leq \prod_{\ell=1}^k \left( 1 + C \sup |f_t''| \frac{|x_\ell - y_\ell|}{|y_\ell - c|} \right)$$

(15) $$\leq \prod_{\ell=1}^\infty (1 + \tilde{C} \sup |f_t''| \ell^{-\beta+\alpha}) < \infty,$$

uniformly in $j$. We used that $|x_\ell - y_\ell| \leq L^{-1}\ell^{-\beta}$ and, if $\ell > H_0$, that $|y_\ell - c| \geq \ell^{-\alpha} - L^{-1}\ell^{-\beta} \geq \ell^{-\alpha}/2$. If $1 \leq \ell \leq H_0$, we used condition (11).

The series $\ell^{-\beta+\alpha}$ is summable since $\beta > \alpha + 1$. Choosing $y_\ell = f^{\ell-1}(y)$ and $x_\ell = f^{\ell-1}(x)$, we get the upper bound in (15). Taking $y_\ell = f^{\ell-1}(x)$ and $x_\ell = f^{\ell-1}(y)$, we obtain the lower bound in (15). Observe that while the last product in (15) blows up when $\beta$ tends to $1 + \alpha$, it does not depend on the constant $L$. $\qquad\square$

The following key estimate is our polynomial version of [12, Proposition 3.7] (the proof is to be found in Appendix A, it uses the tower, and we refer to the last paragraph in the proof regarding the role of $\beta$ and $\delta$):

**Proposition 2.4** (Key estimate for polynomially recurrent maps). *Let $f_t$ be good. There exists $C > 0$ depending only on the goodness constants $(\lambda_c, H_0, \alpha, \rho, C_0)$ such that for every $j \geq 0$ we have*

$$(16) \qquad \sum_{k=j+1}^{\infty} \frac{1}{|(f_t^{k-j})'(f_t^j(c_1))|} \leq Cj^\alpha \,.$$

The following notation will be convenient: For $k \geq 1$, let

$$(17) \qquad f_{t,+}^{-k} := (f_t^k|_{U_{k,t,+}})^{-1} \,, \quad f_{t,-}^{-k} := (f_t^k|_{U_{k,t,-}})^{-1} \,,$$

where $U_{k,t,+}$ is the monotonicity interval of $f_t^k$ containing $c$ located to the right of $c$, and $U_{k,t,-}$ is the monotonicity interval of $f_t^k$ containing $c$, to the left of $c$.

The following polynomial version of the upper and lower bounds in [12, Lemma 3.4 and Lemma 4.1]), about points which climb for exactly $j - 1$ steps, recall (12), will be needed:

**Lemma 2.5** (The $j$-bound intervals $I_{j,t}^{\pm}$). *Let $f_t$ be good and assume that condition (11) is satisfied. Then there exists a constant $C$ depending only on the goodness constants $(\lambda_c, H_0, \alpha)$ and on $\beta$ (in particular $C$ does not depend on $L$), such that we have*

$$(18) \qquad |x - c| \leq C|B_{j-1,t}|^{1/2}|(f_t^{j-2})'(c_{1,t})|^{-1/2} \,, \quad \forall x \in J_{j-1,t}, \ j \geq 1 \,,$$

*and for all $x \in I_{j,t}, \ j \geq H(\delta)$, we have*

$$(19) \qquad |f_t'(x)| \geq \frac{1}{C}\frac{\sqrt{|f_t^j(x) - c_{j,t}|}}{|(f_t^{j-1})'(c_{1,t})|^{1/2}} \geq \frac{1}{CL^{3/2}}j^{-\beta/2}\frac{1}{|(f_t^{j-1})'(c_{1,t})|^{1/2}} \,,$$

$$(20)$$
$$|(f_t^j)'(x)| \geq \frac{1}{C}\sqrt{|f_t^j(x) - c_{j,t}|}|(f_t^{j-1})'(c_{1,t})|^{1/2} \geq \frac{1}{CL^{3/2}}j^{-\beta/2}|(f_t^{j-1})'(c_{1,t})|^{1/2} \,.$$

*In addition, there exists a constant $C$ depending on $L$ and on the constant in Proposition 2.4 so that for all $j \geq H(\delta)$ and $x \in f^j(I_{j,t})$ we have, for $\zeta = +$ or $-$,*

$$(21) \qquad \left|\partial_x \frac{1}{|(f_t^j)'(f_{t,\zeta}^{-j}(x))|}\right| \leq C\frac{j^{\max(1+2\alpha+\beta/2,3\beta/2)}}{|(f_t^{j-1})'(c_1)|^{1/2}} \,,$$

*and, finally,*

$$(22) \qquad \left|\partial_x^2 \frac{1}{|(f_t^j)'(f_{t,\zeta}^{-j}(x))|}\right| \leq C\frac{j^{\max(4\alpha+1+\beta/2,5\beta/2)}}{|(f_t^{j-1})'(c_{1,t})|^{1/2}} \,.$$

An immediate corollary of (18) is exponential decay of the length of $I_{j,t}$ and $J_{j,t}$. More precisely, for any fixed goodness constants and any $L$, there exists a constant $\tilde{C}$ so that for all $j \geq H_0$

$$(23) \qquad |I_{j,t}| \leq |J_{j-1,t}| \leq \tilde{C} j^{-\beta/2} |(f_t^{j-2})'(c_1)|^{-1/2} \leq \tilde{C}^2 j^{-\beta/2} \lambda_c^{-j/2} ,$$

where the second last inequality holds for all $j \geq 2$.

*Proof.* To simplify the writing, we remove the $t$ from the notation and write, e.g., $f$, $I_j$, and $J_j$ instead of $f_t$, $I_{j,t}$, and $J_{j,t}$.

Let $x \in J_{j-1}$, $j \geq 1$. First, our definitions and the mean value theorem imply that there exists $y \in J_{j-1}$ so that

$$|(f^{j-2})'(f(y))||f(x) - c_1| \leq |B_{j-1}|/2 .$$

Therefore, Lemma 2.3 and the fact that $|f(x) - c_1| \geq C^{-1}|x - c|^2$ (recall that $f'(c) = 0$ and $f''(c) \neq 0$) yield (18).

Next, the reverse consequence of the mean value theorem

$$(24) \qquad |(f^{j-1})'(f(y))||f(x) - c_1| \geq C^{-1}|f^j(x) - c_k| \geq C^{-1}L^{-3}j^{-\beta} ,$$

together with $|f(x) - c_1| \leq C|x - c|^2$ and Lemma 2.3 gives

$$(25) \qquad |x - c| \geq C^{-1} \frac{|f^j(x) - c_k|^{1/2}}{|(f^{j-1})'(f(y))|^{1/2}} \geq \frac{1}{C^2 L^{3/2}} j^{-\beta/2} |(f^{j-1})'(c_1)|^{-1/2} ,$$

where in the last inequality we used that $f^j(x) \notin B_j$. The bound (19) then follows from (25).

To show (20), we decompose $|(f^j)'(x)| = |(f^{j-1})'(f(x))||f'(x)|$, and we apply (19), noting that Lemma 2.3, implies

$$(26) \qquad |(f^{j-1})'(f(x))| \geq C^{-1}|(f^{j-1})'(c_1)| .$$

Note that reversing the inequalities in the arguments above also gives

$$(27) \qquad |(f_t^j)'(x)| \leq C\sqrt{|f_t^j(x) - c_{j,t}|}|(f_t^{j-1})'(c_{1,t})|^{1/2} .$$

Assume now that $\zeta = +$ (the other case is similar). To prove (21), recall first the proof of Lemma 2.3 which implies that there is $C \geq 1$ (depending on the goodness and, in a weaker way, on $\beta$ and $L$) so that

$$(28) \qquad |f'(f^j(y))| \geq C^{-1}j^{-\alpha} , \ \forall y \in J_k , \ \forall 1 \leq j \leq k .$$

Next, Lemma 2.3 and Proposition 2.4 give $C > 0$ so that

$$(29) \qquad \sup_{y \in J_k} \frac{1}{|(f^{k-j})'(f^j(y))|} \leq C \sum_{\ell=1}^{\infty} \frac{1}{|(f^\ell)'(f^{j-1}(c_1))|} \leq C j^\alpha , \ \forall 1 \leq j \leq k - 1 .$$

Applying (28) and (29) for $j \geq 1$ and (19) and (20) for $j = 0$, we find $C > 0$ so that

$$(30) \qquad \sup_{y \in I_k} \partial_y \frac{1}{|(f^k)'(y)|} \leq \sup_{y \in I_k} \sum_{j=0}^{k-1} \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))|} \frac{1}{|f'(f^j(y))|} \leq C k^{\max(1+2\alpha,\beta)} .$$

Then, (20) (or its proof) gives

$$(31) \qquad \sup_{y \in I_k} \frac{1}{|(f^m)'(y)|} \leq C L^{3/2} k^{\beta/2} \frac{1}{|(f^{m-1})'(c_1)|^{1/2}} , \quad \forall 1 \leq m \leq k .$$

Finally, if $x \in f^k(I_k)$,

$$(32) \qquad \partial_x \frac{1}{|(f^k)'(f_+^{-k}(x))|} = \frac{1}{|(f^k)'(f_+^{-k}(x))|} \cdot \partial_y \frac{1}{|(f^k)'(y)|} \, .$$

The first factor in (32) is bounded by (31) for $m = k$, the second by (30), so that we have proved (21).

To prove (22), we start from the decomposition (32), and we deduce from (30) that for any $x \in f^k(I_k)$, setting $y = f_+^{-k}(x)$,

$$\partial_x^2 \frac{1}{|(f^k)'(f_+^{-k}(x))|} \leq C k^{\max(2\alpha,\beta)} \partial_x \frac{1}{|(f^k)'(f_+^{-k}(x))|}$$

$$(33) \qquad\qquad + \frac{1}{|(f^k)'(f_+^{-k}(x))|} \cdot \sum_{j=0}^{k-1} \partial_x \left( \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|} \right).$$

By (21), the first term in the right hand side is bounded by a constant times $k^{\max(1+4\alpha+\beta/2,5\beta/2)}|(f^{k-1})'(c_1)|^{-1/2}$. For the second term, we have, for $0 \leq j \leq k-1$,

$$\partial_x \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|} \leq \frac{1}{|(f^k)'(f_+^{-k}(x))|} \partial_y \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|} \, .$$

Since $f$ is $C^3$, the Leibniz formula gives for $0 \leq j \leq k-1$,

$$\frac{1}{|(f^k)'(f_+^{-k}(x))|} \partial_y \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|}$$

$$\leq \frac{1}{|(f^k)'(f_+^{-k}(x))|} \frac{1}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|} \cdot \left[ |f'''(f^j(y))||(f^j)'(y)| \right.$$

$$(34)$$

$$+ |f''(f^j(y))| \Big[ \frac{|f''(f^j(y))||(f^j)'(y)|}{|f'(f^j(y))|} + \sum_{\ell=j}^{k-1} \frac{|f''(f^\ell(y))||(f^\ell)'(y)|}{|f'(f^\ell(y))|} \Big] \Big]$$

$$\leq \frac{C}{|(f^{k-j})'(f^j(y))|} \cdot \left[ \frac{1}{|(f^{k-j})'(f^j(y))|} \left( \frac{1}{|f'(f^j(y))|} + \frac{1}{|f'(f^j(y))|^2} \right) \right.$$

$$\left. + \frac{1}{|(f^k)'(y)|} \sum_{\ell=j}^{k-1} \frac{|(f^\ell)'(y)|}{|f'(f^\ell(y))|} \right].$$

If $j \geq 1$, we may apply (28) and (29), so that (34) implies

$$\frac{1}{|(f^k)'(f_+^{-k}(x))|} \sum_{j=1}^{k-1} \partial_y \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|} \leq C k^{\alpha+1} \cdot \left[ k^{2\alpha} + k^{3\alpha} + k^\alpha \right].$$

If $j = 0$, then (34) together with (19) and (31) for $m = k$ imply (distinguish between $\ell = 0$ and $\ell \geq 1$)

$$\frac{1}{|(f^k)'(f_+^{-k}(x))|} \partial_y \frac{|f''(y)|}{|(f^k)'(y)||f'(y)|}$$

$$\leq C \frac{k^{\beta/2}}{|(f^k)'(c_1)|^{1/2}} (2k^\beta + k^{3\beta/2}|(f^k)'(c_1)|^{1/2} + k^{\alpha+1}).$$

The two above inequalities, together with (33) and (31) for $m = k$, give (22). $\qquad \square$

2.3. **Maps in a neighbourhood of a good map – Admissible pairs** $(M, t)$.
We next state some basic facts about the maps $f_s$ in a neighbourhood of a good
map $f_{t_0}$. To simplify the writing, we assume $t_0 = 0$ and remove the $t_0$ from the
notation. We emphasize that the maps $f_s$ in the following lemma are not necessarily
all Collet–Eckmann. (Indeed, for both our main theorems, we shall apply the mean
value theorem, $B(t) - B(0) = tB'(s_t)$, or the fundamental theorem [2] of calculus
$B(t) - B(0) = \int_0^t B'(s) \, ds$, in parameter space. Even if $0$ and $t$ are good parameters,
the parameters $s_t$ and $s \in [0, t]$ are not all good.) Recall the intervals $I_{k,t}$ and $J_{k,t}$
defined in (12) and (13).

**Lemma 2.6** (Uniformity of goodness and distortion constants for suitable maps
$f_s$ and iterates $M$). *Let $f = f_0$ be good for parameters $(\lambda_c, H_0, \alpha > 1, \rho, C_0)$ and
assume that (11) is satisfied. Then there exist constants $C \geq 1$ and $\epsilon > 0$ (depending
only on $\lambda_c$, $H_0$, $\alpha$, $\rho$, $C_0$, $\beta$, and in particular not on $L$) so that, for any pair $(s, M)$,
$M \geq 1$ and $s \in (-\epsilon, \epsilon)$ satisfying*

$$(35) \qquad |(f^{k-1})'(c_1)||s| \leq k^{-\beta}, \quad \forall 1 \leq k \leq M,$$

*the following holds: We have*

$$(36) \qquad C^{-1} \leq \left| \frac{(f_s^{k-1})'(x)}{(f^{k-1})'(x)} \right| \leq C, \quad \forall x \in f(J_{k-1}), \, \forall 1 < k \leq M.$$

*Furthermore, we have*

$$(37) \qquad |(f_s^k)'(x)| \geq C^{-1} L^{-3/2} k^{-\beta/2} |(f^{k-1})'(c_1)|^{1/2}, \quad \forall x \in I_k, \, \forall 1 \leq k \leq M,$$

*and*

$$(38) \qquad |\partial_s f_s^k(x)| \leq C|(f^{k-1})'(c_1)|, \quad \forall x \in J_{k-1}, \, \forall 1 \leq k \leq M,$$

*and, if $\mathcal{J}_0 \neq 0$ (recall (4)) then for $\delta > 0$ sufficiently small we have*

$$(39) \qquad |\partial_s f_s^k(x)| \geq C^{-1} |\mathcal{J}_0||(f^{k-1})'(c_1)|, \quad \forall x \in J_{k-1}, \, \forall H(\delta) \leq k \leq M,$$

*where $\text{sign}(\partial_s f_s^k(x) \cdot (f^{k-1})'(c_1)) = \text{sign}(\mathcal{J}_0)$. Finally, for $1 \leq k < l \leq M - 1$, we
have*

$$(40) \qquad \frac{|(f_s^k)'(c_{1,s})|}{|(f_s^l)'(c_{1,s})|} \leq Ck^\alpha,$$

*where the constant in (40) depends in addition also on $\delta$.*

*Proof.* As a preliminary step, note that (35) implies that we can assume, up to
taking $\epsilon$ small enough (depending only on $\lambda_c$ and $H_0$) that

$$(41) \qquad |s| < M^{-\beta}.$$

Next, for $x \in J_{M-1}$ and $s$ satisfying (35), let

$$D_k = \sup_{t \in [0,s]} |f^k(x) - f_t^k(x)|.$$

Fix $\alpha + 1 < \beta_0 < \beta$. We claim that

$$(42) \qquad D_k \leq k^{-\beta_0}, \quad \forall 1 \leq k \leq M - 1.$$

---

[2]To prove (115) in [12] one should apply the fundamental theorem of calculus and Fubini
instead of the mean value theorem, see (99) for a similar computation.

We show this by induction over $k$. For $\epsilon > 0$ sufficiently small, (42) obviously holds for all small $k$'s and arbitrary $M$. Assume that $1 < k \leq M - 1$ and that the claim holds for $k - 1$. Observe that

$$(43) \qquad \partial_t f_t^k(y) = \sum_{j=1}^{k} (f_t^{k-j})'(f_t^j(y))(\partial_t f_t)(f_t^{j-1}(y)), \quad \forall y \in I \text{ and } t \in \mathcal{E},$$

and there exists $\tilde{C} > 1$ so that

$$|\partial_t f_t(y)|, \ |f_t''(y)|, \ |\partial_t f_t'(y)| \leq \tilde{C}, \quad \forall y \in I \text{ and } t \in \mathcal{E}.$$

Hence, by applying twice the mean value theorem

$$|f_t'(f_t^k(x)) - f'(f^k(x))| \leq |f_t'(f_t^k(x)) - f_t'(f^k(x))| + |f_t'(f^k(x)) - f'(f^k(x))|$$
$$\leq \tilde{C}D_k + \tilde{C}|t|, \quad \forall t \in [0, s],$$

and we get $|f_t'(f_t^k(x))| \leq |f'(f^k(x))| + CD_k + \tilde{C}|t|$. Combined with (43), it follows

$$(44) \qquad \left| \frac{\partial_t f_t^k(x)}{(f^{k-1})'(f(x))} \right| \leq \sum_{j=1}^{k} \frac{\tilde{C}}{|(f^{j-1})'(f(x))|} \prod_{i=j}^{k-1} \left( 1 + \frac{\tilde{C}|t| + \tilde{C}D_i}{|f'(f^i(x))|} \right).$$

Since $f^{i+1}(J_{M-1}) \subset B_{i+1}$, for $i \leq M - 2$, it follows that $|f^i(x) - c| \geq |f^i(c_1) - c| - i^{-\beta}/L$. By (5) and (11) (maybe increase $\tilde{C}$ in order to control the small $i$'s), we see that $|f^i(x) - c| \geq \tilde{C}^{-1}i^{-\alpha}$, for all $i \leq M - 2$, and since the critical point is nondegenerate, we have $|f'(f^i(x))| \geq \tilde{C}^{-1}|f^i(x) - c| \geq \tilde{C}^{-2}i^{-\alpha}$. Together with the induction assumption on $D_i$, $i \leq k - 1$, and the assumption on $s$, it follows that for all $t \in [0, s]$

$$\frac{\tilde{C}|t| + \tilde{C}D_i}{|f'(f^i(x))|} \leq \tilde{C}^3 i^\alpha(|t| + D_i) \leq \tilde{C}^3 i^\alpha(M^{-\beta} + i^{-\beta_0}) \leq 2\tilde{C}^3 i^{-(\beta_0 - \alpha)}.$$

Since $i^{-(\beta_0 - \alpha)}$ is summable, the product in (44) is uniformly bounded by a constant $C'$ and, by the mean value theorem and the distortion estimate Lemma 2.3 for $t = 0$, we conclude

$$D_k \leq C'\tilde{C}C(\lambda_c - 1)^{-1}|(f^{k-1})'(c_1)||s| \leq C'\tilde{C}C(\lambda_c - 1)^{-1}k^{-\beta},$$

where in the last inequality we used the assumption (35) on $s$. This shows (42).

If $f$ is Misiurewicz–Thurston (with $\alpha = \beta = 0$), then as explained in Section 3.3 below the right hand side of (35) is replaced by a sufficiently small constant $\eta > 0$. Then, we derive by a similar calculation as the one showing (42) that there exists a constant $C'$ (depending only on the goodness parameters of $f$) such that for all sufficiently small $\eta > 0$,

$$(45) \qquad D_k \leq C'\eta|(f^{M-k})'(c_k)|^{-1}, \quad \forall \ 1 \leq k \leq M - 1.$$

The $\eta$ in the above bound is then used to show a positive lower bound for (46) below. The remaining estimates for the polynomial case and the Misiurewicz–Thurston case are the same. For further comments when $f$ is Misiurewicz–Thurston see Section 3.3.

We may now proceed to the estimates. For the distortion estimate (36), we find, similarly as when deriving (44),

$$(46) \qquad \left| \frac{(f_s^{k-1})'(x)}{(f^{k-1})'(x)} \right| \leq \prod_{i=0}^{k-2} \left( 1 + \frac{\tilde{C}|s| + \tilde{C}D_i}{|f'(f^i(x))|} \right), \quad \forall x \in f(J_{k-1}).$$

Using (42) and (41), we can proceed as in the proof of Lemma 2.3 to show that the above product is bounded. The lower bound is obtained in a similar way, where, without loss of generality, it is enough to consider the case of large $M$. (To deal with small $M$, we might decrease $\epsilon$, like when proving (41).)

The estimate (37) follows from $|(f_s^k)'(x)| \geq C^{-1}|x - c||(f_s^k)'(f_s(x))|$ combined with (36), Lemma 2.3, and (25).

By a similar calculation as in deriving (44) (whose right hand side is uniformly bounded, as we have shown above), there exists a constant $C' \geq 1$ so that

$$(47) \qquad |\partial_s f_s^k(x)| \leq C'|(f^{k-1})'(f(x))|, \quad \forall x \in J_{k-1}.$$

By the distortion estimate Lemma 2.3, this shows (38).

Regarding (39), recall (36) and (14), and the fact that $f$ is $(\lambda_c, H_0)$-Collet–Eckmann. For $H_0 \leq k_0 \leq k \leq M - 1$ and $x \in J_{k-1}$, we get

$$(48) \qquad \Big| \sum_{j=k_0}^{k} \frac{(\partial_s f_s)(f_s^j(x))}{(f_s^j)'(f_s(x))} \Big| \leq \sum_{j=k_0}^{k} \frac{\tilde{C}C^2}{|(f^j)'(c_1)|} \leq \frac{\tilde{C}C^2 \lambda_c}{\lambda_c - 1} \lambda_c^{-k_0}.$$

Fix $k_0 \geq H_0$ such that the right hand side of (48) is smaller than $\mathcal{J}_0/4$. Once $k_0$ is fixed, we can take $\epsilon$ and $\delta$ small enough so that

$$\Big| \sum_{j=0}^{k_0-1} \Big( \frac{(\partial_t f_t)(c_j)|_{t=0}}{(f^j)'(c_1)} - \frac{(\partial_s f_s)(f_s^j(x))}{(f_s^j)'(f_s(x))} \Big) \Big| \leq \frac{\mathcal{J}_0}{4}.$$

Recalling (43) and the definition of $\mathcal{J}_0$ in (4), we conclude that

$$\Big| \frac{\partial_s f_s^k(x)}{(f_s^{k-1})'(f_s(x))} \Big| = \Big| \sum_{j=0}^{k-1} \frac{(\partial_s f_s)(f_s^j(x))}{(f_s^j)'(f_s(x))} \Big| \geq \frac{\mathcal{J}_0}{4}.$$

Applying once more (36) and (14), this implies (39), provided $\delta$ is so small such that $H(\delta) \geq k_0$ (observe that the constants in the estimates we used do not depend on $\delta$). The statement about the signs follows immediately.

Finally, by Proposition 2.4 [3] there exists a constant $C'$ such that for all $t$ which is good for the same parameters $\lambda_c$, $H_0$, $\alpha > 1$, $\rho$, $C_0$, we have

$$|(f_t^k)'(c_{1,t})|/|(f_t^l)'(c_{1,t})| \leq C'k^\alpha, \quad \forall l > k \geq 1,$$

and claim (40) follows immediately by (36). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $f_t$ be a smooth one-parameter family of smooth nondegenerate unimodal maps. As usual, we put $f = f_0$. Adapting [12, (107)–(109)] to the polynomial towers of the present work, and in view of Lemma 3.8, we introduce a key definition:

**Definition 2.7** (Admissible pairs). Let $C_a > C$, where $C \geq 1$ is given by Proposition 2.4. Let $\alpha \geq 0$, $\beta \geq 0$, $\epsilon > 0$. A pair $(M, t)$ with $M \in \mathbb{Z}_+$ is called a $(C_a, \alpha, \beta, \epsilon)$-*admissible pair* (or just an admissible pair, if the meaning is clear) if $0 < |t| < \epsilon$ and

$$(49) \qquad |(f^M)'(c_1)||t| \leq C_a^{-1} M^{-(\alpha+\beta)},$$

and $M$ is maximal for this property.

---

[3]The proof of this fact uses properties (7) and (8) in Proposition 2.1.

Observe that if $(M, t)$ is an admissible pair then, by the maximality of $M$, we find a constant $C$ (only dependent on $\sup |f'|$) such that

$$(50) \qquad |(f^M)'(c_1)|^{-1} \leq CC_a M^{\alpha+\beta} |t| \,.$$

To motivate the definition, let $f = f_0$, be good for parameters $\lambda_c$, $H_0$, $\alpha > 1$, $\rho$, $C_0$. Let $\delta > 0$ be so small such that all results in Section 2 hold. Choose $\beta > \alpha + 1$. Let $\epsilon > 0$ be given by Lemma 2.6). (In our application below, $\epsilon$ may be further reduced when invoking Lemma 3.8.) Then, if $t$ is good for the same parameters and $(M, t)$ is a $(C_a, \alpha, \beta, \epsilon)$-admissible pair, we claim that the estimates in Lemma 2.6 hold for $M$ and all $|s| \leq t$ , with constants depending only on $C_a$. (Indeed, (40) holds for $s = 0$ by Proposition 2.4 since $C_a$ is larger than the constant $C$ from that proposition, so that (35) is satisfied by the admissibility condition.) In addition, using (35) again, we may ensure, by Lemma 3.8 below, that the tower of $f_t$ coincides with that of $f$ up to level $M$.

## 3. Banach spaces and transfer operators on the tower

In this section we define the Banach spaces, transfer operators, and truncated transfer operators used to prove our theorems, and we strengthen the results from [12] on these objects: In Sections 3.1 and 3.2 we consider $f_t$ good for parameters $(\lambda_c, H_0, \alpha, \rho, C_0)$, with $\alpha > 1$, using the notation and tower construction from Section 2.2. In Section 3.3, we summarise the changes needed to adapt the constructions of Sections 2, 3, and 3.1, 3.2 to the Misiurewicz–Thurston case (where we take $\alpha = 0$).

3.1. **Banach spaces and transfer operators $\widehat{\mathcal{L}}_t$.** Just like in [12], we shall work with Sobolev spaces. For integer $r \geq 0$, recall that the generalized Sobolev norm of $\psi : I \to \mathbb{C}$ is

$$\|\psi\|_{W_1^r} = \|\partial_x^r \psi(x)\|_{L^1(I)} \,.$$

Note that $\|\psi\|_{L^\infty(I)} \leq C \|\psi\|_{W_1^1}$ if $\psi$ is compactly supported (cf. also inequality (136) below).

Fix $\lambda$ so that

$$(51) \qquad 1 < \lambda < \min(\lambda_c^{1/2}, \sqrt{\rho}) \,.$$

(The square root in $\lambda < \sqrt{\rho}$ is used in (74) below.) Let $\Lambda_t \geq \lambda_c$ be so that, for some constant $C = C_t \geq 1$, [4]

$$(52) \qquad |(f_t^k)'(c_{1,t})| \geq \frac{\Lambda_t^k}{C_t}, \ \forall k \geq 1 \,.$$

We first introduce the Banach space of functions on the tower on which the transfer operator (to be defined next) will act:

**Definition 3.1** (Spaces $\mathcal{B}_t = \mathcal{B}_t^{W_1^1}$, $\mathcal{B}_t^{L^1}$, $\mathcal{B}_t^{L^p}$). Let $\mathcal{B}_t = \mathcal{B}_t^{W_1^1}$ be the space of sequences $\hat{\psi} = (\psi_k : I \to \mathbb{C}, k \in \mathbb{Z}_+)$, so that each $\psi_k$ is $W_1^1$ and, in addition,

$$(53) \qquad \mathrm{supp}(\psi_0) \subset (0, 1) \,, \quad \text{and} \quad \mathrm{supp}(\psi_k) \subset J_{k,t}, \quad \forall k \geq 1 \,,$$

endowed with the norm

$$\|\hat{\psi}\|_{\mathcal{B}_t} = \sum_{k \geq 0} \|\psi_k\|_{W_1^1} \,.$$

---

[4]The supremum of Collet–Eckmann constants $\lambda_{c,t}$ is not always a Collet–Eckmann constant, this is why we introduce $\Lambda_t$. See also Lemma 3.9.

Let $\mathcal{B}_t^{L^1}$ be the space of sequences $\hat{\psi}$ of functions $\psi_k \in L^1(I)$ satisfying (53), with

$$\|\hat{\psi}\|_{\mathcal{B}_t^{L^1}} = \sum_{k \geq 0} \lambda^k \|\psi_k\|_{L^1(I)} \,. \tag{54}$$

For $p > 1$ and $r = r(t, p)$ so [5] that

$$\lambda^{1-r} = \Lambda_t^{\frac{1}{2}(1-\frac{1}{p})} \,, \tag{55}$$

let $\mathcal{B}_t^{L^p}$ be the space[6] of sequences $\hat{\psi}$ of functions $\psi_k \in L^p(I)$ satisfying (53), with

$$\|\hat{\psi}\|_{\mathcal{B}_t^{L^p}} := \sum_{k \geq 0} \lambda^{kr} \|\psi_k\|_{L^p(I)} \,.$$

*Remark* 3.2 (Strong and weak norms). Generally, $\mathcal{B}_t^{W_1^1}$ will be the "strong" norm and $\mathcal{B}_t^{L^1}$ the "weak" norm, in the usual Lasota–Yorke meaning, see e.g. (66). (It is easy to check that $\mathcal{B}_t^{W_1^1}$ is continuously embedded in $\mathcal{B}_t^{L^1}$ using (23) and (53), (51).)

The auxiliary weak norms $\mathcal{B}_t^{L^p}$ for $p > 1$ will only be used in the Misiurewicz–Thurston case (where $\beta = 0$), to get the lower bound in Theorem 1.7. We have, recalling (23),

$$\|\hat{\psi}\|_{\mathcal{B}_t^{L^p}} = \sum_{k \geq 0} \lambda^{kr} \left( \int_I |\psi_k|^p \, dx \right)^{1/p} \leq \sum_{k \geq 0} \lambda^{kr} |\mathrm{supp}(\psi_k)|^{1/p} \|\psi_k\|_{L^\infty}$$

$$\leq C \sum_{k \geq 0} \lambda^{kr} k^{-\beta/2p} \lambda_c^{-k/2p} \|\psi_k\|_{W_1^1} \,.$$

Since $r < 1/p$, we get $\|.\|_{\mathcal{B}_t^{L^p}} \leq C \|.\|_{\mathcal{B}_t}$ for any $p > 1$ by using $\lambda < \sqrt{\lambda_c}$ from (51).

In addition the embedding $\mathcal{B}_t^{L^p} \subset \mathcal{B}^{L^1}$ is bounded for any $p > 1$:

$$\|\hat{\psi}\|_{\mathcal{B}^{L^1}} = \sum_{k \geq 0} \lambda^k \|\psi_k\|_{L^1(I)} \leq \sum_{k \geq 0} \lambda^k |\mathrm{supp}(\psi_k)|^{(p-1)/p} \|\psi_k\|_{L^p(I)} \tag{56}$$

$$\leq C \sum_{k \geq 0} \lambda^k k^{-\beta(p-1)/2p} |(f^{k-1})'(c_1)|^{-(p-1)/2p} \|\psi_k\|_{L^p(I)} \leq C C_t \|\hat{\psi}\|_{\mathcal{B}_t^{L^p}} \,,$$

by the Hölder inequality, and the definitions (55) of $r$ and (52) of $\Lambda_t$.

The definition (53) of $\psi_k$ implies $\psi_k = 1_{J_{k,t}} \cdot \psi_k$. Recalling (17) and noting that $J_{k,t} \subset U_{k,t,+} \cup U_{k,t,-}$, the function $\psi_k \circ f_{t,\varsigma}^{-k} = 1_{f_t^k(J_{k,t})} \cdot (\psi_k \circ f_{t,\varsigma}^{-k})$ is well defined for $k \geq 1$ and $\varsigma = \pm$. The *projection* $\Pi_t(\hat{\psi})$ for a function $\hat{\psi} \in \mathcal{B}_t$ is then defined by

$$\Pi_t(\hat{\psi})(x) = \sum_{k \geq 0, \varsigma \in \{+, -\}} \frac{\lambda^k}{|(f_t^k)'(f_{t,\varsigma}^{-k}(x))|} \psi_k(f_{t,\varsigma}^{-k}(x)) \,. \tag{57}$$

---

[5] Note that (51) implies that $r < 1/p < 1$. If $\lambda \to 1$ then $r \to -\infty$, but it is instead convenient to take $\lambda \sim \min(\lambda_c^{1/4}, \sqrt{\rho})$, in view of (65).

[6] Defining $\mathcal{B}_t^{L^p}$ by interpolation instead would not be appropriate, in view of (88), (118).

For $\hat{\psi} \in \mathcal{B}_t$, the function $f_t^k : [c,1] \cap \mathrm{supp}(\psi_k) \to I$ is injective, and by a change of variables we have

$$\int_0^1 \frac{|\psi_k(f_{t,+}^{-k}(x))|}{|(f_t^k)'(f_{t,+}^{-k}(x))|} \, dx = \int_c^1 |\psi_k(x)| \, dx \, ,$$

and a similar formula holds when considering the branch $f_{t,-}^{-k}$ (instead of integrating over $[c,1]$ we integrate over $[0,c]$ on the right hand side). Thus, we have $\|\Pi_t(\hat{\psi})\|_{L^1(I)} \le \|\hat{\psi}\|_{\mathcal{B}_t^{L^1}}$. The case of $\mathcal{B}_t^{L^p}$ for $p > 1$ is a little less trivial:

**Lemma 3.3.** *For any $p > 1$ and any $1 \le \tilde{p} < p\frac{2}{p+1}$ there exists $C(p,\tilde{p}) \ge 1$ so that* $\|\Pi_t(\hat{\psi})\|_{L^{\tilde{p}}} \le C(p,\tilde{p})C_t\|\hat{\psi}\|_{\mathcal{B}_t^{L^p}}$.

*Proof.* By the Minkowski inequality and a change of variable

$$\|\Pi_t(\hat{\psi})\|_{L^{\tilde{p}}} \le \sum_{k \ge 0, \varsigma \in \{+,-\}} \lambda^k \left( \int_I \frac{|\psi_k(f_{t,\varsigma}^{-k}(x))|^{\tilde{p}}}{|(f_t^k)'(f_{t,\varsigma}^{-k}(x))|^{\tilde{p}}} \, dx \right)^{1/\tilde{p}}$$

$$\le \sum_{k \ge 0} \lambda^k \left( \int_I \frac{|\psi_k(y)|^{\tilde{p}}}{|(f_t^k)'(y)|^{\tilde{p}-1}} \, dy \right)^{1/\tilde{p}} \, .$$

For $q' > 1$ and $p' > 1$ so that $q'^{-1} + p'^{-1} = 1$, applying the Hölder inequality, changing variables again, and using the first inequality in (20) (which holds for any $y \in J_{k,t}$) give [7]

$$\left( \int_I \frac{|\psi_k(y)|^{\tilde{p}}}{|(f_t^k)'(y)|^{\tilde{p}-1}} \, dy \right)^{1/\tilde{p}} \le \left( \int_{\mathrm{supp}(\psi_k)} \frac{1}{|(f_t^k)'(y)|^{(\tilde{p}-1)q'}} \, dy \right)^{1/\tilde{p}q'} \|\psi_k\|_{L^{\tilde{p}p'}(I)}$$

$$\le \sum_{\zeta = \pm} \left( \int \frac{1}{|(f_t^k)'(f_{t,\varsigma}^{-k}(x))|^{(\tilde{p}-1)q'+1}} \, dx \right)^{\frac{1}{\tilde{p}q'}} \|\psi_k\|_{L^{\tilde{p}p'}(I)}$$

$$\le \frac{C}{(|(f^{k-1})'(c_{1,t})|^{1/2})^{(1-1/\tilde{p})+1/(\tilde{p}q')}} \left( \int_I \frac{1_{f_t^k(J_{k,t})}(x)}{\sqrt{|x - c_{k,t}|}^{(\tilde{p}-1)q'+1}} \, dx \right)^{\frac{1}{\tilde{p}q'}} \|\psi_k\|_{L^{\tilde{p}p'}(I)} \cdot$$

Now, if

(58) $$(\tilde{p}-1)q' + 1 < 2$$

then $|x - c_{k,t}|^{-\frac{(\tilde{p}-1)q'+1}{2}}$ is integrable, and we find

$$\left( \int_I \frac{|\psi_k(y)|^{\tilde{p}}}{|(f_t^k)'(y)|^{\tilde{p}-1}} \, dy \right)^{1/\tilde{p}} \le C^2 \frac{1}{(|(f^{k-1})'(c_{1,t})|^{1/2})^{(1-1/\tilde{p})+1/(\tilde{p}q')}} \|\psi_k\|_{L^{\tilde{p}p'}(I)}$$

Set $p = \tilde{p}p'$. Then, $\tilde{p} = p(1 - 1/q')$ so that (58) amounts to

$$q' < \frac{p+1}{p-1} \, ,$$

and the condition on $\tilde{p}$ is $\tilde{p} < p\frac{2}{p+1}$, as announced. Using that our choices give

$$1 - \frac{1}{p} = 1 - \frac{1}{\tilde{p}p'} = 1 - \frac{1}{\tilde{p}} + \frac{1}{\tilde{p}q'} \, ,$$

---

[7]Note that $\max f_t^k(J_{k,t}) = c_{k,t}$ if $f_t^k$ has a local maximum at $c$, while $\min f_t^k(J_{k,t}) = c_{k,t}$ if $f_t^k$ has a local minimum at $c$.

we find (by definition of $r$ and $\Lambda_t$)

$$\frac{\lambda^{k(1-r)}}{|(f^{k-1})'(c_{1,t})|^{[(1-1/\tilde{p})+1/(\tilde{p}q')]/2}} \leq C \frac{\Lambda_t^{k\frac{p-1}{2p}}}{|(f^{k-1})'(c_{1,t})|^{[(1-1/\tilde{p})+1/(\tilde{p}q')]/2}} \leq CC_t \,,$$

which concludes the proof of the lemma. $\qquad\square$

In order to define the transfer operator $\widehat{\mathcal{L}}_t$, we follow [12] and introduce smooth *cutoff functions* $\xi_{k,t}$ defined as follows. (Smooth cutoff function are needed to ensure that the smoothness of functions in $\mathcal{B}_t$ is preserved when applying the transfer operator defined below.) For each $k \geq 0$, let $\xi_{k,t} : I \to [0,1]$ be a $C^\infty$ function, with

$$(59) \qquad \mathrm{supp}(\xi_{0,t}) = [c-\delta, c+\delta]\,, \qquad \xi_{0,t}|_{[c-\frac{\delta}{2},c+\frac{\delta}{2}]} \equiv 1\,,$$

while for $k \geq 1$ we set $\xi_{k,t} \equiv 1$ if $I_{k+2,t} = \emptyset$, and, otherwise we assume

- $\xi_{k,t}$ is unimodal,
- $\mathrm{supp}(\xi_{k,t}) = J_{k+1,t}$,
- $\xi_{k,t}|_{\cup_{\varsigma \in \{+,-\}} f_{t,\varsigma}^{-(k+1)}[c_{k+1,t}-(k+1)^{-\beta}/(2L^3),c_{k+1,t}+(k+1)^{-\beta}/(2L^3)]} \equiv 1$,
- $\sup|\partial_x^j \xi_{k,t}(x)| \leq C|J_{k+1,t}|^{-j}$, for $j=1,2,3$.

(The last property we assume holds also for $k = 0$.) Note that $\xi_{k,t}(y) > 0$ if and only if $\hat{f}_t(f_t^k(y),k) \in B_{k+1,t} \times (k+1)$. Further, observe that if $\xi_{k,t} \not\equiv 1$, then $f_t^{k+2}(J_{k+2,t})$ is adjacent to the boundary of $B_{k+2,t}$ from which follows that $|f_t^{k+2}(J_{k+2,t})| \geq (k+2)^{-\beta}L^{-3}/2$. Hence, we derive similarly as in the estimate (23) above that for some constant $C \geq 1$

$$(60) \qquad |J_{k+1,t}| \geq |J_{k+2,t}| \geq C^{-1}k^{-\beta/2}|(f_t^{k+1})'(c_1)|^{-1/2}\,, \qquad \text{if } I_{k+2,t} \neq \emptyset\,.$$

This will give the estimate (63) below.

**Definition 3.4** (Transfer operator)**.** The transfer operator $\widehat{\mathcal{L}}_t$ is defined for $\hat{\psi} \in \mathcal{B}_t$ by

$$(61) \quad (\widehat{\mathcal{L}}_t\hat{\psi})_k(x) = \begin{cases} \frac{\xi_{k-1,t}(x)}{\lambda} \cdot \psi_{k-1}(x) & k \geq 1, \\ \sum_{j\geq 0,\varsigma\in\{+,-\}} \frac{\lambda^j(1-\xi_{j,t}(f_{t,\varsigma}^{-(j+1)}(x)))}{|(f_t^{j+1})'(f_{t,\varsigma}^{-(j+1)}(x))|} \cdot \psi_j(f_{t,\varsigma}^{-(j+1)}(x)) & k = 0. \end{cases}$$

To define $\psi_j(f_{t,\varsigma}^{-(j+1)}(x)) = \psi_j(f_{t,\varsigma}^{-(j+1)}(x))1_{f_t^{j+1}(J_{j,t})}$ (recall (53)), we used that, since $c \notin f_t^j(J_{j,t})$, for any $x$, and any $j \geq 1$, the set $f^{-1}(x) \cap f_t^j(J_{j,t})$ is either empty (the corresponding $j$-term in the sum for $(\widehat{\mathcal{L}}_t\hat{\psi})_0(x)$ is then zero) or a singleton. In particular, the terms for all $1 \leq j < H_0$ in the sum for $(\widehat{\mathcal{L}}_t\hat{\psi})_0(x)$ vanish, because of our choice of small $\delta$. If $0 < \xi_{j,t}(y) < 1$, then $y$ will contribute to both $(\widehat{\mathcal{L}}_t\hat{\psi})_{j+1}(y)$ and $(\widehat{\mathcal{L}}_t\hat{\psi})_0(f^{j+1}(y))$. In other words, the transfer operator just defined is associated to a multivalued (probabilistic-type) tower dynamics. For this multivalued dynamics, some points may fall from the tower a little earlier than they would for $\hat{f}_t$. However, the conditions on the functions $\xi_{k,t}$ guarantee that they do not fall *too* early. More precisely, if we define "fuzzy" analogues of the intervals $I_{k,t}$ and $J_{k,t}$ from (12) as follows

(62)
$$\widetilde{I}_{k,t} := \{x \in I \mid \xi_{k,t}(x) < 1\,, \xi_{j,t}(x) > 0\,, \forall 0 \leq j < k\}\,, \quad \widetilde{J}_{k,t} = \{c\} \cup_{j\geq k+1} \widetilde{I}_{k,t}\,,$$

then we can replace $I_{k,t}$ and $J_{k,t}$ by $\widetilde{I}_{k,t}$ in the previous estimates, in particular in Lemma 2.6. Indeed, just observe that if a point "falls" according to our fuzzy dynamics, it would have fallen for some choice of intervals $\widetilde{B}_{k,t}$ so that

$$[c_{k,t} - k^{-\beta}/(2L^3), c_{k,t} + k^{-\beta}/(2L^3)] \subset \widetilde{B}_{k,t} \subset B_{k,t} \,.$$

Since we can apply Lemma 2.6 to the fuzzy intervals, we can combine (60) with (20), (21), and (22), and it follows immediately from the conditions on $\xi_{k,t}$ that there is a constant $\tilde{C} \geq 1$ such that

(63) $\qquad \|\xi_{k,t} \circ f_{t,\pm}^{-(k+1)}\|_{C^1} \leq \tilde{C} k^\beta \,, \quad \|\xi_{k,t} \circ f_{t,\pm}^{-(k+1)}\|_{C^2} \leq \tilde{C} k^{\max(1+2\alpha+\beta, 2\beta)},$

$\qquad \|\xi_{k,t} \circ f_{t,\pm}^{-(k+1)}\|_{C^3} \leq \tilde{C} k^{\max(1+4\alpha+\beta, 3\beta)} \,, \quad \text{for all } k \geq 1 \,.$

(This is the polynomial analogue of condition [12, (75)]; the case $j = 3$ is used together with (21), (22) in Appendix B.)

*Remark* 3.5 (Overlap control). In contrast to the intervals $I_k$, the intervals $\widetilde{I}_k$ do not have pairwise disjoint interiors. Nevertheless, it follows from the first paragraph in the proof of Lemma 3.8 (see in particular (80)) that if $L$ is large enough (and thus $\delta$ small enough), we may choose the cutoff functions $\xi_k$ so that for each $k$, the cardinality of those $\widetilde{I}_j$, $j \neq k$, whose interiors intersect the interior of $\widetilde{I}_k$ is bounded by 2. In other words each set $\{x \in \operatorname{supp} \xi_k(x) \mid \xi_k(x) \neq 1\}$ is contained in $I_{k+2}$, and hence these sets are disjoint. (This overlap control is used to get the Lasota–Yorke estimate at the heart of Proposition 3.6. The fact that the overlap is at most two is used to get a good control in (103), which is essential for Theorem 1.7.)

Now, if we introduce the ordinary (Perron–Frobenius) transfer operator

$$\mathcal{L}_t : L^1(I) \to L^1(I), \qquad \mathcal{L}_t \varphi(x) = \sum_{f_t(y)=x} \frac{\varphi(y)}{|f_t'(y)|} \,,$$

then we have

(64) $\qquad\qquad \mathcal{L}_t \Pi_t(\hat{\psi}) = \Pi_t(\widehat{\mathcal{L}}_t \hat{\psi}) \,, \ \forall \hat{\psi} \in \mathcal{B}_t^{L^1} \,.$

(See, e.g., [12] below equation (78).) In particular, if $\widehat{\mathcal{L}}_t \hat{\phi} = \hat{\phi}$ then $\mathcal{L}_t \Pi_t(\hat{\phi}) = \Pi_t(\hat{\phi})$.

Set $w(x, k) = \lambda^k$, for $x \in I$ and $k \geq 0$, and define $\nu$ to be the nonnegative measure on $\cup_{k \geq 0} I \times \{k\}$ whose density with respect to Lebesgue is $w(x, k)$.

**Proposition 3.6** (Spectral properties of $\widehat{\mathcal{L}}_t$). *Let $f_t$ be good for parameters $\lambda_c$, $H_0$, $\alpha > 1$, $\rho$, $C_0$. Choose $\delta > 0$ small, $\beta > \alpha + 1$ and $\lambda > 1$ as in (51). Then the operator $\widehat{\mathcal{L}}_t$ is bounded on $\mathcal{B}_t$, and for any*

(65) $\qquad\qquad 1 < \Theta_0 < \min(\dfrac{\lambda_c^{1/2}}{\lambda}, \lambda) \,,$

*the essential spectral radius of $\widehat{\mathcal{L}}_t$ on $\mathcal{B}_t$ is bounded by $\Theta_0^{-1}$. The spectral radius of $\widehat{\mathcal{L}}_t$ on $\mathcal{B}_t$ is equal to 1, where 1 is a simple eigenvalue for a nonnegative eigenvector $\hat{\phi}_t$. If $f_t$ is mixing, then 1 is the only eigenvalue of modulus 1, otherwise the other eigenvalues of modulus 1 are simple and located at roots of unity $e^{2\iota j\pi/P_t}$, $j = 0, \ldots, P_t - 1$, for $P_t \geq 2$ the renormalisation period of $f_t$. The fixed point of the dual of $\widehat{\mathcal{L}}_t$ is $\nu$. If $\nu(\hat{\phi}_t) = 1$, then $\phi_t := \Pi_t(\hat{\phi}_t)$ is the density of the unique absolutely*

continuous $f_t$-invariant probability measure. Finally,[8] $\hat{\phi}_{t,0} \in W_1^2$, uniformly in the goodness (once $\delta$, $\beta$, $L$, and $\lambda$ are fixed).

*Proof.* The proof is an adaptation of Propositions 4.10 and 4.11 in [12] to our fat tower, using the polynomial recurrence condition. We give it in Appendix B, mentioning here only that the key (Lasota–Yorke) estimate is that there exists a constant $C > 0$, depending only on the goodness of $f_t$, $\delta$, and $L$, such that

$$(66) \qquad \|\widehat{\mathcal{L}}_t^n(\hat{\psi})\|_{\mathcal{B}_t} \leq C\Theta_0^{-n}\|\hat{\psi}\|_{\mathcal{B}_t} + C\|\hat{\psi}\|_{\mathcal{B}_t^{L^1}}, \quad \forall n \geq 1,$$

for all $\hat{\psi} \in \mathcal{B}_t$. $\qquad\qquad\square$

### 3.2. Truncated transfer operators $\widehat{\mathcal{L}}_{t,M}$ on the tower.
We introduce for each $M \geq 0$ the *truncation operator* $\mathcal{T}_M$ defined by

$$(67) \qquad \mathcal{T}_M(\hat{\psi})_k = \begin{cases} \psi_k & k \leq M \\ 0 & k > M. \end{cases}$$

By definition $\mathcal{T}_M$ is a bounded operator on $\mathcal{B}_t$, with $\|\mathcal{T}_M\|_{\mathcal{B}_t} \leq 1$ for any $M$. The *truncated transfer operator* $\widehat{\mathcal{L}}_{t,M} : \mathcal{B}_t \to \mathcal{B}_t$ is the bounded operator defined by

$$\widehat{\mathcal{L}}_{t,M} = \mathcal{T}_M \widehat{\mathcal{L}}_t \mathcal{T}_M.$$

The following proposition lists the basic spectral properties of the truncated transfer operator. For the maximal eigenvector $\hat{\phi}_t$ of $\widehat{\mathcal{L}}_t$ given by Proposition 3.6 we assume always that it is normalised by $\nu(\phi_t) = 1$.

**Proposition 3.7** (Spectral properties of the truncated operator $\widehat{\mathcal{L}}_{t,M}$). *For any $t$ which is good for parameters $\lambda_c$, $H_0$, $\alpha > 1$, $\rho$, $C_0$, the essential spectral radius of $\widehat{\mathcal{L}}_{t,M}$ acting on $\mathcal{B}_t$ is not larger than $\Theta_0^{-1} < 1$, where $\Theta_0$ satisfies condition (65).*

*There exists $M_0 \geq 1$ (depending only on the goodness of $f_t$) so that, for all $M \geq M_0$, the operator $\widehat{\mathcal{L}}_{t,M}$ has a real nonnegative maximal eigenfunction $\hat{\phi}_{t,M}$, for a simple eigenvalue $\Theta_0^{-1} < \kappa_{t,M} \leq 1$, and the dual operator of $\widehat{\mathcal{L}}_{t,M}$ has a nonnegative maximal eigenfunction $\nu_{t,M}$. If we normalise $\nu_{t,M}$ by $\nu_{t,M}(\hat{\phi}_t) = 1$, and $\hat{\phi}_{t,M}$ by $\nu(\hat{\phi}_{t,M}) = 1$ (recall that $\nu = \nu_t$), then we have the bounds $\sup_M \|\nu_{t,M}\|_{(\mathcal{B}_t^{L^1})^*} \leq C_1$, $\sup_M \|\hat{\phi}_{t,M}\|_{\mathcal{B}_t} \leq C_1$, for a constant $C_1$ depending only on the goodness of the family $f_t$.*

*Furthermore, fixing $\upsilon < 1$, and setting*

$$(68) \qquad \tau_{t,M} = M^{(\alpha-\beta)/2}\lambda^M|(f_t^M)'(c_{1,t})|^{-1/2} < 1,$$

*there exists $C_t \geq 1$ so that for all $M \geq M_0$*

$$(69) \quad \|\hat{\phi}_t - \hat{\phi}_{t,M}\|_{\mathcal{B}^{L^1}} \leq C_t\tau_{t,M}^\upsilon, \quad \|\nu - \nu_{t,M}\|_{\mathcal{B}_t^*} \leq C_t\tau_{t,M}^\upsilon, \quad |\kappa_{t,M} - 1| \leq C_t\tau_{t,M}^\upsilon.$$

*In particular,*

$$(70) \qquad 1 \leq \kappa_{t,M}^{-M} \leq C_t, \qquad \forall M \geq M_0.$$

*Finally, $\sup_M \|\hat{\phi}_{t,M,0}\|_{W_1^2} \leq \hat{C}$, for a constant $\hat{C}$ depending only on the goodness of $f_t$ and $\|f_t\|_{C^4}$.*

---

[8]We use here that $f_t$ is $C^4$ and not just $C^3$.

Bootstrapping from the estimates above, Proposition 4.1 will give the more precise control on $\|\hat{\phi}_t - \hat{\phi}_{t,M}\|_{\mathcal{B}^{L^1}}$ and estimates on $\|\hat{\phi}_t - \hat{\phi}_{t,M}\|_{\mathcal{B}^{L^p}}$ ($p > 1$) that are needed for Theorems 1.2 and 1.7. Then Lemma 4.6 will give uniformity of $C_t$ as a function of $t$, needed for Proposition 4.2 and Theorem 1.3

*Proof of Proposition 3.7.* We adapt the proof of [12, Lemma 4.12] to our polynomial tower setting, i.e., we apply the perturbation results of Keller and Liverani [25]. As usual, we assume that $t = 0$ and set $f = f_0$. Uniformity in $t$ of the constant $C_1$ will follow from uniformity of the goodness.

The claim about the essential spectral radius can be obtained by going over the proof of Proposition 3.6, checking that it applies to $\widehat{\mathcal{L}}_M$ and that the constants are uniform in $M$. More precisely, there exists $C \geq 1$ so that for all $n$ and all $M$

$$(71) \qquad \max(\|\widehat{\mathcal{L}}^n(\hat{\psi})\|_{\mathcal{B}^{W_1^1}}, \|\widehat{\mathcal{L}}_M^n(\hat{\psi})\|_{\mathcal{B}^{W_1^1}}) \leq C\Theta_0^{-n}\|\hat{\psi}\|_{\mathcal{B}^{W_1^1}} + C\|\hat{\psi}\|_{\mathcal{B}^{L^1}},$$

and (note that $\nu(|\widehat{\mathcal{L}}_M^n(\hat{\psi})|) \leq \nu(\widehat{\mathcal{L}}_M^n(|\hat{\psi}|)) \leq \nu(\widehat{\mathcal{L}}^n(|\hat{\psi}|)) = \nu(|\hat{\psi}|) = \|\hat{\psi}\|_{\mathcal{B}^{L^1}}$; see also (141) below)

$$(72) \qquad \|\widehat{\mathcal{L}}^n\|_{\mathcal{B}^{L^1}} \leq 1, \quad \|\widehat{\mathcal{L}}_M^n\|_{\mathcal{B}^{L^1}} \leq 1, \quad \forall M, \forall n.$$

To prove the other claims, we shall use that there exists $C$ so that for all large enough $M$

$$(73) \qquad \|(\widehat{\mathcal{L}} - \widehat{\mathcal{L}}_M)(\hat{\psi})\|_{\mathcal{B}^{L^1}} \leq C\tau_M\|\hat{\psi}\|_{\mathcal{B}^{W_1^1}}.$$

This inequality is an easy consequence of

$$\|(\mathrm{id} - \mathcal{T}_M)\hat{\psi}\|_{\mathcal{B}^{L^1}} \leq C\tau_M\|\hat{\psi}\|_{\mathcal{B}^{W_1^1}},$$

which follows from the estimate $\|\psi_k\|_{L^1} \leq |\mathrm{supp}(\psi_k)| \sup |\psi_k|$ combined with (23), Proposition 2.4, and (136) (and recalling the $\lambda^k$ weight in the $\mathcal{B}^{L^1}$ norm). Indeed, recalling (23), we have,

$$\|(\mathrm{id} - \mathcal{T}_M)(\hat{\psi})\|_{\mathcal{B}^{L^1}} = \sum_{k \geq M+1} \lambda^k \int_I |\psi_k(x)|\, dx \leq \sum_{k \geq M+1} \lambda^k |\mathrm{supp}(\psi_k)| \|\psi_k\|_{L^\infty}$$

$$(74) \qquad \leq C^2 \|\hat{\psi}\|_{\mathcal{B}} \lambda^M \frac{M^{-\beta/2}}{|(f^M)'(c_1)|^{1/2}} \sum_{k \geq M+1} \frac{\lambda^{k-M}}{|(f^{k-M-1})'(c_{M+1})|^{1/2}}.$$

Since $\lambda < \sqrt{\rho}$ we derive, as in the proof of Proposition 2.4, that the last sum on the right hand side is bounded by a constant times $M^{\alpha/2}$.

Then, setting $\mathbb{P}(\hat{\psi}) = \hat{\phi}\nu(\hat{\psi})$ and $\mathbb{P}_M(\hat{\psi}) = \hat{\phi}_M\nu_M(\hat{\psi})$ for the respective spectral projectors of $\widehat{\mathcal{L}}$ corresponding to their maximal eigenvalue, [25, Theorem 1, Corollary 1] give for any $\upsilon < 1$ a constant $C_t$ so that $\|(\mathbb{P}_M - \mathbb{P})(\hat{\psi})\|_{\mathcal{B}^{L^1}} \leq C_t\tau_M^\upsilon\|\hat{\psi}\|_{\mathcal{B}}$, which gives, taking $\hat{\psi} = \hat{\phi}$, that $\|\hat{\phi}_M - \hat{\phi}\| \leq C_t\tau_M^\upsilon$.

We may also apply the results of [25] to the dual operators (exchanging the roles of the weak and strong norms): Indeed, for any $\mu \in (\mathcal{B}^{L^1})^*$, we have

$$(75) \qquad \sup_{\|\hat{\psi}\|_{\mathcal{B}^{W_1^1}} \leq 1} |\mu(\widehat{\mathcal{L}}(\hat{\psi}) - \widehat{\mathcal{L}}_M(\hat{\psi}))| \leq C\tau_M \sup_{\|\hat{\psi}\|_{\mathcal{B}^{L^1}} \leq 1} |\mu(\hat{\psi})|,$$

while the Lasota–Yorke estimates for $\widehat{\mathcal{L}}^*$ and $\widehat{\mathcal{L}}_M^*$ are an immediate consequence of those for $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}_M$. Setting $\mathbb{P}^*(\mu) = \nu\mu(\hat{\phi})$ and $\mathbb{P}_M^*(\mu) = \nu_M\mu(\hat{\phi}_M)$, [25, Theorem

1, Corollary 1] give

$$(76) \qquad \sup_{\|\hat{\psi}\|_{\mathcal{B}} \leq 1} |[(\mathbb{P}_M^* - \mathbb{P})(\mu)](\hat{\psi})| \leq C_t \tau_M^\upsilon \sup_{\|\hat{\psi}\|_{\mathcal{B}^{L^1}} \leq 1} |\mu(\hat{\psi})| ,$$

and, recalling our normalisation, we may apply the above bound to $\mu = \nu$. Altogether, this gives the bounds (69) for $\upsilon \in (0, 1)$. The bound $\kappa_M \leq 1$ follows from the fact that $\kappa_M$ is an eigenvalue for $\hat{\phi}_M \in \mathcal{B}^{L^1}$ and since by (72) the spectral radius of $\widehat{\mathcal{L}}_M$ on $\mathcal{B}^{L^1}$ is bounded by 1.

Since $\widehat{\mathcal{L}}_{t,M}$ is nonnegative (see [22, pp. 933-935, Thm 27]), its maximal eigenvector, normalised by $\nu(\hat{\phi}_{t,M}) = 1$, satisfies $\hat{\phi}_{t,M} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \kappa_{t,M}^{-k} \widehat{\mathcal{L}}_{t,M}^k(\hat{\phi})$, because $\nu_t(\hat{\phi}) = \nu(\hat{\phi}) = 1$ and $\lim_{k\to\infty} (\nu(\widehat{\mathcal{L}}_{t,M}^k(\hat{\phi})))^{1/k} = \kappa_{t,M}$. It follows from what has been done up to now that

$$\sup_M \|\hat{\nu}_{t,M}\|_{(\mathcal{B}_t^{L^1})^*} \leq C_1 , \quad \sup_M \|\hat{\phi}_{t,M}\|_{\mathcal{B}^{W_1^1}} \leq C_1 ,$$

uniformly in $t$.

To show $\sup_M \|\hat{\phi}_{t,M,0}\|_{W_1^2} \leq \hat{C}$, we adapt the strategy of [6, Prop. 3.3], like for the analogous statement of Proposition 3.6 (see Appendix B), checking that the bounds are uniform in $M$ (and $t$): Take $\hat{\psi}$ with $\nu(\hat{\psi}) = 1$, so that $\psi_k = 0$ for all $k \geq 1$ and $\psi_0 \in C^\infty$, and use again that $\frac{1}{k} \sum_{n=0}^{k-1} \kappa_{t,M}^{-k} \widehat{\mathcal{L}}_{t,M}^n(\hat{\psi})$ converges to $\hat{\phi}_{t,M}$ in the $\mathcal{B}^{W_1^1}$ norm as $n \to \infty$. We claim that $\sup_{k,M} \|(\kappa_{t,M}^{-k} \widehat{\mathcal{L}}_{t,M}^k(\hat{\psi}))_0\|_{W_1^3} \leq C$, uniformly in $t$: Indeed, we can use (145) from the proof of Proposition 3.6 , and, in view of (5), we can exploit the properties (63) of $\xi_k''$, $\xi_k'''$ to adapt [12, (152) in Appendix B] (noting also that $\hat{\psi}|_\omega = 0$ if the interval $\omega$ is in some level $E_k$ with $k > 0$). To conclude, use Rellich–Kondrachov. $\qquad \square$

Recall the notion (49) of admissible pairs $(M, t)$. In the following lemma, we use the freedom in the choice of the intervals $B_{k,t}$ and cutoff functions $\xi_{k,t}$ in order to, loosely speaking, identify the towers up to some level $M = M(t)$ for $t$ close to $t_0 = 0$ . The result is a counterpart to Proposition 5.9 in [12]. The difference with the horizontal case there is that, in our present transversal case, the distance $|c_{k,t} - c_k|$ grows like $|t||(f^k)'(c_1)|$, i.e., exponentially fast.

**Lemma 3.8** (Identical $M$-truncated towers for $f$ and $f_t$). *Let $f = f_0$ be good for parameters $\lambda_c$, $H_0$, $\alpha > 1$, $\rho$, $C_0$. If the constant $L > 1$ in the definitions of the tower and of the cutoff functions $\xi_k$ is sufficiently large, then we can choose a tower $\hat{f} : \hat{I} \to \hat{I}$, cutoff functions $\xi_k$, and a transfer operator $\widehat{\mathcal{L}}$ for $f$, and $\epsilon > 0$ such that for any $M \geq 1$, and for any $t \in (-\epsilon, \epsilon)$ which is good for the same parameters and so that*

$$(77) \qquad |(f^{k-1})'(c_1)||t| \leq k^{-\beta} \quad \forall 1 \leq k \leq M ,$$

*one can construct the tower $\hat{f}_t : \hat{I}_t \to \hat{I}_t$ and the transfer operator $\widehat{\mathcal{L}}_t$ such that*

$$(78) \qquad J_{k+1,t} = J_{k+1} \quad and \quad \xi_{k,t} = \xi_k , \ \forall 0 \leq k \leq M - 1 .$$

*Proof.* Let $\epsilon > 0$ be so small as in Lemma 2.6, and take $L > \max(2C^3, 4)$ where $C$ is given by Lemma 2.3 and Lemma 2.6. (Observe that the constant $C$ in Lemma 2.3 and Lemma 2.6, respectively, does not depend on $L$ and on $\delta$.) Regarding Remark 3.5 above, we will be a bit careful in choosing the tower for $f$. We fix a tower

$\hat{f} : \hat{I} \to \hat{I}$ where the levels $E_k$, $k \geq 1$, are defined inductively by setting

$$B_k = \left[ c_k - \frac{k^{-\beta}}{2L^2}, c_k + \frac{k^{-\beta}}{2L^2} \right], \quad \text{if } \frac{k^{-\beta}}{L^2} \leq |f^k(J_{k-1})| \leq 2\frac{k^{-\beta}}{L^2},$$

and

$$B_k = \left[ c_k - \frac{k^{-\beta}}{L^2}, c_k + \frac{k^{-\beta}}{L^2} \right], \quad \text{otherwise.}$$

This implies $|f^k(I_k)|/|f^k(J_{k-1})| \geq 1/2$ whenever $I_k \neq \emptyset$, which in turn implies that

$$(79) \qquad \frac{|I_k|}{|J_{k-1}|} \geq C^{-2} \sqrt{\frac{|f(I_k)|}{|f(J_{k-1})|}} \geq C^{-3}/\sqrt{2},$$

where in the second inequality we used Lemma 2.3. Since the length of $I_k$ is comparable to the length of $J_{k-1}$, we can now construct the cutoff function $\xi_{k-2}$ such that

$$(80) \qquad \{x \,|\, 0 < \xi_{k-2}(x) < 1\} \subset I_k, \qquad \forall k \geq 2,$$

and $|\partial_x^r \xi_{k-2}| \leq \tilde{C}|J_{k-1}|^{-r}$, $r = 1, 2, 3$, for some constant $\tilde{C}$, and all the other requirements on cutoff functions are satisfied.

Recall that no point falls from the tower up to level $H(\delta)$. Hence, the assertion of Lemma 3.8 is trivially satisfied for all $0 \leq k \leq H(\delta) - 2$ (for arbitrary choices of towers and transfer operators). Let $H(\delta) - 1 \leq k \leq M - 1$, and assume that (78) is satisfied for all $0 \leq j \leq k - 1$. Assuming $C$ so large that $C^{-1}|x - c| \leq |f_t'(x)| \leq C|x - c|$, we derive from (36) and Lemma 2.3 that

$$(81) \qquad C^{-3}|f^{k+1}(J_k)| \leq |f_t^{k+1}(J_k)| \leq C^3|f^{k+1}(J_k)|,$$

for all $t$ satisfying (77). If $I_{k+1} = \emptyset$ then $f^{k+1}(J_k) \subset B_{k+1}$. Hence, by the upper bound in (81), the choice of $L$, and the definition of $B_{k+1}$, we have $|f_t^{k+1}(J_k)| \leq L^{-1}(k+1)^{-\beta}$. It follows that we can choose $B_{k+1,t}$ satisfying (10) and so that $I_{k+1,t} = \emptyset$. By definition, $\xi_{k-1,t} = \xi_{k-1} \equiv 1$, which implies (78).

If $I_{k+1} \neq \emptyset$ then the interval $f^{k+1}(J_{k+1})$ is adjacent to the boundary of $B_{k+1}$. Set $B_{k+1,t} = [c_{k+1,t}-b, c_{k+1,t}+b]$, where $b = |f_t^{k+1}(J_{k+1})|$. By the choice of $L$ and $B_{k+1}$, and using both inequalities in (81), we get $L^{-3}(k+1)^{-\beta} \leq b \leq L^{-1}(k+1)^{-\beta}$ from which follows that $B_{k+1,t}$ satisfies the condition (10). By construction $J_{k+1,t} = J_{k+1}$ and $I_{k+1,t} = I_{k+1}$. Hence, we can set $\xi_{k-1,t} = \xi_{k-1}$ which concludes the proof of Lemma 3.8. $\qquad \square$

From now on, we use the modified tower given by Lemma 3.8.

We did not claim that the constant $C_t$ in Proposition 3.7 is uniform in $t$, because we have not proved yet that there exists a neighbourhood of 1 which intersects the spectrum $\sigma(\widehat{\mathcal{L}}_t)$ of $\widehat{\mathcal{L}}_t : \mathcal{B}_t \to \mathcal{B}_t$ only at $z = 1$, for all good $t$ close enough to a good $t_0$. Indeed, a priori, the renormalisation period $P_t$ of $f_t$ could be unbounded, and the constants

$$(82) \qquad \theta_t = \sup\{z \in \sigma(\widehat{\mathcal{L}}_t) \,|\, |z| \neq 1\} < 1$$

could accumulate at 1 for (good) $t \to t_0$. (Note that $\theta_t$ gives an upper bound on the rate of decay of correlations of $f^{P_t}$ for $C^1$ functions, e.g.) Uniformity of $P_t$ and $\theta_t$ is given by [9] Lemma 4.6 in § 4.4.

---

[9]This holds a fortiori in the easier setting of [12] where $P_t \equiv P$, proving the claim on $\theta_t$ in [12, §5.2].

**3.3. The Misiurewicz–Thurston case.** In this section we discuss the modifications in the parameter selection, tower construction, and transfer operator properties, which will allow us to get stronger results in the Misiurewicz–Thurston case.

If $f_t$ is Misiurewicz–Thurston, we shall prove next that we may take $\alpha = \beta = 0$ in the definitions in Sections 2 and 3. In particular, the sizes of the levels of the tower are uniformly bounded from below. (The fact that the size of the levels is bounded from below will be essential to get the lower bound of Theorem 1.7 in Section 5, see e.g. (132).) The first remark is that we can take $\beta = \alpha = 0$ in the distortion Lemma 2.3, if we assume that $L$ is large enough so that for all $k \geq 1$, an $L^{-1}$ neighbourhood of $c_k$ does not intersect a fixed neighbourhood of $c$. Next, Proposition 2.4 holds, setting $\alpha = 0$ (its proof is trivial in the Misiurewicz–Thurston case). The exponential decay property (23) also holds, setting $\beta = 0$, and all bounds in Lemma 2.5 are true, setting $\alpha = \beta = 0$, and removing the remaining factor $j$ in the right hand sides of (21) and (22). All claims in Lemma 2.6 are true for $\alpha = \beta = 0$, up to replacing $k^{-\beta}$ by $\eta$ for some small $\eta > 0$ in (35) (see also the paragraph containing (45) in the proof of Lemma 2.6). If $t$ is Misiurewicz–Thurston, we use the definition (49) of admissible pairs $(M, t)$, setting $\alpha = \beta = 0$. Then, in the Misiurewicz–Thurston case, if $C_a$ is large enough then $|(f^M)'(c_1)||t| \leq C_a^{-1}$ implies (35), and we shall assume this throughout.

We now construct the set $\Delta_{MT}$ of sequences $t(n) \to t_0$ which will give Theorem 1.7, by exhibiting Misiurewicz–Thurston parameters with uniform average postcritical expansion. As usual we assume $t_0 = 0$ and we remove the 0 from the notation. (We shall discuss the Banach space construction, transfer operator properties, and Lemma 3.8 in the Misiurewicz–Thurston case after the proof of the following lemma.)

**Lemma 3.9** (Admissible Misiurewicz–Thurston pairs with uniform postcritical multipliers). *Let $f = f_0$ be a mixing Misiurewicz–Thurston map, and assume the family $f_t$ is transversal at $f_0$. Let $\ell_0$ denote the postcritical period of $f$. If $C_a$ in (49) defining admissible pairs for $\alpha = \beta = 0$ is large enough then, defining $\Lambda_t$ in (52) for Misiurewicz–Thurston maps by*

$$\Lambda_t = \lim_{k \to \infty} |(f_t^k)'(c_{1,t})|^{1/k} \,,$$

*there exists $C > 1$ such that for each large enough integer $m$, there exists a Misiurewicz–Thurston map $f_t$ and an integer $M$ with $|M - m| < \ell_0$ such that $(M, t)$ is an admissible pair and*

$$(83) \qquad\qquad C^{-1} \Lambda_t^k \leq |(f_t^{k-1})'(c_{1,t})| \leq C \Lambda_t^k \,, \qquad \forall\, k \geq 1 \,,$$

*furthermore, setting $\Lambda = \Lambda_0$, we have*

$$(84) \qquad\qquad C^{-1} \leq \left( \frac{\Lambda_t}{\Lambda} \right)^M \leq C \,,$$

*and, for $0 \leq k \leq M$, the points $c_{k,t}$ and $c_k$ are either both local maxima or both local minima for $f_t^k$ and $f^k$, respectively, and, for $H(\delta) \leq k \leq M$ and $x \in J_k$, if they are local maxima then $f_t^k(x) < f^k(x)$, and if they are local minima then $f^k(x) < f_t^k(x)$.*

*Finally, the set $\Delta_{MT}$ of parameters satisfying the above properties enjoy uniform goodness constants. (This set is infinite countable and accumulates at $t = 0$.)*

*Proof of Lemma 3.9.* For $m$ large, $c_m$ is in the postcritical periodic orbit of $f$. Let $M \geq m$ be minimal such that $|(f^{M+i})'(c_1)| > |(f^M)'(c_1)|$, for all $i \geq 1$. Obviously $M - m < \ell_0$.

Since the family $f_t$ is transversal at $f_0$, the map $f_0$ is not conjugated to the Ulam–von Neumann map, i.e., $c_2 = f_0^2(c)$ is not equal to the left fixed point $f(0) = 0$. (If it were, by (39), for $k$ large, a neighbourhood of $t = 0$ in $\mathcal{E}$ would be mapped by $t \mapsto f_t^k(c)$ to a neighbourhood of 0 in $\mathbb{R}$. But since 0 is the left endpoint of $I$, this is not possible.) Therefore $c_M$ is contained in the open interval $(c_2, c_1)$. Furthermore, there is at least one point $y$ in the interior of $[c_2, c_1] \setminus \{c_i \mid i \geq 0\}$ which is eventually mapped to $c_M$ but such that $f^i(y) \neq c$, for all $i \geq 0$. [10]

Let then $t_1 = t_1(M) > 0$ be minimal and $t_2 = t_2(M) > t_1$ be maximal such that $(M, t)$ is an admissible pair for all $t \in (t_1, t_2)$. Observe for further use that, by definition of admissible pairs, if $m$ is large enough,

$$\frac{t_2 - t_1}{t_2} = \frac{C_a^{-1}(|(f^M)'(c_1)|^{-1} - \max_{i \geq 1} |(f^{M+i})'(c_1)|^{-1})}{C_a^{-1}|(f^M)'(c_1)|^{-1}}$$

$$= 1 - \max_{1 \leq i \leq \ell_0} |(f^i)'(c_M)|^{-1} \,,$$

so that, by the definition of $M$, and for large enough $m_0$,

$$(85) \qquad \inf_{m \geq m_0} \inf_{M \geq m} \frac{|t_2(M) - t_1(M)|}{|t_2(M)|} > 0 \,.$$

By Lemma 2.6, for $k \geq H(\delta)$, the sign of $(f^{k-1})'(c_1) \cdot \partial_t f_t^k(c)|_{t=0}$ is independent on $k$. We are therefore in one of the following two situations: Either for all $k \geq H(\delta)$, if $c_k$ is a local maxima for $f^k$ then $\partial_t f_t^k(c)|_{t=0} < 0$, while if $c_k$ is a local minima for $f^k$ then $\partial_t f_t^k(c)|_{t=0} > 0$; in this case we set $\Delta(M) = [-t_2, -t_1)$. Or for all $k \geq H(\delta)$, if $c_k$ is a local maxima for $f^k$ then $\partial_t f_t^k(c)|_{t=0} > 0$, while if $c_k$ is a local minima for $f^k$ and $\partial_t f_t^k(c)|_{t=0} < 0$; in this case, we set $\Delta(M) = (t_1, t_2]$. By (14) and by the sign assertion just below (39) (applied to $x \in J_k$), our choice of $\Delta(M)$ implies the assertion below property (84).

By (38), $|c_j - c_{j,t}|$ is bounded from above by a constant times $|(f^{j-1})'(c_1)||t|$, for all $j \leq M$ and all $|t| \leq t_2$. Since $|(f^M)'(c_1)||t| \leq C_a^{-1}$ and since the derivative $|(f^{M-j})'(c_j)|$ of the Misiurewicz–Thurston map $f$ grows exponentially in $M - j$, there exists an integer $\ell_1$, which does not depend on $M$, such that $c_{M-\ell_1,t}$ is contained in $(c_2, c_1)$, for all $t \in [-t_2, t_2]$. On the other hand, by the uniformity of $\ell_1$, the fact that $|(f^{M+1})'(c_1)||t| \geq C_a^{-1}$, and by the transversality property (39), there is a constant $\tilde{C} > 1$ such that

$$\inf_M |\{c_{M-\ell_1,t} \mid t \in \Delta(M)\}| \geq \tilde{C}^{-1} \inf_M \frac{t_2(M) - t_1(M)}{t_2(M)} \geq \frac{1}{\tilde{C}^2} \,,$$

where in the last inequality we used (85). It easily follows that there is a finite collection $\mathcal{J}$ of open intervals contained in $(c_2, c_1)$ such that for each large enough $m$ the interval $\{c_{M(m)-\ell_1,t} \mid t \in \Delta(M(m))\}$ contains at least one interval of $\mathcal{J}$.

Since we assumed that $f_0$ is mixing, the support of its absolutely continuous invariant measure is $[c_2, c_1]$. Recall the point $y \in (c_2, c_1)$ constructed in the beginning of the proof, and let $k_M \geq 1$ be minimal such that $f^{k_M}(y) = c_M$. Fix $m$ large and let $J \in \mathcal{J}$ be covered by $\{c_{M(m)-\ell_1,t} \mid t \in \Delta(M(m))\}$. By ergodicity of $f$ on

---

[10] If $f_0$ is not mixing, we could apply the argument to its deepest renormalisation $f_0^R$ on a mixing interval $\mathcal{R}_c$, if we assumed that $f_0^R$ is not conjugated to an Ulam–von Neumann map.

the support of the absolutely continuous invariant measure, there exists $k = k(J)$ so that $f^k(J)$ contains $y$. Therefore, we find a point $x \in J$ and an iterate $j_M \geq 1$ (with $j_M \leq k(J) + k_M$) such that $f^{j_M}(x) = c_M$ and so that the points $f^i(x)$, $0 \leq i \leq j_M - 1$, avoid a neighbourhood $V$ of $c$. Hence, by the implicit function theorem, if $|t_2|$ is sufficiently small, for all $t \in [-t_2, t_2]$, we find points $x_{M,t} \in J$ and $d_{M,t} \in (c_2, c_1)$ (depending differentiably on $t$), such that $d_{M,0} = c_M$, and $d_{M,t}$ is a (repelling) periodic point for $f_t$ with period $\ell_0$, while $f_t^{j_M}(x_{M,t}) = d_{M,t}$, and the points $f_t^i(x_{M,t})$, $0 \leq i \leq j_M - 1$, avoid a neighbourhood $V' \subset V$ of $c$. Since $\{c_{M-\ell_1,t} \mid t \in \Delta(M)\}$ contains $J$, and $J$ contains the closure of $\{x_{M,t} \mid t \in \Delta(M)\}$, it follows by the intermediate value theorem that there exists $t \in \Delta(M)$ such that $c_{M-\ell_1,t} = x_{M,t}$ (with $f_t^{j_M}(x_{M,t})$ the repelling periodic point $d_{M,t}$).

Since the number of intervals in $\mathcal{J}$ is finite and since $c_M$ can attain maximal $\ell_0$ different values, it follows that $\sup_{J \in \mathcal{J}} k(J) < \infty$ and $\sup_M k_M < \infty$. Hence, there is an integer $j_0$ and a neighbourhood $U$ of $c$, such that for every large $m$, defining $M(m)$ as above, there exist $t = t(M) \in \Delta(M)$ and $j \leq j_0$ such that $c_{M-\ell_1+j,t} = d_{M,t}$ (the repelling periodic point constructed above by considering a suitable $J$) and $c_{i,t} \notin U$, for all $i \geq 1$. By construction, $(M, t(M))$ is an admissible pair. By the admissible pair condition, $|f'(f^i(c_m)) - f_t'(f_t^i(d_{M,t}))|$, $1 \leq i \leq \ell_0$, is bounded from above by a constant times $\Lambda^{-M}$. This immediately implies (84). It remains to show (83), which easily follows from the distortion bounds (36), property (84), and the fact that $c_{M,t}$ is iterated at most $j_0$ steps to the postcritical periodic orbit while these iterations lie outside the neighbourhood $U$ of $c$.

Let $\Delta_{MT}$ be the sequence of Misiurewicz–Thurston parameters

$$\{t(M(m)) \mid m \geq m_0\}$$

just defined. Uniformity of goodness constants for $\Delta_{MT}$ is straightforward: Take $\lambda_c < \Lambda$ and assume that $M$ is sufficiently large. Then, by (83) and (84), we find an integer $H_0$ such that, for all $t \in \Delta_{MT}$, the map $f_t$ is $(\lambda_c, H_0)$-Collet–Eckmann. Properties (7) and (8) can be shown as in the proof of Proposition 2.1 (where we might have to intersect $\Delta_{MT}$ with a $\epsilon(\delta)$-neighbourhood of $t_0 = 0$).

This concludes the proof of Lemma 3.9.                                    $\square$

*Remark* 3.10. The assertion in Lemma 3.9 just below (84) holds only on one side of $t_0$. Hence, the Misiurewicz–Thurston parameters constructed in Lemma 3.9 lie either all to the left or all to the right of $t_0$. This will simplify the proof of the lower bound in Section 5.1 considerably by avoiding potential cancellations. However, it is quite likely that, by doing careful estimates and by possibly adapting the observable $A_D$ below, one would obtain a similar lower bound in Section 5.1, even if the assertion in Lemma 3.9 below (84) is not satisfied (allowing the construction of Misiurewicz–Thurston parameters on both sides).

**Banach spaces, transfer operators, Propositions 3.6–3.7 and Lemma 3.8.**

In the Misiurewicz–Thurston case, we may now construct the Banach spaces, the transfer operators $\widehat{\mathcal{L}}_t$ and the truncated transfer operators $\widehat{\mathcal{L}}_{t,M}$ exactly like in Sections 3.1 and 3.2, setting $\alpha = \beta = 0$. (The only difference is that (63) is replaced by $\|\xi_k \circ f_\pm^{-(k+1)}\|_{C^3} \leq C$ for all $k \geq 1$.) In the definition of the space $\mathcal{B}^{L^p}$, we take $\Lambda_t$ in (52) equal to $\Lambda_t$ in (83). We claim that for each $p > 1$ there exists $C$ so that

(86) $$\sup_n \|\widehat{\mathcal{L}}_t^n\|_{\mathcal{B}_t^{L^p}} \leq C.$$

Indeed, for $k \geq 1$, the definition of $\widehat{\mathcal{L}}_t$ gives

$$\lambda^{kr} \|[\widehat{\mathcal{L}}_t \hat{\psi}]_k\|_{L^p(I)} \leq \lambda^{(k-1)r} \lambda^{r-1} \|\hat{\psi}_{k-1}\|_{L^p(I)},$$

and we only have to consider $[\widehat{\mathcal{L}}_t \hat{\psi}]_0$. Using Minkowski's inequality, a change of variable, the bound (20), and the definition of $\Lambda_t$, we find a constant $C$ such that

$$\|[\widehat{\mathcal{L}}_t \hat{\psi}]_0\|_{L^p(I)} \leq \sum_{j=0 \text{ or } j \geq H(\delta)} \lambda^j \Big( \int_I \frac{|(1-\xi_j)\psi_j|^p}{|(f_t^{j+1})'(y)|^{p-1}} dy \Big)^{1/p}$$

$$\leq C \sum_{j=0 \text{ or } j \geq H(\delta)} \lambda^j \Lambda_t^{-\frac{p-1}{2p}j} \|\psi_j\|_{L^p(I)}.$$

By the definition (55) of $r$, the right hand side is bounded by $C\|\hat{\psi}\|_{\mathcal{B}_t^{L^p}}$. We have proved $\|\widehat{\mathcal{L}}_t\|_{\mathcal{B}_t^{L^p}} \leq C$. The proof of $\sup_n \|\widehat{\mathcal{L}}_t^n\|_{\mathcal{B}_t^{L^p}} \leq C$ using the above remarks is straightforward under the Misiurewicz–Thurston assumption, exploiting the overlap control of fuzzy intervals in Remark 3.5 (simplifying greatly Appendix B of [12]). The Lasota–Yorke inequality

$$(87) \qquad \max(\|\widehat{\mathcal{L}}_t^n(\hat{\psi})\|_{\mathcal{B}}, \|\widehat{\mathcal{L}}_{t,M}^n(\hat{\psi})\|_{\mathcal{B}}) \leq C\Theta_0^{-n}\|\hat{\psi}\|_{\mathcal{B}} + C\|\hat{\psi}\|_{\mathcal{B}^{L^p}},$$

for $p > 1$ follows from the Lasota–Yorke inequality for $\mathcal{B}$ and $\mathcal{B}^{L^1}$ and the embedding of $\mathcal{B}^{L^p}$ in $\mathcal{B}^{L^1}$ given by (56). Note that Rellich–Kondrachov gives that the embedding $\mathcal{B}^{W_1^1} \subset \mathcal{B}^{L^p}$ is compact since $p < \infty$ and using (23). Finally,

$$(88) \qquad \|(\mathrm{id} - \mathcal{T}_M)\hat{\psi}\|_{\mathcal{B}^{L^p}} = \sum_{k \geq M+1} \lambda^{kr} \left( \int_I |\psi_k(x)|^p \, dx \right)^{1/p}$$

$$\leq C \sum_{k \geq M+1} \frac{\lambda^k \sup|\psi_k|}{\lambda^{k(1-r)}|(f_t^{k-1})'(c_{1,t})|^{1/2p}}$$

$$\leq C^2 \lambda^M \sum_{k \geq M+1} \frac{1}{\Lambda_t^{(1-1/p)k/2}\Lambda_t^{k/2p}}$$

$$\leq C^3 \lambda^M |(f_t^M)'(c_1)|^{-1/2}.$$

(In the second inequality, we used the definition (55) of $r$. In the last inequality we used the *upper bound* $|(f_t^{k-1})'(c_{1,t})| \leq C\Lambda_t^k$ for all $k$.)

Therefore, the spectral properties stated in Propositions 3.6 and 3.7 (setting $\alpha = \beta = 0$) hold, with the same proofs. In fact, by (86) and (87), (88), we may use the norm of $\mathcal{B}^{L^p}$ for all $p \geq 1$ as a weak norm.

Finally, if $(M, t)$ is an admissible pair furnished by Lemma 3.9, then by (83) and (84), we have

$$C^{-3}\Lambda^k \leq |(f_t^{k-1})'(c_{1,t})| \leq C^3\Lambda^k, \qquad \forall \, 1 \leq k \leq 2M.$$

Using this estimate one can easily adapt the proof of Lemma 3.8 to ensure that the tower of $f_t$ coincides with the tower of $f$ up to level $2M$ (instead of $M$). Thus, in the Misiurewicz–Thurston case we have

$$(89) \qquad J_{k+1,t} = J_{k+1} \quad \text{and} \quad \xi_{k,t} = \xi_k, \qquad \forall \, 0 \leq k \leq 2M - 1.$$

## 4. Whitney–Hölder upper bounds — Proof of Theorem 1.2

4.1. **The main decomposition – Upper bounds.** For $f_t$ either good with $\alpha > 1$ or Misiurewicz–Thurston, we proved in Sections 3.1 and 3.3, respectively, that the invariant density of $f_t$ can be written as $\phi_t = \Pi_t(\hat{\phi}_t)$, where $\hat{\phi}_t$ is the nonnegative and normalised fixed point of $\widehat{\mathcal{L}}_t$ on $\mathcal{B}_t$. Writing $f = f_{t_0}$ (as usual we assume $t_0 = 0$ and we remove the 0 from the notation), as in [12, §6], our starting point is the decomposition

(90)
$$\phi_t - \phi = \left[\Pi_t(\hat{\phi}_t - \hat{\phi}_{t,M}) + \Pi(\hat{\phi}_M - \hat{\phi})\right] + \left[\Pi_t(\hat{\phi}_{t,M} - \hat{\phi}_M)\right] + \left[(\Pi_t - \Pi)(\hat{\phi}_M)\right].$$

We shall state three propositions giving upper bounds on the three terms in the right hand side of the above decomposition. The proof of Theorem 1.2 will easily follow. The upper bounds for the first two square brackets in (90) have a stronger form in the Misiurewicz–Thurston case, and they will be used in combination with Proposition 5.1 below (which gives a lower bound for the third square bracket in the decomposition (90)) to show Theorem 1.7 in Section 5.

We start with the effect of truncation, i.e., the contribution of the terms in the first square bracket of (90). For any $\upsilon < 1$, Proposition 3.7 and the definition of admissible pairs $(M, t)$ give a constant $C$ so that

$$\|\hat{\phi}_{t,M} - \hat{\phi}_t\|_{\mathcal{B}^{L^1}} \leq C|t|^{\upsilon/2}.$$

The first proposition, given below, contains the improvement of the above bound needed for both our main theorems:

**Proposition 4.1** (Weak norm control of $M \mapsto \hat{\phi}_{t,M}$). *If $f = f_0$ is good, with $\alpha > 1$, constructing the tower and transfer operators as in Sections 2.2 and 3.1–3.2, there exists $C$ such that for each admissible pair $(M, t)$, with $t$ good for the same parameters and $|t|$ sufficiently small, we have*

(91)
$$\max\{\|\hat{\phi}_{t,M} - \hat{\phi}_t\|_{\mathcal{B}^{L^1}}, \|\nu_{t,M} - \nu_t\|_{\mathcal{B}^*}\} \leq C|t|^{1/2}M^{2+\alpha}.$$

*If $f = f_0$ is Misiurewicz–Thurston, mixing, and transversal, recalling the construction in Section 3.3 (in particular, $\alpha = \beta = 0$) and the set $\Delta_{MT}$ of Misiurewicz–Thurston parameters accumulating at 0 given by Lemma 3.9, there exists a constant $C$ such that for each admissible pair $(M, t)$ with $t \in \Delta_{MT}$, we have for any $p > 1$,*

(92)
$$\max\{\|\hat{\phi}_{t,2M} - \hat{\phi}_t\|_{\mathcal{B}^{L^p}}, \|\nu_{t,2M} - \nu_t\|_{\mathcal{B}^*}|\} \leq C|t|^{1/2}.$$

Proposition 4.1 is proved in Section 4.3.

We next discuss the effect of parameter change on the truncated eigenvalues, i.e., the contribution of the second square bracket in the right hand side of (90). The next proposition shows that our choice of admissible pairs $(M, t)$ is indeed optimal:

**Proposition 4.2** (Strong norm control of $t \mapsto \hat{\phi}_{t,M}$). *If $f = f_0$ is good, with $\alpha > 1$, constructing the tower and transfer operators as in Sections 2.2 and 3.1–3.2 (in particular $\beta > \alpha + 1$, and we use Lemma 3.8), there exists $C$ such that for each admissible pair $(M, t)$, with $t$ good for the same parameters and $|t|$ sufficiently small, we have*

(93)
$$\|\hat{\phi}_{t,M} - \hat{\phi}_M\|_{\mathcal{B}} \leq CM^{\max(2+2\alpha, 1+\beta)}|t|^{1/2}.$$

*If $f = f_0$ is Misiurewicz–Thurston, mixing, and transversal, recalling the tower and transfer operator construction in Section 3.3 ($\alpha = \beta = 0$, recalling also (89))*

*and the set $\Delta_{MT}$ of Misiurewicz–Thurston parameters accumulating at $0$ given by Lemma 3.9, there exists a constant $C$ such that for each admissible pair $(M, t)$ with $t \in \Delta_{MT}$ and $|t|$ sufficiently small, we have* [11] *for each $p \geq 1$*

$$(94) \qquad \|\hat{\phi}_{t,2M} - \hat{\phi}_{2M}\|_{\mathcal{B}^{L^p}} \leq \|\hat{\phi}_{t,2M} - \hat{\phi}_{2M}\|_{\mathcal{B}} \leq C|t|^{1/2} \, .$$

The proof of Proposition 4.2 is given in Section 4.4.

The last ingredient for the proof of Theorem 1.2 is the following elementary but crucial lemma, which takes care of the last contribution in (90), i.e., the displacement of the "spikes" (the square root singularities $1/\sqrt{|x - c_{k,t}|}$ in the invariant densities):

**Proposition 4.3** (Upper bounds on spike displacement). *If $f = f_0$ is good, with $\alpha > 1$, taking $\beta > \alpha + 1$ and constructing the tower and transfer operators as in Sections 2.2 and 3.1–3.2, there exists a constant $C$ such that for each admissible pair $(M, t)$, with $t$ good for the same parameters and $|t|$ sufficiently small, and for all $A \in C^{1/2}$,*

$$|\int_I A(x)(\Pi_t - \Pi)(\hat{\phi}_M)(x)dx| \leq C|t|^{1/2}\|A\|_{C^{1/2}} \, .$$

*Remark* 4.4. The proof of Proposition 4.3 applied to the Misiurewicz–Thurston setting $\alpha = \beta = 0$ would give an additional factor $|\log|t||$ in the upper bound, since the size of the $B_k$'s does not converge to $0$ when $k \to \infty$. If the observable $A$ is $C^1$, this log-factor vanishes. (See the upper bound in Proposition 5.1 and its proof.)

*Proof of Theorem 1.2.* Let $\Delta \subset \mathcal{E}$ be the set given by Proposition 2.1. For given $\Gamma > 4$, we can choose $\alpha > 1$ and $\beta > 1 + \alpha$, so that $4 < \max(2 + 2\alpha, 1 + \beta) \leq \Gamma$. For this choice of $\alpha$, by Proposition 2.1, we find for each $t_0 \in \Delta$ a set $\Delta_0 \subset \Delta$ of good parameters having the same goodness constants as $t_0$, and $\Delta_0$ contains $t_0$ as a Lebesgue density point. Recall the constant $\epsilon(\delta)$ in Proposition 2.1, and the choice of $\epsilon > 0$ in Lemmas 3.8 and 2.6. Now we can choose $\delta > 0$ and $\epsilon(\delta) \geq \epsilon > 0$ small enough (and $L > 1$ in Section 2.2 large enough) so that all corresponding assertions in Sections 2–3 and in Propositions 4.3–4.2 hold (with uniform constants) for all $t \in \Delta_0 \cap (t_0 - \epsilon, t_0 + \epsilon)$. (In Lemma 2.6, we allow of course $t \in (t_0 - \epsilon, t_0 + \epsilon)$.) Redefining the set $\Delta_0$ as $\Delta_0 \cap (t_0 - \epsilon, t_0 + \epsilon)$, we obtain the set $\Delta_0$ in Theorem 1.2.

In the following, assume $t \in \Delta_0$. As remarked after the definition (57) of the projection $\Pi_t$, we have $\|\Pi_t(\hat{\phi}_t - \hat{\phi}_{t,M})\|_{L^1(I)} \leq \|\hat{\phi}_t - \hat{\phi}_{t,M}\|_{\mathcal{B}^{L^1}}$. Thus, we can apply Propositions 4.1 and 4.2 to bound the $L^1(I)$ norm of the two first square brackets on the right hand side of (90). Regarding the last square bracket we apply Proposition 4.3. Altogether, we derive

$$\Big|\int_I A(x)\phi_t(x)dx - \int_I A(x)\phi_{t_0}(x)dx\Big| \leq C|t - t_0|^{1/2}M^{\max(2+\alpha,1+\beta)}\|A\|_{C^{1/2}} \, .$$

Since, by (49), $M$ is bounded from above by a constant times $|\log|t-t_0||$ and since $\max(2 + 2\alpha, 1 + \beta) \leq \Gamma$, this concludes the proof of Theorem 1.2. $\qquad\square$

It remains to prove Proposition 4.3.

---

[11] The proof of (92) shows that truncating at $M + C\log M$ would be sufficient; but the proof for $2M$ in (94) is not harder.

*Proof of Proposition 4.3.* Let $1 \leq k \leq M$, and focus on the branch $f_+^{-k}$ (the other one is handled in a similar way). Assume that $c_k$ and $c_{k,t}$ are local maxima for $f^k$ and $f_t^k$, respectively (the other possibility is treated similarly and left to the reader).

For $A \in C^{1/2}$, we need to consider

$$(95) \qquad \left| \int_0^{c_{k,t}} \lambda^k A(x) \frac{\phi_{M,k}(f_{t,+}^{-k}(x))}{|(f_t^k)'(f_{t,+}^{-k}(x))|} \, dx - \int_0^{c_k} \lambda^k A(x) \frac{\phi_{M,k}(f_+^{-k}(x))}{|(f^k)'(f_+^{-k}(x))|} \, dx \right|$$

$$= \lambda^k \left| \int_c^1 (A(f_t^k(x)) - A(f^k(x)))\phi_{M,k}(x) \, dx \right|$$

$$\leq \lambda^k \|A\|_{C^{1/2}} \int_c^1 |f_t^k(x) - f^k(x)|^{1/2} \phi_{M,k}(x) \, dx$$

By Proposition 3.7 it follows that $\sup |\phi_{M,k}| \leq \lambda^{-k} \kappa_M^k \sup |\phi_{M,0}| \leq C\lambda^{-k}$. By the second inequality in (23), we have $|\operatorname{supp}(\phi_{M,k})| \leq C|(f^{k-1})'(c_1)|^{-1/2}k^{-\beta/2}$, and for $|t|$ sufficiently small we have, by (36),

$$\sup_{x \in \operatorname{supp}(\phi_{M,k})} |f_t^k(x) - f^k(x)| \leq C|(f^{k-1})'(c_1)||t| \, .$$

It follows that

$$\left| \int_0^1 A(x)(\Pi_t - \Pi)(\hat{\phi}_M)(x)dx \right| \leq C^3 \|A\|_{C^{1/2}} |t|^{1/2} \sum_{k=0}^M k^{-\beta/2} \, .$$

Since $\beta/2 > 1$ this concludes the proof of Proposition 4.3. $\qquad \square$

4.2. **Two key lemmas for Propositions 4.2 and 4.1.** The next lemma is the key to Proposition 4.2:

**Lemma 4.5** (Strong norm estimates for the maximal eigenvector)**.** *In the setting of Proposition 4.2, there exists a constant $C$ such that the following holds. If $f_t$, $f$, and $M$ are as in (93) then*

$$(96) \qquad \|(\widehat{\mathcal{L}}_{t,M} - \widehat{\mathcal{L}}_M)\hat{\phi}_M\|_{\mathcal{B}} \leq CM^{\max(2+2\alpha,1+\beta)}|t|^{1/2} \, .$$

*If $f_t$, $f$, and $M$ are as in (94) then*

$$(97) \qquad \|(\widehat{\mathcal{L}}_{t,2M} - \widehat{\mathcal{L}}_{2M})\hat{\phi}_{2M}\|_{\mathcal{B}} \leq C|t|^{1/2} \, .$$

Note that we cannot apply directly the results of Galatolo and Nisoli [18] to deduce Proposition 4.2 from the above lemma, because the eigenvectors are not fixed points.

Next, Lemma 4.6 will also be useful to prove Proposition 4.1 (and it gives Theorem 1.3):

**Lemma 4.6** (Uniformity of basic spectral data)**.** *In the setting of Proposition 4.1, there exists a constant $C$ such that the following holds. If $f_t$, $f$, and $M$ are as either like in (91) or like in (92), up to taking smaller $|t|$, the following hold:*

- *the renormalisation period $P_t$ of $f_t$ is not larger than $P_0$;*
- *the constant $C_t$ from Proposition 3.7 is uniformly bounded if $(M,t)$ is an admissible pair (recall (49));*
- *there is $\Theta_1 > 1$ so that, if in addition $P_t = P_0$, then $\theta_t < \Theta_1^{-1}$.*

The proofs of both lemmas follow:

*Proof of Lemma 4.5.* This argument is similar to [12, App C], with the very important difference that we must now deal with a much larger dominant term which arises from the transversality assumption, recall (39). (See also Lemma 2.6 and the comment above it.) We provide a detailed proof:

The main part of the proofs of (96) and (97) can be done simultaneously. When there is a difference, we shall refer as usual to the setting in (96) as the *polynomial case* and to the setting in (97) as the *Misiurewicz–Thurston case*. In the estimates below, the constants $\alpha$ and $\beta$ will appear, and in the Misiurewicz–Thurston case, except if otherwise mentioned, these estimates are to be read by setting $\alpha = \beta = 0$. Henceforth, let $\bar{M} := M$ in the polynomial case and $\bar{M} := 2M$ in the Misiurewicz–Thurston case. Observe first that

$$[(\widehat{\mathcal{L}}_{t,\bar{M}} - \widehat{\mathcal{L}}_{\bar{M}})\hat{\phi}_{\bar{M}}]_j \equiv 0\,, \quad \text{for all } j \geq 1\,.$$

For $j > \bar{M}$ this follows immediately by the truncation. For $1 \leq j \leq \bar{M}$, this follows from the fact that we constructed the tower for $f_t$ to coincide with the tower for $f$ up to level $\bar{M}$, i.e., $\xi_{j,t} \equiv \xi_j$, for $1 \leq j \leq \bar{M}$ (see Lemma 3.8) and recall that in the Misiurewicz–Thurston case we used very special properties of $f_t$ and $f$ in order to obtain identical towers up to the higher level $2M$ (see also (89)). Henceforth we consider only level 0.

For $1 \leq k \leq \bar{M} + 1$, set $\varphi_k = (1 - \xi_{k-1})\phi_{\bar{M},k-1}$. In order to prove Lemma 4.5, we have to show that the term

$$(98) \qquad \left\| \partial_x \sum_{\substack{k=1 \\ \varsigma \in \{+,-\}}}^{\bar{M}+1} \lambda^{k-1} \left[ \frac{\varphi_k(f_{t,\varsigma}^{-k}(x))}{|(f_t^k)'(f_{t,\varsigma}^{-k}(x))|} - \frac{\varphi_k(f_\varsigma^{-k}(x))}{|(f^k)'(f_\varsigma^{-k}(x))|} \right] \right\|_{L^1}.$$

is bounded above (up to a constant) by $M^{\max(2+2\alpha,1+\beta)}|t|^{1/2}$ in the polynomial case and by $|t|^{1/2}$ in the Misiurewicz–Thurston case. We consider first the indices $k \leq M + 1$, i.e., the terms which correspond to a fall from a level below $M$. For the polynomial case, this includes all terms we have to study. Regarding the Misiurewicz–Thurston case, the terms corresponding to a fall from levels between $M+1$ and $2M$ are easier to deal with, and they are treated at the end of this proof.

Since, for $k \leq M + 1$ the signs of $(f_t^k)'$ and $(f^k)'$ are identical in the domains we are interested in, we can skip writing the absolute values in the following estimates.

Henceforth, let $k \leq M$, and consider only $\varphi_k$ such that $\varphi_k \not\equiv 0$. In order to estimate (98), recall that $\phi_{\bar{M},0} \in W_1^2$ and note that, by Fubini and the fundamental theorem of calculus, we have

$$(99) \quad \int_I \partial_x \left[ \frac{\varphi_k(f_{t,\varsigma}^{-k}(x))}{(f_t^k)'(f_{t,\varsigma}^{-k}(x))} - \frac{\varphi_k(f_\varsigma^{-k}(x))}{(f^k)'(f_\varsigma^{-k}(x))} \right] dx = \int_0^t \int_I \partial_x \partial_s \frac{\varphi_k(f_{s,\varsigma}^{-k}(x))}{(f_s^k)'(f_{s,\varsigma}^{-k}(x))} dx ds\,.$$

Observe that if $(x,t) \mapsto \Phi_t(x) \in I$ is a $C^1$ map on $I \times \mathcal{E}$ so that $x \mapsto \Phi_t(x)$ is invertible, then we have

$$\partial_t \Phi_t^{-1}(x)|_{t=s} = -\frac{(\partial_t \Phi_t|_{t=s}) \circ \Phi_s^{-1}(x)}{(\partial_x \Phi_s) \circ \Phi_s^{-1}(x)}\,.$$

We consider only the branch $f_+^{-k}$. The branch $f_-^{-k}$ is handled similarly. For $s \in [0, t]$, we derive

(100)  $\partial_s \dfrac{\varphi_k(f_{s,+}^{-k}(x))}{(f_s^k)'(f_{s,+}^{-k}(x))}$

$$= \frac{(\partial_s f_s^k)(f_{s,+}^{-k}(x))}{(f_s^k)'(f_{s,+}^{-k}(x))} \partial_x(\varphi_k(f_{s,+}^{-k}(x))) + \partial_x \frac{(\partial_s f_s^k)(f_{s,+}^{-k}(x))}{(f_s^k)'(f_{s,+}^{-k}(x))} \varphi_k(f_{s,+}^{-k}(x)) .$$

Setting $y = f_{s,+}^{-k}(x)$, and taking the $x$-derivative we get

(101)  $\partial_x \partial_s \dfrac{\varphi_k(f_{s,+}^{-k}(x))}{(f_s^k)'(f_{s,+}^{-k}(x))}$

$$= \frac{\partial_s f_s^k(y)}{(f_s^k)'(y)} \partial_x^2(\varphi_k(f_{s,+}^{-k}(x))) + \frac{2}{(f_s^k)'(y)} \partial_y \frac{\partial_s f_s^k(y)}{(f_s^k)'(y)} \partial_x(\varphi_k(f_{s,+}^{-k}(x)))$$

$$+ \frac{\varphi_k(y)}{[(f_s^k)'(y)]^2} \partial_y^2 \frac{\partial_s f_s^k(y)}{(f_s^k)'(y)}$$

By the assumptions on the cutoff functions $\xi$, including Remark 3.5, and since $\hat{\phi}_{\bar{M}}$ is the eigenfunction of $\widehat{\mathcal{L}}_{\bar{M}}$ for the eigenvalue $\kappa_{\bar{M}}$, we have

(102)  $$\varphi_k(y) = \lambda^{-k} \kappa_{\bar{M}}^k (1 - \xi_{k-1}(y)) \xi_{k'-1}(y) \phi_{\bar{M},0}(y) ,$$

where $1 \leq k' < k$ is maximal such that $\xi_{k'-1} \not\equiv 1$. Recall that, by Proposition 3.7, there is $C \geq 1$ such that for all $\bar{M}$, we have $\max\{\|\phi_{\bar{M},0}\|_{L^\infty}, \|\phi'_{\bar{M},0}\|_{L^\infty}\} \leq \|\phi_{\bar{M},0}\|_{W_1^2} \leq C$. Recall also the property (63) of the cutoff functions. (Observe that in order to derive (63), we used the estimates (20), (21), and (22) in Lemma 2.5 which we a priori cannot apply for the map $f_s$ since $s$ might not be good. However, using the estimates provided by Lemma 2.6, we deduce that these estimates still hold for $f_s$, and, thus, the property (63) holds also for $\xi_{k-1} \circ f_{s,\pm}^{-k}$.) We obtain

(103)  $$\|\partial_x^r(\varphi_k(f_{s,+}^{-k}(x)))\|_{L^1} \leq C^2 \lambda^{-k} k^{(r-1)\beta} , \quad \text{for } r = 0, 1 , \text{ and}$$

$$\|\partial_x^2(\varphi_k(f_{s,+}^{-k}(x)))\|_{L^1} \leq C^2 \lambda^{-k} k^{\max(1+2\alpha, \beta)} ,$$

where the appearance of the factor $k^{-\beta}$ is explained as follows: The size of the support of $\varphi_k(f_{s,+}^{-k}(x))$ is bounded above by a constant times $k^{-\beta}$. Hence, all terms in the derivative in (103) not containing $\phi''_{\bar{M},0}$ can be estimated by taking the supremum times the size of the support. It is easy to see that the term containing $\phi''_{\bar{M},0}$ is bounded by a constant times $\lambda^{-k}$ (there is no $k^\beta$ factor here). In the Misiurewicz–Thurston case, there is only a constant times $\lambda^{-k}$ on the right hand sides in (103).

To estimate the $L^1$ norm of (101), we must consider the factor $\partial_s f_s^k(y)/(f_s^k)'(y)$ and its $y$-derivatives. By (38) and (37) in Lemma 2.6, we get

(104)  $$\left| \frac{\partial_s f_s^k(y)}{(f_s^k)'(y)} \right| \leq C^2 k^{\beta/2} |(f^{k-1})'(c_1)|^{1/2} , \quad \forall y \in \text{supp}(\varphi_k) .$$

Observe that (104) and (103) give a polynomial factor $k^{\max(1+2\alpha+\beta/2, 3\beta/2)}$ in the upper bound of the $L^1$ norm of the first term on the right hand side of (101) (this

explains the corresponding factor in the bound of (110) below). Regarding the $y$-derivative, observe first that, by (43), we have

$$\frac{\partial_s f_s^k(y)}{(f_s^k)'(y)} = \sum_{j=1}^{k} \frac{(\partial_s f_s)(f_s^{j-1}(y))}{(f_s^j)'(y)} = \sum_{j=1}^{k} \frac{X_s(f_s^j(y))}{(f_s^j)'(y)} \, .$$

Therefore, we obtain

$$(105) \quad \partial_y \frac{\partial_s f_s^k(y)}{(f_s^k)'(y)} = \sum_{j=1}^{k} X_s'(f_s^j(y)) - \sum_{j=1}^{k} X_s(f_s^j(y)) \sum_{\ell=0}^{j-1} \frac{f_s''(f_s^\ell(y))}{(f_s^{j-\ell})'(f_s^\ell(y))f_s'(f_s^\ell(y))} \, .$$

We shall use again the estimates in Lemma 2.6. For $1 \le m \le k$ and $y \in \mathrm{supp}(\varphi_k)$, we get, by (36) and (37),

$$(106) \quad \frac{1}{|(f_s^m)'(y)|} = \left| \frac{(f_s^{k-1})'(f_s(y))}{(f_s^k)'(y)(f_s^{m-1})'(f_s(y))} \right| \le C^3 \frac{k^{\beta/2}|(f^{k-1})'(c_1)|^{1/2}}{|(f^{m-1})'(c_1)|} \, ,$$

and, by (14), (36), and (40),

$$\frac{1}{|(f_s^{k-m})'(f_s^m(y))|} = \left| \frac{(f_s^{m-1})'(f_s(y))}{(f_s^{k-1})'(f_s(y))} \right| \le C^5 m^\alpha \, .$$

The dangerous factors in estimating (105) (even more dangerous when estimating the terms (108) and (109) below) are powers of $f_s'(y)$ in the denominator and factors $(f_s^m)'(y)$ for large $m$ in the numerator. The dominant terms on the right hand side of (105) appear when $\ell = 0$ (summing over $j$ when $\ell = 0$ gives a geometrical series). We derive that there is a constant $\tilde{C}$ so that

$$(107) \quad \left| \partial_y \frac{\partial_s f_s^k(y)}{(f_s^k)'(y)} \right| \le \tilde{C} k^\beta |(f^{k-1})'(c_1)| \, .$$

Regarding the $y$-derivative of order two (appearing in the last term in (101)), observe that

$$\partial_y^2 \frac{\partial_s f_s^k(y)}{(f_s^k)'(y)} =$$

$$(108) \quad \sum_{j=1}^{k} X_s''(f_s^j(y))(f_s^j)'(y) - \sum_{j=1}^{k} X_s'(f_s^j(y)) \sum_{\ell=0}^{j-1} \frac{f_s''(f_s^\ell(y))(f_s^\ell)'(y)}{f_s'(f_s^\ell(y))}$$

$$(109) \quad - \sum_{j=1}^{k} X_s(f_s^j(y)) \sum_{\ell=0}^{j-1} \left[ \frac{f_s'''(f_s^\ell(y))(f_s^\ell)'(y)}{(f_s^{j-\ell})'(f_s^\ell(y))f_s'(f_s^\ell(y))} \right.$$

$$\left. - \frac{f_s''(f_s^\ell(y))^2(f_s^\ell)'(y)}{(f_s^{j-\ell})'(f_s^\ell(y))[f_s'(f_s^\ell(y))]^2} - \frac{f_s''(f_s^\ell(y))}{f_s'(f_s^\ell(y))} \sum_{i=\ell}^{j-1} \frac{f_s''(f_s^i(y))}{(f_s^{j-i})'(f_s^i(y))f_s'(f_s^i(y))} \right] \, .$$

The dominant terms in (108) appear when $\ell = 0$. Summing over $j$ gives the upper bound $k^{1+\beta/2}|(f^{k-1})'(c_1)|^{1/2}$ (up to a constant). The last expression (109) contains the largest terms. The dominant terms appear in the last line when $i = \ell = 0$. Summing over $j$, which gives a geometrical series, we derive that (109) is bounded from above by a constant times $k^{3\beta/2}|(f^{k-1})'(c_1)|^{3/2}$. It follows that there is a constant $\tilde{C}$ so that

$$\left| \partial_y^2 \frac{\partial_s f_s^k(y)}{(f_s^k)'(y)} \right| \le \tilde{C} k^{3\beta/2}|(f^{k-1})'(c_1)|^{3/2} \, .$$

We have bounded all terms for the $L^1$ norm of (101) (observe that $|(f_s^k)'(y)|^{-1} \leq C\frac{k^{\beta/2}}{|(f^{k-1})'(c_1)|^{1/2}}$, by setting $m = k$ in (106)). Recalling (99), we conclude that

$$
(110) \qquad \left\| \lambda^{k-1} \partial_x \left( \frac{\varphi_k(f_{t,+}^{-k}(x))}{(f_t^k)'(f_{t,+}^{-k}(x))} - \frac{\varphi_k(f_+^{-k}(x))}{(f^k)'(f_+^{-k}(x))} \right) \right\|_{L^1}
$$

is bounded above by a constant times $k^{\max(1+2\alpha+\beta/2, 3\beta/2)}|(f^{k-1})'(c_1)|^{1/2}|t|$, where in the Misiurewicz–Thurston case the polynomial factor vanishes. Thus, in the polynomial case, by (40), we get

$$
\|\partial_x[(\widehat{\mathcal{L}}_{t,M} - \widehat{\mathcal{L}}_M)\hat{\phi}_M]_0(x)\|_{L^1} \leq 2\tilde{C}M^{\max(2+(5\alpha+\beta)/2, 1+(\alpha+3\beta)/2)}|(f^M)'(c_1)|^{1/2}|t|.
$$

Applying the admissible pair condition (49), it follows that $\|(\widehat{\mathcal{L}}_{t,M} - \widehat{\mathcal{L}}_M)\hat{\phi}_M\|_{\mathcal{B}}$ is bounded from above by a constant times $|t|^{1/2}M^{\max(2+2\alpha, 1+\beta)}$. This proves inequality (96), i.e., the polynomial case of Lemma 4.5.

In the Misiurewicz–Thurston case, instead of applying (40), we can use (83), and we derive

$$
\|\partial_x[(\widehat{\mathcal{L}}_{t,M} - \widehat{\mathcal{L}}_M)\hat{\phi}_{2M}]_0(x)\|_{L^1} \leq C|(f^M)'(c_1)|^{1/2}|t|\sum_{k=0}^{M}|(f^{M-k})'(c_{k+1})|^{-1/2}
$$

$$
\leq C^2|(f^M)'(c_1)|^{1/2}|t|\sum_{k=0}^{M}\Lambda^{-(M-k)/2} \leq C^4|t|^{1/2},
$$

where in the last inequality we used the admissible pair condition (49) for $\alpha = \beta = 0$.

It only remains to consider the terms involving $\varphi_k$'s for $M + 2 \leq k \leq 2M + 1$ in the Misiurewicz–Thurston case, i.e., terms in the level 0 which correspond to a fall from a level between $M + 1$ and $2M$. The bounds here are similar but easier than those above. We do not look at differences as above, but estimate each term individually. More precisely,

$$
\|\partial_x[(\widehat{\mathcal{L}}_{t,2M} - \widehat{\mathcal{L}}_{2M})(\mathrm{id} - \mathcal{T}_M)(\hat{\phi}_{2M})]_0(x)\|_{L^1}
$$
$$
\leq \|\partial_x[\widehat{\mathcal{L}}_{t,2M}(\mathrm{id} - \mathcal{T}_M)(\hat{\phi}_{2M})]_0(x)\|_{L^1} + \|\partial_x[\widehat{\mathcal{L}}_{2M}(\mathrm{id} - \mathcal{T}_M)(\hat{\phi}_{2M})]_0(x)\|_{L^1}.
$$

Consider the second term on the right hand side. (The first term is estimated similarly.) Observe that

$$
\|\partial_x[\widehat{\mathcal{L}}_{2M}(\mathrm{id} - \mathcal{T}_M)(\hat{\phi}_{2M})]_0(x)\|_{L^1} \leq \sum_{\substack{k=M+2 \\ \varsigma\in\{+,-\}}}^{2M+1} \lambda^{k-1}\|\partial_x \frac{\varphi_k(f_\varsigma^{-k}(x))}{|(f^k)'(f_\varsigma^{-k}(x))|}\|_{L^\infty}.
$$

By (102) and (103), and observing that, in the Misiurewicz–Thurston case, we have

$$
|\partial_x((f^k)'(f_\varsigma^{-k}(x)))^{-1})| \leq C|(f^k)'(f_\varsigma^{-k}(x)))|^{-1} \leq C^2|(f^{k-1}(c_1)|^{-1/2},
$$

(see for example the computation for the last term in (105) when $j = k$ and $\ell = 0$), we easily deduce that this sum is bounded from above by a constant times $|(f^M)'(c_1)|^{-1/2}$. This, in turn, is bounded by a constant times $|t|^{1/2}$, by the consequence (50) of the admissible pair condition. This concludes the proof of (97), and hence the proof of Lemma 4.5. □

*Proof of Lemma 4.6.* We are going to use again the arguments in [25]. Recall that there exist $\epsilon > 0$, $C \geq 1$ so that for all good $|t| \leq \epsilon$ (with the same goodness

parameters) and all $M$ we have (71), (72), and (73), in particular the essential spectral radius of $\widehat{\mathcal{L}}_{t,M}$ acting on $\mathcal{B}$ is not larger than $\Theta_0^{-1}$ for some $\Theta_0 > 1$.

Fix $(M, t)$ as in the statement of the lemma and let $\bar{M} := M$ in the setting of (93) (polynomial recurrence), and $\bar{M} := 2M$ in the setting of (94) (Misiurewicz–Thurston). Set

$$(111) \qquad \widehat{\mathcal{Q}}_{t,\bar{M}} = \widehat{\mathcal{Q}}_{t,\bar{M}}(z) = z - \widehat{\mathcal{L}}_{t,\bar{M}}\,.$$

(If $t = 0$ we remove $t$ from the notation as usual, writing $\mathcal{Q}_{\bar{M}}$ instead of $\mathcal{Q}_{0,\bar{M}}$.) We claim that there exist a small circle $\gamma$ centered at 1 and a constant $C \geq 1$ such that for all sufficiently large $\bar{M}$ (or equivalently $|t| \leq \epsilon$ sufficiently small) we have the strong norm control

$$(112) \qquad \sup_{z \in \gamma} \|\widehat{\mathcal{Q}}_{t,\bar{M}}(z)^{-1}\|_{\mathcal{B}} \leq C\,,$$

and the intersection of the disc $D_\gamma$ bounded by $\gamma$ with the spectrum of $\widehat{\mathcal{L}}_{t,\bar{M}}$ is reduced to $\kappa_{t,\bar{M}}$, which is a simple eigenvalue. To see this, we apply first [25, Theorem 1, Corollary 1] (just like in Proposition 3.7) to the operators $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}_{\bar{M}}$. Let $V_{r,\Theta_0} = \{z \in \mathbb{C} \mid |z| \leq \Theta_0^{-1}$ or $\operatorname{dist}(z, \sigma(\widehat{\mathcal{L}})) < r\}$. For any $r > 0$, we find an integer $M_0 \geq 1$ and a constant $\mathcal{H} \geq 1$ such that

$$(113) \qquad \|\widehat{\mathcal{Q}}_M(z)^{-1}\|_{\mathcal{B}} \leq \mathcal{H}\,, \qquad \forall M \geq M_0\,, \forall z \in \mathbb{C} \setminus V_{r,\Theta_0}\,.$$

In particular there exists a small circle $\gamma$ centered at 1 and $M_1 \geq 1$ so that the intersection of the disc $D_\gamma$ bounded by $\gamma$ with the spectrum of $\widehat{\mathcal{L}}_M$ is reduced to the simple eigenvalue $\kappa_M$ for all $M \geq M_1$. Then, if $r$ is small enough, we have $\gamma \subset \mathbb{C} \setminus V_{r,\Theta_0}$.

To get (112), we will apply [25, Theorem 1, Corollary 1] to the operators $\widehat{\mathcal{L}}_{t,\bar{M}}$ and $\widehat{\mathcal{L}}_{\bar{M}}$. Since we have a "moving target" (just like in [12]), we must be careful. We shall use that there are constants $C \geq 1$ and $0 < \eta < 1/2$ such that

$$(114) \qquad \|(\widehat{\mathcal{L}}_{t,\bar{M}} - \widehat{\mathcal{L}}_{\bar{M}})\hat{\psi}\|_{\mathcal{B}L^1} \leq C|t|^\eta \|\hat{\psi}\|_{\mathcal{B}} \quad \forall\, \hat{\psi} \in \mathcal{B}\,.$$

(We show (114) at the end of the proof of this lemma.) The estimate (114) replaces condition (5) in [25]. Recalling (113), if we assume that $z \in \mathbb{C} \setminus V_{r,\Theta_0}$ when applying the proof of [25, Theorem 1], the constant $H$ in [25, Equality (13)] is bounded from above by $\mathcal{H}$. Since all other constants are uniform in $t$ and $M$, this implies (112).

The uniformity claims on the [12] renormalisation period $P_t$ of $f_t$ and on $\theta_t$ follow from (112), using appropriate curves $\gamma_j$. Uniformity of $C_t$ then follows from the proof of Proposition 3.7.

It remains to prove (114). We can use the estimates in the proof of Lemma 4.5. For $1 \leq k \leq \bar{M} + 1$, set $\varphi_k = (1 - \xi_{k-1})\psi_{k-1}$. We have to estimate the term (98), but without taking the $x$-derivative, since on the left hand side of (114), we are only considering the weak norm $\|.\|_{\mathcal{B}L^1}$. For $k \leq M + 1$, recall (99) (without the $x$-derivative). Hence, it is enough to estimate the $L^1$ norm of (100). Similarly as in (103), and using (136) below, note that

$$\|\partial_x^r (\varphi_k(f_{s,+}^{-k}(x)))\|_{L^1} \leq Ck^{(r-1)\beta}\|\psi_{k-1}\|_{W_1^1}\,, \quad \text{for } r = 0, 1\,,$$

---

[12]The renormalisation period can drop a priori, because eigenvalues $\neq 1$ on the unit circle could move inside the open unit disc by perturbation.

where $C\lambda^{-k}$ in (103) is replaced by $\|\psi_{k-1}\|_{W_1^1}$ (since $\hat{\psi}$ is not necessarily an eigenvector). By (104) and (107), and by the estimate (106) when $m = k$, we derive that $\|(100)\|_{L^1}$ is bounded from above by a constant times $k^{\beta/2}|(f^{k-1})'(c_1)|^{1/2}$. In the polynomial case, combined with the consequence (50) of the admissible pair condition, this gives the bound $M\lambda^M|t|^{1/2}\|\hat{\psi}\|_{\mathcal{B}}$ (up to some constant) of the term (98) (without the $x$-derivative). In the Misiurewicz–Thurston case, for $M + 2 \le k \le 2M + 1$, we can apply the same comments as in the last paragraph of the proof of Lemma 4.5, and we get the upper bound $\lambda^M|t|^{1/2}\|\hat{\psi}\|_{\mathcal{B}}$ of the term (98) (without the $x$-derivative). By (51) and (49), we find constants $C$ and $0 < \vartheta < 1/2$ such that $M\lambda^M \le C|t|^{-\vartheta}$ which concludes the proof of (114), and, hence, the proof of Lemma 4.6. □

4.3. **The effect of truncation on $\hat{\phi}_t$: Proof of Proposition 4.1.** To obtain (91) in Proposition 4.1, we cannot apply the Keller–Liverani perturbation [25] result directly, because, in the weak norm of $\mathcal{B}^{L^1}$ the difference between $\widehat{\mathcal{L}}_t$ and $\widehat{\mathcal{L}}_{t,M}$ is not $M^\Gamma|(f^M)'(c_{1,t})|^{-1/2}$ but $e^{M/\Gamma}|(f^M)'(c_{1,t})|^{-1/2}$ (due to the $\lambda^k$ factor in the definition of the weak norm; cf. (73)). In [12] there were other exponential losses, but in the polynomially recurrent case of Theorem 1.2, we can afford to lose (at most) powers (which give the logarithmic factor there). [13] Inspired by Baladi–Young [14, §2], which only requires to control the difference between the operator and its perturbation *applied to the maximal eigenvector,* we develop (see also (125) below), a variant [14] of the Keller–Liverani argument to show Proposition 4.1. (In our application, the maximal eigenvectors are weighted by negative powers of $\lambda$, which ensures that the $\lambda^k$ factor in the definition of the weak norm does not create problems.)

*Proof of Proposition 4.1 .* We shall consider $t = 0$ and arbitrary $M$ and show

$$(115) \qquad \max(\|\hat{\phi}_M - \hat{\phi}\|_{\mathcal{B}^{L^1}}, \|\nu_M - \nu\|_{\mathcal{B}^*}) \le CM^2 M^{(\alpha-\beta)/2}|(f^M)'(c_1)|^{-1/2} .$$

In the polynomial case, the proof of (91) for $(M, t)$ satisfying the admissible pair condition (49) then follows from (50) and the fact that the distortion estimates in Lemma 2.6 hold for $(M, t)$ so that $|(f_t^M)'(c_{1,t})|$ is comparable to $|(f^M)'(c_1)|$ (see inequality (36)). We also use the uniformity in $t$ of $P_t \le P$, and the constant $C_t < C$ in Proposition 3.7 given by Lemma 4.6. In particular, it follows from the uniform bounds on $P_t$ and $C_t$ and the proof of Proposition 3.7 that the distance between $\sigma(\widehat{\mathcal{L}}_{t,M}) \setminus \kappa_{t,M}$ and the point $z = 1$ is bounded away from zero, uniformly in $M$ and $t$.

In the Misiurewicz–Thurston case, we consider $(N, t)$ admissible, and take $M = 2N$ (with $\alpha = \beta = 0$) in (115). Therefore, there is a constant $C$ such that

$$\|\hat{\phi}_{t,2N} - \hat{\phi}_t\|_{\mathcal{B}_t^{L^p}} \le C(2N)^2|(f_t^{2N})'(c_{1,t})|^{-1/2} \le C^2(2N)^2|(f^{2N-N})'(c_N)|^{-1/2}|t|^{1/2} ,$$

where in the last inequality we used (50) for $\alpha = 0 = \beta$. Since $|(f^{2N-N})'(c_N)|^{-1/2}$ is bounded from above by a constant times $\Lambda^{-N/2}$ by (83), this gives (92).

---

[13]In Theorem 1.7 we cannot afford to lose a logarithmic factor, however in this setting we have the flexibility of using a cutting time a bit higher than $M$ in (92).

[14]C. Liverani has explained to us a simpler but less general variant, which would give a slightly better bound, where $\tau|\log \tau|^2$ after (123) would be replaced by $\tau|\log \tau|$. This would however not improve our final statement and it only applies to the mixing case.

We start with some preliminary bounds. Set

$$\tau_M = \tau_M(\lambda) = \lambda^M M^{(\alpha-\beta)/2} |(f^M)'(c_1)|^{-1/2}$$

(recalling that $\alpha = \beta = 0$ in the Misiurewicz–Thurston case). Let $\hat{\phi}_t$ be the fixed point of $\widehat{\mathcal{L}}_t$. Clearly, there exists $C \geq 1$, which by Proposition 3.6 depends only on the goodness[15] of $t$ (once $\delta$, $\beta$, $L$, and $\lambda$ are fixed), such that for all good $t$ close enough to 0

$$(116) \qquad \|\phi_{t,k}\|_{L^\infty} \leq \lambda^{-k}\|\phi_{t,0}\|_{L^\infty} \leq C\lambda^{-k}\|\phi_0\|_{L^\infty} \,.$$

In the polynomial case, injecting (116) into (74) for $\hat{\psi} = \hat{\phi}_t$, we get

$$(117) \qquad \|(\mathrm{id} - \mathcal{T}_M)\hat{\phi}_t\|_{\mathcal{B}^{L^1}} \leq C\lambda^{-M}\tau_M \,.$$

In the Misiurewicz–Thurston case, injecting (116) into (88) for $\hat{\psi} = \hat{\phi}_t$, from (23) and (116), we derive, for any $p \geq 1$

$$(118) \qquad \|(\mathrm{id} - \mathcal{T}_M)\hat{\phi}_t\|_{\mathcal{B}^{L^p}} \leq C\lambda^{-M}\tau_M \,.$$

Next, using again the fact that $\hat{\phi}_t$ is the fixed point, we derive (recalling Remark 3.5 and (80))

$$(119) \qquad \|\phi_{t,M+k}\|_{W_1^1} = \lambda^{-M}\|\xi_{k',t}\phi_{t,k}\|_{W_1^1} = \lambda^{-M}\|\xi'_{k',t}\phi_{t,k} + \xi_{k',t}\phi_{t,k'}\|_{L^1}$$
$$\leq \lambda^{-M}(2\sup|\phi_k| + \|\phi_{t,k'}\|_{L^1}) \leq 3\lambda^{-M}\|\phi_{t,k}\|_{W_1^1} \,,$$

where $k' < M+k$ is maximal such that $\xi_{k',t} \not\equiv 1$. (Regarding the use of Remark 3.5 and (80), observe that in the proof of Lemma 3.8, property (80) is only guaranteed for the functions $\xi_{k,0}$, i.e., when $t = 0$. However, it is straightforward to adapt the construction behind (80) so that this property holds for all good $t$ sufficiently close to 0.) Hence, we get

$$(120) \qquad \|(\widehat{\mathcal{L}}_{t,M} - \widehat{\mathcal{L}}_t)(\hat{\phi}_t)\|_{\mathcal{B}} \leq C\lambda^{-M}\|\hat{\phi}_t\|_{\mathcal{B}} \,.$$

(The estimate for the level 0 gives much smaller contributions.)

Set $\|.\| := \|.\|_{\mathcal{B}^{W_1^1}}$, and let $|.| := \|.\|_{\mathcal{B}^{L^1}}$ in the polynomial case, and $|.| := \|.\|_{\mathcal{B}^{L^p}}$ in the Misiurewicz–Thurston case. We now move to the main part of the proof. For $\Theta_0 > 1$ given by Proposition 3.6, set $C_\theta = 1/\log\Theta_0$, which implies that $\Theta_0^{-C_\theta \log(1/\tau_M)} \leq \tau_M$. By Proposition 3.7, for all large enough $M$, the circle $\gamma_M$ centered at 1 and[16] of radius

$$\frac{\log\log(1/\tau_M)}{C_\theta \log(1/\tau_M)}$$

contains exactly one (simple) eigenvalue of $\widehat{\mathcal{L}}$ (at $z = 1$) and one (simple) eigenvalue of $\widehat{\mathcal{L}}_M$ (at $z = \kappa_M$). Recall that $\nu_M$ and $\nu$ are normalised so that $\nu_M(\hat{\phi}) = \nu(\hat{\phi}) = 1$. Since $\widehat{\mathcal{Q}}^{-1}(\hat{\phi}) = \frac{\hat{\phi}}{z-1}$, we have, like in (125),

$$(121) \qquad (\mathbb{P}_M - \mathbb{P})(\hat{\phi}) = \hat{\phi}_M - \hat{\phi} = -\frac{1}{2i\pi}\int_{\gamma_M} \frac{\widehat{\mathcal{Q}}_M^{-1}(z)}{z-1}(\widehat{\mathcal{L}}_M - \widehat{\mathcal{L}})(\hat{\phi})\,dz \,.$$

---

[15]Note that uniformity in the goodness holds.
[16]Use that $\log(1/\delta) < 1/\delta$ for small $\delta > 0$.

Then, for $n \geq 1$ to be chosen later, inspired by [24], see also [25, p.147], we write

$$\widehat{\mathcal{Q}}_M^{-1}(z) = (z - \widehat{\mathcal{L}}_M)^{-1}\mathbb{P}_M + (z - \widehat{\mathcal{L}}_M)^{-1}(\mathrm{id} - \mathbb{P}_M)$$

$$(122) \qquad = \frac{\mathbb{P}_M}{z - \kappa_M} + z^{-n}\widehat{\mathcal{Q}}_M^{-1}(z)(\mathrm{id} - \mathbb{P}_M)\widehat{\mathcal{L}}_M^n + \sum_{j=0}^{n-1} z^{-j-1}(\mathrm{id} - \mathbb{P}_M)\widehat{\mathcal{L}}_M^j \,.$$

Recalling the spectral observation made after (115), Proposition 3.7 implies that the distance between $z \in \gamma_M$ and $\kappa_M$ is $\geq (\log\log(1/\tau_M)/(2C_\theta \log(1/\tau_M))$, while the distance between $z \in \gamma_M$ and the rest of the spectrum of $\widehat{\mathcal{L}}_M$ is bounded from below uniformly in $M$. Therefore, for $z \in \gamma_M$, recalling (120), the Lasota–Yorke estimate (71) and the uniform weak-norm bounds (72) and (86) (using also $|\widehat{\mathcal{L}}_M| \leq |\widehat{\mathcal{L}}|$) give

$$|\widehat{\mathcal{Q}}_M^{-1}(z)(\widehat{\mathcal{L}}_M - \widehat{\mathcal{L}})\hat{\phi}|$$

$$\leq |\frac{1}{z - \kappa_M}\mathbb{P}_M((\widehat{\mathcal{L}}_M - \widehat{\mathcal{L}})(\hat{\phi}))| + |z|^{-n}\|\widehat{\mathcal{Q}}_M^{-1}(z)(\mathrm{id} - \mathbb{P}_M)\|\|\widehat{\mathcal{L}}_M^n(\widehat{\mathcal{L}}_M - \widehat{\mathcal{L}})(\hat{\phi})\|$$

$$+ \sum_{j=0}^{n-1} |z|^{-j-1}|\widehat{\mathcal{L}}_M^j(\widehat{\mathcal{L}}_M - \widehat{\mathcal{L}})(\hat{\phi})|$$

$$\leq \frac{|\hat{\phi}_M|}{|z - \kappa_M|}|\nu_M((\widehat{\mathcal{L}}_M - \widehat{\mathcal{L}})\hat{\phi})|+$$

$$+ C\left|1 - \frac{\log\log(1/\tau_M)}{C_\theta \log(1/\tau_M)}\right|^{-n}\left[\Theta_0^{-n}\|(\widehat{\mathcal{L}}_M - \widehat{\mathcal{L}})(\hat{\phi})\| + C|(\widehat{\mathcal{L}}_M - \widehat{\mathcal{L}})(\hat{\phi})|\right]$$

$$+ \sum_{j=0}^{n-1}\left|1 - \frac{\log\log(1/\tau_M)}{C_\theta \log(1/\tau_M)}\right|^{-j-1}\lambda^{-M}\tau_M$$

$$\leq 2C\lambda^{-M}\tau_M\frac{C_\theta \log(1/\tau_M)}{\log\log(1/\tau_M)} + C\left|1 - \frac{\log\log(1/\tau_M)}{C_\theta \log(1/\tau_M)}\right|^{-n}\lambda^{-M}(\Theta_0^{-n} + 2\tau_M)\,.$$

(We used that $|\nu_{t,M}(\hat{\psi})| \leq C|\hat{\psi}|$, uniformly in $t$ and $M$, from Proposition 3.7.) Then, taking $n = C_\theta \log(1/\tau_M)$, and using $\lim_{n\to\infty}(1 - x/n)^{-n} = e^x$, we find $C \geq 1$ such that for any $z \in \gamma_M$

$$(123) \qquad |\widehat{\mathcal{Q}}_M^{-1}(z)(\widehat{\mathcal{L}}_M - \widehat{\mathcal{L}})\hat{\phi}| \leq CC_\theta\lambda^{-M}\tau_M \log(1/\tau_M)\,.$$

Multiplying by $|z - 1|^{-1} \leq \frac{\log(1/\tau_M)}{\log\log(1/\tau_M)}$, and applying (121), we have shown that $|(\mathbb{P}_M - \mathbb{P})(\hat{\phi})| \leq C\lambda^{-M}|\log\tau_M|^2\tau_M/|\log\log\tau_M|$.
    We have proved

$$(124) \qquad |\hat{\phi}_M - \hat{\phi}| \leq \lambda^{-M}|\log\tau_M|^2\tau_M \leq CM^2 M^{(\alpha-\beta)/2}|(f^M)'(c_1)|^{-1/2}\,.$$

We can apply the same argument to the dual operators $\widehat{\mathcal{L}}_M^*$ and $\widehat{\mathcal{L}}^*$, up to exchanging the role of the weak and the strong norm. Then, just like after (76), specialising to $\mu = \nu$, we get, in the polynomial case, (115), and thus (91) of Proposition 4.1. The Misiurewicz–Thurston case is parallel. $\qquad\square$

## 4.4. The effect of parameter change on $\hat{\phi}_{t,M}$: Proof of Proposition 4.2.
We will deduce Proposition 4.2 from Lemma 4.5, using also Proposition 4.1 and the proof of Lemma 4.6.

*Proof of Proposition 4.2.* We are going to prove (93) and (94) in Proposition 4.2 simultaneously. Denote by $\mathbb{P}_{t,\bar{M}}(\hat{\psi}) = \hat{\phi}_{t,\bar{M}}\nu_{t,\bar{M}}(\hat{\psi})$ and $\mathbb{P}_t(\hat{\psi}) = \hat{\phi}_t\nu(\hat{\psi})$ the rank-one spectral projectors corresponding to the (simple) eigenvalues $\kappa_{t,\bar{M}}$ and $1$ of $\widehat{\mathcal{L}}_{t,\bar{M}}$ and $\widehat{\mathcal{L}}_t$, respectively (recall Propositions 3.6 and 3.7). Since $\widehat{\mathcal{Q}}_{t,\bar{M}}^{-1} - \widehat{\mathcal{Q}}_{\bar{M}}^{-1} = \widehat{\mathcal{Q}}_{t,\bar{M}}^{-1}(\widehat{\mathcal{L}}_{t,\bar{M}} - \widehat{\mathcal{L}}_{\bar{M}})\widehat{\mathcal{Q}}_{\bar{M}}^{-1}$ and $\widehat{\mathcal{Q}}_{\bar{M}}^{-1}(\hat{\phi}_{\bar{M}}) = \frac{\hat{\phi}_{\bar{M}}}{z - \kappa_{\bar{M}}}$, we have

$$(125) \qquad (\mathbb{P}_{t,\bar{M}} - \mathbb{P}_{\bar{M}})(\hat{\phi}_{\bar{M}}) = \hat{\phi}_{t,\bar{M}}\nu_{t,\bar{M}}(\hat{\phi}_{\bar{M}}) - \nu_{\bar{M}}(\hat{\phi}_{\bar{M}})\hat{\phi}_{\bar{M}}$$

$$= -\frac{1}{2i\pi}\int_\gamma \frac{\widehat{\mathcal{Q}}_{t,\bar{M}}^{-1}(z)}{z - \kappa_{\bar{M}}}(\widehat{\mathcal{L}}_{t,\bar{M}} - \widehat{\mathcal{L}}_{\bar{M}})\hat{\phi}_{\bar{M}}\,dz\,.$$

Recall that $\kappa_{\bar{M}}$ tends to $1$ as $\bar{M} \to \infty$. Hence, if $|t|$ is sufficiently small, by (112) from the proof of Lemma 4.6, we find a constant $\tilde{C} \geq 1$ such that

$$\|\hat{\phi}_{t,\bar{M}}\nu_{t,\bar{M}}(\hat{\phi}_{\bar{M}}) - \nu_{\bar{M}}(\hat{\phi}_{\bar{M}})\hat{\phi}_{\bar{M}}\|_{\mathcal{B}} \leq \tilde{C}\|(\widehat{\mathcal{L}}_{t,\bar{M}} - \widehat{\mathcal{L}}_{\bar{M}})\hat{\phi}_{\bar{M}}\|_{\mathcal{B}}\,.$$

By Lemma 4.5, the right hand side of this inequality is bounded (up to a constant) by $M^{\max(2+2\alpha, 1+\beta)}|t|^{1/2}$ in the polynomial case and by $|t|^{1/2}$ in the Misiurewicz–Thurston case. Hence, it is only left to estimate the coefficients $\nu_{t,\bar{M}}(\hat{\phi}_{\bar{M}})$ and $\nu_{\bar{M}}(\hat{\phi}_{\bar{M}})$, which can be done by Proposition 4.1: Recall the normalisation $\nu_{\bar{M}}(\hat{\phi}) = 1$ in Proposition 3.7. Since $|1 - \nu_{t,\bar{M}}(\hat{\phi}_{\bar{M}})| \leq |1 - \nu_{\bar{M}}(\hat{\phi}_{\bar{M}})| + |\nu_{\bar{M}}(\hat{\phi}_{\bar{M}}) - \nu_{t,\bar{M}}(\hat{\phi}_{\bar{M}})|$, it is sufficient to estimate

$$\max(|\nu_{\bar{M}}(\hat{\phi}) - \nu_{\bar{M}}(\hat{\phi}_{\bar{M}})|, |\nu_{\bar{M}}(\hat{\phi}_{\bar{M}}) - \nu_{t,\bar{M}}(\hat{\phi}_{\bar{M}})|)\,.$$

By Proposition 3.7 and Proposition 4.1, the first term is bounded (up to a constant) by $M^{2+\alpha}|t|^{1/2}$ in the polynomial case and by $|t|^{1/2}$ in the Misiurewicz–Thurston case. Regarding the second term (using that $\nu_t = \nu_0$), we have

$$|\nu_{\bar{M}}(\hat{\phi}_{\bar{M}}) - \nu_{t,\bar{M}}(\hat{\phi}_{\bar{M}})| \leq |\nu_t(\hat{\phi}_{\bar{M}}) - \nu_{t,\bar{M}}(\hat{\phi}_{\bar{M}})| + |\nu_{\bar{M}}(\hat{\phi}_{\bar{M}}) - \nu_0(\hat{\phi}_{\bar{M}})|$$

$$\leq (\|\nu_t - \nu_{t,\bar{M}}\|_{\mathcal{B}^*} + \|\nu_{\bar{M}} - \nu_0\|_{\mathcal{B}^*})\|\hat{\phi}_{\bar{M}}\|_{\mathcal{B}},$$

which is, by Proposition 4.1, bounded (up to a constant) by $M^{2+\alpha}|t|^{1/2}$ in the polynomial case and by $|t|^{1/2}$ in the Misiurewicz–Thurston case. This concludes the proof of Proposition 4.2. $\qquad\square$

## 5. Whitney–Hölder lower bounds in the Misiurewicz–Thurston case — Proof of Theorem 1.7

To prove Theorem 1.7 using the decomposition (90), we shall combine the $L^p$ version of the Misiurewicz–Thurston upper bounds in Propositions 4.1 and 4.2 with the following statement (the proof of which is to be found in Section 5.1):

**Proposition 5.1** (Upper and lower bounds on spike displacement in the Misiurewicz–Thurston case)**.** *If $f = f_0$ is transversal, mixing, and Misiurewicz–Thurston, then, recalling the set $\Delta_{MT}$ from Lemma 3.9, there exists a constant $C > 1$ such that for each sufficiently small $D > 0$ and each admissible pair $(M, t)$, where $M$ is sufficiently large and $t \in \Delta_{MT}$, the following holds. There is $A_D \in C^\infty(I)$, such that $\|A_D\|_{L^{\tilde{q}}(I)} \leq 2D^{1/\tilde{q}}$, for all $\tilde{q} \geq 1$, and*

$$C^{-1}|t|^{1/2} \leq \left|\int_I A_D(x)(\Pi_t - \Pi)(\hat{\phi}_{2M})(x)\,dx\right| \leq C|t|^{1/2}\,,$$

*where $\hat{\phi}_{2M}$ is the maximal eigenvector of $\widehat{\mathcal{L}}_{2M}$ from Proposition 3.7 (see Section 3.3).*

*Proof of Theorem 1.7.* Let $p > 1$. Fix $1 < \tilde{p} < p\frac{2}{p+1}$, and let $1 \le \tilde{q} < \infty$ be such that $\tilde{p}^{-1} + \tilde{q}^{-1} = 1$. By Lemma 3.3 and the Hölder inequality, there is a constant $C = C(p, \tilde{p}) > 1$ such that

$$(126) \qquad \left| \int_I A(x)\Pi_t(\hat{\psi})(x)\, dx \right| \le C\|A\|_{L^{\tilde{q}}(I)}\|\hat{\psi}\|_{\mathcal{B}_t^{L^p}}, \quad \forall\, \hat{\psi} \in \mathcal{B}_t^{L^p}.$$

By the decomposition (90) (replacing $M$ by $2M$), the estimates (94) in Proposition 4.2 and (92) in Proposition 4.1 combined with (126), and Proposition 5.1, we conclude that there is a constant $C > 1$ and, for each sufficiently small $D > 0$, a $C^\infty$ function $A_D$ satisfying

$$\left| \int_I A_D(x)\phi_t(x)\, dx - \int_I A_D(x)\phi(x)\, dx \right| \ge C^{-1}|t|^{1/2} - 2CD^{1/\tilde{q}}|t|^{1/2},$$

for all $t \in \Delta_{MT}$ sufficiently close to 0. Since the constant $C$ does not depend on $D$, this implies the lower bound in Theorem 1.7.

The upper bound in Proposition 5.1 and the same reasoning as for the lower bound give $|\int_I A_D\phi_t\, dx - \int_I A_D\phi\, dx| \le C|t|^{1/2} + CD^{1/\tilde{q}}|t|^{1/2}$, and thus the upper bound in Theorem 1.7. $\qquad\qquad\Box$

5.1. **Proof of Proposition 5.1.** We shall need the following property of the eigenvector $\hat{\phi}_{\bar{M}}$ of the truncated operator $\widehat{\mathcal{L}}_{\bar{M}}$.

**Lemma 5.2** (Lower bound for truncated maximal eigenvectors). *Let $f$ be a Misiurewicz–Thurston map. Then there exist a neighbourhood $V$ of $c$ and constants $M_0 \ge 1$ and $C_1 \ge 1$ such that*

$$\inf_{x \in V} \phi_{M,0}(x) \ge C_1^{-1}, \quad \text{for all } M \ge M_0.$$

*Proof of Lemma 5.2.* Since the density $\phi$ of the absolutely continuous probability measure is $C^1$ away from the (finite) postcritical orbit of $f$, we find an interval $J$ such that $\phi|_J \in C^1$ and $\inf_J \phi > 0$. By ergodicity of $f$ on the support of the absolutely continuous probability measure, there exists $\ell \ge 0$ such that $c$ lies in the interior of $f^\ell(J)$ (we use that $c$ lies in the interior of the support of the absolutely continuous probability measure). Thus, using $(\mathcal{L}^\ell\phi)(c) = \phi(c)$, we find a neighbourhood $V$ around $c$ such that $\inf_{x \in V} \phi(x) > 0$. Since $\phi = \Pi(\hat{\phi})$, and the union over $k \ge 1$ of the supports of $\phi_k \circ f_\pm^{-k}$ is disjoint from a neighbourhood of $c$, this implies $\inf_{x \in V} \phi_0(x) > 0$, up to shrinking $V$. We conclude by using that $\phi_{\bar{M},0} \in W_1^2$ converges to $\phi_0$ in the $L^1$ topology, and $\sup_{\bar{M}} \|\partial_x\phi_{\bar{M},0}(x)\|_{L^\infty} < \infty$. $\quad\Box$

*Proof of the lower bound in Proposition 5.1 (simplest case).* As a warmup, we consider the case where $f$ has the kneading sequence $RLLR^\infty$, i.e., the critical point is mapped after 4 iterations to the fixed point $c_4$ at the right hand side. This is the simplest possible combinatorics. (If the critical point were mapped after 3 iterations into the fixed point at the right hand side, the map $f$ would be renormalisable which is excluded by assumption. In fact, in this case the renormalisation of $f$ would be conjugated to the Ulam–von Neumann map which is an obstruction to construct the set $\Delta_{MT}$ in Lemma 3.9; see also remark below Theorem 1.7.) In order to have a good mental picture, note that for $f$ with the combinatorics as above the construction of the set $\Delta_{MT}$ in the proof of Lemma 3.9 could be done so

that, for all $t \in \Delta_{MT}$, the Misiurewicz–Thurston map $f_t$ has the kneading sequence $RLLR...RLR^\infty$ (where the middle block of $R$'s has odd length). In other words the fourth iteration $c_{4,t}$ lies close to the fixed point of $f_t$, where we repel until we are mapped to the left of $c$, whereafter we are immediately mapped to the fixed point of $f_t$.

The observable which will give us a lower bound is concentrated around the fixed point $c_4$ of $f$. For $D > 0$ small, let $A_D \in C^\infty([0,1])$ satisfy the following properties.

- $\mathrm{supp}(A_D) \subset [c_4 - D, c_4 + D]$ and $\|A_D'\|_{L^\infty} \leq D^{-1}$;
- $A_D$ is monotonously increasing in $[c_4 - D, c_4]$ and monotonously decreasing in $[c_4, c_4 + D]$;
- $A_D(x) \geq 1/3$ if $x \in [c_4 - D/2, c_4 + D/2]$.

It follows immediately from the construction that $\|A_D\|_{L^{\tilde q}(I)} \leq 2D^{1/\tilde q}$, for all $\tilde q \geq 1$. Let $(M, t)$ be an admissible pair with $|t|$ sufficiently small. For simplicity assume that (39) holds for all $4 \leq k \leq M$. This is for example the case when $f_t$ is the logistic family (2) (and $f$ is the map in this family with kneading sequence $RLLR^\infty$). This assumption implies that the assertion below (84) in Lemma 3.9 is satisfied for all $4 \leq k \leq M$ (this follows immediately from the proof of Lemma 3.9).

Recall the definition (57) for $\Pi_t$ and $\Pi = \Pi_0$. By definition, $\hat\phi_{2M}$ is supported in levels $k \leq 2M$. We consider first the terms which come from levels of the tower not higher than $M$ (they will give the dominant term for the lower bound), and we show at the end of the argument that the levels between $M + 1$ and $2M$ give a smaller contribution. Since the observable $A_D$ is concentrated around $c_4$, we only have to consider iterations $4 \leq k \leq M$. Recall that, by Lemma 3.9, the iterates $c_{k,t}$ and $c_k$ are either both local maxima or both local minima for $f_t^k$ and $f^k$, respectively (observe that this is only true if $k \leq M$).

Consider first the case when both are local maxima and focus on the branch $f_+^{-k}$. By a simple change of variables, we obtain

$$(127) \qquad \int_0^{c_4} \lambda^k A_D(x) \frac{\phi_{2M,k}(f_+^{-k}(x))}{|(f^k)'(f_+^{-k}(x))|}\, dx - \int_0^{c_{k,t}} \lambda^k A_D(x) \frac{\phi_{2M,k}(f_{t,+}^{-k}(x))}{|(f_t^k)'(f_{t,+}^{-k}(x))|}\, dx$$

$$= \lambda^k \int_c^1 (A_D(f^k(x)) - A_D(f_t^k(x)))\phi_{2M,k}(x)\, dx\,.$$

The assertion just after (84) in Lemma 3.9 implies that $f_t^k(x) < f^k(x) \leq c_4$, for $x \in \mathrm{supp}(\phi_{2M,k})(\subset J_k)$, and since $A_D$ is monotonously increasing to the left of $c_4$, it follows that the integrand in the right hand side of (127) is everywhere nonnegative. Recall the constant $C_a$ in the admissible pair condition (49). For $0 < D \ll C_a^{-1}$ let $\widetilde M = \widetilde M(D) \geq 4$ be minimal such that

$$|c_4 - c_{\widetilde M, t}| > D\,.$$

By the mean value theorem, (38), the transversality estimate (39), (36), (83) (for $t = 0$), and the admissibility condition (49), we derive that $\widetilde M$ exists and $\widetilde M < M$. (Observe that $M - \widetilde M$ is of the order $|\log D|$; we shall not need this fact.) We claim that there is a constant $C \geq 1$ so that

$$C^{-1}|(f^{\widetilde M - 1})'(c_1)||t| \leq |c_4 - c_{\widetilde M, t}| \leq CD\,.$$

Indeed, we can argue similarly as just above. By the mean value theorem the inequality on the left hand side follows by transversality estimate (39), while the

inequality on the right hand side follows essentially from the minimality of $\widetilde{M}$. We derive that

$$(128) \qquad |(f^{\widetilde{M}-1})'(c_1)|^{-1/2} \geq C^{-1}D^{-1/2}|t|^{1/2}.$$

Since the sizes of the levels $E_k$ of the tower for $f$ are uniformly bounded away from 0, we can choose $D$ so small so that $\phi_{2M,k} \circ f_{\pm}^{-k}|_{[c_4-D,c_4]} \equiv \lambda^{-k}\kappa_{2M}^k(\phi_{2M,0} \circ f_{\pm}^{-k})$, for all $k \leq 2M$. Hence, by Lemma 5.2, we derive that

$$\phi_{2M,\widetilde{M}}(f_{\pm}^{-\widetilde{M}}(x)) \geq \lambda^{-\widetilde{M}}\kappa_{2M}^{\widetilde{M}}C_1^{-1} \geq \lambda^{-\widetilde{M}}C^{-1}C_1^{-1}, \qquad \forall x \in [c_4 - D, c_4],$$

where in the last inequality we used (70) combined with Lemma 4.6. Observe that, by (27) and (128), we have

$$|f_{+}^{-\widetilde{M}}([c_4 - D/2, c_4])| \geq C^{-1}\sqrt{D/2}|(f^{\widetilde{M}-1})'(c_1)|^{-1/2} \geq C^{-2}2^{-1/2}|t|^{1/2}.$$

By the definition of $\widetilde{M}$, it follows that $A_D(f_t^{\widetilde{M}}(x)) = 0$, for all $x \in \operatorname{supp}(\phi_{2M,\widetilde{M}})$, while $A_D(f^{\widetilde{M}}(x)) \geq 1/3$, for $x \in f_{+}^{-\widetilde{M}}([c_4 - D/2, c_4])$. Therefore, there is a constant $\tilde{C} > 1$, so that the following lower bound for (127) holds when $k = \widetilde{M}$:

$$(129) \quad \lambda^{\widetilde{M}}\int_c^1 A_D(f^{\widetilde{M}}(x))\phi_{2M,\widetilde{M}}(x)\,dx \geq \lambda^{\widetilde{M}}\int_{f_{+}^{-\widetilde{M}}([c_4-D/2,c_4])} \frac{1}{3}\phi_{2M,\widetilde{M}}(x)\,dx$$

$$\geq \tilde{C}^{-1}|t|^{1/2}.$$

Observe that the constant $\tilde{C}$ does not depend on $D$ by our choice of $\widetilde{M}$.

If we consider the branches $f_{-}^{-k}$, or the case when $c_{k,t}$ and $c_k$ are both local minima for $f_t^k$ and $f^k$, respectively, then we derive, similarly as above, that the term corresponding to (127) is still nonnegative, for all $4 \leq k \leq M$.

It remains to show that the terms corresponding to $M < k \leq 2M$ can be neglected. Recall that $\sup|\phi_{2M,k}| \leq \lambda^{-k}\sup|\phi_{2M,0}| \leq C\lambda^{-k}$ (see Proposition 3.7 and Section 3.3). We can estimate each term separately, and we get

$$(130) \quad \left|\int_I A_D(x)(\Pi_t - \Pi)(\mathrm{id} - \mathcal{T}_M)(\hat{\phi}_{2M})(x)dx\right|$$

$$\leq 2C\sum_{k=M+1}^{2M}\|A_D\|_{L^\infty}\max_{s\in\{0,t\}}|\operatorname{supp}((A_D \circ f_s^k)\cdot\phi_{2M,k})|.$$

Since, by (20) and (84) (note that we use the Misiurewicz–Thurston assumption here), $|\operatorname{supp}((A_D \circ f_s^k)\cdot\phi_{2M,k})|$ is not larger than a constant multiple of $D^{1/2}\Lambda^{-k/2}$, we derive that (130) is bounded from above by a constant times $D^{1/2}|(f^M)'(c_1)|^{-1/2}$ which is in turn, by (50), bounded from above by a constant $C \geq 1$ times $D^{1/2}|t|^{1/2}$.

Hence, for $D$ sufficiently small, we conclude from (130) and (129) that

$$-\int_I A_D(x)(\Pi_t - \Pi)(\hat{\phi}_{2M})(x)dx \geq \tilde{C}^{-1}|t|^{1/2} - CD^{1/2}|t|^{1/2} \geq \tilde{C}^{-1}|t|^{-1/2}/2,$$

whenever $|t|$ is sufficiently small. $\qquad\qquad\square$

*Proof of Proposition 5.1 (general case).* The first and main part of the proof is dedicated to the lower bound, the upper bound is given at the end of the proof. Fix a periodic point $c_j$ in the postcritical orbit of $f$ and, for $D > 0$ small, let $A_D \in C^\infty([0,1])$ satisfy the following properties:

- $\operatorname{supp}(A_D) \subset [c_j - D, c_j + D]$ and $\|A_D'\|_{L^\infty} \leq D^{-1}$;

- $A_D$ is monotonously increasing in $[c_j - D, c_j]$ and monotonously decreasing in $[c_j, c_j + D]$;
- $A_D(x) \geq 1/3$ if $x \in [c_j - D/2, c_j + D/2]$.

The construction immediately implies $\|A_D\|_{L^{\tilde{q}}(I)} \leq 2D^{1/\tilde{q}}$. Let $(M, t)$ be an admissible pair such that $|t|$ is sufficiently small. We can assume that $1 \leq j \leq M$. As in the argument for the simplest case, we consider only the terms which come from levels of the tower not higher than $M$ (the terms coming from levels between $M$ and $2M$ can be handled just like around (130) above). Since the observable $A_D$ is concentrated around $c_j$, and since, by the admissible pair condition (49), $c_{k,t}$ stays very close to $c_k$, for $1 \leq k \leq M$, by possibly increasing the constant $C_a$ in the admissible pair condition, we have only to consider iterations $1 \leq k \leq M$ when $c_k = c_j$ (i.e., if $c_k \neq c_j$ we shall have no contributions). Given such an iteration $k$, note that, by the admissible pair condition, $c_{k,t}$ and $c_k$ are either both local maxima or both local minima for $f_t^k$ and $f^k$, respectively (observe that this is only true if $k \leq M$). Consider first the case when both are local maxima and focus on the branch $f_+^{-k}$. Recall (127). If $H(\delta) \leq k \leq M$ then, by Lemma 3.9, we have $f_t^k(x) < f^k(x) \leq c_j$, for $x \in \text{supp}(\phi_{2M,k})$, and since $A_D$ is monotonously increasing to the left of $c_j$, it follows that the integrand in the right hand side of (127) is nonnegative. For each $D > 0$ sufficiently small let $\widetilde{M} = \widetilde{M}(D)$ be minimal such that

$$c_{\widetilde{M}} = c_j \qquad \text{and} \qquad |c_{\widetilde{M}} - c_{\widetilde{M},t}| > D \,.$$

The admissible pair condition ensures that $\widetilde{M} < M$ (as in the simplest case), and we can assume that $M$ is large enough (making $|t|$ smaller) so that $\widetilde{M} > H(\delta)$. (In fact, $M - \widetilde{M}$ is of the order $|\log D|$.) Let $\ell_0$ be the period of $c_j$. By the mean value theorem and by (38) and (39) combined with the fact that $|(f_t^{\ell_0})'(c_{\widetilde{M}-\ell_0,t})| \approx \Lambda^{\ell_0}$ (see (83) and (84)), we find a constant $C$ such that

$$|c_{\widetilde{M}} - c_{\widetilde{M},t}| \leq C|c_{\widetilde{M}-\ell_0} - c_{\widetilde{M}-\ell_0,t}| \leq CD \,,$$

where the last inequality follows from the minimality of $\widetilde{M}$. By the transversality estimate (39), we have that $|c_{\widetilde{M}} - c_{\widetilde{M},t}|$ is bounded from below by a constant times $|(f^{\widetilde{M}-1})'(c_1)||t|$. Hence, we find a constant $\tilde{C} > 1$ such that

$$(131) \qquad |(f^{\widetilde{M}-1})'(c_1)|^{-1/2} \geq \tilde{C}^{-1} D^{-1/2} |t|^{1/2} \,.$$

By Lemma 3.9 and the definition of $\widetilde{M}$, it follows immediately that $A_D(f_t^{\widetilde{M}}(x)) = 0$, for all $x \in \text{supp}(\phi_{2M,\widetilde{M}})$. Since the sizes of the levels $E_k$ of the tower for $f$ are uniformly bounded away from 0, we can choose $D$ so small so that

$$(132) \qquad \phi_{2M,k} \circ f_{\pm}^{-k}|_{[c_j-D,c_j]} \equiv \lambda^{-k} \kappa_{2M}^k \phi_{2M,0} \circ f_{\pm}^{-k}, \quad \forall 1 \leq k \leq 2M \,.$$

By Lemma 5.2, if $M$ in the admissible pair $(M, t)$ is sufficiently large, it follows that

$$\phi_{2M,\widetilde{M}} \circ f_{\pm}^{-\widetilde{M}}(x) \geq \lambda^{-\widetilde{M}} \kappa_{2M}^{\widetilde{M}} C_1^{-1} \,, \qquad \forall x \in [c_j - D, c_j] \,.$$

Thus, by possibly slightly increasing the constant $\tilde{C} > 1$, by the construction of $A_D$, we get the following lower bound for (127) when $k = \widetilde{M}$:

$$\lambda^{\widetilde{M}} \int_c^1 A_D(f^{\widetilde{M}}(x))\phi_{2M,\widetilde{M}}(x)\,dx \geq \lambda^{\widetilde{M}} \int_{f_+^{-\widetilde{M}}([c_j-D/2,c_j])} \frac{1}{3}\phi_{2M,\widetilde{M}}(x)\,dx$$

$$\geq \frac{1}{3}\kappa_{2M}^{\widetilde{M}}C_1^{-1}|f_+^{-\widetilde{M}}([c_j-D/2,c_j])| \geq \frac{1}{3}\kappa_{2M}^{\widetilde{M}}C_1^{-1}C^{-1}\sqrt{D/2}|(f^{\widetilde{M}-1})'(c_1)|^{-1/2}$$

$$\geq \tilde{C}^{-2}|t|^{1/2}\,,$$

where in the last inequality we used the lower bounds (131) and (70). Observe that the constant $\tilde{C}$ does not depend on $D$.

Next, consider the case when $c_{k,t}$ and $c_k$ are both local minima for $f_t^k$ and $f^k$, respectively (and $1 \leq k \leq M$ satisfies $c_k = c_j$). By a similar reasoning as above, we see that (127) is nonnegative, for all $H(\delta) \leq k \leq M$, and we find $H(\delta) < \widetilde{M} < M$ such that (127), when $k = \widetilde{M}$, is bounded from below by a constant (independent on $D$) times $|t|^{1/2}$. If we consider the branches $f_-^{-k}$ then, we see that the terms corresponding to (127) are still nonnegative for all $H(\delta) \leq k \leq M$.

For the lower bound, it only remains to show that the terms corresponding to $0 \leq k < H(\delta)$ can be neglected. For $0 \leq k < H(\delta)$, we see immediately that the absolute values of (127) are bounded from above by a constant times

$$(133) \quad \lambda^k\|A_D'\|_{L^\infty}|c_k - c_{k,t}|\|\phi_{2M,k}\|_{L^\infty}|\cup_{s\in\{0,t\}} \text{supp}(A_D \circ f_s^k\phi_{2M,k})|$$

$$\leq C\Lambda^{H(\delta)/2}\|\phi_{2M,0}\|_{L^\infty}D^{-1/2}|t|\,.$$

Hence, if $|t|$ is sufficiently small, the terms corresponding to $0 \leq k < H(\delta)$ can be neglected.

Regarding the upper bound in Proposition 5.1, we need only to consider the terms when $H(\delta) \leq k \leq M$. For $H(\delta) \leq k \leq \widetilde{M}$, doing a similar estimate as in (133), we derive that the absolute values of (127) are bounded from above by a constant times $D^{-1/2}\Lambda^{k/2}|t|$. Thus, by (131) and (83), the sum of these terms is bounded above by a constant times $|t|^{1/2}$. If $\widetilde{M} \leq k \leq M$ (and $c_k = c_j$), then only the first term on the left hand side of (127) is non-zero. As in the estimate (130), the sum over these terms can be estimated from above by a constant times $\Lambda^{-M/2}D^{1/2}\Lambda^{(M-\widetilde{M})/2}$. By the definitions of $\widetilde{M}$ and $M$, we see that $D\Lambda^{M-\widetilde{M}}$ is bounded from above by a constant. Thus, the contribution of these last terms is also of the order $|t|^{1/2}$. $\square$

## APPENDIX A. PROOF OF THE KEY ESTIMATE PROPOSITION 2.4

In order to prove Proposition 2.4, we recall useful notations from [12] needed to show variants of tower estimates in [12]. Let $f_t$ be a good map, and let $\hat{f}_t$ be the associated tower map as defined in § 2.2. To simplify the writing we assume $t = 0$ and remove $t$ from the notation. For each $x \in I$ we define inductively an infinite non decreasing sequence

$$0 = S_0(x) \leq T_1(x) < S_1(x) \leq \cdots < S_i(x) \leq T_{i+1}(x) < S_{i+1}(x) \leq \ldots\,,$$

with $S_i(x)$ and $T_i(x) \in \mathbb{N} \cup \{\infty\}$ as follows: Put $T_0(x) = S_0(x) = 0$ for every $x \in I$. Let $i \geq 1$ and assume recursively that $S_j(x)$ and $T_j(x)$ have been defined for $j \leq i-1$. Then, we set (as usual, we put $\inf \emptyset = \infty$)

$$T_i(x) = \inf\{j \geq S_{i-1}(x) \mid |f^j(x)| \leq \delta\}\,.$$

If $T_i(x) = \infty$ for some $i \geq 1$, then we set $S_i(x) = \infty$. Otherwise, either $f^{T_i(x)}(x) = c$, and then we put $S_i(x) = \infty$, or $f^{T_i(x)}(x) \in I_j$ for some $j \geq H(\delta)$, and we put $S_i(x) = T_i(x) + j$.

Note that if $T_i(x) < \infty$ for some $i \geq 1$ then

$$\hat{f}^j(x, 0) \notin E_0 \,, \ T_i(x) + 1 \leq j \leq S_i(x) - 1 \,,$$
$$\hat{f}^\ell(x, 0) \in E_0 \,, \ S_{i-1}(x) \leq \ell \leq T_i(x) \,.$$

If $T_{i_0}(x) = \infty$ for $i_0 \geq 1$, minimal with this property, then $\hat{f}^\ell(x, 0) \in E_0$ for all $\ell \geq S_{i_0 - 1}$ (that is, $|f^\ell(x)| > \delta$ for all $\ell \geq S_{i_0 - 1}$).

In other words, $T_i$ is the beginning of the $i$-th *bound period* and $S_i - 1$ is the end of the $i$-th bound period, [17] and if $S_i < T_{i+1}$ then $S_i$ is the beginning of the $i+1$-th *free period* (which ends when the $i + 1$-th bound period starts).

For consistency, we also set $S_i - T_i = 0$ if $S_i = T_i = \infty$, and $T_i - S_{i-1} = 0$ if $S_{i-1} = T_i = \infty$, and, for all $x \in I$, we put $(f^\infty)'(x) := \infty$ and $f^\infty(x) := c_1$.

The following lemma gives expansion at the end of the free period $T_i - 1$ (just before climbing the tower), at the end $S_i - 1$ of the bound period (after falling from the tower), and during the free period (when staying at level zero).

**Lemma A.1** (Tower expansion for good maps). *Let $f$ be a $(\lambda_c, H_0)$-Collet–Eckmann map, polynomially recurrent of exponent $\alpha > 0$ for the same $H_0$. For every small enough $\delta_0 > 0$, if $\delta < \delta_0$, $\rho > 1$, and $C_0$ are so that (7) and (8) hold, letting $S_i(x)$ and $T_i(x)$ be the times associated to the tower for $\delta$ and $L$, then*

(134) $\quad |(f^{S_i(x)})'(x)| \geq \rho^{S_i(x)} \,, \qquad |(f^{T_i(x)})'(x)| \geq C_0 \rho^{T_i(x)} \,, \qquad \forall x \in I \,, \ \forall i \geq 0 \,,$

*and*

$$|(f^{S_i(x)+j})'(x)| \geq C_0 \delta \rho^{S_i(x)} \rho^j \,, \qquad \forall x \in I \,, \ \forall i \geq 0 \,, \ \forall 0 \leq j < T_{i+1}(x) - S_i(x) \,.$$

*Proof of Lemma A.1.* We may and shall assume that condition (11) is satisfied. Choose $\delta_0 > 0$ small enough so that $H(\delta)$, for all $\delta < \delta_0$, is large enough so that

(135) $$C_0 \frac{1}{CL^{3/2}} j^{-\beta/2} \lambda_c^{\frac{j-1}{2}} \geq \rho^j \,, \quad \forall j \geq H(\delta) \,,$$

where $C$ is the constant in (20). The rest of the proof is exactly like for [12, Lemma 3.5], we recall it for the convenience of the reader: Let $x \in I$. For any $\ell \geq 1$, the definitions imply $f^{S_{\ell-1}(x)+k}(x) \in I \setminus [-\delta, \delta]$ for all $0 \leq k < T_\ell(x) - S_{\ell-1}(x)$ and $f^{T_\ell(x)}(x) \in I_j$ with $j = S_\ell(x) - T_\ell(x) \geq H(\delta)$. Therefore, (20), combined with (135), and (8) give for all $i \geq 0$

$$|(f^{S_i})'(x)| = \prod_{\ell=1}^i |(f^{S_\ell(x) - T_\ell(x)})'(f^{T_\ell(x)} x)||(f^{T_\ell(x) - S_{\ell-1}(x)})'(f^{S_{\ell-1}(x)} x)| \geq \rho^{S_i(x)} \,,$$

and

$$|(f^{T_i})'(x)| = |(f^{T_i(x) - S_{i-1}(x)})'(f^{S_{i-1}(x)} x)||(f^{S_i})'(x)| \geq C_0 \rho^{T_i(x) - S_i(x)} \rho^{S_i(x)} \,.$$

Using in addition (7), we get, for $0 \leq j \leq T_{i+1}(x) - S_i(x)$,

$$|(f^{S_i(x)+j})'(x)| = |(f^j)'(f^{S_i(x)}(x))||(f^{S_i})'(x)| \geq C_0 \delta \rho^j \rho^{S_i(x)} \,.$$

$\square$

---

[17]Bound period refers to the fact that the orbit is bound to the postcritical orbit.

*Proof of Proposition 2.4.* This proof is very similar to that of [12, Proposition 3.7], we give it for the convenience of the reader. Let $\delta > 0$ be sufficiently small (see (135), and so that condition (11) is satisfied for all $\delta < \delta_0$).

Fix $j \geq 1$. Since the summands are all positive, we may (and shall) group them in a convenient way, using the times $T_i := T_i(c_{j+1})$ and $S_i := S_i(c_{j+1})$ for a small enough $\delta$. We have

$$\sum_{k=j+1}^{\infty} \frac{1}{|(f^{k-j})'(f^j(c_1))|}$$
$$= \sum_{i=0}^{\infty} \frac{1}{|(f^{S_i})'(c_{j+1})|} u_{T_{i+1}-S_i}\left(f^{S_i}(c_{j+1})\right) + \sum_{i=1}^{\infty} \frac{1}{|(f^{T_i})'(c_{j+1})|} u_{S_i-T_i}\left(f^{T_i}(c_{j+1})\right),$$

where we use the notation $u_n(y) = \sum_{\ell=1}^{n} \frac{1}{|(f^\ell)'(y)|}$. (In particular $u_0 \equiv 0$.) Since $T_{i+1} - S_i = T_1(f^{S_i}(c_{j+1}))$, Lemma A.1 implies

$$u_{T_{i+1}-S_i}\left(f^{S_i}(c_{j+1})\right) \leq \frac{C}{C_0\delta(1 - \rho^{-1})},$$

(in particular the series converges if $n = T_{i+1} - S_i = \infty$). Since $f''(c) \neq 0$, the polynomial recurrence assumption (5) implies for all $i$

$$|f'(f^{T_i}(c_{j+1}))| = |f'(f^{T_i+j}(c_1))| \geq C^{-1}(T_i + j)^{-\alpha}.$$

Therefore, the bounded distortion estimate (15) in the proof of Lemma 2.3 gives, together with the Collet–Eckmann assumption, [18]

$$u_{S_i-T_i}\left(f^{T_i}(c_{j+1})\right) \leq \frac{1}{|f'(f^{T_i}(c_{j+1}))|} \sum_{\ell=0}^{\infty} \frac{C}{|(f^\ell)'(c_1)|} \leq \frac{C^3}{(1 - \lambda_c^{-1})}(T_i + j)^{\alpha}.$$

By Lemma A.1 we have $|(f^{S_i})'(c_{j+1})| \geq \rho^{S_i}$ and $|(f^{T_i})'(c_{j+1})| \geq C_0\rho^{T_i}$. Therefore, there exists constants $C_1$, $C_2$ so that

$$\sum_{k=j+1}^{\infty} \frac{1}{|(f^{k-j})'(f^j(c_1))|} \leq C_1\delta^{-1}j^{\alpha}\left[\sum_{i=0}^{\infty} \rho^{-S_i} + \sum_{i=1}^{\infty} \rho^{-T_i}T_i^{\alpha}\right] \leq C_2\delta^{-1}j^{\alpha}.$$

This proof gives a constant $C$ in (16) which depends on $\beta > \alpha + 1$ and $\delta$. We can simply choose $\beta = \alpha + 1.1$. The dependence on $\delta$ is less innocent, since $C$ blows up when $\delta \to 0$, and we require smaller $\delta$ if $L$ is large, but considering a fixed map $f_{t_0}$ as in Theorems 1.2 and 1.7, both parameters may be chosen once and for all, depending on the goodness parameters of $f_{t_0}$ and the $C^2$ norm of $t \mapsto f_t$.          $\square$

## Appendix B. Spectral properties of the transfer operators $\widehat{\mathcal{L}}_t$

Recall that $f_t$ is assumed good. We prove Proposition 3.6:

*Proof.* Let $c(\delta)$ be as in (7). For $\hat{\psi} \in \mathcal{B}$, our assumptions on the $\xi_j$ ensure that $(\widehat{\mathcal{L}}(\hat{\psi}))_k \in W_1^1$ for all $k \geq 1$, with $(\widehat{\mathcal{L}}(\hat{\psi}))_k$ supported in the desired interval, and that $(\widehat{\mathcal{L}}(\hat{\psi}))_0$ is supported in the desired interval.

---

[18]The constant $C$ above depends on $[\sup_{1 \leq j < H_0} \lambda_c/|(f^j)'(c_1)|^{1/j}]^{H_0}$. By Proposition 2.1, this expression is uniform for suitable families $f_t$.

Note that for any interval $U$ (not necessarily containing the support of $\psi_j$), using the Sobolev embedding,

$$(136) \qquad \sup_U |\psi_j| \le \min(C\|\psi'_j\|_{L^1}, \int_U |\psi'_j| \, dx + |U|^{-1} \int_U |\psi_j| \, dx) \, .$$

Since $\xi_\ell$ is unimodal if it is not $\equiv 1$, for each $\ell \ge 1$ there exist $v_\ell > u_\ell$ in $B_\ell$ so that, setting $K_\ell = \{x \in \text{supp}(\psi_\ell) \mid x \le u_\ell\} \cup \{x \in \text{supp}(\psi_\ell) \mid x \ge v_\ell\}$,

$$(137) \qquad \int_{B_\ell} |\xi'_\ell \psi_\ell| \, dx = \int_{x \le u_\ell} \xi'_\ell |\psi_\ell| \, dx - \int_{x \ge v_\ell} \xi'_\ell |\psi_\ell| \, dx \le 2 \sup_{K_\ell} |\psi_\ell|$$

Therefore, for all $k \ge 1$, using also (136),

$$(138) \qquad \|(\widehat{\mathcal{L}}(\hat\psi))'_k\|_{L^1} \le \frac{3C}{\lambda} \|\psi'_{k-1}\|_{L^1} \, .$$

More generally, for $1 \le n \le k$,

$$(139) \qquad \|(\widehat{\mathcal{L}}^n(\hat\psi))'_k\|_{L^1} \le \frac{3Cn}{\lambda^n} \|\psi'_{k-n}\|_{L^1} \, .$$

If $|\psi_k(y)| > 0$ then $|f^j(y) - c_j| \le j^{-\beta}/L$ for all $j \le k$. If $\xi_k(f_\pm^{-(k+1)}(x)) < 1$ then $|x - c_{k+1}| \ge L^{-3}(k+1)^{-\beta}$. Thus, changing variables in the integrals, using (137) for the terms involving $\xi'_k$ for $k \ge 0$, and recalling (20), (21), (30) and (32) from Lemma 2.5 and its proof, we see that $(\widehat{\mathcal{L}}(\hat\psi))_0$ belongs to $W_1^1$ and

$$(140) \quad \|(\widehat{\mathcal{L}}(\hat\psi))'_0\|_{L^1} \le Cc(\delta)^{-1}(\|\psi'_0\|_{L^1} + \|\psi_0\|_{L^1} + \sup|\psi_0|)$$

$$+ C \sum_{k \ge H(\delta)} \frac{\lambda^k k^{1+2\alpha+\beta/2}}{|(f^{k+1})'(c_1)|^{1/2}} (\|\psi'_k\|_{L^1} + \sup|\psi_k| + \|\psi_k\|_{L^1}) \, .$$

In view of (51) and (136), we have proved that $\widehat{\mathcal{L}}$ is bounded on $\mathcal{B}$.

Observe that $\nu(\hat\psi) = \sum_k \int_{B_k} |\psi_k(x)| w(x,k) \, dx$ is finite if $\hat\psi \in \mathcal{B}$: Indeed, $\psi_k$ is supported in $J_k$ which decays exponentially according to (23), and we may use the bound $\lambda < \lambda_c^{1/2}$ from (51). So $\nu$ is an element of the dual of $\mathcal{B}$. The fact that $\widehat{\mathcal{L}}^*(\nu) = \nu$ can easily be proved using the change of variables formula (see [12, (85)]). Note for further use that $\widehat{\mathcal{L}}^*(\nu) = \nu$ implies

$$(141) \qquad \nu(|\widehat{\mathcal{L}}^N(\hat\psi)|) \le \nu(\widehat{\mathcal{L}}^N(|\hat\psi|)) = \nu(|\hat\psi|) \, .$$

Furthermore, note that $\nu(|\hat\psi|) = \|\hat\psi\|_{\mathcal{B}^{L^1}}$.

We next estimate the spectral and essential spectral radii of $\widehat{\mathcal{L}}$ on $\mathcal{B}$. Using (137) and the overlap control of fuzzy intervals (see Remark 3.5), it is not very difficult (this may be done exactly like in Appendix B of [12], since our overlap control is in fact much better) to adapt the proof of [13, Sublemma in Section 4] to show inductively that for any $\Theta < \Theta_0$, there exists $C$, and for all $n$ there exists $C(n)$, so that

$$(142) \qquad \|(\widehat{\mathcal{L}}^n(\hat\psi))'_0(x)\|_{L^1(I)} \le C\Theta^{-n} \|\hat\psi\|_{\mathcal{B}} + C(n) \|\hat\psi\|_{\mathcal{B}^{L^1}} \, .$$

Recalling (139), and using (141), up to slightly decreasing $\Theta$, one then finds $C'$ so that for all $n \ge 1$ (see the proof of [13, Variation Lemma in Section 4])

$$(143) \qquad \|\widehat{\mathcal{L}}^n(\hat\psi)\|_{\mathcal{B}} \le C'\Theta^{-n} \|\hat\psi\|_{\mathcal{B}} + C' \|\hat\psi\|_{\mathcal{B}^{L^1}} \, .$$

Since (53), (23), and (51) imply that the length of the support of $\psi_k$ is (much) smaller than $\lambda^{-2k}$, we find $\|\hat{\psi}\|_{\mathcal{B}^{L^1}} \leq \|\hat{\psi}\|_{\mathcal{B}^{L^1}} + C\lambda^{-M}\|\hat{\psi}\|_{\mathcal{B}^{W_1^1}}$ for all $M \geq 1$ (we used again the Sobolev embedding to estimate the supremum by the $W_1^1$ norm).

Finally, since Rellich–Kondrachov implies that $\mathcal{B}^{W_1^1}$ is compactly included in $\mathcal{B}^{L^1}$ (the total length is bounded, even up to $\lambda^k$-expansion at level $k$, by (23)), the Lasota–Yorke estimate (143) together with Hennion's theorem [20] give the claimed bound on essential spectral radius of $\widehat{\mathcal{L}}$ on $\mathcal{B} = \mathcal{B}^{W_1^1}$. This ends the proof of the claims on the essential spectral radius in Proposition 3.6.

We now describe the eigenvalues of modulus 1:

The bound (143) implies that the spectral radius of $\widehat{\mathcal{L}}$ on $\mathcal{B}$ is at most one, and thus equal to one (since the essential spectral radius is strictly smaller than one while 1 is an eigenvalue of the dual operator). Since $\widehat{\mathcal{L}}_t$ is a nonnegative operator with spectral radius equal to 1 and essential spectral radius strictly smaller than 1, classical results of Karlin [22, pp. 933-935, Thm 27] (using for $\mathcal{K}$ the lattice of continuous functions, with $\hat{\psi}_1 \geq \hat{\psi}_2$ if $\psi_{1,k}(x) \geq \psi_{2,k}(x)$ for all $x$ and $k$), imply that its eigenvalues of modulus one are a finite set of roots of unity $e^{2ij\pi/P}$, $j = 0, \ldots, P-1$ for some $P \geq 1$. In addition, the eigenfunctions for the eigenvalue 1 are nonnegative. If $\hat{\phi}$ is a (nonnegative) fixed point normalised by $\nu(\hat{\phi}) = 1$, we have $\int_I \phi\, dx = \int_I \Pi(\hat{\phi})\, dx = 1$. Recalling (64), we get that $\mathcal{L}(\Pi(\hat{\phi})) = \phi$, so that $\Pi(\hat{\phi}) \in L^1$ is indeed the invariant density of $f$. Since this density is unique and ergodic, the eigenvalue at 1 is simple, and therefore also the eigenvalues of modulus 1 are also simple. If $f_t$ is mixing, then $f_t^j$ is ergodic for all $j \geq 1$, so the only eigenvalue of modulus one is 1. Otherwise, let $P_t \geq 2$ be the renormalisation period and let $\phi_{j,t}$ and $\nu_{j,t}$ be the eigenvectors of $\widehat{\mathcal{L}}_t$ and $\widehat{\mathcal{L}}_t^*$, respectively for $e^{2ij\pi/P_t}$. It is not difficult to check that $\max_j \|\hat{\phi}_{j,t}\|_{\mathcal{B}^{W_1^1}}$ and $\max_j \|\nu_{j,t}\|_{(\mathcal{B}^{L^1})^*}$ are bounded uniformly in $t$ (using for example the special structure of maximal eigenvectors of a positive operator [22, p. 933–934]).

It only remains to show that $\hat{\phi}_0 \in W_1^2$ if $f$ is $C^4$. For this, take $\hat{\psi}$ so that $\psi_k = 0$ for all $k \geq 1$ and $\psi_0$ is $C^\infty$, of Lebesgue average 1 (we can even take $\psi_0$ constant in a neighbourhood of $[c_2, c_1]$), and use that

$$(144) \qquad \frac{1}{k}\sum_{n=0}^{k-1} \widehat{\mathcal{L}}^n(\hat{\psi})$$

converges to $\hat{\phi}$ in the $\mathcal{B}^{W_1^1}$ norm as $n \to \infty$. We claim that $\|(\widehat{\mathcal{L}}^n(\hat{\psi}))_0\|_{W_1^3} \leq C$ for all $n$. Adapting the proof of (22), one shows

$$(145) \qquad \sup_{x \in f^k(I_k)} \left| \partial_x^3 \frac{1}{|(f^k)'(f_\pm^{-k}(x))|} \right| \leq C \frac{k^{8\alpha}}{|(f^{k-1})'(c_1)|^{1/2}}$$

if $\alpha > 1$ and $\beta < 2\alpha$. Then, in view of (5), one can exploit (in addition to the properties already used in the proof of the Lasota–Yorke estimates for the $W_1^1$ norm in Proposition 3.6) the properties (63) of $\xi_k''$, $\xi_k'''$ to adapt (152) in [12, Appendix B] (noting also that $\hat{\psi}|_\omega = 0$ if the interval $\omega$ is in some level $E_k$ with $k > 0$). Note that (137) is not needed, since we only look at the component of $\widehat{\mathcal{L}}^n(\hat{\psi})$ at level 0. Details are straightforward and left to the reader. To conclude, use that if a sequence converging to $\hat{\phi}_0$ in $W_1^1(I)$ has bounded $W_1^3(I)$ norms, then $\hat{\phi}_0 \in W_1^2(I)$ by Rellich–Kondrachov (using (23) again). $\square$

## References

[1] Jose F. Alves, Maria Carvalho, and Jorge Milhazes Freitas. Statistical stability and continuity of SRB entropy for systems with Gibbs-Markov structures. *Comm. Math. Phys.*, 296(3):739–767, 2010.

[2] Jose F. Alves, Maria Carvalho, and Jorge Milhazes Freitas. Statistical stability for Hénon maps of the Benedicks-Carleson type. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(2):595–637, 2010.

[3] Artur Avila, Mikhail Lyubich, and Welington de Melo. Regular or stochastic dynamics in real analytic families of unimodal maps. *Invent. Math*, 154(3):451–550, 2003.

[4] Artur Avila and Carlos Gustavo Moreira. Statistical properties of unimodal maps: smooth families with negative Schwarzian derivative. *Astérisque*, 286:81–118, 2003.

[5] Artur Avila and Carlos Gustavo Moreira. Statistical properties of unimodal maps: the quadratic family. *Ann. of Math*, 161(2):831–881, 2005.

[6] Viviane Baladi. On the susceptibility function of piecewise expanding interval maps. *Comm. Math. Phys.*, 275(3):839–859, 2007.

[7] Viviane Baladi. Linear response despite critical points. *Nonlinearity*, 21(6):T81–T90, 2008.

[8] Viviane Baladi, Stefano Marmi, and David Sauzin. Natural boundary for the susceptibility function of generic piecewise expanding unimodal maps. *Ergodic Theory Dynam. Systems*, 10:1–24, 2013.

[9] Viviane Baladi and Daniel Smania. Linear response formula for piecewise expanding unimodal maps. *Nonlinearity*, 21(4):677–711, 2008. (Corrigendum, *Nonlinearity* 25(7):2203–2205, 2012).

[10] Viviane Baladi and Daniel Smania. Analyticity of the SRB measure for holomorphic families of quadratic-like Collet-Eckmann maps. *Proc. Amer. Math. Soc.*, 137(4):1431–1437, 2009.

[11] Viviane Baladi and Daniel Smania. Smooth deformations of piecewise expanding unimodal maps. *Discrete Contin. Dyn. Syst.*, 23(3):685–703, 2009.

[12] Viviane Baladi and Daniel Smania. Linear response for smooth deformations of generic nonuniformly hyperbolic unimodal maps. *Ann. Sci. École Norm. Sup. (4)*, 45(6):861–926, 2012.

[13] Viviane Baladi and Marcelo Viana. Strong stochastic stability and rate of mixing for unimodal maps. *Ann. Sci. École Norm. Sup. (4)*, 29(4):483–517, 1996.

[14] Viviane Baladi and Lai-Sang Young. On the spectra of randomly perturbed expanding maps. *Comm. Math. Phys.*, 156:355–385, 1993.

[15] Michael Benedicks and Lennart Carleson. The dynamics of the Hénon map. *Ann. of Math. (2)*, 133(1):73–169, 1991.

[16] Pierre Collet and Jean-Pierre Eckmann. Positive Liapunov exponents and absolute continuity for maps of the interval. *Ergodic Theory Dynam. Systems*, 3(1):13–46, 1983.

[17] Neil Dobbs and Mike Todd. Entropy jumps up. 2014. Work in progress.

[18] Stefano Galatolo and Isaia Nisoli. An elementary approach to rigorous approximation of invariant measures. *arXiv:1109.2342*.

[19] Bing Gao and Weixiao Shen. Summability implies Collet–Eckmann almost surely. *Ergodic Theory Dynam. Systems*. doi:10.1017/etds.2012.173, to appear.

[20] Hubert Hennion. Sur un théorème spectral et son application aux noyaux lipchitziens. *Proc. Amer. Math. Soc.*, 118:627–634, 1993.

[21] Miaohua Jiang. Differentiating potential functions of SRB measures on hyperbolic attractors. *Ergodic Theory Dynam. Systems*, 32:1350–1369, 2012.

[22] Samuel Karlin. Positive operators. *J. Math. Mech.*, 8:907–937, 1959.

[23] Anatole Katok, Gerhard Knieper, Mark Pollicott, and Howard Weiss. Differentiability and analyticity of topological entropy for anosov and geodesic flows. *Invent. Math.*, 98(3):581–597, 1989.

[24] Gerhard Keller. Stochastic stability in some dynamical systems. *Monatshefte Math.*, 94:313–333, 1982.

[25] Gerhard Keller and Carlangelo Liverani. Stability of the spectrum for transfer operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(1):141–152, 1999.

[26] Gerhard Keller and Tomasz Nowicki. Spectral theory, zeta functions and the distribution of periodic points for Collet–Eckmann maps. *Comm. Math. Phys.*, 149(1):633–680, 1992.

[27] Genadi Levin. On an analytic approach to the Fatou conjecture. *Fund. Math.*, 171:177–196, 2002.

[28] Mikhail Lyubich. Almost every real quadratic map is either regular or stochastic. *Ann. of Math.*, 156(1):1–78, 2002.

[29] Marco Mazzolena. Dinamiche espansive unidimensionali: dipendenza della misura invariante da un parametro. *Master's Thesis, Roma 2*, 2007.

[30] Tomasz Nowicki and Duncan Sands. Non-uniform hyperbolicity and universal bounds for $S$-unimodal maps. *Invent. Math.*, 132(3):633–680, 1998.

[31] Juan Rivera-Letelier and Weixiao Shen. Statistical properties of one-dimensional maps under weak hyperbolicity assumptions. *arXiv:1004.0230*.

[32] David Ruelle. Differentiation of SRB states. *Comm. Math. Phys.*, 187(1):227–241, 1997.

[33] David Ruelle. General linear response formula in statistical mechanics, and the fluctuation-dissipation theorem far from equilibrium. *Phys. Lett. A*, 245:220–224, 1998.

[34] David Ruelle. Differentiation of SRB states: Corrections and complements. *Comm. Math. Phys.*, 234(1):185–190, 2003.

[35] David Ruelle. Application of hyperbolic dynamics to physics: some problems and conjectures. *Bull. Amer. Math. Soc.*, 41(3):275–278, 2004.

[36] David Ruelle. Differentiating the absolutely continuous invariant measure of an interval map $f$ with respect to $f$. *Comm. Math. Phys.*, 258(2):445–453, 2005.

[37] David Ruelle. Structure and $f$-dependence of the A.C.I.M. for a unimodal map $f$ of Misiurewicz type. *Comm. Math. Phys.*, 287(3):1039–1070, 2009.

[38] David Ruelle. Singularities of the susceptibility of a sinai-ruelle-bowen measure in the presence of stable-unstable tangencies. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 369:482–493, 2011.

[39] Marek Rychlik and Eugene Sorets. Regularity and other properties of absolutely continuous invariant measures for the quadratic family. *Comm. Math. Phys.*, 150(2):217–236, 1992.

[40] Weixiao Shen. On stochastic stability of nonuniformly expanding interval maps. *Proc. London. Math. Soc.*, 107(5):1091–1134, 2013.

[41] Hans Thunberg. Unfolding of chaotic unimodal maps and parameter dependence of natural measures. *Nonlinearity*, 14(2):323–338, 2001.

[42] Masato Tsujii. Positive Lyapunov exponents in families of one-dimensional dynamical systems. *Invent. Math.*, 111(1):113–137, 1993.

[43] Masato Tsujii. On continuity of Bowen-Ruelle-Sinai measures in families of one-dimensional maps. *Comm. Math. Phys.*, 177(1):1–11, 1996.

[44] Marcelo Viana. *Stochastic Dynamics of Deterministic Systems*. 1997. 21 Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro (1997), available on `http://w3.impa.br/∼viana` (under Lecture Notes).

[45] Lai-Sang Young. Decay of correlations for certain quadratic maps. *Comm. Math. Phys.*, 146(1):123–138, 1992.

[46] Lai-Sang Young. What are SRB measures, and which dynamical systems have them? Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. *J. Statist. Phys.*, 108(5–6):733–754, 2002.

MATHEMATICAL SCIENCES, COPENHAGEN UNIVERSITY, 2100 COPENHAGEN, DENMARK;
CURRENT ADDRESS: DMA, UMR 8553, ÉCOLE NORMALE SUPÉRIEURE, 75005 PARIS, FRANCE
*E-mail address*: `viviane.baladi@ens.fr`

KTH, DEPARTMENT OF MATHEMATICS, S-100 44 STOCKHOLM, SWEDEN
*E-mail address*: `michaelb@kth.se`

DMA, UMR 8553, ÉCOLE NORMALE SUPÉRIEURE, 75005 PARIS, FRANCE
*E-mail address*: `daniel.schnellmann@gmail.com`