Counting zeros of zeta functions

Maciej Zworski

The purpose of the talks is to explain the bounds on the dynamical zeta functions in strips in terms of the dimension of the limit set (in the case of hyperbolic quotients) and the Julia set (in the case of hyperbolic rational maps), δ:

$$|Z(s)| \leq C_K \exp(C_K |s|^\delta \,, \quad \operatorname{Re} s \geq -K \,, \tag{1}$$

see (4) and (6) for definitions of $Z(s)$. Bounds like this immediately imply upper bounds on the number of zeros of $Z(s)$ (see (5) below). So far only the simplest cases have been studied [1],[2]. Numerical evidence [2],[5] suggests that the bounds are optimal. The motivation for this work comes from the study of quantum resonances – see [3],[4] for related recent results and references.

More precisely the talks will

- explain motivation for the study of such bounds; that motivation comes from semiclassical potential scattering, obstacle scattering, and most recently quantum dots;
- present a complete proof [1] in the case of hyperbolic rational maps; from the analysis point of view it has the same features as the geometrically more complicated case of Schottky quotients;
- discuss the numerical results and possible new directions.

We conclude this abstract by reviewing the definitions of the two zeta functions for which (1) is proved. We consider quotients $\mathbb{H}/\Gamma$, where Γ is a discrete finitely generated subgroup of isometries of $\mathbb{H}$, the hyperbolic plane (for simplicity we present the two dimensional case only), such that all elements $\gamma \in \Gamma$ are hyperbolic, and $\mathbb{H}/\Gamma$ is non compact. The hyperbolicity of $\gamma \in \Gamma$ means that its action on $\mathbb{H}$ can be represented as

$$\alpha \circ \gamma \circ \alpha^{-1}(x, y) = e^{\ell(\gamma)}(x, y) \,, \quad (x, y) \in \mathbb{H} \simeq \mathbb{R}_+ \times \mathbb{R} \,, \quad \alpha \in \operatorname{Aut}(\mathbb{H}) \,. \tag{2}$$

The non-compactness implies that if $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$, and $\Lambda(\Gamma) \subset \partial\mathbb{H}$ is the limit set of Γ, that is the set of limit points of $\{\gamma(z) : \gamma \in \Gamma\}$, $z \in \mathbb{H}$, then $\pi(\text{convex hull } \Lambda(\Gamma))$ is compact.

The trapped set is determined by $\Lambda(\Gamma)$: trapped trajectories are given by geodesics connecting two points of $\Lambda(\Gamma)$ at infinity, and

$$\dim K_E = 2\delta_\Gamma + 1 \,, \quad \delta_\Gamma = \dim \Lambda(\Gamma) \,.$$

The limit set is always of pure dimension, which coincides with its Hausdorff dimension.

A nice feature of this model is the exact correspondence between the resonances of

$$H = h^2(-\Delta_{\mathbb{H}/\Gamma} - 1/4) \,,$$

and the zeros of the Selberg zeta function, $Z_\Gamma(s)^\ddagger$:

$$z \in \mathrm{Res}(H) \iff Z_\Gamma(s) = 0\,, \quad z = h^2(s(1-s) - 1/4)\,, \quad \mathrm{Re}\, s \leq \delta_\Gamma\,, \tag{3}$$

where the multiplicities of zeros and resonances agree. The Selberg zeta function is defined by the analytic continuation of

$$Z_\Gamma(s) = \prod_{\{\gamma\}} \prod_{k\geq 0} \left(1 - e^{-(s+k)\,\ell(\gamma)}\right)\,, \quad \mathrm{Re}\, s > \delta_\Gamma\,, \tag{4}$$

where $\{\gamma\}$ denotes a conjugacy class of a primitive element $\gamma \in \Gamma$ (an element which is not a power of another element), and we take a product over distinct primitive conjugacy classes (each of which corresponds to a primitive closed orbit). The length $\ell(\gamma)$ of the corresponding closed orbit appears in (2). The estimate (1) obtained for convex co-compact Schottky groups in any dimension,implies

$$\#\{s \;:\; Z_\Gamma(s) = 0\,, \quad \mathrm{Re}\, s > -C_0\,, \quad r < \mathrm{Im}\, s < r + C_1\} \leq C_2\, r^{\delta_\Gamma}\,, \tag{5}$$

In the context of rational maps on the complex plane, we take f a uniformly expanding rational map on $\mathbb{C}$ (for instance $z \mapsto z^2 + c$, $c < -2$), and call f^n its n-fold composition. The zeta function associated with this map is given by

$$Z(s) = \exp\left(-\sum_{n=1}^{\infty} n^{-1} \sum_{f^n(z)=z} \frac{|(f^n)'(z)|^{-s}}{1 - |(f^n)'(z)|^{-1}} \right)\,. \tag{6}$$

Then the number or resonances in a strip is also given by a law of the type (5), where δ_Γ is replaced by the dimension of the Julia set:

$$J = \overline{\bigcup_{n\geq 1} \{z \;:\; f^n(z) = z\}}\,.$$

Note that this set is also made of "trapped orbits".

References

[1] H. Christianson, *Growth and zeros of zeta functions for hyperbolic rational maps.* to appear in Can. J. Math.

[2] L. Guillopé, K. Lin, and M. Zworski, *The Selberg zeta function for convex co-compact Schottky groups,* Comm. Math. Phys. **245**(2004), 149-176.

[3] S. Nonnenmacher and M. Zworski, *Distribution of Resonances of open quantum maps.* preprint 2005, `www.math.berkeley.edu/ zworski/nz.ps.gz`.

[4] J. Sjöstrand and M. Zworski, *Fractal upper bounds on the density of semiclassical resonances.* preprint 2005, `www.math.berkeley.edu/ zworski/sz10.ps.gz`.

[5] J. Strain and M. Zworski, *Growth of the zeta function for a quadratic map and the dimension of the Julia set,* Nonlinearity **17** (2004), 1607-1622.

‡See [2] for detailed references. The term $\frac{1}{4}$ in the definition of the Hamiltonian H comes from requiring that the bottom of the spectrum of H is 0, so that Green's function $(H - \lambda^2)^{-1}$ is meromorphic in $\lambda \in \mathbb{C}$