COUNTING ZEROS OF ZETA FUNCTIONS IN TERMS OF FRACTAL DIMENSIONS

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1. Introduction

The purpose of these notes is to

- motivate the study of fractal Weyl laws for resonances
- review existing mathematical and numerical results in the subject
- outline the proof of the most recent semiclassical result [41]
- present a complete proof of a fractal upper bound in the case of dynamical zeta functions for hyperbolic rational maps in $\mathbb{C}$ [6].

Acknowledgments. MZ is grateful to his collaborators, Stéphane Nonnenmacher and Johannes Sjöstrand for the permission to use parts of [20] and [41] in the preparation of these notes. Generous support of both authors by the National Science Foundation under the grant DMS-0200732 is also gratefully acknowledged.

2. Fractal Weyl laws in different settings

In this section we will present mathematical and numerical results on the fractal Weyl laws. We will also mention some preliminary experimental results.

2.1. **Schrödinger operators.** The original motivation comes from the study of resonances in potential scattering. The simplest case is given by considering the following quantum Hamiltonian:

$$H = -h^2 \Delta + V(x) \,, \quad V \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n; \mathbb{R}) \,. \tag{2.1}$$

By assuming that the potential vanishes near infinity and that it is infinitely differentiable, we eliminate the need for technical assumptions — see [16], and [41] for more general settings, in the analytic and $\mathcal{C}^\infty$ categories respectively. For instance, as noted in [38, (c.32)-(c.33)] the theory of [16] applies to arbitrary homogeneous polynomial potentials at non-degenerate energy levels.

Before discussing open systems we recall the well known results for closed systems, obtained for instance by considering H above on a compact Riemannian manifold M. Then the spectrum, Spec(H), of H is discrete and, at a non-degenerate energy level E its density is described by the celebrated *Weyl law*:

$$\# \{ \operatorname{Spec}(H) \cap [E-\delta, E+\delta] \} = \frac{1}{(2\pi h)^n} \iint_{|\xi^2+V(x)-E|<\delta} dq\, dp + \mathcal{O}(h^{1-n}) \,, \tag{2.2}$$

see [9, 19] and references given there. We note that this implies a precise upper bound

$$\# \{ \operatorname{Spec}(H) \cap [E-Ch, E+Ch] \} = \mathcal{O}(h^{1-n}) \,. \tag{2.3}$$

This can be improved further by making assumptions on the classical flow of the Hamiltonian $p(x,\xi) = |\xi_g^2 + V(x)$ on Ω. Thus assume that E is a nondegenerate energy level,

$$p(x,\xi) = 0 \implies dp(x,\xi) \neq 0 \,. \tag{2.4}$$

Assume further that the union of periodic orbits of the Hamilton flow on $p = E$ has measure zero with respect to the Liouville measure. Then the spectrum of the quantized Hamiltonian,

$$H = -h^2 \Delta_g + V(x) \,, \quad x \in M \,, \tag{2.5}$$

near E satisfies,

$$|\operatorname{Spec}(\cap [E-\rho h, E+\rho h]| = \frac{2\rho h}{(2\pi h)^n} \int_{p(x,\xi)=0} d\mathcal{L}(x,\xi) + o(h^{-n+1}) \,, \tag{2.6}$$

see [9],[19] for references to mathematical literature on this subject.

For open systems, with the simplest example given by the Hamiltonian in (2.1), real eigenvalues are replaced by complex *resonances*. The simplest definition (easily made rigorous in the case (2.1)) comes from considering the meromorphic continuation of the resolvent. Defining the Green's function $G(z, x', x)$ for $\operatorname{Im} z > 0$ through

$$(z-H)^{-1} u(x') = \int_{\mathbb{R}^n} G(z, x', x)\, u(x)\, dx \,, \quad u \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n) \,,$$

then $G(z, x', x)$ admits a meromorphic continuation in z across the real axis. Its poles for $\operatorname{Im} z < 0$ (which do not depend on (x', x)) are the *quantum resonances* of H.

We denote the set of resonances by $\operatorname{Res}(H)$. In general, the Weyl law (2.6) does not hold in this case but for nondegenerate energy levels (2.4) we still have the following bound [4],[29],

$$| \operatorname{Res}(H) \cap [E-\rho h, E+\rho h] - i[0, \gamma h]| \leq C(\rho, \gamma) h^{-n+1} \,. \tag{2.7}$$

When the interaction region is separated from infinity by a barrier, this bound is optimal since resonances are well approximated by eigenvalues of a closed system [26] and (2.6) can be used.

Counting of resonances is affected by the dynamical structure of the scatterer much more dramatically than counting of eigenvalues of closed systems. Since we are now counting points in the complex plane we need to make geometric choices dictated by dynamical and physical considerations. Here we consider scatterers and energies exhibiting a hyperbolic classical flow, and regions in the lower half-plane which lie at a distance proportional to h from the real axis. This choice is motivated as follows. Quantum mechanics interprets a resonance at $z = E - i\gamma$ in terms of a *metastable state*, which decays proportionally to $\exp(-t\gamma/h)$. Hence for $\gamma \gg h$ the decay is so rapid that the state is invisible. On the other hand, for many chaotic scatterers there are no resonances with $\gamma \ll h$. One class for which this is known rigorously consists in the Laplacian on co-compact quotients $\mathbb{H}^n/\Gamma$, $H = -h^2\Delta_{\mathbb{H}^n/\Gamma}$, when the dimension of the limit set satisfies $\delta(\Gamma) < (n-1)/2$. This follows from the work of Patterson and Sullivan — see the discussion below and [27].

After a complex deformation (see [41] and references given there) the long living quantum states should semiclassically concentrate on the set of phase space points which do not escape to infinity, that is on the *trapped set* K_E defined as follows: let Ξ_H be the Hamilton vector field of the Hamiltonian $H(q,p) = p^2/2 + V(q)$:

$$\Xi_H = \sum_{j=1}^{n} 2p_j \partial_{q_j} - \partial_{q_j} V(q) \partial_{p_j} \, .$$

Then

$$\begin{aligned} K_E &\stackrel{\text{def}}{=} \Gamma_+(E) \cap \Gamma_-(E) \, , \\ \Gamma_\pm(E) &\stackrel{\text{def}}{=} \{\rho \in \Sigma_E \; : \; \exp t\,\Xi_H(\rho) \not\to \infty \, , \; \mp t \to \infty\} \, . \end{aligned} \tag{2.8}$$

Suppose that the flow generated by Ξ_H is hyperbolic near $K_{E'}$ for E' close to a non-degenerate energy E. That means that the field Ξ_H does not vanish on the energy surfaces $\Sigma_{E'} = \{p^2 + V(q) = E'\} \subset \mathbb{R}^n \times \mathbb{R}^n$ for $E' \approx E$, and that for $\rho \in \Sigma_{E'}$ near $K_{E'}$,

$$\begin{aligned} & T_\rho \Sigma_{E'} = \mathbb{R}\,\Xi_H(\rho) \oplus E_+(\rho) \oplus E_-(\rho) \, , \\ & \Sigma_{E'} \ni \rho \longmapsto E_\pm(\rho) \subset T_\rho \Sigma_{E'} \;\; \text{is continuous,} \\ & d(\exp t\,\Xi_H)(E_\pm(\rho)) = E_\pm(\exp t\,\Xi_H(\rho)) \, , \\ & \|d(\exp t\,\Xi_H)(X)\| \leq C e^{\pm\lambda t} \|X\| \, , \quad \text{for all } X \in E_\pm(\rho), \, \mp t \geq 0. \end{aligned} \tag{2.9}$$

Weaker assumptions are possible — see [38, §5] and [41, §7].

Typically, the set K_E has a fractal structure and in the semiclassical estimates the Minkowski dimension naturally appears:

$$\dim K_E = 2n - 1 - \sup\left\{c \; : \; \limsup_{\epsilon \to 0} \epsilon^{-c} \operatorname{vol}\{\rho \in \Sigma_E \; : \; \operatorname{dist}(K_E, \rho) < \epsilon\} < \infty\right\} .$$

We say that K_E is of *pure dimension* if the supremum is attained. For simplicity of the presentation we assume that this is the case.

The first indication that fractal dimensions enter into counting laws for quantum resonances of chaotic open systems appears in a result of Sjöstrand [38]:

$$\#\big\{ \operatorname{Res}(H) \cap \{ z \ : \ |z-E| < \delta \,, \ \ \operatorname{Im} z > -\gamma \} \big\} \leq C_1 \delta \left(\frac{h}{\gamma} \right)^{-n} \gamma^{-\frac{1}{2}\widetilde{m}} \,, \tag{2.10}$$
$$Ch \leq \gamma \leq 1/C \,, \ \ \max(h^{\frac{1}{2}}, h/\gamma) \leq \delta \leq 2/C \,,$$

where $\widetilde{m}$ is any number greater than the dimension of the trapped set in the shell $H^{-1}(E - 1/C_2, E + 1/C_2)$. In a homogeneous situation, such as for instance obstacle scattering, the dimension of K_E, $2\mu_E + 1$, is independent of E, so that $\widetilde{m} > 2(\mu_E + 1)$. Heuristic arguments suggesting that the estimate (2.10) should be optimal were given in [22] and [23].

The more precise estimate which is an analogue of (2.3) has recently been established Sjöstrand and the second author [41]. For $C_0 > 0$ there exists C_1 such that

$$\#\big\{ \operatorname{Res}(H) \cap \{ z \ : \ |z-E| < C_0 h \} \big\} \leq C_1 h^{-\mu_E} \,, \ \ \dim K_E = 2\mu_E + 1 \,. \tag{2.11}$$

We notice that for a closed system the trapped set is the entire energy surface, so that in that case $\mu_E = n - 1$, hence (2.11) is consistent with (2.3).

The improvement over (2.10) lies in providing a bound for the number of resonances in a smaller region $D(E, Ch)$.

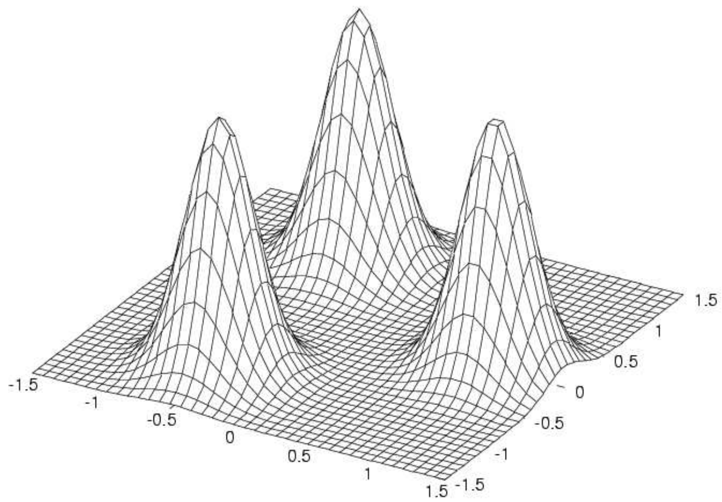


FIGURE 1. A three bump potential exhibiting hyperbolic dynamics in a range of energies.

2.2. **Quantum Maps.** Quantum maps are obtained from quantizing symplectic transformations, or more generally Lagrangian relations, between compact symplectic manifolds, in our case here tori, $\mathbb{T}^2$ – see [3],[8],[33],[34],and for a precise mathematical discussion [20].

We consider $\mathbb{T}^2 = [0,1) \times [0,1)$, the two-torus, as our classical phase space. Classical observables are functions on $\mathbb{T}^2$ and classical dynamics is given in terms of relations, $B \subset \mathbb{T}^2 \times \mathbb{T}^2$, which are unions of truncated graphs of symplectic (area preserving) maps $\mathbb{T}^2 \to \mathbb{T}^2$. An example is given by the baker's relation

$$(\rho';\rho) = (q',p';q,p) \in B \Longleftrightarrow \begin{cases} q' = 3q\,, & p' = p/3\,, \quad 0 \leq q \leq 1/3 \\ q' = 3q-2\,, & p' = (p+2)/3\,, \quad 2/3 \leq q < 1\,. \end{cases} \tag{2.12}$$

This is a "rectangular horseshoe" modeling a Poincaré map of a chaotic open system: some points (here $\{\,1/3 < q < 2/3\,\}$) are thrown out at each iteration.

For relations such as B we can define the *incoming and outgoing tails* (see (2.8) for the definition in the case of flows):

$$\rho \in \Gamma_- \;\Leftrightarrow\; \exists \{\rho_j\}_{j=0}^{\infty}\,, \quad \rho_0 = \rho\,, \;\; (\rho_j;\rho_{j-1}) \in B\,,\;\; j > 0\,,$$
$$\rho \in \Gamma_+ \;\Leftrightarrow\; \exists \{\rho_j\}_{j=-\infty}^{0}\,, \quad \rho_0 = \rho\,, \;\; (\rho_j;\rho_{j-1}) \in B\,,\;\; j \leq 0\,.$$

In the example (2.12), $\Gamma_- = C \times [0,1)$, $\Gamma_+ = [0,1) \times C$, where C is the usual $\frac{1}{3}$−Cantor set – see Fig.2.

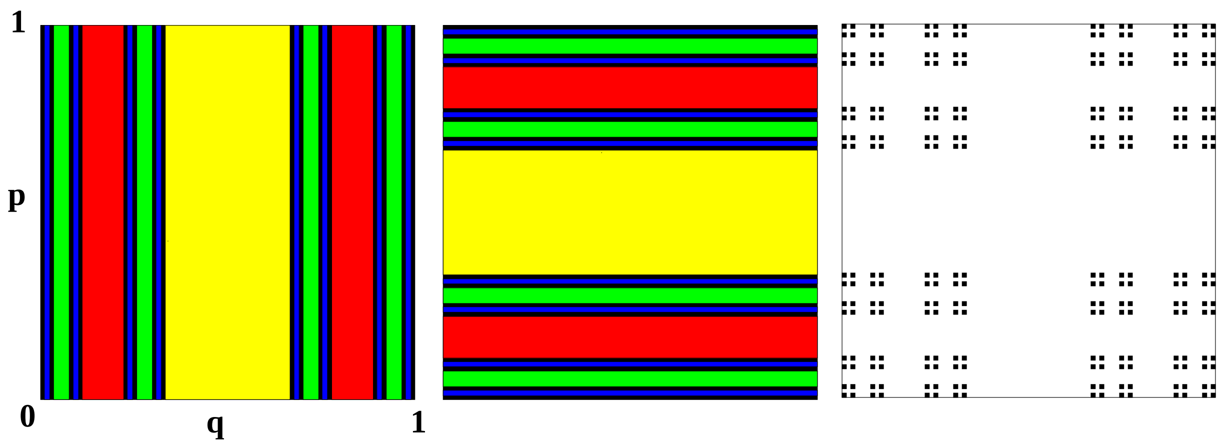


FIGURE 2. We show, from left to right, approximations of the in/outcoming tails Γ_-, Γ_+ and the trapped set K for the open 3-baker B_3. On the left and central plots, each color corresponds to points escaping at the same time.

We also define the *trapped set* $K = \Gamma_+ \cap \Gamma_-$ and, at points of K, the stable and unstable manifolds, $W_\pm$. In the case of the above baker's relation,

$$\frac{1}{2}\dim K = \dim \Gamma_+ \cap W_- = \dim \Gamma_- \cap W_+ = \frac{\log 2}{\log 3}\,,$$

but for general (possibly multivalued) relations these equalities do not hold.

A quantization (in the sense made rigorous in [20, Sect.4]) of B is given by

$$B_h = \mathcal{F}_N^* \begin{pmatrix} \mathcal{F}_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{F}_{N/3} \end{pmatrix} , \quad h = (2\pi N)^{-1} , \ 3|N , \tag{2.13}$$

where $\mathcal{F}_M$ is the discrete Fourier transform on $\mathbb{C}^M$.

$N = 3^k$	$r = 0.1$	$r = 0.2$	$r = 0.3$	$r = 0.4$	$r = 0.5$	$r = 0.6$	$r = 0.7$	$r = 0.8$
$k = 1$	5	5	5	5	5	4	3	3
$k = 2$	14	14	10	9	8	8	7	6
$k = 3$	32	26	23	19	16	16	14	5
$k = 4$	63	53	45	40	33	33	30	6
$k = 5$	124	103	85	78	71	65	63	11
$k = 6$	237	196	161	150	142	131	128	12

TABLE 1. Number of eigenvalues of B_h in the regions $\{ |\lambda| > r \}$, for $2\pi h = 1/N$, N given by powers of 3.

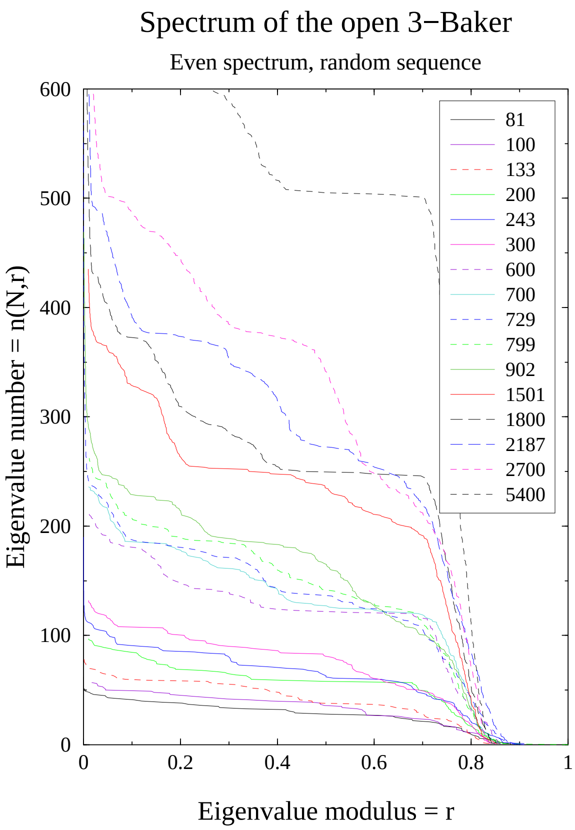
Table 2 shows the analogies between the eigenvalues of this subunitary quantum map and the resonances of a Schrödinger operator for a scattering situation (see §2.1).

For B_h given by (2.13) we are unable to prove the fractal Weyl law presented in the last line of Table 2, but numerical results, an aspect of them shown in Fig.3 strongly support its validity. A striking illustration is given by tripling N in which case the number of eigenvalues approximately doubles, in agreement with the fractal Weyl law — see Table 1.

To obtain a fractal Weyl law we simplify B_h, as follows

$$\begin{gathered} \widetilde{B}_h = \widetilde{B}_N = \frac{1}{\sqrt{3}} [\widetilde{B}_{1,h}, 0, \widetilde{B}_{2,h}] , \\ (\widetilde{B}_{1,h})_{k\,l} = \begin{cases} 1 & \text{if } l = \lfloor k/3 \rfloor \\ 0 & \text{if } l \neq \lfloor k/3 \rfloor \end{cases} , \qquad (\widetilde{B}_{2,h})_{k\,l} = \omega_3^{2k} \, (\widetilde{B}_{1,h})_{k\,l} , \end{gathered} \tag{2.14}$$

It turns out that $\widetilde{B}_h$ is a quantization of a more complicated multivalued relation for which $\Gamma_+ = \mathbb{T}^2$, $\Gamma_- = C \times [0, 1)$, and $\Gamma_- \cap W_+ \simeq C$ – see [20, §6]. We then have an exact fractal



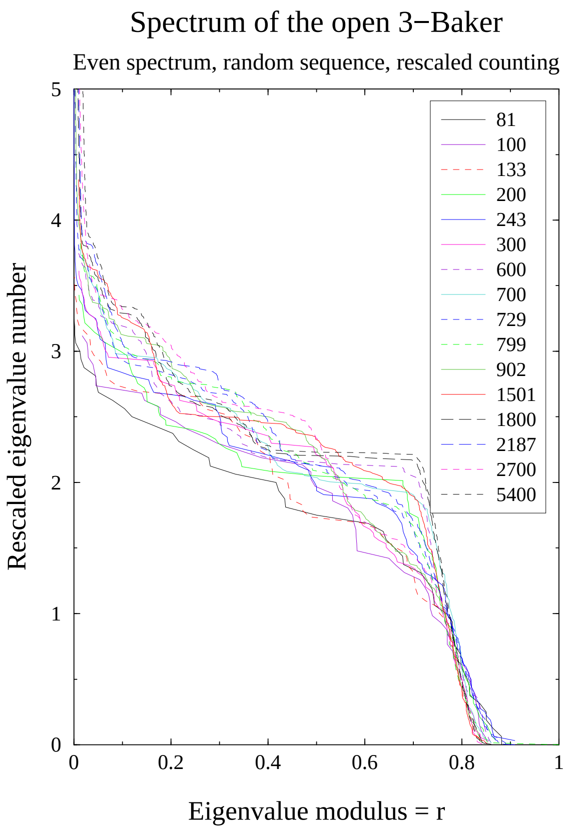


FIGURE 3. On the left, we plot the number $n(N, r)$ of even eigenvalues of modulus $\geq r$ of the 3-baker B_h (the numbers in the box is the rank $N/3$ of the desymmetrized matrices). On the right we have rescaled that number by the factor $N^{-\log 2/\log 3}$.

Weyl law:

$$\# \operatorname{Spec}(\widetilde{B}_h) \setminus D(0,r) = c(r)\, h^{-\nu} + o(h^{-\nu}) \,,$$
$$r > 0 \,, \quad h = h_k \to 0, \quad k \to \infty \,,$$
$$\nu = \dim \left(\Gamma_-(\widetilde{B}) \cap W_+(\widetilde{B})\right) ,$$
$$c(r) = (2\pi)^{-\nu} \chi_{[0, r_0(\widetilde{B})]}(r) \,, \quad 0 < r_0(\widetilde{B}) < 1 \,,$$

where $h_k = 3^k/(2\pi)$.

2.3. **Hyperbolic quotients.** Another class of Hamiltonians with chaotic classical flows and fractal trapped sets is given by Laplacians on convex co-compact quotients, $\mathbb{H}/\Gamma$, where Γ is a discrete subgroup of isometries of $\mathbb{H}$, the hyperbolic plane, such that

$h \to 0$	$N = (2\pi h)^{-1} \to \infty$
$\chi \exp\big(-it(-h^2\Delta + V)/h\big)\chi\,, \quad t \geq 0\,,$ χ a cut-off to an interaction region	B_h^t, $t = 0, 1, \cdots$ B_h a subunitary matrix
$e^{-itz/h}$, z a resonance of $H = -h^2\Delta + V$	λ^t, λ an eigenvalue of $B_h \in \mathcal{L}(\mathbb{C}^N)$
$z \in [E-h, E+h] - i[0, \gamma h]$	$1 \geq \|\lambda\| > r > 0$
$\#\{z \in [E-h, E+h] - i[0,\gamma h]\} \simeq C(\gamma)\, h^{-\mu_E}$	$\#\{\lambda, \|\lambda\| > r\} \simeq C(r)\, N^{\frac{\log 2}{\log 3}}$

Table 2. Analogies between Schrödinger propagators and quantum maps.

- All elements $\gamma \in \Gamma$ are hyperbolic, which means that their action on $\mathbb{H}$ can be represented as

$$\alpha \circ \gamma \circ \alpha^{-1}(x, y) = e^{\ell(\gamma)}(x, y)\,, \quad (x, y) \in \mathbb{H} \simeq \mathbb{R}_+ \times \mathbb{R}\,, \quad \alpha \in \mathrm{Aut}(\mathbb{H})\,. \tag{2.15}$$

- If $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$, and $\Lambda(\Gamma) \subset \partial\mathbb{H}$ is the limit set of Γ, that is the set of limit points of $\{\gamma(z) : \gamma \in \Gamma\}$, $z \in \mathbb{H}$, then $\pi(\text{convex hull } \Lambda(\Gamma))$ is compact.

An example is shown in Fig. 4. The trapped set is determined by $\Lambda(\Gamma)$: trapped trajectories are given by geodesics connecting two points of $\Lambda(\Gamma)$ at infinity, and

$$\dim K_E = 2\delta_\Gamma + 1\,, \quad \delta_\Gamma = \dim \Lambda(\Gamma)\,.$$

The limit set is always of pure dimension, which coincides with its Hausdorff dimension.

A nice feature of this model is the exact correspondence between the resonances of

$$H = h^2(-\Delta_{\mathbb{H}/\Gamma} - 1/4)\,,$$

and the zeros of the Selberg zeta function, $Z_\Gamma(s)$[‡]:

$$z \in \mathrm{Res}(H) \iff Z_\Gamma(s) = 0\,, \quad z = h^2(s(1-s) - 1/4)\,, \quad \mathrm{Re}\, s \leq \delta_\Gamma\,, \tag{2.16}$$

[‡]We refer to [28] for this and a general treatment. The term $\frac{1}{4}$ in the definition of the Hamiltonian H comes from requiring that the bottom of the spectrum of H is 0, so that Green's function $(H - \lambda^2)^{-1}$ is meromorphic in $\lambda \in \mathbb{C}$

where the multiplicities of zeros and resonances agree. The Selberg zeta function is defined by the analytic continuation of

$$Z_\Gamma(s) = \prod_{\{\gamma\}} \prod_{k\geq 0} \left(1 - e^{-(s+k)\,\ell(\gamma)}\right) , \quad \operatorname{Re} s > \delta_\Gamma ,$$

where $\{\gamma\}$ denotes a conjugacy class of a primitive element $\gamma \in \Gamma$ (an element which is not a power of another element), and we take a product over distinct primitive conjugacy classes (each of which corresponds to a primitive closed orbit). The length $\ell(\gamma)$ of the corresponding closed orbit appears in (2.15). The exact analogue of (2.11) is given by

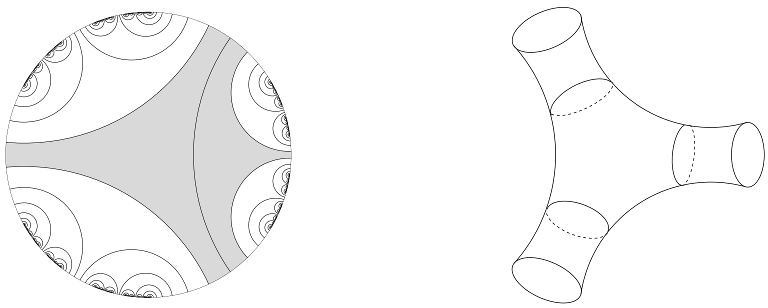


FIGURE 4. An example of $\mathbb{H}^2/\Gamma$ where Γ is generated by compositions of reflections in three discs.

$$\# \{ s \ : \ Z_\Gamma(s) = 0 , \quad \operatorname{Re} s > -C_0 , \quad r < \operatorname{Im} s < r + C_1 \} \leq C_2 \, r^{\delta_\Gamma} , \tag{2.17}$$

which is a consequence of an estimate established by Guillopé-Lin-Zworski [15] in a more general setting of convex co-compact Schottky groups in any dimension,

$$|Z_\Gamma(s)| \leq C_K \, e^{C_K \, |s|^{\delta_\Gamma}} , \quad \operatorname{Re} s \geq -K , \quad \text{for any } K. \tag{2.18}$$

This improved earlier estimates of [45], the proof of which was largely based on [38].

2.4. **Survey of numerical results.** The first model investigated numerically was perhaps the hardest to give definitive results. Lin [21, 22] studied the semiclassical Schrödinger Hamiltonian (2.1) with the potential given in Fig. 1. The semiclassical resonances were computed using the method of complex scaling and were counted in boxes of type $[E - \delta, E + \delta] - i[0, h]$ with δ fixed. The purpose was to verify optimality of Sjöstrand's estimate (2.10) with these parameters. The results were encouraging but not conclusive. Since for small values of h the method of [21] required the use of large matrices to discretize the Hamiltonians, the range of h's was rather limited.

A different point of view was taken by Lu-Sridhar-Zworski [23] where resonances for the three discs scatterer in the plane were computed using the semiclassical zeta function of Eckhardt-Cvitanović, Gaspard, and others (see for instance [7, 12, 44] and references therein). The zeta function is computed using the cycle expansion method loosely based on the Ruelle theory of dynamical zeta functions.

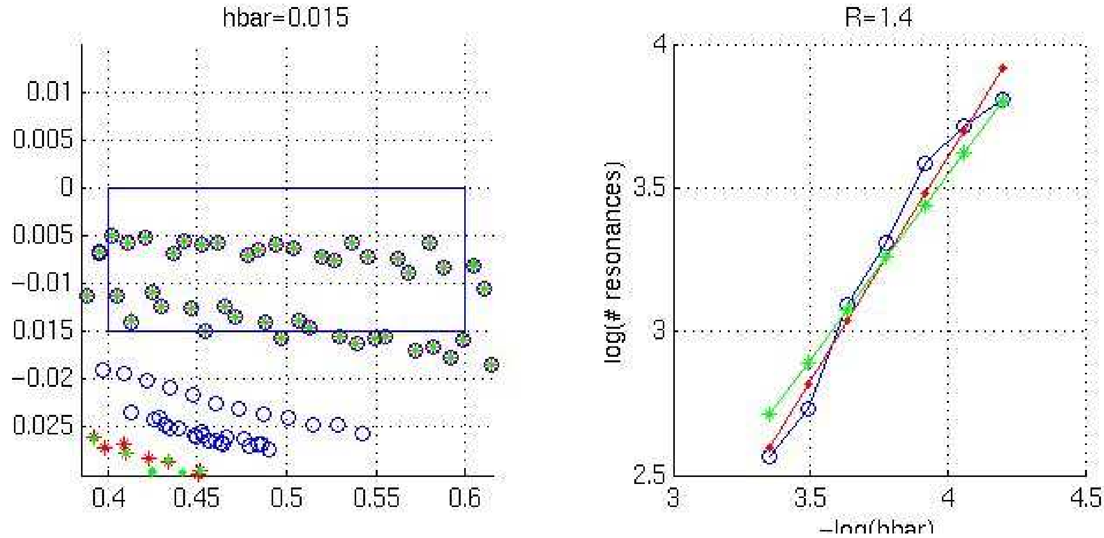


FIGURE 5. A sample of results of [21]: the plot on the left shows resonances for $h = 0.015$, and the plot on the right is the log-log plot of the number of resonances vs. h with $\circ$ denoting numerical data, $*$ the density predicted by the fractal Weyl law, and $\bullet$, the least square interpolants.

Although it is not rigorously known if the resonances computed by this method approximate resonances of the Dirichlet Laplacian in the exterior of the discs, or even if the semiclassical zeta function has an analytic continuation, proceeding this way is widely accepted in the physics literature. Resonances $z = h^2 k^2$ were counted in regions

$$\{ k \in \mathbb{C} \; : \; 1 \leq \operatorname{Re} k \leq r \, , \quad \operatorname{Im} k \geq -\gamma \} \; , \quad r \to \infty \, , \tag{2.19}$$

which under semiclassical rescaling correspond to counting in $[1/2, 2] - i[0, \gamma h/2]$, $h \to 0$. Let us denote the number of resonances (zeros of the semiclassical zeta function) in (2.19) by $N(r, \gamma)$. The fractal Weyl law corresponds to the claim that for γ large enough,

$$N(r, \gamma) \sim C(\gamma) r^{\mu} \, , \quad r \to \infty \, , \tag{2.20}$$

where $2\mu + 1$ is the dimension of the trapped set in the three dimensional energy shell (for such homogeneous problems, all energy shells are equivalent). In [23] the prediction (2.20) was tested by linear fitting of $\log N(r, \gamma)$ as a function of $\log r$:

$$\log N(r, \gamma) = \alpha(\gamma) \log r + \mathcal{O}(1) \, .$$

We found that the coefficient $\alpha(\gamma)$ was independent of γ for γ large enough, and that it agreed with μ. The counting was done for three different equilateral disc configurations, parametrized by $\rho = R/a$ where a is the radius of each disc, and R the distance between

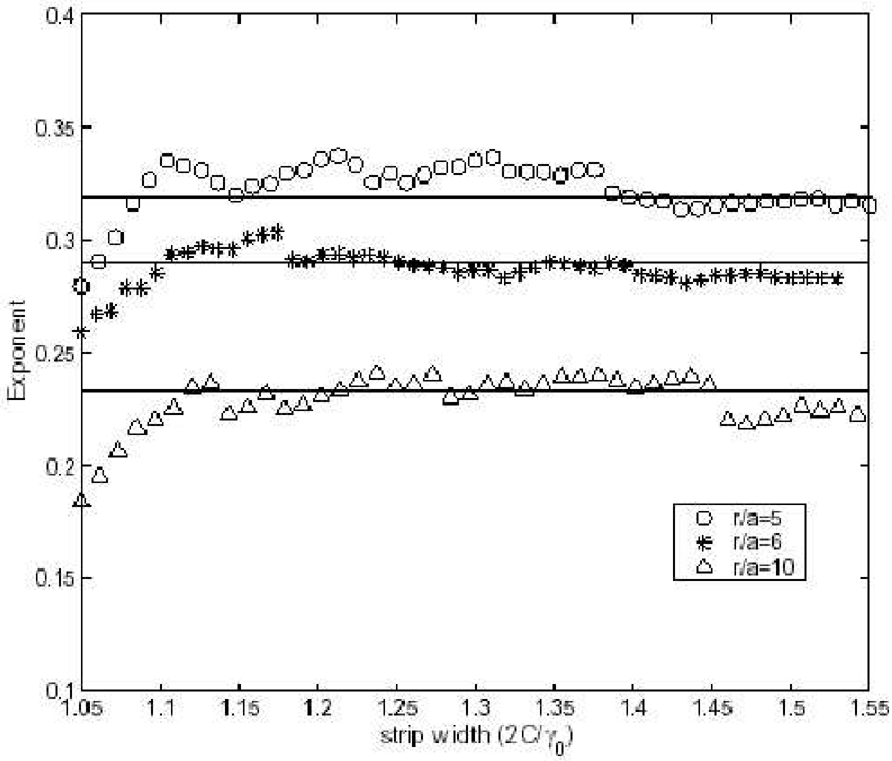


FIGURE 6. The least square fit exponents for different widths γ for three different disc configurations. The dimensions are shown by solid lines.

them. We also noticed that if γ_ρ is the classical rate of decay for the ρ configuration, then

$$\frac{\alpha_\rho(x\gamma_\rho/2)}{\mu_\rho}$$

is essentially independent of ρ for $1 < x < 1.5$. This corresponds to a numerical observation that for each ρ the distribution of resonance widths (imaginary parts) peaks near $\gamma = \gamma_\rho/2$.

Encouraged by the results of [23], the cycle method was used in [15] to count the zeros of the Selberg zeta function for the quotient depicted in Fig. 4, but the results were not definitive. For the dynamical zeta function (4.4) with $f(z) = z^2 + c$, $c < -2$ the resonances were computed by Strain-Zworski [42], using a different method based on the theory of the transfer operator on Hilbert spaces of holomorphic functions introduced in [15]. The zeros were counted in a region of the same type as in (2.19),

$$\{s \ : \ \operatorname{Re} s > -K \, , \ 0 \leq \operatorname{Im} s \leq r\} \, ,$$

where real parts and imaginary parts exchange their meaning due to different conventions[§]. By reaching very high values of r we saw a very good agreement of the log-log fit with the fractal Weyl, with μ given by the dimension of the Julia set. The results are shown in Fig.7.

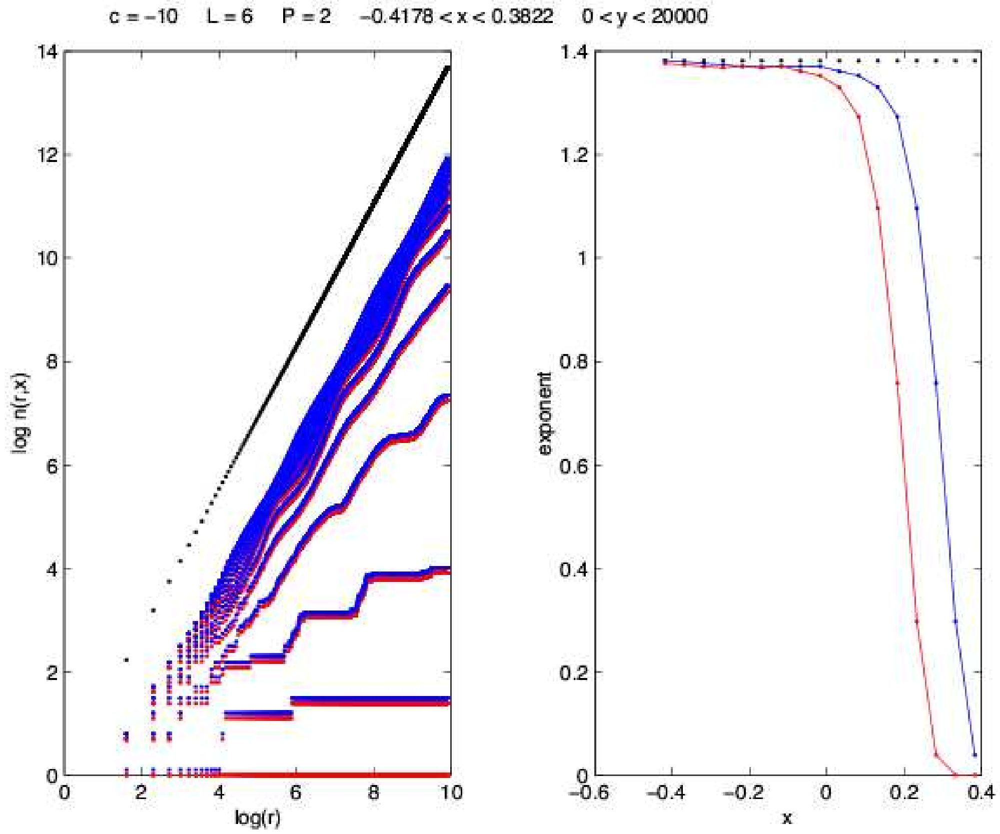


FIGURE 7. The upper and lower bounds on the number of zeros of the zeta function in strips for $z \mapsto z^2 - 10$. The plots on the left correspond to different width. The least squares exponents for different width converge to $1 + \delta$, where $\delta = 0.3822 \cdots$ is the dimension of the Julia set.

3. Outline of the proof of the semiclassical estimate

To prove the main result on fractal upper bounds (2.11) we first develop methods for proving the natural results on the absence of resonances and on general upper bounds at

[§]Although frustrating, the different conventions of semiclassical, obstacle, and hyperbolic scattering show how the same phenomenon appears in historically different fields.

non-degenerate energies (2.7). This outline comes from [41, §2] and we refer to that paper for details and for pointers to the literature.

The absence of resonances for operators with $\mathcal{C}^\infty$ coefficients in domains of size $h\log(1/h)$ around an energy level hold under a nontrapping condition:

$$\exists\, \epsilon_0 > 0\,,\ \forall\, K \Subset p^{-1}(0)\,,\ \exists\, T_K\,,\quad (x,\xi) \in K \implies \exp(tH_p(x,\xi)) \notin K\,,\ t > T_K\,.$$

This implies the existence of an escape function in a neighbourhood of $p^{-1}(0)$:

$$\exists\, G_1 \in \mathcal{C}^\infty(T^*X)\,,\quad H_pG_1(x,\xi) \geq c_0 > 0\,,\ \text{for } |p(x,\xi)| < \epsilon_0.$$

The resonances of P are given by the eigenvalues of the deformed operator P_θ. In the case of $P = -h^2\Delta + V(x)$ with V analytic in a conic neighbourhood of $\mathbb{R}^n$,

$$V(x) + 1 \longrightarrow 0, \quad |x| \longrightarrow \infty\,,$$

the scaled operator is simply

$$P_\theta = -h^2 e^{-2i\theta}\Delta + V(e^{i\theta}x)\,,$$

and it behaves as $-h^2e^{-2i\theta}\Delta - 1$ near infinity. For $\theta > 0$ that last operator is clearly invertible.

We can introduce a modified $G = \chi G_1$, $\chi \in \mathcal{C}^\infty_{\rm c}(X)$, so that for

$$\theta \sim \epsilon \sim Mh\log(1/h)\,,$$

we have

$$|\operatorname{Re} p_\theta| < \delta \implies -\operatorname{Im} p_\theta + \epsilon H_pG \geq c_0\epsilon\,,\quad p_\theta = \sigma(P_\theta)\,.$$

The operators $\exp(\pm\epsilon G^w(x,hD)/h)$ are now pseudodifferential operators in a mildly exotic class and we consider

$$P_{\theta,\epsilon} = e^{-\epsilon G^w/h}P_\theta e^{\epsilon G^w/h}\,.$$

The spectrum of P_θ in $D(0, Mh\log(1/h))$ is the same as that of $P_{\theta,\epsilon}$ but the properties of G imply that

$$\|P_{\theta,\epsilon}^{-1}\| \leq C/\epsilon$$

showing that in fact there is no spectrum in $D(0, M'h\log(1/h))$. This approach allows us to obtain the absence of resonances very directly.

The estimate (2.7) shows that if 0 is a non-critical energy level then

$$|\operatorname{Res} P \cap D(0, Ch)| = \mathcal{O}(h^{-n+1})\,.$$

The proof follows from a "robust" proof of the same estimate for an operator with a compact resolvent (for instance, an elliptic operator on a compact manifold). Let P be such an operator, say, $P = -h^2\Delta_g - 1$, on a compact Riemannian manifold. We would like to consider a modified operator

$$\widetilde{P}(h) \stackrel{\rm def}{=} P(h) - iMh\psi(MP(h)/h)\,,\quad \psi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R})\,,$$

whose "symbol", $p - iMh\psi(Mp/h)$, has the absolute value bounded from below by $Mh/2$ everywhere. That does not make sense at first since

$$\psi(MP(h)/h)$$

is *not* an h-pseudodifferential operator. To remedy this we construct a second microlocal calculus with a new Planck constant $\tilde{h} \sim 1/M$. The new operator $\widetilde{P}(h)$ becomes elliptic in this calculus and for $\tilde{h}$ small enough, independent of h, it is invertible. We then have

$$(P(h) - z)^{-1} = (I + K(z))^{-1}(\widetilde{P}(h) - z)^{-1}, \quad K(z) \stackrel{\text{def}}{=} i(\widetilde{P}(h) - z)^{-1} Mh\psi(P(h)/h)$$

and the eigenvalues of $P(h)$ near 0 coincide with the zeros of $\det(I + K(z))$. The zeros of this determinant are the same as the zeros of a determinant $\det(I + R(z))$ where $R(z)$ is a finite rank operator with the rank proportional to h^{-n} times the volume of the support of $\psi(Mp/h)$. That gives estimates on the deteminant which imply (2.6). A slight modification of this argument is needed to obtain (2.7).

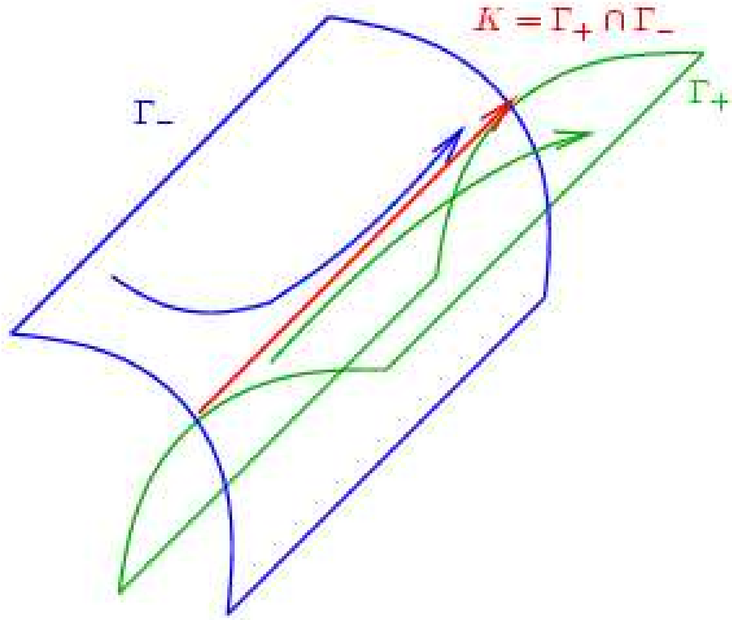


FIGURE 8. Outgoing and incoming sets in the case of one orbit in a three dimensional energy hypersurface.

We now assume that the flow of H_p is hyperbolic and introduce the sets

$$\Gamma_\pm \stackrel{\text{def}}{=} \{(x,\xi) \in T^*X \; : \; p(x,\xi) = 0\,, \quad \exp(tH_p)(x,\xi) \not\to \infty\,, \; t \to \mp\infty\}\,, \tag{3.1}$$

depicted in a simple case in Fig.8. The trapped set at zero energy is

$$K = \Gamma_+ \cap \Gamma_-\,. \tag{3.2}$$

If we assume that $K \subset \overline{\Gamma_\pm \setminus K}$, that is K has no component isolated from infinity, then K is a set of Liouville measure 0.

To prove an upper bound involving the dimension of K we combine the methods used to prove the absence of resonance and to obtain the general upper bound (2.7). There exist functions $\varphi_\pm \in \mathcal{C}^{1,1}(T^*X)$ such that, uniformly on compact sets,

$$H_p\varphi_\pm \sim \mp\varphi_\pm\,, \quad \varphi_\pm \sim d(\Gamma_\pm, \bullet)^2\,, \quad \varphi_+ + \varphi_- \sim d(K, \bullet)^2\,.$$

A local model for the simplest case of one trajectory is given by $p = \xi_1 + x_2\xi_2$, $(x,\xi) \in T^*\mathbb{R}^2$, so that

$$H_p = \partial_{x_1} + x_2\partial_{x_2} - \xi_2\partial_{\xi_2}\,, \quad \varphi_+ = \xi_2^2\,, \quad \varphi_- = x_2^2\,, \quad K = \{(t,0;0,0) \; : \; t \in \mathbb{R}\}\,. \tag{3.3}$$

A new escape function is given by

$$G \stackrel{\text{def}}{=} \Big(\log(C\epsilon + \widehat{\phi}_-) - \log(C\epsilon + \widehat{\phi}_+)\Big)\,, \quad \epsilon \sim Mh\,, \quad M \gg 1\,, \tag{3.4}$$

where $\widehat{\phi}_\pm$ are suitable h-dependent regularizations of $\varphi_\pm$.

The logarithmic flattening of the more straightforward escape function $\varphi_- - \varphi_+$ is forced by the requirement that $G = \mathcal{O}(\log(1/h))$ so that the conjugation used above can be applied. However, even for uniformly smooth $\widehat{\varphi}_\pm$ the regularization of G is essentially in the symbolic class $S_{\frac{1}{2}}$ and the situation becomes more complicated in general. Nevertheless we obtain the following estimates:

$$\partial^\alpha_{(x,\xi)} H_p^k G = \mathcal{O}(\epsilon^{-\frac{|\alpha|}{2}})\,, \quad \text{for } |\alpha| + k \geq 1, \text{ uniformly on compact sets,}$$

and

$$d((x,\xi), K)^2 \geq C\epsilon \implies H_pG \geq 1/C\,.$$

As in the proof of resonance free regions (but with very different parameters and escape functions) we introduce a conjugated operator,

$$P_{\theta,t}(h) \stackrel{\text{def}}{=} e^{-tG^w} P_\theta(h) e^{tG^w}\,,$$

which now is in an exotic $\frac{1}{2}$-class, with the second Planck constant $\tilde{h} \sim 1/M$ playing the rôle of the asymptotic parameter. The escape function used here, G, has compact support.

We now build a second microlocal calculus which combines this exotic class with the one used in the proof of (2.7). The first allows us the use of the irregular escape function and

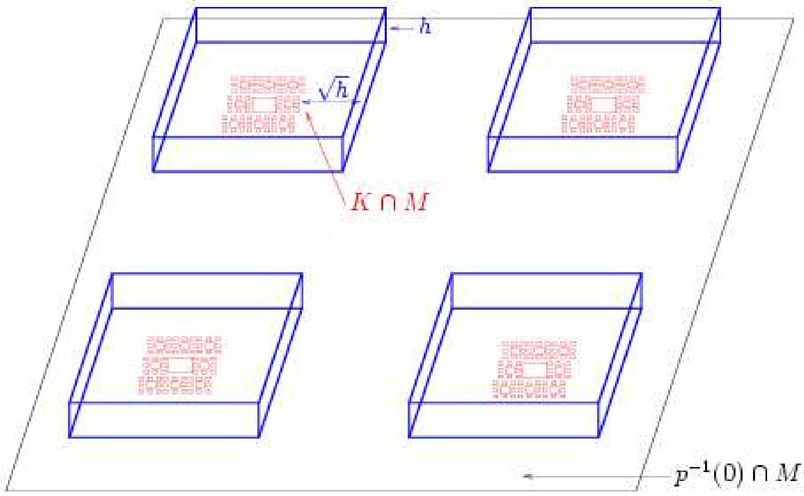


FIGURE 9. The trapped set $K \subset p^{-1}(0)$ intersected with a hypersurface $M \subset T^*X$ transversal to the flow and surrounded by an h-dependent neighbourhood admissible in the second microlocal calculus. For K of dimension less than $2\nu + 1$, the volume of this neighbourhood is bounded by $h^{n-\nu}$, y$n = \dim X$.

the second allows a localization to an h-neighbourhood of the energy surface. In the new calculus the operator

$$\widetilde{P}_{\theta,t} = P_{\theta,t} - iMh\widetilde{\operatorname{Op}}(a)\,, \quad a(x,\xi) \stackrel{\text{def}}{=} \chi\left(\frac{p(x,\xi)}{K_1 h}\right)\chi\left(\frac{C_0 H_p G(x,\xi)}{K_1 h}\right)\,,$$

is globally elliptic (here $\widetilde{\operatorname{Op}}$ describes a second microlocal quantization operator). As in the proof of (2.7) the number of eigenvalues of $P_{\theta,t}$, and hence P_θ, near 0, is estimated by h^{-n} times the volume of the support of a. A cross-section of that support with a hypersuface transversal to the flow is illustrated in Fig.9. That volume is bounded by $h^{1+(2n-2-2\nu)/2}$, where $\nu > \nu_0$, $2\nu_0 + 1$ is the dimension of K. That gives (2.7).

4. UPPER BOUNDS FOR DYNAMICAL ZETA FUNCTIONS FOR HYPERBOLIC RATIONAL FUNCTIONS

4.1. **Dynamical zeta functions.** The motivation for the estimates described in the remaining sections comes from scattering resonances. In the case where the underlying fractal set is the limit set of a convex co-compact Schottky group there is a correspondence between zeros of the zeta function and scattering resonances of the classically trapped set (see [15, 31, 42, 46] for details).

In the case of the Julia set, we are primarily interested in counting zeros of the zeta function Z, which may be interpreted as the Pollicott-Ruelle resonances of this dynamical system. The most interesting case is the number of zeros in regions $\operatorname{Re} s > -C,\ |\operatorname{Im} s| < r$, which we will show is bounded above by $Cr^{1+\delta}$, with δ the dimension of the Julia set. In [6] lower bounds are considered. There is a weak, sublinear lower bound on the number of zeros in this region, as well as an honest linear lower bound in logarithmic neighbourhoods of the imaginary axis.

Similar to [42, 15], we consider the dynamical system associated to a hyperbolic rational map f when the Julia set $\mathcal{J}$ associated to this map is a totally disconnected set.

We think of $\mathcal{J}$ as a subset of the sphere $\widehat{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ naturally identified with $\widehat{\mathbb{R}}^2 = \mathbb{R}^2\bigcup\{\infty\}$. Then $|f'(z)|$ can be thought of as a map $|f'(z)| : U \subset \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}$, analytic in a neighbourhood U of $\mathcal{J}$. If $[f'(z)]$ is the holomorphic extension of $|f'(z)|$ to a map $[f'(z)] : \widetilde{U} \subset \widehat{\mathbb{C}}^2 = \mathbb{C}^2\bigcup\{\infty\} \to \widehat{\mathbb{C}}$, where $\widetilde{U} \subset \widehat{\mathbb{C}}^2$ is a neighbourhood of $\mathcal{J} \times \{0\}$, we can define the transfer operator:

$$\mathcal{L}(s)u(z) = \sum_{w\in f^{-1}(z)} [f'(w)]^{-s}\, u(w). \tag{4.1}$$

We will show in the remainder of this paper that $\mathcal{L}(s)$ is a trace class operator on an appropriately chosen class of functions, and the dynamical zeta function Z will be defined as

$$Z(s) = \det\left(I - \mathcal{L}(s)\right). \tag{4.2}$$

The zeta function defined in a similar context was first studied in the famous work of Ruelle [32]. We will prove the following bound of Z in terms of δ, the Hausdorff dimension of $\mathcal{J}$:

Theorem 1. *Suppose $Z(s)$ is the zeta function defined by (4.2) for the function f. Then for any C_0, there exists C_1 such that for $|\operatorname{Re} s| \leq C_0$ we have*

$$|Z(s)| \leq C_1 \exp(C_1\, |s|^{\delta}), \quad \delta = \dim \mathcal{J} \tag{4.3}$$

where δ is the dimension of the Julia set of f.

In [42] the same result is given for the case when $f(z) = z^2 + c$ for c real, $c < -2$, in which case the Julia set is a real Cantor-type set. Numerical results in [42] suggest this theorem is sharp. Using Theorem 1 and a dynamical formula for $Z(s)$ from Proposition 4.7, we derive an upper bound on the number of zeros in strips.

We remark that for a simple case of $f(z) = z^2 + c$, $c < -2$, the zeta function is given by

$$Z(s) = \exp\left(-\sum_{n=1}^{\infty} n^{-1} \sum_{f^n(z)=z} \frac{|(f^n)'(z)|^{-s}|}{1 - |(f^n)'(z)|^{-1}} \right), \quad \operatorname{Re} s \gg 0, \tag{4.4}$$

see Proposition 4.7 below for the well known general statement.

4.2. **The transfer operator on L^2 spaces.** It is more convenient to define the Ruelle transfer operator in terms of the inverse branches to f. Suppose f is an m to 1 function, and let $g_i(z)$ for $i = 1, 2, \ldots, m$ be the branches of f^{-1}. Now we interpret $\mathcal{J}$ as a subset of $\widehat{\mathbb{R}}^2$ instead of $\widehat{\mathbb{C}}$ and view $g_i : U \subset \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}^2$ real analytic and $|g_i'| : U \subset \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}$ analytic in a neighbourhood U about $\mathcal{J}$. Then it is clear that both g_i and $|g_i'|$ extend holomorphically to functions $g_i : \widetilde{U} \subset \widehat{\mathbb{C}}^2 \to \widehat{\mathbb{C}}^2$ and $[g_i'] : \widetilde{U} \subset \widehat{\mathbb{C}}^2 \to \widehat{\mathbb{C}}$ for $\widetilde{U} \subset \widehat{\mathbb{C}}^2$ a neighbourhood of $\mathcal{J} \times \{0\}$. The Ruelle transfer operator can then be defined as

$$\mathcal{L}(s)u(z) = \sum_{i=1}^{m} [g_i'(z)]^s \, u(g_i(z)). \tag{4.5}$$

We will show that with an appropriately chosen neighbourhood about $\mathcal{J} \subset \widehat{\mathbb{C}}^2$ and an appropriately chosen class of functions u, $\mathcal{L}$ is trace class. Now we want to define the Hilbert space we will be working with. For $D \subset \widehat{\mathbb{C}}^2$ open, let

$$H^2(D) := \left\{ u \text{ holomorphic in } D : \int_D |u(z)|^2 dm(z) < \infty \right\} ,$$

We can take for D disjoint neighbourhoods of $\mathcal{J}_i = g_i(\mathcal{J})$ for $i = 1, 2, \ldots, m$ and we get the following:

Proposition 4.1. *Suppose that $\mathcal{L}(s) : H^2(D) \to H^2(D)$ is defined by (4.5), with g_i the m inverse branches of f hyperbolic rational. Then for all $s \in \mathbb{C}$, $\mathcal{L}(s)$ is trace class and*

$$|\det(I - \mathcal{L}(s))| \leq C \exp(C|s|^3) \tag{4.6}$$

for some constant C.

Proof. We write $H^2(D) = \bigoplus_{j=1}^{m} H^2(D_j)$ and $\mathcal{L}(s) = \bigoplus_{i,j=1}^{m} \mathcal{L}_{ij}(s)$, where

$$\mathcal{L}_{ij}(s)u(z) := [g_i'(z)]^s \, u(g_i(z)), \;\; z \in D_j. \tag{4.7}$$

Note that from (A.8) and (A.10) we have

$$\nu_k(\mathcal{L}(s)) \leq m^2 \max_{1 \leq i,j \leq m} \nu_{[k/2m]}(\mathcal{L}_{ij}(s)).$$

Let $r_0 > 0$ be the minimum radius for which $|Dg_i(z)| < 1$, $i = 1, 2, \ldots, m$, on a ball of radius r_0 centered at a point of $\mathcal{J}$. Let

$$U = \bigcup_{i=1}^{m} U_i := \bigcup_{i=1}^{m} \{ \mathcal{J}_i + B_{\widehat{\mathbb{C}}^2}(0, r) \}$$

for $r < r_0/2$. Let $M = \max_{\overline{U}} |Dg_i(z)| < 1$, and pick for D a finite cover of $\mathcal{J}$, $D = \bigcup_{i=1}^m D_i$ made up of balls of radius r centered at points of $\mathcal{J}$ as above, so that for each $z \in \mathcal{J}$, $d_{\widehat{\mathbb{C}}^2}(z, \partial D) \geq \frac{1+M}{2} r$, and D_i covers $\mathcal{J}_i = g_i(\mathcal{J})$. Then for any point $z \in D_i$, $|z - w| < r$

for some $w \in \mathcal{J}$, so $|g_j(z) - g_j(w)| \leq Mr$, so that $d_{\widehat{\mathbb{C}}^2}(g_j(D_i), \partial D_j) \geq \left(\frac{1+M}{2} - M\right) r > 0$. Lemmas 4.2 and 4.3 together with the estimate $|[g_i'(z)]^s| \leq e^{C|s|}$ now give for some C_1:

$$\nu_l(\mathcal{L}_{ij}(s)) \leq C_1 e^{C|s| - l^{1/2}/C_1}.$$

With this in hand, we see that (A.7) implies

$$\det(I - \mathcal{L}(s)) \leq \prod_{l=0}^{\infty} \left(1 + Ce^{C|s| - l^{1/2}/C}\right) \leq Ce^{C^5|s|^3}$$

so that $\mathcal{L}(s)$ is trace class as claimed. To finish with the proposition, we need the following two lemmas, taken almost directly from [15]. □

Lemma 4.2. *Let $\rho \in (0,1)$ and $R^\rho : H^2(B_{\mathbb{C}^2}(0,1)) \to H^2(B_{\mathbb{C}^2}(0,\rho))$ induced by the restriction map of $B_{\mathbb{C}^2}(0,1)$ to $B_{\mathbb{C}^2}(0,\rho)$. Then for any $\widetilde{\rho} \in (\rho, 1)$ there exists a constant C such that*

$$\nu_l(R^\rho) \leq C\widetilde{\rho}^{l^{1/2}}.$$

Proof. Using (A.2) with the standard basis $(x_\alpha)_{\alpha \in \mathbb{N}^2}$ for $H^2(B_{\mathbb{C}^2}(0,1))$ given by

$$x_\alpha(z) = c_\alpha z_1^{\alpha_1} z_2^{\alpha_2}, \quad \int_{B_{\mathbb{C}^2}(0,1)} |x_\alpha(z)|^2 dz = 1, \ \ \alpha \in \mathbb{N}^2 \tag{4.8}$$

for which we have

$$\|R^\rho(x_\alpha)\|^2 = \int_{B_{\mathbb{C}^2}(0,\rho)} |x_\alpha(z)|^2 dz = \rho^{2|\alpha|+4}.$$

The number of α's for which $|\alpha| \leq m$ is bounded by $(m+1)^2$, so by (A.2)

$$\nu_l(R^\rho) \leq \sum_{|\alpha| \geq l} \rho^{|\alpha|+2} \leq C \sum_{k \geq l^{1/2}} (k+1)^2 \rho^k \leq C\widetilde{\rho}^{l^{1/2}}.$$

□

Lemma 4.3. *Suppose $\Omega_j \subset \mathbb{C}^2$, $j = 1, 2$ are open sets and $\Omega_1 = \bigcup_{k=1}^K B_{\mathbb{C}^2}(z_k, r_k)$. Let g be a holomorphic mapping defined on a neighbourhood $\widetilde{\Omega}_1$ of Ω_1 taking values in Ω_2 satisfying*

$$d_{\mathbb{C}^2}(g(\Omega_1), \partial\Omega_2) > \frac{1}{C_0} > 0, \quad 0 < \|Dg(z)\| < 1, \ z \in \widetilde{\Omega}_1.$$

If

$$A : H^2(\Omega_2) \to H^2(\Omega_1), \quad Au(z) := u(g(z)), \ z \in \Omega_1$$

then for some C_1 depending only on K, $d_{\mathbb{C}^2}(g(\Omega_1), \partial\Omega_2)$, $\sup_{\widetilde{\Omega}_1} \|Dg(z)\|$, we have

$$\nu_l(A) \leq C_1 e^{-l^{1/2}/C_1}$$

where $\nu_l(A)$'s are the characteristic values of A.

Proof. Define a new Hilbert space

$$\mathcal{H} := \bigoplus_{k=1}^{K} H^2(B_k), \quad B_k = B_{\mathbb{C}^2}(z_k, r_k),$$

and a natural operator

$$J : H^2(\Omega_1) \to \mathcal{H}, \quad (Ju)_k = u|_{B_k}$$

We claim $J^*J : H^2(\Omega_1) \to H^2(\Omega_1)$ is invertible, with constants depending only on K. To see this, note that for any $u \in H^2(\Omega_1)$,

$$\|u\|^2 \leq \langle Ju, Ju \rangle_{\mathcal{H}} \leq K \|u\|^2 .$$

Hence

$$\|J^*Ju\|^2 = \langle Ju, JJ^*Ju \rangle_{\mathcal{H}} \leq K \|u\| \|J^*Ju\| \tag{4.9}$$

and

$$\|u\|^2 \leq \langle J^*Ju, u \rangle_{H^2(\Omega_1)} \leq \|J^*Ju\| \|u\| \tag{4.10}$$

for any $u \in H^2(\Omega_1)$. The estimate (4.9) implies J^*J is bounded, while the estimate (4.10) implies J^*J is one-to-one. Since any one-to-one self-adjoint operator is also onto, J^*J is invertible, and furthermore,

$$\frac{1}{K} \|u\|^2 \leq \left\| (J^*J)^{-1} u \right\| \leq \|u\|^2.$$

Thus we calculate,

$$\nu_l(A) = \nu_l((J^*J)^{-1} J^*JA) \leq \left\| (J^*J)^{-1} \right\| \|J^*\| \, \nu_l(JA).$$

Note then that

$$\nu_l(JA) \leq K \max_{1 \leq k \leq K} \nu_{[l/K]}(A_k),$$

where

$$A_k : H^2(\Omega_2) \to H^2(B_k), \;\; A_k u(z) = u(g_k(z)), \;\; g_k = g|_{B_k}$$

In order to estimate the characteristic values for A_k, note we can extend g_k to a larger ball in $\widetilde{\Omega}_1$, $\widetilde{B}_k$ such that the image of its closure is still in Ω_2. That gives us the operators $R_k : H^2(\widetilde{B}_k) \to H^2(B_k)$, $R_k u = u|_{B_k}$, and $\widetilde{A}_k$ defined similarly to A_k with B_k's replaced with $\widetilde{B}_k$'s. Now we have $A_k = R_k \widetilde{A}_k$ which implies

$$\nu_l(A_k) \leq \left\| \widetilde{A}_k \right\| \nu_l(R_k).$$

Lemma 4.2 gives $\nu_l(R_k) \leq C_2 e^{-l^{1/2}/C}$. To see these constants don't depend on the r_k's, note that the proof of Lemma 4.2 scales to the case of $R^\rho : B(0, r_k) \to B(0, \rho r_k)$ without modifying the constants, and this completes the proof. ∎

4.3. **Estimates in terms of the dimension of $\mathcal{J}$.** In order to prove Theorem 1 we need a few more important facts. Recall that the diameter of a set E is defined as $\operatorname{diam}(E) = \sup\{|x-y| : x, y \in E\}$.

Proposition 4.4. *Let $\mathcal{J} \in \widehat{\mathbb{C}}$ be the Julia set for f hyperbolic rational, and assume $\mathcal{J}$ is totally disconnected. Then there exist constants $K = K(c)$ and δ_0 such that for $\delta < \delta_0$ the connected components of $\mathcal{J} + \overline{B}_{\widehat{\mathbb{C}}}(0, \delta)$ have diameter at most $K\delta$.*

Proof. Let c and r_0 be as in Proposition B.1. Since $\mathcal{J}$ is totally disconnected, there exists $\epsilon_0 > 0$ such that $\widehat{\mathcal{J}} = \mathcal{J} + B(0, \epsilon_0)$ has more than one connected component, and every connected component of $\widehat{\mathcal{J}}$ has diameter at most $(4c)^{-1}$. Then we apply Proposition B.1 with $r = c\delta\epsilon_0^{-1}$, with $\delta \leq \delta_0 < r_0\epsilon_0 c^{-1}$. The function g guaranteed by Proposition B.1 takes points in $\mathcal{J}$ to points in $\mathcal{J}$, so if $z \in \mathcal{J}$, $g(B(z, \delta)) \subset B(g(z), \epsilon_0)$. Thus a connected component of $\mathcal{J} + B(0, \delta)$ is mapped into a connected component of $\widehat{\mathcal{J}}$. Now suppose $d(z, w) > r/2$. Then

$$d(g(z) - g(w)) \geq \frac{1}{cr} d(z, w) \geq \frac{1}{2c} > \frac{1}{4c}$$

so that $g(z)$ is in a different connected component from $g(w)$. Hence z and w must have been in separate connected components, and we conclude the diameter of the connected component containing z is at most $K\delta = r$. □

We have a bound on the diameter of the connected components of $\mathcal{J} + \overline{B}_{\widehat{\mathbb{C}}}(0, \delta)$, but eventually we will need to cover $\mathcal{J}$ by balls, uniformly finite in δ so that we may again apply Lemma 4.3.

Lemma 4.5. *Suppose $D \subset \widehat{\mathbb{C}}^2$ is a compact set with the property that all connected components of $E = D + B_{\widehat{\mathbb{C}}^2}(0, \delta)$ have diameter bounded by $K\delta$. Then for any $\lambda \in (0, 1)$ and any connected component E_i of E, there exists a cover $U_i = U_i(\delta) \subset E_i$ of $D_i = E_i \bigcap D$ by at most $K' = K'(\lambda)$ balls of radius δ centered at points of D_i such that $d_{\widehat{\mathbb{C}}^2}(z, \partial U_i) \geq \lambda\delta$ for $z \in D_i$.*

Proof. Let $l = (1-\lambda)/2$. If E_i is a connected component of E, then it fits in a closed ball of diameter $K\delta$ by hypothesis. A ball of diameter $K\delta$ is contained in a closed cube Q of side length $K\delta$, which can be covered by $K'(\lambda)$ closed cubes of side length $l\delta$ by starting at one corner of Q and covering it with cubes $\{q_k\}$ of side length $l\delta$ intersecting only on their boundaries. For each k, if $D \bigcap q_k \neq \emptyset$, select any point $p_k \in D \bigcap q_k$; if the intersection is empty, select nothing. Then set $U_i = \bigcup_{k=1}^{K'} B_{\widehat{\mathbb{C}}^2}(p_k, \delta)$. A simple calculation gives for any $z \in D_i$, $z \in q_k$ for some k, giving

$$d_{\widehat{\mathbb{C}}^2}(z, \partial U_i) \geq d_{\widehat{\mathbb{C}}^2}(p_k, \partial U_i) - d_{\widehat{\mathbb{C}}^2}(p_k, z) \geq \lambda\delta$$

since $z \in q_k$ implies, in particular, that $d_{\widehat{\mathbb{C}}^2}(p_k, z) \leq 2l\delta$. □

Proof of Theorem 1. As in [42] choose $h = |s|^{-1}$ where $|s|$ is large, but $|\mathrm{Re}\,(s)|$ is bounded. Now viewing $\mathcal{J}$ as a subset of $\widehat{\mathbb{R}}^2$ instead of $\widehat{\mathbb{C}}$, form $\widetilde{\mathcal{J}}(h) = \mathcal{J} + B_{\widehat{\mathbb{C}}^2}(0,h)$. Proposition 4.4 tells us the diameter of each connected component of $\widetilde{\mathcal{J}}(h)$ has diameter less than Kh. Since g_i (now thought of as the holomorphic extension $g_i : \widehat{\mathbb{C}}^2 \to \widehat{\mathbb{C}}^2$) is a contraction near $\mathcal{J}$, there is some h_0 so that if $h < h_0$, $M = \max |Dg_i(z)| < 1$ for z in the closure of $\mathcal{J} + B_{\widehat{\mathbb{C}}^2}(0,h)$. Let $\beta = \frac{1}{2}(M+1)$, and suppose there are $P(h)$ connected components of $\widetilde{\mathcal{J}}(h)$. Using Lemma 4.5 we can pick a subcover $U(h) = \bigcup_{i=1}^m U_i(h)$ of at most $K'P(h)$ balls contained in $\widetilde{\mathcal{J}}$ and centered at points of $\mathcal{J}$ satisfying

$$d_{\widehat{\mathbb{C}}^2}(z, \partial U) \geq \beta h, \quad z \in \mathcal{J}.$$

Since any point of U is within h of some $z \in \mathcal{J}$ and we know $g_i : \mathcal{J}_j \to \mathcal{J}_i$,

$$d_{\widehat{\mathbb{C}}^2}(g_i(U_j(h)), \partial U_i(h)) \geq (\beta - M)h > C^{-1}h$$

for some constant C independent of h.

It is classical that the Hausdorff measure of the Julia set is finite (see [43] and the references therein) and that the Hausdorff dimension equals the box-counting dimension. Using the setup of the box-counting dimension, let $N(\epsilon)$ be the number of sets of diameter ϵ needed to cover $\mathcal{J}$. With

$$\delta = -\lim_{\epsilon \to 0^+} \frac{\log N(\epsilon)}{\log \epsilon},$$

$P(h) = N(Kh)$ implies $P(h) = \mathcal{O}(h^{-\delta})$. We write $\mathcal{L}(s)$ as a sum of m^2 operators $\mathcal{L}_{ij}(s)$ as before, $\mathcal{L}_{ij}(s) : H^2(U_i) \to H^2(U_j)$, but now we will see we have a better bound on the weight independent of h. Recall $[g_i'(z)] : \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}$, and we are only interested in values of $[g_i'(z)]$ on $U_j(h)$, so $|\arg[g_i'(z)]| \leq |\mathrm{Im}\, z| \leq h = |s|^{-1}$. Hence,

$$\begin{aligned} |[g_i'(z)]^s| &\leq C \exp(|s| |\arg [g_i'(z)]|) \\ &\leq \exp\left(C_1 |s| \left(|\mathrm{Im}\,(z_1)|^2 + |\mathrm{Im}\,(z_2)|^2\right)^{1/2}\right) \\ &\leq C_2, \quad z \in U_j(h). \end{aligned} \tag{4.11}$$

Each $\mathcal{L}_{ij}(s)$ is a sum of no more than $P(h)$ operators, each of which satisfies $\nu_l \leq C\alpha^{l^{1/2}/C}$ for some $0 < \alpha < 1$ by Lemma 4.3. Thus using again (A.8) and (A.10) we get the estimate

$$\log |\det (I - \mathcal{L}(s))| \leq CP(h) = \mathcal{O}(h^{-\delta}) \tag{4.12}$$

which is (4.3). □

4.4. **Counting Zeros in Strips.** In this section we prove the following corollary to Theorem 1. The methods used here are similar to those used in [42] and [18].

Corollary 4.6. *Let $m(s)$ denote the multiplicity of a zero of $Z(s)$ at s. Then*

$$\sum \{m(s) : r \leq |\mathrm{Im}\, s| \leq r+1, \quad \mathrm{Re}\, s > -C_0\} \leq C_1 r^\delta \tag{4.13}$$

where $\delta = \dim \mathcal{J}$.

In order to prove this corollary, we will need to bound $Z(s)$ away from zero for $\operatorname{Re} s \geq C_0$. We do this by employing a dynamical formula for $Z(s)$ which is interesting in its own right. For the development of this dynamical formula, we take D_i to be $\widehat{\mathbb{C}}^2$-balls containing $\mathcal{J} \subset \widehat{\mathbb{R}}^2$. We again view f as a map $f : \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}^2$ and then extend to a holomorphic function $\widehat{\mathbb{C}}^2 \to \widehat{\mathbb{C}}^2$ and write f for this extension whenever unambiguous.

Proposition 4.7. *For* $\operatorname{Re}(s) \gg 0$,

$$\det(I - \mathcal{L}(s)) = \exp\left(- \sum_{n=1}^{\infty} \frac{1}{n} \sum_{f^n(z)=z} \frac{[(f^n)'(z)]^{-s}}{\left|\det(I - (d\,(f^n)\,(z))^{-1})\right|} \right). \tag{4.14}$$

Proof. For $|\lambda|$ sufficiently small, $\log(I - \lambda\mathcal{L}(s))$ is well defined and

$$\det(I - \lambda\mathcal{L}(s)) = \exp\left(- \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \operatorname{tr}(\mathcal{L}(s)^n) \right).$$

In order to evaluate the traces, we write

$$\operatorname{tr} \mathcal{L}(s)^n = \sum_{(i_1,\dots,i_{n+1})} \operatorname{tr}\left(\mathcal{L}_{i_1 i_2}(s) \circ \cdots \circ \mathcal{L}_{i_n i_{n+1}}(s) \right),$$

where $\mathcal{L}_{ij}(s)$ is given by (4.7). If the target space is different from the domain space, there are no eigenvalues, so that

$$\sum_{(i_1,\dots,i_{n+1})} \operatorname{tr}\left(\mathcal{L}_{i_1 i_2}(s) \circ \cdots \circ \mathcal{L}_{i_n i_{n+1}}(s) \right) = \sum_{(i_1,\dots,i_n)} \operatorname{tr}\left(\mathcal{L}_{i_1 i_2}(s) \circ \cdots \circ \mathcal{L}_{i_n i_1}(s) \right).$$

We have

$$\mathcal{L}_{i_1 i_2}(s) \circ \cdots \circ \mathcal{L}_{i_n i_1}(s) u(z) = [(g_{i_1} \circ \cdots \circ g_{i_n})'(z)]^s \, u(g_{i_1} \circ \cdots \circ g_{i_n}(z)),$$

and Lemma C.1 in the appendix shows that

$$\operatorname{tr}\left((g_{i_1} \circ \cdots \circ g_{i_n})^* \right) = \frac{1}{|\det(I - d\,(g_{i_1} \circ \cdots \circ g_{i_n})(z))|}$$

which completes the proof once we put $\lambda = 1$. □

Proof of Corollary 4.6. Using (4.14) it is clear that for $\operatorname{Re} s \geq C_0$ we have

$$|[(f^n)'(z)]^{-s}| \leq C C_1^{-n\operatorname{Re}(s)}$$

with $C_1 > 1$ since z is a periodic repeller and $[(f^n)'(z)]$ is real on $\mathcal{J}$. Then the convergence of the double series in (4.14) is immediate and gives for $\operatorname{Re} s \geq C_0$, $Z(s) \geq 1/2$. With Z

zero free for $\operatorname{Re} s \geq C_0$, an application of the Jensen formula shows the left hand side of (4.13) is bounded by

$$\begin{aligned} \sum \{m(s) : |s - ir - C_0| \leq 2(C_2 + C_0)\} &\leq C_3 \max_{|s-ir-C_0|\leq 4C_2} \log|Z(s)| \\ &\leq C \max_{\substack{|\operatorname{Re} s|\leq C_3 \\ |s|\leq 4C_2+r}} \log|Z(s)| \\ &\leq C_1 r^\delta. \end{aligned}$$

□

4.5. **Final Comments.** Experimental evidence in [42] suggests Corollary 4.6 is sharp. However, as is common with this type of estimates, sharp lower bounds have remained elusive. In order to illustrate the subtlety of this question, we will look at the following example.

Assume for simplicity that $f(z) = z^2 + c$ for c real, $c < -2$, and that A/B is irrational, with $1 < A = \min_{\mathcal{J}} |f'(z)| < B = \max_{\mathcal{J}} |f'(z)|$ as before. Then $\mathcal{J}$ is a Cantor-like set in the real line and all the proofs above go through by complexifying to $\mathbb{C}$ instead of $\mathbb{C}^2$. It is well known that the distribution of the $L_n(z)$s is Gaussian with concentration at $\log(AB)^{1/2}$. This suggests a simple model for the zeta function. With A and B as above, we model the distribution of the $L_n(z)$s in the following fashion. We write $L_n(z) = kl_1 + (n-k)l_2$ with multiplicity $\binom{n}{k}$, where we have set $l_1 = \log A$ and $l_2 = \log B$. Using (4.14) as a basis, we calculate

$$\begin{aligned} & -\sum \frac{1}{n} \sum_{f^n(z)=z} \frac{\exp(-sL_n(z))}{(1-\exp(-L_n(z)))} = \\ &= -\sum_n \frac{1}{n} \sum_{f^n(z)=z} \sum_k \exp(-(s+k)L_n(z)) \\ &= -\sum_n \frac{1}{n} \sum_k \sum_m \binom{n}{m} (\exp(-(s+k)l_1))^m (\exp(-(s+k)l_2)^{n-m} \\ &= -\sum_n \frac{1}{n} \sum_k (\exp(-(s+k)l_1) + \exp(-(s+k)l_2))^n \\ &= -\sum_k \log\left(1 - e^{-(s+k)l_1} - e^{-(s+k)l_2}\right) \\ &= \log \prod_k \left(1 - e^{-(s+k)l_1} - e^{-(s+k)l_2}\right) \end{aligned}$$

so we set

$$\widetilde{Z}(s) = \prod_k \left(1 - A^{-(s+k)} - B^{-(s+k)}\right). \tag{4.15}$$

This model shares some important features with $Z(s)$. First, it has one zero at $s = \delta$, where δ, solving $A^\delta + B^\delta = 1$, is the "dimension". Second, it is easy to see that, since A/B is irrational, there are no other zeros on $\operatorname{Re} s = \delta$. However, if $\operatorname{Re} s > -C$, we can take

$$\widetilde{Z}(s) \sim C \prod_{k=0}^{K} \left(1 - A^{-(s+k)} - B^{-(s+k)}\right)$$

for some K. Then as $|s| \to \infty$, $\widetilde{Z}(s) \sim Ce^{C|s|}$, whence the number of zeros in $\{\operatorname{Re} s > -C, |\operatorname{Im} s| \leq r\}$ grows linearly.

Appendix A

In this appendix we recall some general facts about determinants and characteristic values and present the proof of Weyl inequalities.

Let H_j's be complex Hilbert spaces and $A : H_1 \to H_2$ a compact operator. Define

$$\|A\| = \nu_0(A) \geq \nu_1(A) \geq \cdots \geq \nu_l(A) \to 0$$

to be the eigenvalues of $(A^*A)^{\frac{1}{2}} : H_1 \to H_1$. The min-max principle shows that

$$\nu_l(A) = \min_{\substack{V \subset H_1 \\ \operatorname{codim} V = l}} \max_{\substack{v \in V \\ \|v\|_{H_1} = 1}} \|Av\|_{H_2} \tag{A.1}$$

Now suppose $\{x_j\}_{j=0}^{\infty}$ is an orthonormal basis of H_1, then

$$\nu_l(A) \leq \sum_{j=l}^{\infty} \|Ax_j\|_{H_2}. \tag{A.2}$$

To see this we use $V_l = \operatorname{span}\{x_j\}_{j=l}^{\infty}$ in (A.1): for $v \in V_l$ we have, by the Cauchy-Schwartz inequality, and the inequality $\| \cdot \|_{\ell_2} \leq \| \cdot \|_{\ell_1}$,

$$\|Av\|_{H_2}^2 = \left\| \sum_{j=l}^{\infty} \langle v, x_j \rangle_{H_1} Ax_j \right\|^2 \leq \|v\|_{H_1}^2 \left(\sum_{j=l}^{\infty} \|Ax_j\|_{H_2} \right)^2,$$

from which (A.1) gives (A.2).

The followig Lemma is the *Weyl inequality*. The proof is from [13]; see [37] for a slightly different approach.

Lemma A.1. *Suppose $H_1 = H_2 = H$ and $\lambda_j(A)$ are the eigenvalues of A,*

$$|\lambda_0(A)| \geq |\lambda_1(A)| \geq \cdots \geq |\lambda_l(A)| \to 0. \tag{A.3}$$

Then for any N,

$$\prod_{l=0}^{N} (1 + |\lambda_l(A)|) \leq \prod_{l=0}^{N} (1 + |\nu_l(A)|). \tag{A.4}$$

The proof of Lemma A.1 will follow from the following two lemmas.

Lemma A.2. *With the notation above,*

$$|\lambda_1\lambda_2\cdots\lambda_N| \leq |\nu_1\nu_2\cdots\nu_N| \,. \tag{A.5}$$

Proof. Let $\{\phi_j\}_{j=1}^N$ be the corresponding eigenvectors for the λ_j, and let V_N be the sum of the generalized eigenspaces of the ϕ_j. Let $A_N = A|_{V_N}$. Then

$$|\lambda_1\cdots\lambda_N| = \left|\det_H A_N\right| = \left|\det_H (A_N^* A_N)\right|^{1/2}. \tag{A.6}$$

If μ_l is a characteristic value of A_N, then

$$\begin{aligned} \mu_l &= \min_{\substack{V\subset H \\ \text{codim } V = l}} \max_{\substack{v\in V \\ \|v\|=1}} \|(A_N^* A_N)^{1/2} v\| \\ &\leq \min_{\substack{V\subset H \\ \text{codim } V = l}} \max_{\substack{v\in V \\ \|v\|=1}} \|(A^* A)^{1/2} v\| \\ &= \nu_l(A). \end{aligned}$$

Thus the right hand side of (A.6) is bounded by the right hand side of (A.5). □

Lemma A.3. *Suppose* $\Phi : \mathbb{R} \to \mathbb{R}$ *is a* C^2 *convex function satisfying* $\lim_{x\to-\infty}\Phi(x) = \lim_{x\to-\infty}\Phi'(x) = 0$. *Let* $\{a_j\}$ *and* $\{b_j\}$ *be two nonincreasing sequences of real numbers satisfying*

$$\sum_{j=1}^N a_j \leq \sum_{j=1}^N b_j$$

for each N. *Then for each* N,

$$\sum_{j=1}^N \Phi(a_j) \leq \sum_{j=1}^N \Phi(b_j).$$

Proof. Using integration by parts, we can write

$$\Phi(x) = \int_{-\infty}^{\infty} (x-u)_+ \Phi''(u)du.$$

Then

$$\sum_1^N \Phi(a_j) = \int_{-\infty}^{\infty} A_N(x)\Phi''(x)dx,$$

where $A_N(x) = \sum_1^N (a_j - x)_+$, and similarly

$$\sum_1^N \Phi(b_j) = \int_{-\infty}^{\infty} B_N(x)\Phi''(x)dx,$$

for $B_N(x) = \sum_1^N (b_j - x)_+$. We claim $A_N(x) \leq B_N(x)$. If $x \leq \min(a_N, b_N)$ or $x \geq b_1$ this is clear, so suppose $a_{q+1} \leq x \leq a_q$ and $b_{p+1} \leq x \leq b_p$. If $p \geq q$, then

$$A_N(x) = \sum_1^q a_j - qx \leq \sum_1^q b_j - qx + (b_{q+1} - x) + \cdots + (b_p - x) = B_N(x).$$

If $p < q$, then

$$\begin{aligned} A_N(x) &= \sum_1^q a_j - qx \\ &\leq \sum_1^q a_j - qx - (b_{p+1} - x) - (b_{p+2} - x) - \cdots - (b_q - x) \\ &\leq \sum_1^q b_j - px - \sum_{p+1}^q b_j \\ &= B_N(x). \end{aligned}$$

Then the lemma follows after recalling $\Phi''(x) \geq 0$. □

Proof of Lemma A.1. The function $\Phi(x) = \log(1 + e^x)$ satisfies the hypotheses of Lemma A.3, and with $a_j = \log|\lambda_j|$ and $b_j = \log \nu_j$, we get for any N

$$\sum_1^N \log(1 + |\lambda_j|) \leq \sum_1^N \log(1 + \nu_j).$$

Thus

$$\log \prod_1^N (1 + |\lambda_j|) \leq \log \prod_1^N (1 + |\lambda_j|),$$

and the lemma is proved. □

If A is trace class, i.e. if $\sum_l \nu_l(A) < \infty$, then the determinant

$$\det(I + A) := \prod_{l=0}^\infty (1 + \lambda_l(A)),$$

is well defined and

$$|\det(I + A)| \leq \prod_{l=0}^\infty (1 + \nu_l(A)). \tag{A.7}$$

We also need the following standard inequality about characteristic values.

Lemma A.4. *With the notation as above,*

$$\nu_{l_1+l_2}(A + B) \leq \nu_{l_1+1}(A) + \nu_{l_2}(B). \tag{A.8}$$

Proof. Fix $\epsilon > 0$, and let V_1 and V_2 be two subspaces of H so that

$$\nu_{l_1}(A) + \nu_{l_2}(B) + \epsilon \geq \max_{\substack{v \in V_1 \\ \|v\|=1}} \|Av\| + \max_{\substack{v \in V_2 \\ \|v\|=1}} \|Bv\| \geq \max_{\substack{v \in V_1 \cap V_2 \\ \|v\|=1}} (\|Av\| + \|Bv\|). \tag{A.9}$$

Note $V_1 \cap V_2$ has codimension greater than or equal to $l_1 + l_2$, so that the right hand side of (A.9) is greater than or equal to

$$\min_{\substack{V \subset H \\ \operatorname{codim} V = l_1 + l_2}} \max_{\substack{v \in V \\ \|v\|=1}} \|(A+B)v\|,$$

and as ϵ was arbitrary, the lemma is proved. □

Finally, we finish with an obvious equality: suppose $A_j : H_{1j} \to H_{2j}$ and we form $\bigoplus_{j=1}^J A_j : \bigoplus_{j=1}^J H_{1j} \to \bigoplus_{j=1}^J H_{2j}$, then

$$\sum_{l=0}^{\infty} \nu_l \left(\bigoplus_{j=1}^{J} A_j \right) = \sum_{j=1}^{J} \sum_{l=0}^{\infty} \nu_l(A_j). \tag{A.10}$$

Appendix B

In this appendix we present basic results about Julia sets for hyperbolic rational maps. All of them can be deduced from the following classical theorem which we recall without proof:

Montel's Theorem. *Let $\mathcal{F}_{a,b,c}$, $a, b, c \in \widehat{\mathbb{C}}$, be the family of all meromorphic functions on $\Omega \to \widehat{\mathbb{C}} \setminus \{a, b, c\}$, $\Omega \subset \widehat{\mathbb{C}}$. Then $\mathcal{F}_{a,b,c}$ is* normal.

The Julia set $\mathcal{J}$ for a rational map can be defined to be the closure of the set of repelling periodic points, hence $\mathcal{J}$ is compact in the sphere. It is easy to see [5] that $\mathcal{J}$ is backward and forward invariant: $\mathcal{J} = f(\mathcal{J}) = f^{-1}(\mathcal{J})$, and in fact $f^p(\mathcal{J}) = \mathcal{J}$ for $p = 1, 2, \ldots$. We are interested in the case where $\mathcal{J}$ is disconnected (and hence totally disconnected). The hypothesis that $\mathcal{J}$ be totally disconnected is necessary in what follows, as in the proof of the essential Proposition 4.4. In the simple setup where $f(z) = z^2 + c$ for $c \notin \mathcal{M}$, where $\mathcal{M}$ is the Mandelbrot set, it would be interesting to determine the behaviour of the zeta function as $\operatorname{dist}(c, \mathcal{M}) \to 0$. The assumption that f be hyperbolic means there exists an $n \geq 1$ such that $\inf\{|(f^n)'(z)| : z \in \mathcal{J}\} > 1$. In other words, some iterate of f is expanding on the whole set. A sometimes useful fact (see [43]) is that a rational function is hyperbolic if and only if $\overline{\mathrm{PCV}(f)} \bigcap \mathcal{J} = \emptyset$, where $\mathrm{PCV}(f) = \bigcup_{n \geq 0} f^n(\operatorname{Crit} f)$ is the forward propagation of the set of critical points of f. Note since f is hyperbolic, we can replace f with an appropriate iterate and assume that f is strictly expanding near $\mathcal{J}$, so we will do this throughout.

The most important properties of $\mathcal{J}$ are the those making it a "cookie-cutter" set in the sense of [11]. Roughly speaking, this is to mean that a small neighbourhood intersected with $\mathcal{J}$ looks more or less like $\mathcal{J}$. This is made precise in the following proposition:

Proposition B.1. *$\mathcal{J}$ is a cookie-cutter set, that is, there exist constants $c > 0$, $r_0 > 0$ such that for each $r < r_0$ and $z_0 \in \mathcal{J}$ there is a map $g : B(z_0, r) \to \widehat{\mathbb{C}}$ such that $g(B(z_0, r) \bigcap \mathcal{J}) \subset \mathcal{J}$ satisfying*

$$c^{-1} r^{-1} |z - w| \leq |g(z) - g(w)| \leq c r^{-1} |z - w| . \tag{B.1}$$

To prove this, we will need the Koebe distortion theorem (see [5] for a slightly different statement).

Lemma B.2 (Koebe distortion theorem)**.** *Let S be the set of univalent (analytic and one-to-one) functions g on the unit disk in $\mathbb{C}$ with $g(0) = 0$ and $g'(0) = 1$. Then S is normal.*

Proof. The proof is from [25]. Let $\{f_n\} \subset S$. Note that by Schwarz's Lemma, $f_n(D(0, 1))$ cannot contain $D(0, R)$ for any $R > 1$. For each n, select a_n and b_n such that $|a_n| = 1$, $|b_n| = 2$, and $a_n, b_n \notin f_n(D(0, 1)$. Set

$$g_n(z) = \frac{f_n(z) - a_n}{b_n - a_n},$$

and note that $g_n(z)$ cannot be equal to 0 or 1. Montel's theorem them implies we can pass to a subsequence convergent on $D(0, 1)$. Select a further subsequence so that $a_n \to a$ and $b_n \to b$, and note that then $g_n(z) \to g(z)$ for $g(z) = (f(z) - a)/(b - a)$ with $f \in S$. Thus $f_n \to f$. □

Corollary B.3. *There is $r > 0$ so that $f \in S$ implies $D(0, r) \subset f(D(0, 1))$.*

Proposition B.1 now follows directly from the Koebe distortion theorem and Corollary B.3.

Appendix C

We need the following well known result:

Lemma C.1. *Suppose $\Omega \subset \mathbb{R}^2$ is an open, bounded, convex domain, and $f : \Omega \to \Omega$ is an analytic contraction obtained from a holomorphic function on $\mathbb{C}$ identified with $\mathbb{R}^2$. Let $\tilde{f} : \widetilde{\Omega} \to \widetilde{\Omega}$ denote the extension of f to a holomorphic contraction on a bounded, pseudoconvex domain $\widetilde{\Omega} \subset \mathbb{C}^2$. Suppose z_1 is the unique fixed point of $\tilde{f}$. Then the pullback operator $\tilde{f}^* : H^2(\tilde{f}(\widetilde{\Omega})) \to H^2(\widetilde{\Omega})$ has trace*

$$\operatorname{tr} \tilde{f}^* = \frac{1}{\left| \det \left(I - d\tilde{f}(z_1) \right) \right|}. \tag{C.1}$$

We first prove this result in the case $\widetilde{\Omega}$ is a ball.

Lemma C.2. *Suppose $f : B_{\mathbb{R}^2}(z_0, r) \to B_{\mathbb{R}^2}(z_0, r')$ is an analytic contraction obtained from a holomorphic function on $\mathbb{C}$ identified with $\mathbb{R}^2$, and let $\tilde{f} : B_{\mathbb{C}^2}(z_0, r) \to B_{\mathbb{C}^2}(z_0, r')$ be the holomorphic extension of f to $\mathbb{C}^2$. If z_1 is the unique fixed point of $\tilde{f}$, then the pullback by $\tilde{f}$, $\tilde{f}^* : H^2(B_{\mathbb{C}^2}(z_0, r')) \to H^2(B_{\mathbb{C}^2}(z_0, r))$ has trace*

$$\operatorname{tr} \tilde{f}^* = \frac{1}{\left|\det\left(I - d\tilde{f}(z_1)\right)\right|}.$$

Proof. Without loss of generality, $z_0 = 0$, and $\tilde{f} : B_{\mathbb{C}^2}(0, 1) \to B_{\mathbb{C}^2}(0, \rho)$ for some $\rho \in (0, 1)$. Since the group $SU_{\mathbb{C}}(2, 1)$ acts transitively on the unit ball in $\mathbb{C}^2$, by composing with appropriate Möbius transformations we may also assume $z_1 = 0$ (see [1]). We first consider $f : B_{\mathbb{R}^2}(0, 1) \to B_{\mathbb{R}^2}(0, \rho)$. The assumption that f is obtained from a holomorphic function on $\mathbb{C}$ means for $z \in \mathbb{C}$,

$$d f(0) z = (a + ib)(x + iy) = (ax - by) + i(bx + ay)$$

for some $a, b \in \mathbb{R}$. But this implies $d f_{\mathbb{R}^2}(0)$ and hence $d\tilde{f}(0)$ has the very special form

$$d\tilde{f}(0) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad a, b \in \mathbb{R}.$$

Thus $d\tilde{f}(0)$ is always diagonalizable. Note then that if

$$d\tilde{f}(0) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 - bz_2 \\ bz_1 + az_2 \end{pmatrix},$$

then the change of variables

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} =$$

$$= \frac{1}{a^2 + b^2} \begin{pmatrix} a(a - ib) & -b(a - ib) \\ b(a + ib) & a(a + ib) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{C.2}$$

makes $d\tilde{f}(0)$ diagonal, and further, $\det A = 1$.

We have an orthonormal basis for $H^2(B_{\mathbb{C}^2}(0, 1))$ in the form $\{c_\alpha z^\alpha\}_{\alpha \in \mathbb{N}^2}$ for constants c_α. We can use the Bergman kernel to write the kernel for the pullback operator on $H^2(B_{\mathbb{C}^2}(0, 1))$,

$$K_{\tilde{f}^*}(z, s) = \sum_\alpha |c_\alpha|^2 (\tilde{f}(z))^\alpha \overline{s}^\alpha,$$

so that for each $u \in H^2(B_{\mathbb{C}^2}(0, 1))$,

$$\tilde{f}^* u(z) = \int_{B_{\mathbb{C}^2}(0,1)} K_{\tilde{f}^*}(z, s) u(s) dm(s).$$

Here $dm(s)$ denotes the usual Lebesgue measure on $\mathbb{C}^2$. We will use the change of variables (C.2) and the fact that $\tilde{f}^*$ is trace class to exchange the integral and sum in the following to get:

$$\begin{aligned}
\operatorname{tr} \tilde{f}^* &= \int_{B_{\mathbb{C}^2}(0,1)} K_{\tilde{f}^*}(z,z)dm(z) \\
&= \int_{B_{\mathbb{C}^2}(0,1)} \sum_\alpha |c_\alpha|^2 (\tilde{f}(z))^\alpha \overline{z}^\alpha dm(z) \\
&= \int_{B_{\mathbb{C}^2}(0,1)} \sum_\alpha |c_\alpha|^2 (d\tilde{f}(0)z + \mathcal{O}(|z|^2))^\alpha \overline{z}^\alpha dm(z) \\
&= \int_{B_{\mathbb{C}^2}(0,1)} \sum_\alpha |c_\alpha|^2 \left(\begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix} w + \mathcal{O}(|w|^2) \right)^\alpha \overline{w}^\alpha dm(w) \\
&= \sum_\alpha (a+ib)^{\alpha_1}(a-ib)^{\alpha_2} \\
&= \left| \det(I - d\tilde{f}(0)) \right|^{-1}.
\end{aligned}$$

□

Proof of Lemma C.1. Let B be the largest open ball with center at z_1 so that $B \subset \widetilde{\Omega}$. Since $\tilde{f}$ is a contraction and we can always replace f with an appropriate iterate if necessary, we may assume without loss of generality that $\tilde{f}(\widetilde{\Omega}) \subset B$. Now suppose u is a generalized eigenfunction of $\tilde{f}^*$ acting on $H^2(\tilde{f}(\widetilde{\Omega}))$ with nonzero eigenvalue λ. That is,

$$\left(\tilde{f}^* - \lambda\right)^k u = 0, \quad \text{but} \quad \left(\tilde{f}^* - \lambda\right)^{k-1} u \neq 0 \tag{C.3}$$

for some $k \in \mathbb{Z}_+$ and $\lambda \neq 0$. We claim u can be extended to an eigenfunction of $\tilde{f}^*$ acting on $H^2(B)$ with the same eigenvalue. Indeed, if (C.3) holds, we have

$$\left(\tilde{f}^* - \lambda\right)^k u = \left[\sum_{j=0}^k \binom{k}{j} (-1)^j \lambda^j \left(\tilde{f}^*\right)^{k-j} \right] u = 0,$$

which motivates setting

$$\tilde{u} := (-1)^{k+1} \lambda^{-k} \left[\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j \left(\tilde{f}^*\right)^{k-j} \right] u. \tag{C.4}$$

The lowest order pullback on the right hand side of (C.4) is order 1, and since $\tilde{f}(B) \subset \tilde{f}(\widetilde{\Omega})$, $\tilde{u}$ is in $H^2(B)$. As $(\tilde{f}^* - \lambda)$ commutes with $\lambda^j(\tilde{f}^*)^{k-j}$, we have

$$\left(\tilde{f}^* - \lambda\right)^k \tilde{u} = 0, \quad \text{but} \quad \left(\tilde{f}^* - \lambda\right)^{k-1} \tilde{u} \neq 0.$$

Lastly, if u is a generalized eigenfunction of $\tilde{f}^*$ acting on $H^2(B)$, clearly the restriction of u to $\tilde{f}(\widetilde{\Omega})$ is a generalized eigenfunction with the same eigenvalue for $\tilde{f}^*$ acting on $H^2(\tilde{f}(\widetilde{\Omega}))$. Then the trace of $\tilde{f}^*$ acting on $H^2(\tilde{f}(\widetilde{\Omega}))$ and $H^2(B)$ are the same and we can apply Lemma C.2 to get (C.1). □

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