Anosov Actions and Resonances. II.

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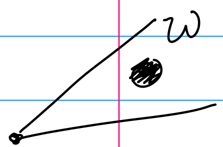
⇒ What information can we get from the first resonance?

Tobías: $\psi \in C_c^\infty(\mathcal{W})$ $\quad\mathcal{W}$ the regular cone.

$\hat{\psi}$ Laplace of ψ



$$R_\psi(\lambda) := \int_{\mathcal{W}} e^{-(X_A+\lambda(A))}\psi(A)\,dA \quad \left(= \hat{\psi}(X+\lambda)\right)$$

⇒ On some anisotropic spaces, $R_\psi(\lambda)$ is quasi-compact.

((⇒ $\underset{\substack{\uparrow\\ \mathbb{C}^k}}{\lambda} \in \mathrm{Res}(X) \iff \lambda \in \mathrm{spectrum}\left(X|_{R_\psi(\lambda)u=u}\right)$.

⇒ Even better: ((if $\lambda_0 \in \mathbb{C}^k$, $\exists\, \varepsilon > 0$ $\lambda \in \mathrm{Res}(X)$ & $|\lambda-\lambda_0| < \varepsilon$
$\Updownarrow$
$\lambda \in \mathrm{spectrum}\left(X|_{R_\psi(\lambda_0)u=u}\right)$

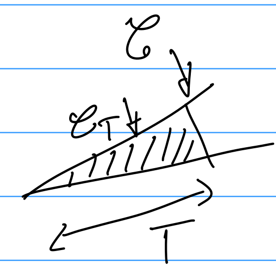
$$X|\{u \in \mathcal{H}_G \mid R_\psi(\lambda)u = u\}.$$

Since $X 1 = 0$, $\underline{0}$ is certainly a resonance.

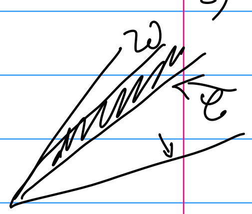
$\Rightarrow \{ u \mid R_\psi(0)\, u = u \}$

$$\underline{\Pi_\psi} := \frac{1}{2i\pi} \int_{\gamma_\varepsilon} (R_\psi(0) - s)^{-1}\, ds \qquad (1)^{\varepsilon}$$

$\Rightarrow$ Detailed study of Π_ψ.



<u>First result</u> 1) $\Pi := \Pi_\psi$ does not depend on ψ.



2) if $\mathcal{C} \subset \mathcal{W}$ is an open cone with $\overline{\mathcal{C}} \subset \mathcal{W}$, and $\underline{\mu, v} \in C^\infty(M) \longrightarrow \mu \in \mathcal{H}_G \quad v \in \mathcal{H}_G^*$

$$\frac{1}{\mathrm{Vol}\,\mathcal{C}_T} \int_{\substack{A \in \mathcal{C} \\ |A| < T}} v \cdot e^{-X_A} \mu \, dA \xrightarrow[T \to +\infty]{} \langle \Pi u, v \rangle .$$

3) The $\mu_v : u \longmapsto \langle \Pi u, v \rangle$ are the flow-invariant finite measures with $\underline{WF \subset E_s^*} = (E_0 \oplus E_u)^\perp$

$\Rightarrow$ This is contained in the first article.

2^{nd} article, still work in progress.

→ More on the resonance at zero.

⇒ Trying to extend results for flows.

<u>Second Result</u>: the measures μ_V are SRB-like:

SRB – the desintegrations w.r.t stable foliation are C^∞.

Physical – There exist $U_1 \dots U_m$ open sets in M $U_i \cap U_j = \emptyset$

- $\left| \underline{M \setminus (U_1} \cup U_2 \dots \cup U_m) \right|_{Leb} = 0$
- Leb-pp $x \in U_j$, $\forall f \in C^0(M)$, $\forall \mathcal{C} \subset \overline{\mathcal{C}} \subset \mathcal{W}$

$$\int_{\substack{A \in \mathcal{C} \\ |A| < T}} f\left(e^{-X_A}(x)\right) dA \longrightarrow \frac{\mu_1(f \mathbb{1}(U_j))}{\mu_1(U_j)}.$$

⇒ <u>Main idea</u>: distributions of the form $\delta_{W^s_{loc}(x)} f(x)$ (with $f \in C^\infty$) are in the anisotropic space.
+ usual Dyn. arguments.

Trace Formulae.

$\Rightarrow$ express the measure from before with periodic orbits.

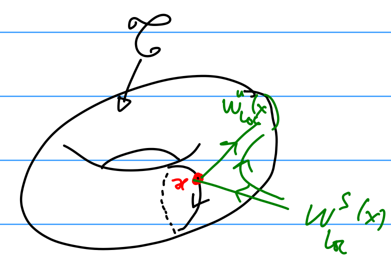
if $A \in \mathcal{W}$, $x \in M$ $\underline{e^{-X_A}(x) = x}$ then $\exists\, \underline{\mathcal{T}} \subset M$,

$\mathcal{T} \simeq \mathbb{R}^{r}/\mathbb{Z}^{k}$, $x \in \mathcal{T}$, $\underline{e^{-X_A}}\big|_{\mathcal{T}} = id$.

$\underline{\mathcal{P}_A} := d\left(e^{-X_A}\big|_{N\mathcal{T}}\right)$ hyperbolic

($\Rightarrow$ such orbits are isolated).

($\mathcal{P}_A$ well defined up to conjugation)



To each such periodic Torus $\rightarrow \begin{cases} \text{Lattice in } \mathfrak{a}.\ \mathcal{L}(\mathcal{T}) \\ \text{dirac measure on the torus.} \end{cases}$

$\rightarrow$ Guillemin trace formula: for $f \in C^0(M)$,

$$\underset{\mathcal{W}}{\overset{A}{\pitchfork}} \mapsto \operatorname{Tr}\left(e^{-X_A} f\right)\Big/\!\!\Big/_{\mathcal{W}} = \sum_{\mathcal{T}} \sum_{\substack{B \in \mathcal{L}(\mathcal{T}) \\ B \in \mathcal{W}}} \frac{1}{|\det(1-\mathcal{P}_B)|} \int_{\mathcal{T}} f \ \delta(A-B).$$

Guillemin trace for a flow:

$$\operatorname{Tr}\left(e^{-tX_1}\right) = \sum_{\gamma \text{ per.}} \frac{\ell(\gamma)}{|\det(1-\mathcal{P}_\gamma)|}\, \delta(t-\ell(\gamma)).$$

$\mathcal{L}(\mathcal{T}) = \{A \in \mathfrak{a} \mid e^{-X_A}\big|_{\mathcal{T}} = 1\}$ a lattice in $\mathfrak{a}$

⇒ obviously no operator here is really trace class...

Lemma: for $f \in C^\infty$, ψ, λ, s $\begin{cases} \lambda \in \mathbb{C}^k \\ s \in \mathbb{C}^* \end{cases}$.

$\psi, \lambda \longrightarrow C_c^\infty(\mathcal{W})$

$f \; \underline{R_\psi(\lambda)\,(s - R_\psi(\lambda))^{-1}}$ $\longrightarrow \mathrm{WF} \cap N^*\Delta_{M\times M} = \emptyset$

has a flat trace.

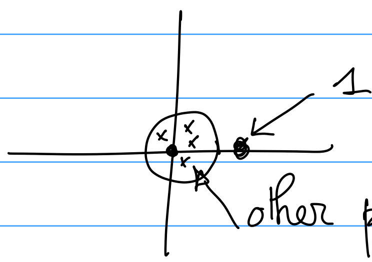
$\longrightarrow Z_{\psi, f}(\lambda, s) = \mathrm{Tr}^\flat\left(f\, R_\psi(\lambda)\,(s - R_\psi(\lambda))^{-1} \right)$

⇒ Dyatlov Zworski "zeta function" '16

$Z_{\psi, f}(0, s)$ has a pole at $s = 1$ with residue $\mu_1(f)$.

(of order 1)

$\mu_1(f) \longrightarrow \langle \Pi f, 1 \rangle$

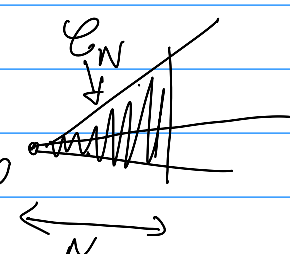


other poles of $Z_{\psi, f}(0, \cdot)$.

⇒ we deduce that:

3rd result: Let $\mathcal{C} \subset \overline{\mathcal{C}} \subset \mathcal{W}$ be an open cone.

$$\frac{1}{|\mathcal{C}_N|} \sum_{\substack{\gamma \\ A \in \mathcal{C}_N}} \sum_{A \in \mathcal{L}(\gamma)} \frac{\int_\gamma f}{|\det(1 - P_A)|} \xrightarrow{N \to +\infty} \mu_1(f)$$



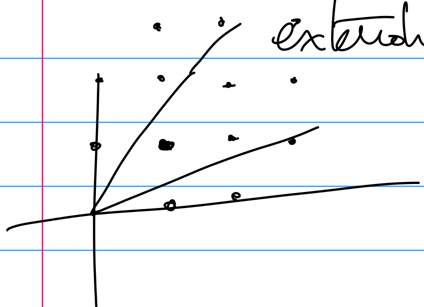
Bowen type formula. $\begin{cases} \text{if algebraic } \sim e^{-P(A)} \\ \text{use formulas } \left(\mathrm{Tr}\left(e^{-tX_A}|_{\Omega_1}\right) = \sum_B \frac{\mathrm{Vol}}{1} \delta(A - B) \right) \end{cases}$

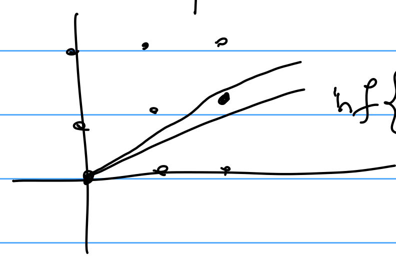
Taking $f = 1$, this leads to counting results of the form

$$\sum_{\underline{\text{regsys } \mathcal{C} < N}} \mathrm{Vol}(\mathcal{C}) \underbrace{\sim}_{} g(N) \quad \lessgtr$$

still to be worked out

extending results of Spatzier et. al to this general setup.





$$\left.\inf\{|A| \mid \begin{array}{l} A \in Z(\mathcal{C}) \\ A \in 2D \end{array}\right\} = \underline{\text{regsys stab}}$$

$\underline{\text{Finally}}$: $d_\psi(\lambda) := \underline{\det}^b\left(1 - R_\psi(\lambda)\right)$ is holomorphic on $\mathbb{C}^\kappa$.

$$\det^b(1-K) = \exp\left(\sum \frac{1}{n} \mathrm{Tr}^b K^n\right)$$

$\underline{\lambda} \in \mathrm{Res}(X) \Rightarrow d_\psi(\lambda) = 0$.

$$d_\psi(\lambda) = \exp\left(\sum_{\mathcal{C}} \sum_{A \in \mathfrak{A}(\mathcal{C})} \frac{-\mathrm{Vol}(T)}{|\det(1-P_A)|} \underbrace{e^{-\lambda(A)} \sum_{k \geq 1} \frac{\overbrace{\psi * \psi \cdots * \psi}^{\ell \text{ times}}(A)}{\ell}}_{}\right)$$

is a sort of zeta function.

$$R_\psi(\lambda) = \hat{\psi}(X + \lambda) \ldots$$

if $\hat{\psi}$ is the Laplace transform of ψ,

$\underline{d_\psi(\lambda)}$ is a regularized version of

$$R_\psi(\lambda) R_\varphi(\lambda) = R_{\psi * \varphi}(\lambda).$$

$$\longrightarrow \boxed{\prod_{\mu \in \mathrm{Res}(-X)} \left(1 - \hat{\psi}(\lambda - \mu)\right)}$$

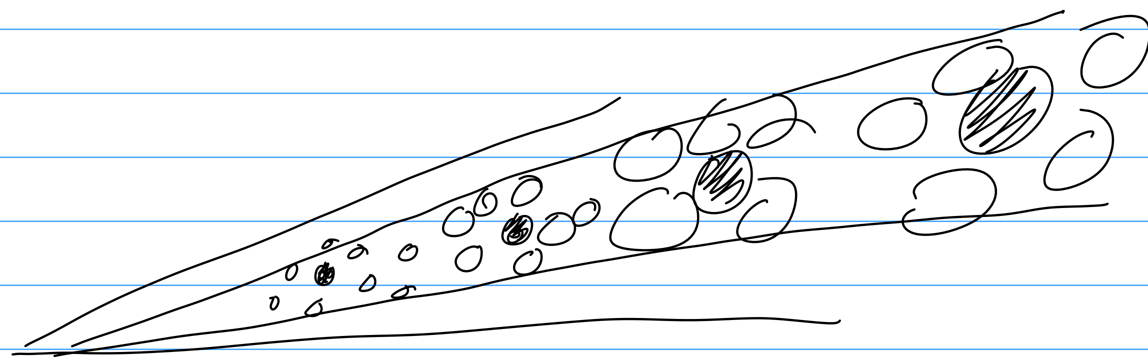
$$R_\psi(\lambda)^N = R_{\underbrace{\psi * \psi \cdots * \psi}_{N \text{ times}}}(\lambda)$$

$\zeta(\lambda) \overset{?}{:=}$ "det" $\Big((X_1-\lambda_1) \cdots (X_k-\lambda_k) \Big).$

$$\text{"="} \prod_{\mu \in \mathrm{Res}(-X)} (\mu_1-\lambda_1) \cdots (\mu_k-\lambda_k).$$

$\mu_1 = \lambda_1 \to 0$

if $\mu_n \in \mathrm{Res}(-X)$ with $(\mu_n)_1$ bounded.



$\int \psi_\sigma \, d\sigma$