The Ruelle zeta function at zero for nearly hyperbolic 3-manifolds

Gabriel P. Paternain

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UNIVERSITY OF CAMBRIDGE

The Ruelle zeta function

- (Σ, g) is a closed oriented Riemannian 3-manifold of negative curvature.
- The Ruelle zeta function

$$\zeta_{\mathrm{R}}(\lambda) = \prod_{\gamma} \left(1 - e^{-\lambda \ell_\gamma}\right), \quad \mathrm{Re}\, \lambda \gg 1 \tag{0.1}$$

  is a converging product for $\mathrm{Re}\, \lambda$ large enough. Here the product is taken over all primitive closed geodesics γ on (Σ, g) and ℓ_γ is the length of γ.
- It can be meromorphically continued to $\lambda \in \mathbb{C}$: Giulietti–Liverani–Pollicott (2013) and Dyatlov–Zworski (2016, microlocal methods).

The result in a nutshell

We are interested in the order of vanishing $n(g)$ of ζ_{R} at $\lambda = 0$.

In other words $n(g)$ is the unique integer such that $\lambda^{-n(g)}\zeta_{\mathrm{R}}(\lambda)$ is holomorphic near zero and has a non-zero value at zero.

Fried (1986, Selberg trace formula): for the hyperbolic metric g_H we have $n(g_H) = 4 - 2b_1$, where b_1 is the first Betti number of Σ.

Main Result: for a generic conformal perturbation g of g_H the order of vanishing is $n(g) = 4 - b_1$ and thus if $b_1 \neq 0$, $n(g)$ jumps relatively to the hyperbolic metric.

This is in stark contrast with the 2-dimensional case where $n(g)$ was shown to be $-\chi(\Sigma)$ for *any* negatively curved surface, Dyatlov–Zworski (2017).

This is the first known result to exhibit instability of the order of vanishing of ζ_{R} for Riemannian metrics.

For 3D volume preserving Anosov flows a jump in the order of vanishing was observed by Cekić–P 2019 when deforming an Anosov contact flow to a volume preserving flow with non-zero winding cycle.

The quantity $n(g)$ is closely related to resonant spaces of distributions invariant under the geodesic flow with values in the bundle of exterior forms.

Some basic facts:

- Sectional curvature < 0 implies the geodesic flow φ_t is Anosov;
- If $\pi : M := S\Sigma \to \Sigma$ is the unit sphere bundle,

$$\alpha_{(x,v)}(\xi) = \langle d\pi(\xi), v \rangle_g$$

  is a contact form with geodesic vector field as Reeb vector field.
- $b_1(M) = b_1(\Sigma), \quad b_2(M) = b_1(\Sigma) + 1.$

Contact Anosov flows

Suppose M is a closed 5-manifold with contact form α and let X be the Reeb vector field, i.e., $\iota_X \alpha = 1$ and $\iota_X d\alpha = 0$.

We assume that the flow φ_t generated by X is Anosov with splitting

$$TM = \mathbb{R}X \oplus E_u \oplus E_s.$$

Let $E_0 = \mathbb{R}X$, we have a dual decomposition

$$T^*M = E_0^* \oplus E_u^* \oplus E_s^*,$$

where E_0^* annihiliates $E_u \oplus E_s$, E_u^* annihilates $E_0 \oplus E_u$, and E_s^* annihilates $E_0 \oplus E_s$.

We let $\mathcal{D}'_{E_u^*}(M; \Omega^k)$ be the space of distributions with values in the bundle of exterior k-forms and with wave front set contained in E_u^*.

Resonant states

$$\mathrm{Res}_0^k := \{u \in \mathcal{D}'_{E_u^*}(M;\Omega^k) : \ \iota_X u = 0,\ \iota_X du = 0\}.$$

We call the dimension of Res_0^k the *geometric multiplicity*.

Generalised resonant spaces:

$$\mathrm{Res}_0^{k,\ell} := \{u \in \mathcal{D}'_{E_u^*}(M;\Omega^k) : \ \iota_X u = 0,\ \mathcal{L}_X^\ell u = 0\},$$

where $\mathcal{L}_X u = d\iota_X u + \iota_X du$ is the Lie derivative.

Set

$$\mathrm{Res}_0^{k,\infty} := \bigcup_{\ell \geq 1} \mathrm{Res}_0^{k,\ell}$$

and call $m_{k,0} := \dim \mathrm{Res}_0^{k,\infty}$ the *algebraic multiplicity*.

Obviously $\mathrm{Res}_0^k \subseteq \mathrm{Res}_0^{k,\infty}$ and when equality holds the geometric and algebraic multiplicities coincide. In this case we say that *k-semisimplicity* holds.

Key facts:

All these resonant spaces are finite dimensional. This is a consequence of the fact that $\mathcal{L}_X$ acting on suitable anisotropic Sobolev spaces has good Fredholm properties: Faure-Sjöstrand (2011), Dyatlov-Zworski (2016) (cf. also Blank-Keller-Liverani (2002), Baladi (2005), Baladi-Tsujii (2007), Butterley-Liverani (2007), and Gouëzel-Liverani (2006)).

The order of vanishing $n(X)$ for the Ruelle zeta function of X is given by

$$n(X) = m_{0,0} - m_{1,0} + m_{2,0} - m_{3,0} + m_{4,0}.$$

This follows from a factorization formula for ζ_{R}.

The challenge

Easy observation: semisimplicity for $k = 0, 4$ always holds and $m_{0,0} = m_{4,0} = 1$.

The real challenge will be to understand $m_{1,0}$ and $m_{2,0}$ and semisimplicity for $k = 1, 2$.

Resonant 1-forms and resonant 3-forms are isomorphic via wedging with $d\alpha$ and thus $m_{1,0} = m_{3,0}$.

The main result

Theorem 1 (Cekić-Dyatlov-Küster-P 2020)
Let $\Sigma = \Gamma \setminus \mathbb{H}^3$ be a closed hyperbolic 3-manifold with hyperbolic metric g_H. There exists an open and dense set $\mathcal{O} \subset C^\infty(\Sigma)$ such that if $h \in \mathcal{O}$, there exists an $\varepsilon > 0$ such that for $s \in (-\varepsilon, \varepsilon)$ and $s \neq 0$, the conformal metric $g_s = e^{-2sh} g_H$ has

$$m_{1,0}^{(s)} = b_1(\Sigma), \quad m_{2,0}^{(s)} = b_1(\Sigma) + 2$$

and semisimplicity holds for $k = 1, 2$. Thus, the order of vanishing $n(g_s)$ of the Ruelle zeta function ζ_{R} at zero is $4 - b_1(\Sigma)$.

The constant curvature picture

Before perturbing, we need to understand the full picture for g_H. Even here there are some surprises.

Proposition (Cekić-Dyatlov-Küster-P 2020)

For g_H we have

$$m_{1,0} = 2b_1(\Sigma), \quad m_{2,0} = 2b_1(\Sigma) + 2.$$

Moreover, semisimplicity holds for $k = 1$, but fails for $k = 2$ and $\dim \mathrm{Res}_0^2 = b_1(\Sigma) + 2$.

Of course, with this proposition one recovers Fried's result that $n(g_H) = 4 - 2b_1$, but this gives a lot more information.

Dang-Guillarmou-Rivière-Shen (2020) computed $m_{1,0}$ and $m_{2,0}$ using Selberg's trace formula (interested in Fried's conjecture about torsion).

Summary table

Dimension of	Hyperbolic	Perturbation
$Res_0^0 = Res_0^{0,\infty}$	1	1
$Res_0^1 = Res_0^{1,\infty}$	$2b_1(\Sigma)$	$b_1(\Sigma)$
Res_0^2	$b_1(\Sigma) + 2$	$b_1(\Sigma) + 2$
$Res_0^{2,2} = Res_0^{2,\infty}$	$2b_1(\Sigma) + 2$	$b_1(\Sigma) + 2$
$Res_0^3 = Res_0^{3,\infty}$	$2b_1(\Sigma)$	$b_1(\Sigma)$
$Res_0^4 = Res_0^{4,\infty}$	1	1

We may introduce natural linear maps

$$\pi_k : \mathrm{Res}_0^k \cap \ker d \to H^k(M; \mathbb{C}).$$

We show that π_1 is always an isomorphism and thus $m_{1,0} \geq \dim \mathrm{Res}_0^1 \geq b_1(M)$.

A priori, there is no reason to expect additional forms in Res_0^1 except the closed ones, but the hyperbolic metric g_H is exceptional in this regard.

The perturbations in the main theorem remove this "excess" of 1-forms and restore semisimplicity for 2-forms.

The pairing

An important ingredient in our proofs is the natural pairing between resonant and coresonant states.

Coresonant states and the spaces $\mathrm{Res}_{0*}^{k,\ell}$ are defined just as the resonant ones but requiring the wavefront set of the distribution to be contained in E_s^* (or resonant for $-X$).

Since E_u^* and E_s^* intersect only at the zero section, we can define the product (Hörmander's condition is satisfied), $u \wedge v$ for any $u \in \mathrm{Res}_0^{k,\infty}$ and $v \in \mathrm{Res}_{0*}^{4-k,\infty}$ and thus a pairing

$$\langle\!\langle u, v \rangle\!\rangle = \int_M \alpha \wedge u \wedge v.$$

Perturbation lemma

Lemma (Perturbation Lemma)

Let $h \in C^\infty(\Sigma)$. If the bilinear form $\langle\langle h\bullet, \bullet\rangle\rangle$ is non-degenerate in $d(\mathrm{Res}^1_0) \times d(\mathrm{Res}^1_{0*})$, then for s small enough and not zero, we have $m^{(s)}_{1,0} = b_1(\Sigma)$ and $m^{(s)}_{2,0} = b_1(\Sigma) + 2$ (as claimed in the main theorem).

Thus, the bulk of the work for proving the main theorem lies in establishing that for generic h, the pairing above is non-degenerate.

This is challenging, because h only depends on the configuration space variable and hence the pairing reduces to an integral on Σ.

Reduced pairing

Given $du \in d(\mathrm{Res}_0^1)$ and $dv \in d(\mathrm{Res}_{0*}^1)$ define $F \in \mathcal{D}'(\Sigma)$ by

$$\pi_*(\alpha \wedge du \wedge dv) = F\, d\mathrm{vol}_{g_H},$$

where π_* stands for push–forward to Σ.

The pairing reduces to

$$\int_\Sigma h\, F\, d\mathrm{vol}_{g_H}.$$

We need to show $F \neq 0$ when a non-zero du is given and $dv = \mathcal{J}^* du$, where $\mathcal{J}$ is the flip in $S\Sigma$ given by $\mathcal{J}(x, v) = (x, -v)$.

Regularizing the push-forward F

For $s \in \mathbb{R}$, let us consider the kernels $k_s : \mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{R}$ given by

$$k_s(x,z) = (\cosh d_{\mathbb{H}^3}(x,z))^{-s}.$$

Associated with these kernels, we have integral operators

$$Q_s : C_0^\infty(\mathbb{H}^3) \to C^\infty(\mathbb{H}^3), \quad Q_s f(z) := \int_{\mathbb{H}^3} k_s(x,z)\, f(x)\, d\mathrm{vol}(x).$$

The integral converges absolutely for f bounded and any $s > 2$ and Q_s is Γ-equivariant, so that it induces an operator on Σ. In fact, it defines a smoothing operator

$$Q_s : \mathcal{D}'(\Sigma) \to C^\infty(\Sigma).$$

The next result unlocks the proof of the main theorem.

Theorem 2 (Regularization theorem)
Define the push forwards

$$\sigma_- := \pi_*(\alpha \wedge du), \quad \sigma_+ := \pi_*(\alpha \wedge dv),$$

which are harmonic 1-forms on Σ. The following relation holds:

$$Q_4 F = \frac{-1}{6}\Delta_{g_H}(\sigma_+ \cdot \sigma_-),$$

where Δ_{g_H} denotes the Laplacian of g_H and $\cdot$ denotes the inner product on $T^*\Sigma$.

Completing the proof

When $dv = \mathcal{J}^* du$, it is not hard to check that $\sigma_+ = \sigma_-$.

If the pushforward $F = 0$, the Regularization Theorem gives that the harmonic form σ_- has constant norm.

But this can only happen if $\sigma_- = 0$! This is because there are no non-zero harmonic 1-forms with constant norm in Σ. This in turn implies $du = 0$.

An extension of this argument supplemented by the appropriate linear algebra gives the main theorem in full.

Contact perturbations

If one is interested only in contact perturbations of the contact structure of a hyperbolic metric, then it is possible to give a more elementary proof which does not require the Regularization Theorem.

For contact perturbations, we are only required to show that the form $\langle\langle h\bullet, \bullet\rangle\rangle$ is non-degenerate in $d(\mathrm{Res}^1_0) \times d(\mathrm{Res}^1_{0*})$, where $h \in C^\infty(S\Sigma)$, thus to produce a perturbation giving order of vanishing $4 - b_1(\Sigma)$ for ζ_{R} it is enough to check that $\alpha \wedge du \wedge dv = 0$ implies $du = 0$ or $dv = 0$.

Conjectures

Conjecture 1

For a generic contact Anosov flow on an 5-dimensional manifold M we have:

1. the semisimplicity condition holds in all degrees;
2. $d(\mathrm{Res}_0^k) = 0$ for all k;
3. for $k = 0, 1, 2$, π_k is onto, $\ker \pi_k = d\alpha \wedge \mathrm{Res}_0^{k-2}$, and $\dim \ker \pi_k = \dim \mathrm{Res}_0^{k-2}$.

In particular, the order of vanishing of the Ruelle zeta function at zero is $3 - 2b_1(M) + b_2(M)$.

One can also hope for similar genericity results in the Riemannian category (harder to come by).

Conjecture 2

Let (Σ, g) be a generic negatively curved closed 3-manifold. Then

1. the semisimplicity condition holds in all degrees;
2. $d(\mathrm{Res}_0^k) = 0$ for all k;
3. for $k = 0, 1, 2$, π_k is onto, $\ker \pi_k = d\alpha \wedge \mathrm{Res}_0^{k-2}$, and $\dim \ker \pi_k = \dim \mathrm{Res}_0^{k-2}$.

In particular, for a generic negatively curved metric, the order of vanishing of the Ruelle zeta function at zero is $4 - b_1(\Sigma)$.