Holonomy inverse problem and generic injectivity of twisted X-ray transform

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1 Introduction
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2 Sketch proof of II

3 Perspectives
Let \((M, g)\) be a compact Riemannian manifold without boundary; \(\mathcal{E} \to M\) a vector bundle over \(M\) equipped with a connection \(\nabla^\mathcal{E}\). We address the following inverse problem:

**Question**

*To what extent does the holonomy of \(\nabla^\mathcal{E}\) over closed geodesics determine the gauge-equivalence class of \(\nabla^\mathcal{E}\)？*

- We assume: \((M, g)\) has Anosov geodesic flow, \(\mathcal{E}\) is Hermitian and \(\nabla^\mathcal{E}\) is unitary.
Recall: connections on vector bundles

- Connection $\nabla^E$ is a map $\nabla^E : C^\infty(M,E) \to C^\infty(M, T^*M \otimes E)$ that locally looks like $d + A$ for a matrix $A$ of 1-forms.
- If $\gamma : [a, b] \to M$ a curve, $e \in E_a$, $s : [a, b] \to E$ is the parallel transport of $e$ along $\gamma$ if $\nabla_\gamma^E s = 0$ (first order ODE) and $s(a) = e$, $\pi \circ s = \gamma$. Denote $P_\gamma e : = s(b) \in E_b$.
- $\nabla^E$ is unitary if compatible with the inner product on $E$; it follows $P_\gamma : E_a \to E_b$ is unitary.
- Unitary connections $\nabla_1^E, \nabla_2^E$ are gauge equivalent if there is a unitary $p : E \xrightarrow{\cong} E$ such that $p^* \nabla_2^E = \nabla_1^E$, where $p^* \nabla_2^E : = p^{-1} \nabla_2^E(p\bullet)$.
- Parallel transport of $p^* \nabla^E$ along $\gamma$ is $p(b)^{-1} P_\gamma p(a)$. 

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The geodesic flow lives on the unit sphere bundle
\[ SM = \{(x, v) \in TM : |v|_x = 1\}; \varphi_t(x, v) := (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) \]. Say 
\((M, g)\) is Anosov if its geodesic flow is Anosov.

Then \(\exists\) bijection between free homotopy classes \(c \in C\) and closed 
geodesics \(\gamma_g(c)\) of length \(L_g(c)\) in the class \(c\).

Examples:
- If \((M, g)\) has negative sectional curvature, then it is Anosov.
- There \(\exists\) examples with portions of positive curvature (Eberlein, Donnay-Pugh).
Modes of conjugacy

Let $\nabla_1^\mathcal{E}, \nabla_2^\mathcal{E}$ be unitary connections; write $C_i((x, v), t) : \mathcal{E}_x \to \mathcal{E}_{\gamma_{x,v}(t)}$ for the parallel transport maps along geodesics wrt $\nabla_i^\mathcal{E}$. Say $\nabla_1^\mathcal{E}$ and $\nabla_2^\mathcal{E}$ are:

**Definition**

- **strongly conjugate** if there is a smooth map $q \in C^\infty(SM, U(\mathcal{E}))$ such that for all $c \in \mathcal{C}$ and $(x, v) \in SM$ generating $\gamma_g(c)$, we have:

  $$C_2((x, v), L_g(c)) = q^{-1}(x, v) C_1(x, L_g(c)) q(x, v),$$

- **weakly conjugate** if for all $c \in \mathcal{C}$, there is an $(x, v) \in SM$ generating $\gamma_g(c)$ and $q_c \in U(\mathcal{E}_x)$ such that:

  $$C_2((x, v), L_g(c)) = q_c^{-1} C_1((x, v), L_g(c)) q_c.$$

Clearly: strong conjugacy $\implies$ weak conjugacy. Maps $C_i$ are cocycles.
To study the problem locally, we will make the following assumptions:

(A) The connection $\nabla^\mathcal{E}$ is **opaque**. By definition, this means that there are no non-trivial sub-bundles $\mathcal{F}$ preserved by parallel transport along geodesics (generic by C-Lefeuvre [2020]).

(B) The **generalized X-ray transform** $\Pi_1$ on twisted 1-forms with values in $\text{End}(\mathcal{E})$ is **solenoidal injective** (generic by Theorem II).

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**Theorem (I. C-Lefeuvre ’21)**

Assume $\nabla^\mathcal{E}$ satisfies (A) and (B). Then, there exists $\delta > 0$ and $N \in \mathbb{N}$ such that the following holds. For every $A$ with $\|A\|_{C^N} < \delta$, if $\nabla^\mathcal{E}$ and $\nabla^\mathcal{E} + A$ have strongly conjugate holonomies they are gauge-equivalent.

**Remark.** Stability estimates are also provided, measuring the distance between connections up to gauge in terms of a distance between cocycles.
Previous results

- Paternain [2009, 2010 2012, 2013] initiated this problem on surfaces. He classified transparent connections (trivial holonomy on closed geodesics) in negative curvature and transparent pairs (connection + potential). Also gave examples of continuous families of non-gauge-equivalent transparent connections (in rank $\mathcal{E} = 2$). If rank $\mathcal{E} = 1$, he proved the result globally and any dim $M \geq 2$.

- Guillarmou-Paternain-Salo-Uhlmann [2016] showed the result in negative curvature under the condition of non-existence of Conformal Killing Tensors of a certain connection. (We study such condition generically in C-Lefeuvre [2020].)

- The remarkably similar problem on manifolds with boundary has been well-studied for: manifolds with a foliation condition by PSU-Zhou [2018], simple surfaces PSU [2012].
Analogous problem: determine the isometry class of $g$ from the **marked length spectrum** $\mathcal{C} \ni c \mapsto L_g(c)$. Our approach reminiscent of Guillarmou-Lefeuvre [2020] who solve this problem locally.

<table>
<thead>
<tr>
<th>Object</th>
<th>Length Spectrum</th>
<th>Holonomy Inverse Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>metric $g$</td>
<td>connection $\nabla^\mathcal{E}$</td>
<td></td>
</tr>
<tr>
<td>diffeomorphisms $\text{Diff}_0(M)$</td>
<td>gauge group $\mathcal{G}(\mathcal{E})$</td>
<td></td>
</tr>
<tr>
<td>$c \mapsto L_g(c)$</td>
<td>$c \mapsto \mathcal{P}_{\gamma_g(c)}/\text{conj.}$</td>
<td></td>
</tr>
<tr>
<td>$DL_g(c)(\beta) = \int_{\gamma_g(c)} \beta(\dot{\gamma}, \dot{\gamma})$</td>
<td>“X-ray on $\text{End}\mathcal{E}$-1-forms”</td>
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To introduce the equivalent of $X$-ray in the twisted case, need to introduce the resolvent.
Twisted Fourier Analysis and X-ray transform

- Consider the bundle $\pi^*\mathcal{E} \to SM$ and the pullback connection $\pi^*\nabla^\mathcal{E}$. Define $X := \pi^*\nabla^\mathcal{E}$ and study cohomological equations: $Xu = f$, where $u, f \in C^\infty(SM, \mathcal{E})$.

- Any $f \in C^\infty(SM, \mathcal{E})$ can be decomposed into spherical harmonics i.e., $f = \sum_{m \geq 0} f_m$, where $f_m \in \Omega_m \otimes \mathcal{E} := \ker(\Delta^\mathcal{E} + m(m + n - 2))$. Define degree $\deg(f) = m$, where $m$ is largest so that $f_m \neq 0$.

- Twisted tensors $f \in C^\infty(M, \mathcal{T}^*M \otimes \mathcal{E})$ can be pulled back to $SM$ via $\pi^*_m f(x, v) := f_x(v, \ldots, v) \in \mathcal{E}_x$. Extend the action of $\nabla^\mathcal{E}$ to symmetric tensors via the Levi-Civita connection $\nabla^{LC}$. Result is a symmetrised operator $D^\mathcal{E}$ such that $\pi^*_m D^\mathcal{E} = X \pi^*_m - 1$.

- Remarkable property of $X$ is that: $X : C^\infty(M, \Omega_m \otimes \mathcal{E}) \to C^\infty(M, \Omega_{m-1} \otimes \mathcal{E}) \oplus C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})$. 

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Twisted tensor tomography on Anosov manifolds:

\[ Xu = f \text{ and } \deg(f) < \infty \implies \deg(u) \leq \max(\deg(f) - 1, 0). \]

Any tensor can be uniquely decomposed as \( u = D^E p + h \) into potential and solenoidal parts (so \((D^E)^* h = 0\)).

Equivalent formulation of tensor tomography: if \( f = \pi^*_m g \) with \((D^E)^* g = 0\), then \( g = 0 \). Say that s-injectivity (solenoidally) holds.

**Theorem (II. C-Lefeuvre 2021)**

There is an \( N > 0 \) such that the following holds. For each \( m \in \mathbb{N}_0 \), there is an open and dense set of connections \( \nabla^E \) in \( C^N \) such that the twisted tensor tomography holds for \( \deg(f) \leq m \).

**Remark.** We believe the analogous result for metrics is also true (work in progress).
Summary

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The resolvent $R_\pm(z) := (\mp X - z)^{-1} : C^\infty(SM, \mathcal{E}) \to \mathcal{D}'(SM, \mathcal{E})$ admits meromorphic continuation to $\mathbb{C}$, with poles PR resonances. (Faure-Sjöstrand, Dyatlov-Zworski, Butterley-Liverani, Baladi, etc).

Since 0 a simple pole, expand $R_\pm(z) = -R_{\pm,0} - \frac{\Pi_\pm}{z} + \mathcal{O}(z)$ close to zero, where $\Pi_+ = \Pi_-$ are spectral projectors to $\ker X \subset C^\infty(SM, \mathcal{E})$.

Define $\pi_m^*$ dually to $\pi_m$. Following Guillarmou [2017], we define the generalized $X$-ray transform $\Pi_m^{\nabla \mathcal{E}} := \pi_m^* (R_{+,0} + R_{-,0} + \Pi_+) \pi_m^*$. Properties the twisted generalized $X$-ray transform:

- $\Pi_m^{\nabla \mathcal{E}} f = 0 \iff \pi_m^* f = X u$ for some $u \in C^\infty(SM, \mathcal{E})$;
- $\Pi_m^{\nabla \mathcal{E}} \in \Psi^{-1}(M, \otimes^m_S T^* M \otimes \mathcal{E})$;
- it is elliptic on $\ker(D^\mathcal{E})^*$, i.e. $\exists Q, R \in \Psi(M, \otimes^m_S T^* M \otimes \mathcal{E})$ of orders $1, -\infty$ with $Q \Pi_m^{\nabla \mathcal{E}} = \pi_{\ker(D^\mathcal{E})^*} + R$. 
Ellipticity of $\Pi^E_m$ $\implies$ ker $\Pi^E_m |_{\ker(D^E)^*}$ is finite dim. and smooth.

Consider the linear perturbation $\nabla^E + \tau \Gamma$ for $\Gamma \in C^\infty(M, \text{End}_{sk}(E))$, where $\tau$ is small; denote its symmetrised derivative by $D_\tau$ and generalized transform by $\Pi_\tau^m$.

$\Pi_\tau^m$ is compact so we take $P_\tau := \pi_{\ker D^*_\tau} \Lambda_k \Pi^T_m \Lambda_k \pi_{\ker D^*_\tau}$ for some $\Lambda = \Lambda_k \in \Psi^k(M, \otimes^m_S T^*M \otimes E)$ self-adjoint, elliptic, diagonal principal symbol, $k > 1/2$, isomorphism on Sobolev spaces.

$P_\tau \in \Psi^{2k-1}$ has discrete spectrum $\subset \mathbb{R}_{\geq 0}$ accumulating at $\infty$.

By C-Lefeuvre [2021] ker $X = \{0\}$ is a generic condition, which we assume from now on.

It follows $\tau \mapsto \pi_{\ker D^*_\tau} = I - D_\tau (D^*_\tau D_\tau)^{-1} D^*_\tau \in \Psi^0$ is continuous and we have: for $\tau$ small enough, dim ker $P_\tau = \dim \ker \Pi_\tau^m$. 

First and second variation

- To show \( \text{ker} \, \Pi^\tau_m = \{0\} \) generically need to show openness and density. Openness follows from continuity of \( \nabla^E \mapsto \Pi_m^{\nabla_E} \in \Psi^{-1} \) (can use the same anisotropic space!).
- For density, consider \( \Pi^\tau := \frac{1}{2\pi i} \oint_\gamma (z - P^\tau)^{-1} \) and \( \lambda^\tau := \text{Tr}(P^\tau \Pi^\tau) \). Here \( \gamma \) a small circle around \( 0 \).
- Since \( \lambda^\tau \geq 0 \) and \( \lambda^0 = 0 \), we have \( \dot{\lambda}_{\tau=0} = 0 \).
- Take \( L^2 \)-ON basis \((u_i)_{i=1}^d \subset \ker(D^E)^* \) of \( \ker P_0 \) and write \( \pi^*_m \Lambda u_i = X v_i \) for some \( v_i \in C^\infty(SM,E) \) (as \( \Pi_m^{\nabla E} \Lambda u_i = 0 \)).
- Long and tedious computation (...) \( \implies \) ugly formula:

\[
\ddot{\lambda}_0 = 2 \sum_{i=1}^d \left( \left\langle (R^+,0 + R^-,0)\pi^*_1 \Gamma v_i, \pi^*_1 \Gamma v_i \right\rangle_{L^2} - \left\langle \pi^*_m (\Pi_m^{\nabla E})^{-1} \pi_m^* (R^+,0 + R^-,0)\pi^*_1 \Gamma v_i, (R^+,0 + R^-,0)\pi^*_1 \Gamma v_i \right\rangle_{L^2} \right).
\]
Not all is lost, set $M_v A := A \cdot v$ and define

$$Q_v := \pi_m \mathcal{I}^\mathcal{E} M_v \pi_1^* \in \Psi^{-1}(M, T^* M \otimes \text{End}(\mathcal{E}) \to \otimes_S^m T^* M \otimes \mathcal{E}),$$

$$L_v := \pi_1^* M_v \mathcal{I}^\mathcal{E} M_v \pi_1^* \in \Psi^{-1}(M, T^* M \otimes \text{End}(\mathcal{E}) \to T^* M \otimes \text{End}(\mathcal{E})).$$

Re-write the formula as

$$\ddot{\lambda}_0 = 2 \sum_{i=1}^d \langle (L_v - Q_v^* \prod_m Q_v) \Gamma, \Gamma \rangle_{L^2}.$$

If $\xi \in S_x^* M$, identify $S_x^* M = S^{n-1}$ and write $S^n_{\xi} := \ker \langle \xi, \cdot \rangle \cap S^{n-1}$. Use coordinates $(0, \pi) \times S^n_{\xi} \to S^{n-1}, (\varphi, u) \mapsto \cos(\varphi) \xi^\# + \sin(\varphi) u$.

The degree $m$ extension operator $E^m_\xi : L^2(S^{n-2}_{\xi}, \mathcal{E}) \to L^2(S^{n-1}, \mathcal{E})$ is defined by the formula $E^m_\xi f(\varphi, u) := \sin^m(\varphi) f(u)$.

Easy to show:

$$E^m_0 L^2(S^{n-2}_{\xi_0}, \pi^* \mathcal{E}_{x_0}) = \ker(\pi_{\ker i_{\xi_0}^* \pi_m^*} \big| L^2(S^{n-1})) \oplus (\otimes_S^m \ker i_{\xi_0}^* \otimes \mathcal{E}_{x_0}).$$
The map $P_m := \pi^*_m \pi_{\ker \xi_0^\#} \left[ \pi_{\ker \xi_0^\#} \pi^*_m \pi_{\ker \xi_0^\#} \right]^{-1} \pi_{\ker \xi_0^\#} \pi^*_m : E^m_{\xi_0} L^2(\mathbb{S}^{n-2}, \pi^* \mathcal{E}_{x_0}) \cap$ is orthogonal projection on the second factor.

Assume $\ddot{\lambda}_0 = 0$ for all $\Gamma$. Recall $\psi$DO's act diagonally on Gaussian states, $e_h(x_0, \xi_0) = (\pi h)^{-n/4} e^{i \frac{\xi_0}{h} \cdot (x-x_0) - \frac{|x-x_0|^2}{2h}}$, to first order.

Set $w_i := E^m_{\xi_0} \left( (\pi^*_1 \Gamma \cdot v_i) \big|_{\mathbb{S}^{n-2}} \right)$ and apply the identity for $\Gamma_h = \Re(e_h(x_0, \xi_0)) \cdot \Gamma$ as $h \to 0$. Major simplification:

$$\sum_{i=1}^{d} \|w_i\|_{L^2(\mathbb{S}^{n-1}, \mathcal{E})}^2 = \sum_{i=1}^{d} \|P_m w_i\|_{L^2(\mathbb{S}^{n-1}, \mathcal{E})}^2.$$ 

Contradiction if we show $\deg(w_1) \geq m + 1$ for some $(x_0, \xi_0)$.

Pick $x_0$ such that $\deg v_1 \geq m + 1$. Reduces to a claim on the sphere $\mathbb{S}^{n-1}$, which (essentially) reduces to representation theory of $SO(n)$. 

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Sandwich operators

- Swept under the rug: pseudodifferential nature of “sandwich” operators. For $P_L, P_R$ differential operators of order $m_L$ and $m_R$:

$$\pi_{m_1} P_L \mathcal{I}_\nabla \varepsilon P_R \pi_{m_2}^* \in \Psi^{m_R + m_L - 1} (M, \otimes^m_S T^* M \otimes \mathcal{E} \rightarrow \otimes^m_S T^* M \otimes \mathcal{E}).$$

$$=: A_{PL, PR}$$

- Elaborate use of stationary phase lemma $\Rightarrow$ principal symbol:

$$\langle \sigma_{P_R, P_L} (x, \xi) f, f' \rangle \otimes_{\mathcal{S}^{m_1} S} T^* M_x \otimes \mathcal{E}_x = \frac{2\pi}{|\xi|} \int_{S^{n-2}_\xi} \langle \sigma_{P_R} ((x, u), \xi_H (x, u)) (\pi_{m_2}^* f (u)), \sigma_{P_L^*} ((x, u), \xi_H (x, u)) (\pi_{m_1} f' (u)) \rangle \mathcal{E}_x \, dS_\xi (u).$$

Here $\xi_H (x, u) := \xi (d_{x,u} \pi (\cdot))$. See also Thibault’s thesis.
Summary

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2. Sketch proof of II

3. Perspectives
We propose a few interesting open questions:

- Are weakly conjugate connections also strongly conjugate? (Partial answer available in our paper, if the weak conjugacies do not oscillate too much.)

- Are vector bundles with weakly conjugate connections necessarily isomorphic?

- Are there transparent Levi-Civita connections? (work in progress)
Thank you for your attention!