Holonomy inverse problem and generic injectivity of twisted X-ray transform

Mihajlo Cekić

Joint work with Thibault Lefeuvre

Laboratoire Mathematiques d'Orsay, Université Paris Saclay

Ruelle-Pollicott Resonances in dynamics and semiclassical analysis (online), March 5, 2021

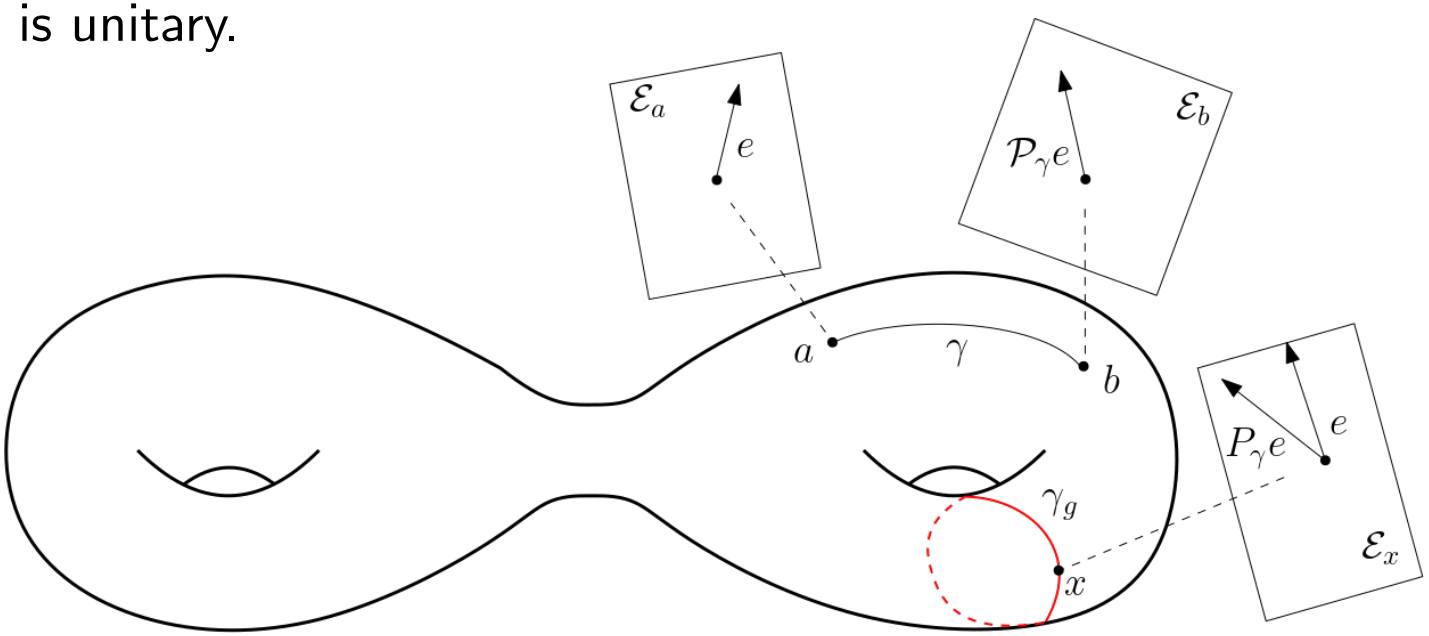
Summary

- Let (M, g) be a compact Riemannian manifold without boundary; $\mathcal{E} \to M$ a vector bundle over M equipped with a connection $\nabla^{\mathcal{E}}$. We address the following inverse problem:

Question

To what extent does the holonomy of $\nabla^{\mathcal{E}}$ over closed geodesics determine the gauge-equivalence class of $\nabla^{\mathcal{E}}$?

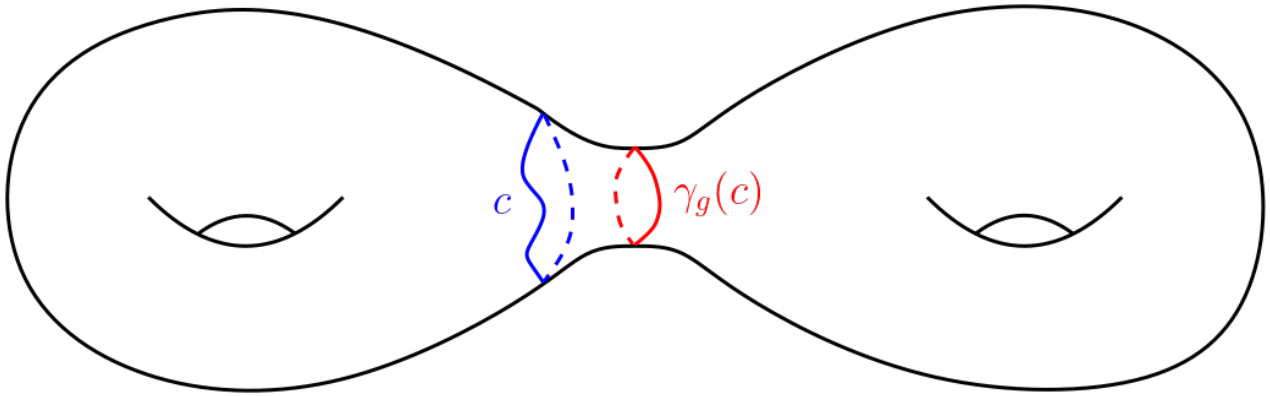
- We assume: (M, g) has Anosov geodesic flow, $\mathcal{E}$ is Hermitian and $\nabla^{\mathcal{E}}$ is unitary.



Recall: connections on vector bundles

- Connection $\nabla^{\mathcal{E}}$ is a map $\nabla^{\mathcal{E}} : C^\infty(M, \mathcal{E}) \to C^\infty(M, T^*M \otimes \mathcal{E})$ that locally looks like $d + A$ for a matrix A of 1-forms.
- If $\gamma : [a, b] \to M$ a curve, $e \in \mathcal{E}_a$, $s : [a, b] \to \mathcal{E}$ is the parallel transport of e along γ if $\nabla^{\mathcal{E}}_{\dot\gamma} s = 0$ (first order ODE) and $s(a) = e$, $\pi \circ s = \gamma$. Denote $\mathcal{P}_\gamma e := s(b) \in \mathcal{E}_b$.
- $\nabla^{\mathcal{E}}$ is unitary if compatible with the inner product on $\mathcal{E}$; it follows $\mathcal{P}_\gamma : \mathcal{E}_a \to \mathcal{E}_b$ is unitary.
- Unitary connections $\nabla^{\mathcal{E}}_1, \nabla^{\mathcal{E}}_2$ are gauge equivalent if there is a unitary $p : \mathcal{E} \xrightarrow{\cong} \mathcal{E}$ such that $p^*\nabla^{\mathcal{E}}_2 = \nabla^{\mathcal{E}}_1$, where $p^*\nabla^{\mathcal{E}}_2 := p^{-1}\nabla^{\mathcal{E}}_2(p\bullet)$.
- Parallel transport of $p^*\nabla^{\mathcal{E}}$ along γ is $p(b)^{-1}\mathcal{P}_\gamma p(a)$.

- The geodesic flow lives on the unit sphere bundle $SM = \{(x, v) \in TM : |v|_x = 1\}$; $\varphi_t(x, v) := (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$. Say (M, g) is Anosov if its geodesic flow is Anosov.
- Then $\exists$ bijection between free homotopy classes $c \in \mathcal{C}$ and closed geodesics $\gamma_g(c)$ of length $L_g(c)$ in the class c.



- Examples:
  - If (M, g) has negative sectional curvature, then it is Anosov.
  - There $\exists$ examples with portions of positive curvature (Eberlein, Donnay-Pugh).

Modes of conjugacy

Let $\nabla_1^{\mathcal{E}}, \nabla_2^{\mathcal{E}}$ be unitary connections; write $C_i((x,v),t) : \mathcal{E}_x \to \mathcal{E}_{\gamma_{x,v}(t)}$ for the parallel transport maps along geodesics wrt $\nabla_i^{\mathcal{E}}$. Say $\nabla_1^{\mathcal{E}}$ and $\nabla_2^{\mathcal{E}}$ are:

Definition

- *strongly conjugate* if there is a smooth map $q \in C^\infty(SM, \mathrm{U}(\mathcal{E}))$ such that for all $c \in \mathcal{C}$ and $(x,v) \in SM$ generating $\gamma_g(c)$, we have:

$$C_2((x,v), L_g(c)) = q^{-1}(x,v) C_1(x, L_g(c)) q(x,v),$$

- *weakly conjugate* if for all $c \in \mathcal{C}$, there is an $(x,v) \in SM$ generating $\gamma_g(c)$ and $q_c \in \mathrm{U}(\mathcal{E}_x)$ such that:

$$C_2((x,v), L_g(c)) = q_c^{-1}\ C_1((x,v), L_g(c))\, q_c.$$

Clearly: strong conjugacy $\implies$ weak conjugacy. Maps C_i are cocycles.

To study the problem locally, we will make the following assumptions:

(A) The connection $\nabla^{\mathcal{E}}$ is **opaque**. By definition, this means that there are no non-trivial sub-bundles $\mathcal{F}$ preserved by parallel transport along geodesics (generic by C-Lefeuvre [2020]).

(B) The *generalized X-ray transform* Π_1 on twisted 1-forms with values in $\mathrm{End}(\mathcal{E})$ is **solenoidal injective** (generic by Theorem II).

Theorem (I. C-Lefeuvre '21)

Assume $\nabla^{\mathcal{E}}$ satisfies **(A)** *and* **(B)**. *Then, there exists $\delta > 0$ and $N \in \mathbb{N}$ such that the following holds. For every A with $\|A\|_{C^N} < \delta$, if $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{E}} + A$ have strongly conjugate holonomies they are gauge-equivalent.*

Remark. Stability estimates are also provided, measuring the distance between connections up to gauge in terms of a distance between cocycles.

Previous results

- Paternain [2009, 2010 2012, 2013] initiated this problem on surfaces. He classified transparent connections (trivial holonomy on closed geodesics) in negative curvature and transparent pairs (connection + potential). Also gave examples of continuous families of non-gauge-equivalent transparent connections (in $\operatorname{rank} \mathcal{E} = 2$). If $\operatorname{rank} \mathcal{E} = 1$, he proved the result globally and any $\dim M \geq 2$.
- Guillarmou-Paternain-Salo-Uhlmann [2016] showed the result in negative curvature under the condition of non-existence of Conformal Killing Tensors of a certain connection. (We study such condition generically in C-Lefeuvre [2020].)
- The remarkably similar problem on manifolds with boundary has been well-studied for: manifolds with a foliation condition by PSU-Zhou [2018], simple surfaces PSU [2012].

Analogous problem: determine the isometry class of g from the marked length spectrum $\mathcal{C} \ni c \mapsto L_g(c)$. Our approach reminiscent of Guillarmou-Lefeuvre [2020] who solve this problem locally.

	Length Spectrum	Holonomy Inverse Problem
Object	metric g	connection $\nabla^{\mathcal{E}}$
Group Action	diffeomorphisms $\mathrm{Diff}_0(M)$	gauge group $\mathcal{G}(\mathcal{E})$
Data	$c \mapsto L_g(c)$	$c \mapsto \mathcal{P}_{\gamma_g(c)}/\text{conj.}$
Linearisation	$DL_g(c)(\beta) = \int_{\gamma_g(c)} \beta(\dot\gamma, \dot\gamma)$	"X-ray on $\mathrm{End}(\mathcal{E})$-1-forms"

To introduce the equivalent of X-ray in the twisted case, need to introduce the resolvent.

Twisted Fourier Analysis and X-ray transform

- Consider the bundle $\pi^*\mathcal{E} \to SM$ and the pullback connection $\pi^*\nabla^{\mathcal{E}}$. Define $\mathbf{X} := \pi^*\nabla^{\mathcal{E}}_X$ and study cohomological equations: $\mathbf{X}u = f$, where $u, f \in C^\infty(SM, \mathcal{E})$.
- Any $f \in C^\infty(SM, \mathcal{E})$ can be decomposed into spherical harmonics i.e., $f = \sum_{m\geq 0} f_m$, where $f_m \in \Omega_m \otimes \mathcal{E} := \ker(\Delta^{\mathcal{E}}_v + m(m+n-2))$. Define degree $\deg(f) = m$, where m is largest so that $f_m \neq 0$.
- Twisted tensors $f \in C^\infty(M, \otimes^m_S T^*M \otimes \mathcal{E})$ can be pulled back to SM via $\pi^*_m f(x,v) := f_x(v, \dots, v) \in \mathcal{E}_x$. Extend the action of $\nabla^{\mathcal{E}}$ to symmetric tensors via the Levi-Civita connection ∇^{LC}. Result is a symmetrised operator $D^{\mathcal{E}}$ such that $\pi^*_m D^{\mathcal{E}} = \mathbf{X}\pi^*_{m-1}$.
- Remarkable property of $\mathbf{X}$ is that: $\mathbf{X} : C^\infty(M, \Omega_m \otimes \mathcal{E}) \to C^\infty(M, \Omega_{m-1} \otimes \mathcal{E}) \oplus C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})$.

- Twisted tensor tomography on Anosov manifolds:

$$\mathbf{X}u = f \text{ and } \deg(f) < \infty \overset{?}{\Longrightarrow} \deg(u) \leq \max(\deg(f) - 1, 0).$$

- Any tensor can be uniquely decomposed as $u = D^{\mathcal{E}} p + h$ into potential and solenoidal parts (so $(D^{\mathcal{E}})^* h = 0$).
- Equivalent formulation of tensor tomography: if $f = \pi_m^* g$ with $(D^{\mathcal{E}})^* g = 0$, then $g = 0$. Say that s-injectivity (solenoidally) holds.

Theorem (II. C-Lefeuvre 2021)

There is an $N > 0$ such that the following holds. For each $m \in \mathbb{N}_0$, there is an open and dense set of connections $\nabla^{\mathcal{E}}$ in C^N such that the twisted tensor tomography holds for $\deg(f) \leq m$.

Remark. We believe the analogous result for metrics is also true (work in progress).

Summary

Tensor Tomography and Pollicott-Ruelle resonances

- The resolvent $\mathbf{R}_{\pm}(z) := (\mp\mathbf{X} - z)^{-1} : C^{\infty}(SM, \mathcal{E}) \to \mathcal{D}'(SM, \mathcal{E})$ admits meromorphic continuation to $\mathbb{C}$, with poles PR resonances. (Faure-Sjöstrand, Dyatlov-Zworski, Butterley-Liverani, Baladi, etc).
- Since 0 a simple pole, expand $\mathbf{R}_{\pm}(z) = -\mathbf{R}_{\pm,0} - \frac{\Pi_{\pm}}{z} + \mathcal{O}(z)$ close to zero, where $\Pi_{+} = \Pi_{-}$ are spectral projectors to $\ker \mathbf{X} \subset C^{\infty}(SM, \mathcal{E})$.
- Define π_{m*} dually to π_m^*. Following Guillarmou [2017], we define the generalized X-ray transform $\Pi_m^{\nabla^{\mathcal{E}}} := \pi_{m*}(\underbrace{\mathbf{R}_{+,0} + \mathbf{R}_{-,0}}_{=:\mathcal{I}^{\nabla^{\mathcal{E}}} = `\int_{-\infty}^{\infty} \varphi_t^* \, dt'} + \Pi_{+})\pi_m^*$.
- Properties the twisted generalized X-ray transform:
  - $\Pi_m^{\nabla^{\mathcal{E}}} f = 0 \iff \pi_m^* f = \mathbf{X} u$ for some $u \in C^{\infty}(SM, \mathcal{E})$;
  - $\Pi_m^{\nabla^{\mathcal{E}}} \in \Psi^{-1}(M, \otimes_S^m T^*M \otimes \mathcal{E})$;
  - it is elliptic on $\ker(D^{\mathcal{E}})^*$, i.e. $\exists Q, R \in \Psi(M, \otimes_S^m T^*M \otimes \mathcal{E})$ of orders $1, -\infty$ with $Q\Pi_m^{\nabla^{\mathcal{E}}} = \pi_{\ker(D^{\mathcal{E}})^*} + R$.

- Ellipticity of $\Pi_m^{\nabla^{\mathcal{E}}} \implies \ker \Pi_m^{\nabla^{\mathcal{E}}}|_{\ker(D^{\mathcal{E}})^*}$ is finite dim. and smooth.
- Consider the linear perturbation $\nabla^{\mathcal{E}} + \tau\Gamma$ for $\Gamma \in C^\infty(M, \mathrm{End}_{\mathrm{sk}}(\mathcal{E}))$, where τ is small; denote its symmetrised derivative by D_τ and generalized transform by Π_m^τ.
- Π_m^τ is compact so we take $P_\tau := \pi_{\ker D_\tau^*} \Lambda_k \Pi_m^\tau \Lambda_k \pi_{\ker D_\tau^*}$ for some $\Lambda = \Lambda_k \in \Psi^k(M, \otimes_S^m T^*M \otimes \mathcal{E})$ self-adjoint, elliptic, diagonal principal symbol, $k > 1/2$, isomorphism on Sobolev spaces.
- $P_\tau \in \Psi^{2k-1}$ has discrete spectrum $\subset \mathbb{R}_{\geq 0}$ accumulating at ∞.
- By C-Lefeuvre [2021] $\ker \mathbf{X} = \{0\}$ is a generic condition, which we assume from now on.
- It follows $\tau \mapsto \pi_{\ker D_\tau^*} = \mathbb{1} - D_\tau(D_\tau^* D_\tau)^{-1} D_\tau^* \in \Psi^0$ is continuous and we have: for τ small enough, $\dim \ker P_\tau = \dim \ker \Pi_m^\tau$.

First and second variation

- To show $\ker \Pi_m^\tau = \{0\}$ generically need to show openness and density. Openness follows from continuity of $\nabla^{\mathcal{E}} \mapsto \Pi_m^{\nabla^{\mathcal{E}}} \in \Psi^{-1}$ (can use the same anisotropic space!).
- For density, consider $\Pi_\tau := \frac{1}{2\pi i} \oint_\gamma (z - P_\tau)^{-1}$ and $\lambda_\tau := \mathrm{Tr}(P_\tau \Pi_\tau)$. Here γ a small circle around 0.
- Since $\lambda_\tau \geq 0$ and $\lambda_0 = 0$, we have $\dot{\lambda}_{\tau=0} = 0$.
- Take L^2-ON basis $(u_i)_{i=1}^d \subset \ker(D^{\mathcal{E}})^*$ of $\ker P_0$ and write $\pi_m^* \Lambda u_i = \mathbf{X} \mathrm{v}_i$ for some $\mathrm{v}_i \in C^\infty(SM, \mathcal{E})$ (as $\Pi_m^{\nabla^{\mathcal{E}}} \Lambda u_i = 0$).
- Long and tedious computation (...) $\implies$ ugly formula:

$$\ddot{\lambda}_0 = 2 \sum_{i=1}^d \Big(\big\langle (\mathbf{R}_{+,0} + \mathbf{R}_{-,0}) \pi_1^* \Gamma \mathrm{v}_i, \pi_1^* \Gamma \mathrm{v}_i \big\rangle_{L^2}$$
$$- \big\langle \pi_m^* (\Pi_m^{\nabla^{\mathcal{E}}})^{-1} \pi_{m*} (\mathbf{R}_{+,0} + \mathbf{R}_{-,0}) \pi_1^* \Gamma \mathrm{v}_i, (\mathbf{R}_{+,0} + \mathbf{R}_{-,0}) \pi_1^* \Gamma \mathrm{v}_i \big\rangle_{L^2} \Big).$$

- Not all is lost, set $M_{\mathrm{v}}A := A \cdot \mathrm{v}$ and define

$$Q_{\mathrm{v}} := \pi_{m*}\mathcal{I}^{\nabla^{\mathcal{E}}} M_{\mathrm{v}}\pi_1^* \in \Psi^{-1}(M, T^*M \otimes \mathrm{End}(\mathcal{E}) \to \otimes_S^m T^*M \otimes \mathcal{E}),$$

$$L_{\mathrm{v}} := \pi_{1*}M_{\mathrm{v}}^*\mathcal{I}^{\nabla^{\mathcal{E}}} M_{\mathrm{v}}\pi_1^* \in \Psi^{-1}(M, T^*M \otimes \mathrm{End}(\mathcal{E}) \to T^*M \otimes \mathrm{End}(\mathcal{E})).$$

- Re-write the formula as $\ddot{\lambda}_0 = 2\sum_{i=1}^d \left\langle (L_{\mathrm{v}_i} - Q_{\mathrm{v}_i}^*\Pi_m^{-1}Q_{\mathrm{v}_i})\Gamma, \Gamma \right\rangle_{L^2}$.
- If $\xi \in S_x^*M$, identify $S_xM = \mathbb{S}^{n-1}$ and write $\mathbb{S}_\xi^{n-2} := \ker\langle \xi, \cdot \rangle \cap \mathbb{S}^{n-1}$. Use coordinates $(0,\pi) \times \mathbb{S}_\xi^{n-2} \to \mathbb{S}^{n-1}$, $(\varphi, u) \mapsto \cos(\varphi)\xi^\sharp + \sin(\varphi)u$.
- The degree m extension operator $E_\xi^m : L^2(\mathbb{S}_\xi^{n-2}, \mathcal{E}) \to L^2(\mathbb{S}^{n-1}, \mathcal{E})$ is defined by the formula $E_\xi^m f(\varphi, u) := \sin^m(\varphi) f(u)$.
- Easy to show:

$$E_{\xi_0}^m L^2(\mathbb{S}_{\xi_0}^{n-2}, \pi^*\mathcal{E}_{x_0}) = \ker(\pi_{\ker \imath_{\xi_0^\sharp}} \pi_{m*}\big|_{L^2(\mathbb{S}^{n-1})}) \oplus^\perp \pi_m^*(\otimes_S^m \ker \imath_{\xi_0^\sharp} \otimes \mathcal{E}_{x_0}).$$

- The map $P_m := \pi_m^* \pi_{\ker \imath_{\xi_0^\sharp}} [\pi_{\ker \imath_{\xi_0^\sharp}} \pi_{m*} \pi_m^* \pi_{\ker \imath_{\xi_0^\sharp}}]^{-1} \pi_{\ker \imath_{\xi_0^\sharp}} \pi_{m*} : E_{\xi_0}^m L^2(\mathbb{S}_{\xi_0}^{n-2}, \pi^* \mathcal{E}_{x_0}) \circlearrowleft$ is orthogonal projection on the second factor.
- Assume $\ddot{\lambda}_0 = 0$ for all Γ. Recall ψDO's act diagonally on Gaussian states, $\mathrm{e}_h(x_0, \xi_0) = (\pi h)^{-n/4} e^{\frac{i}{h}\xi_0 \cdot (x - x_0) - \frac{|x - x_0|^2}{2h}}$, to first order.
- Set $\mathrm{w}_i := E_{\xi_0}^m \big((\pi_1^* \Gamma \cdot \mathrm{v}_i)|_{\mathbb{S}_{\xi_0}^{n-2}} \big)$ and apply the identity for $\Gamma_h = \Re(\mathrm{e}_h(x_0, \xi_0)) \cdot \Gamma$ as $h \to 0$. Major simplification:

$$\sum_{i=1}^{d} \|\mathrm{w}_i\|^2_{L^2(\mathbb{S}^{n-1}, \mathcal{E})} = \sum_{i=1}^{d} \|P_m \mathrm{w}_i\|^2_{L^2(\mathbb{S}^{n-1}, \mathcal{E})}.$$

- Contradiction if we show $\deg(\mathrm{w}_1) \geq m + 1$ for some (x_0, ξ_0).
- Pick x_0 such that $\deg \mathrm{v}_1 \geq m + 1$. Reduces to a claim on the sphere $\mathbb{S}^{n-1}$, which (essentially) reduces to representation theory of $SO(n)$.

Sandwich operators

- Swept under the rug: pseudodifferential nature of "sandwich" operators. For P_L, P_R differential operators of order m_L and m_R:

$$\underbrace{\pi_{m_1 *} P_L \mathcal{I}_{\nabla^\varepsilon} P_R \pi^*_{m_2}}_{=:A_{P_L,P_R}} \in \Psi^{m_R+m_L-1}(M, \otimes_S^{m_2} T^*M \otimes \mathcal{E} \to \otimes_S^{m_1} T^*M \otimes \mathcal{E}).$$

- Elaborate use of stationary phase lemma $\implies$ principal symbol:

$$\langle \sigma_{A_{P_R,P_L}}(x,\xi) f, f' \rangle_{\otimes_S^{m_1} T^*M_x \otimes \mathcal{E}_x} = \frac{2\pi}{|\xi|} \int_{\mathbb{S}^{n-2}_\xi} \big\langle \sigma_{P_R}\left((x,u), \xi_{\mathbb{H}}(x,u)\right)(\pi^*_{m_2} f(u)),$$
$$\sigma_{P_L^*}\left((x,u), \xi_{\mathbb{H}}(x,u)\right)(\pi^*_{m_1} f'(u))\big\rangle_{\mathcal{E}_x} \mathrm{d}S_\xi(u).$$

Here $\xi_{\mathbb{H}}(x,u) := \xi\left(\mathrm{d}_{x,u}\pi(\cdot)\right)$. See also Thibault's thesis.

Summary

We propose a few interesting open questions:

- Are weakly conjugate connections also strongly conjugate? (Partial answer available in our paper, if the weak conjugacies do not oscillate too much.)
- Are vector bundles with weakly conjugate connections necessarily isomorphic?
- Are there transparent Levi-Civita connections? (work in progress)

Thank you for your attention!